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ABSTRACT

Essays for Equilibrium Implementation in Monetary Models and International Trade

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We tackle two important theoretical problems in macroeconomics and international economics. First, in macroeconomics, especially monetary economics, the models with a standard Taylor rule have multiple equilibria. This multiplicity is problematic since we do not have a theory to determine a price level. We propose a theory to pin down a price level, and discuss how our policy selects a unique equilibrium.

Second, we propose a unified framework which nests quantitative gravity models used in international trade. The unified approach allows us to derive counterfactual predictions independent of micro-foundations. We characterize our framework based on two key parameters, estimate them, and quantify the cost of trade war between China and US.

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Table of Contents

ABSTRACT 3

Acknowledgements 4

Table of Contents 5

List of Tables 7

List of Figures 8

Chapter 1. Deviant Behavior in Monetary Economics 10

1.1. Introduction 11

1.2. Model With Taylor Rule and No Escape Clause 17

1.3. Model With Escape Clause 24

1.4. Unique Implementation of the Desired Equilibrium Without Blowing up the World 27

1.5. Where did Cochrane (2011) Go Wrong? 39

1.6. The Atkeson et al. (2010) Argument 40

1.7. Conclusion 44

1.8. Proofs 44

1.9. Arbitrary history 47

Chapter 2. A Mathematical Background and A Sophisticated Equilibrium Approach 52

2.1. Introduction 52

2.2. Households 53

2.3. Stochastic Standard Equilibrium 66

2.4. Setting for Sophisticated Equilibrium 73

2.5. Sophisticated Equilibrium 91

2.6. Implementation by the Taylor rule with Escape Clause 98

2.7. Conclusion 114

Chapter 3. Universal Gravity 116

3.1. Introduction 116

3.2. A universal gravity framework 124

3.3. Existence, uniqueness, and interiority of equilibria 135

3.4. The network effects of a trade shock 139

3.5. Estimating the gravity constants 144

3.6. The impact of a U.S.-China trade war 154

3.7. Conclusion 157

3.8. Tables and Figures 158

3.9. Proofs 162

3.10. Additional materials 168

References 220

List of Tables

3.8.1 Examples of models in the universal gravity framework 158

3.8.2 Estimating the gravity constants 159

List of Figures

1.2.1 Fisher Equation and Taylor Rule 23

2.4.1 Illustration of fixed point 87

2.5.1 Unique Implementation 96

3.8.1 Existence and uniqueness 160

3.8.2 The network effect of a U.S.-China trade war 161

3.10.1 Examples of multiplicity and uniqueness in two locations 211

3.10.2 Correlation between observed income and own expenditure shares and the equilibrium values from the gravity model 212

3.10.3 The network effect of a U.S.-China trade war: Degree 0 213

3.10.4 The network effect of a U.S.-China trade war: Degree 1 214

3.10.5 The network effect of a U.S.-China trade war: Degree 2 215

3.10.6 The network effect of a U.S.-China trade war: Degrees >2 216

3.10.7 The network effect of a U.S.-China trade war: Total effect 217

3.10.8 Local versus global effects of a U.S.-China trade war 218

3.10.9 The effect of a U.S.-China trade war on real output prices in the U.S. and China: Robustness 218

3.10.10 Excess non-monotonic demand function for 1, $Z_1(p_2)$ 219

CHAPTER 1

Deviant Behavior in Monetary Economics

with Lawrence Christiano[1]

We consider a model in which monetary policy is governed by a Taylor rule. The model has a unique equilibrium near the steady state, but also has other equilibria. By using an equilibrium concept from Bassetto (2005) and Atkeson et al. (2010), we show how incorporating an escape clause rules out the other equilibria. It does so by a mechanism that resembles the intuition underlying the Taylor Principle. For example, an exploding inflation equilibrium is ruled out because everyone understands that *if it did happen*, the escape clause would be activated and the government would engineer a Volcker-style recession by raising the real interest rate. That would lead to a reduction in marginal cost and cause price setters to post low prices. Knowing this, no one expects high inflation and it therefore cannot happen. We reconcile our finding about how the escape clause excludes equilibria with the very different conclusion reached in Cochrane (2011). Atkeson et al. (2010) study a different version of the escape clause policy. They use that policy to argue that the Taylor Principle is not necessary for equilibrium uniqueness, a conclusion that is misleading in light of the observations above. Moreover, we show that their version of the escape clause policy is fragile in that it lacks a crucial robustness property.

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1.1. Introduction

Monetary models are notorious for having multiple equilibria. The standard New Keynesian model, which assumes that fiscal policy is passive and monetary policy is set by a Taylor rule is no exception. By a Taylor rule, we mean an interest rate rule that satisfies the Taylor Principle (i.e., has a big coefficient on inflation). An influential sequence of papers shows that such monetary models have two steady states. In simpler models that allow for an analytic characterization of the global set of equilibria, it is found that there are deflation equilibria, hyperinflation equilibria, equilibria in which inflation exhibits cycles and even chaos (see Benhabib, et al. 2001b; 2001a; 2002a; 2002b) (BSGU).

The literature on the standard New Keynesian model has generally ignored the equilibrium multiplicity issue by focusing on the unique local-to-steady-state equilibrium. We refer to this equilibrium as the *desired equilibrium* because in many models that equilibrium is first best or nearly so. Critics correctly argue that until the multiplicity issue is resolved, the New Keynesian model cannot be thought of as a model that determines the price level or anything else.[2] We study one proposal for dealing with the multiplicity issue.[3]

The proposal that we consider, which was suggested by Benhabib et al. (2002a) and Christiano and Rostagno (2001), is studied in a particular model.[4] Ideally, that model would be the New Keynesian model, but a global analysis of equilibrium in that model appears to be infeasible at this time. We require a model which has the flavor of the New Keynesian model, but is tractable. In addition, our conclusions contradict those reached

[2]See, for example, Cochrane (2011).

[3]Perhaps the best known approach for addressing multiplicity is one based on learning. See Christiano et al. (2017) and the extensive literature they cite.

[4]Both are related to the work of Obstfeld and Rogoff (1983) (see Obstfeld and Rogoff (2017))

in Cochrane (2011) and Atkeson et al. (2010). So, to highlight the reason for the difference in results, we work with a model that is comparable to theirs.

The proposal that we study is motivated by the observation in our model that the undesired equilibria associated with a Taylor rule in effect require the complicity of the government. For example, the exploding hyperinflation equilibria that are possible under the Taylor rule can only happen if it is accommodated by exploding money growth. This observation suggests a simple modification to the Taylor rule: follow that rule as long as the economy remains in a neighborhood of the desired equilibrium and implement an *escape clause* in the event that a non-desired equilibrium appears to form. The escape clause could specify that if inflation moves outside a particular monitoring range, then the government deviates from the Taylor rule in favor of some other policy that directly moves inflation back into the monitoring range. The policy to which the government deviates under the escape clause might not be optimal in normal times, but the mere existence of the escape clause prevents undesired equilibria from forming in the first place. A policy in which the government has one set of rules in normal times, but is prepared to deviate to another set of rules under exigent circumstances is not unprecedented. For example, many governments which respect individual liberty in normal times have the authority to impose martial law, an entirely different regime, in the event of extreme disorder.[5]

Our discussion pushes back against the conclusions reached in Cochrane (2011) and in Atkeson et al. (2010) about the type of escape clause policy considered here. Cochrane

[5]Other examples include the 'unusual and exigent circumstances' clause, section 13.3, in the Federal Reserve Act (see `https://www.federalreserve.gov/aboutthefed/section13.htm`) used to rationalize the use of unconventional monetary policy in the wake of the 2008 Financial Crisis. Another example is the exigent circumstances under which the Fourth Amendment to the US Constitution's prohibition against a Warrantless search and seizure may be ignored (see `https://www.law.cornell.edu/wex/fourth_amendment`).

(2011) agrees that the escape clause rules out the non-desired equilibria. However, he argues that it does so by a government commitment to do something infeasible (*blow up the world*) in case the undesired allocations occur. Cochrane (2011) argues, reasonably, that a policy that rules out equilibria by a blow-up-the-world threat is not economically interesting. This is because it is hard to imagine an actual government making such a commitment or private agents believing it. A problem with Cochrane (2011)'s argument is that he makes it within the framework of a standard equilibrium concept. That concept does not allow for off-equilibrium events such as the non-desired allocations. So, it is silent about the economic reason that such allocations are not chosen in equilibrium. We follow Bassetto (2005) and Atkeson et al. (2010) by defining the concept of a *strategy equilibrium,* which makes it possible to study how it is that the escape clause excludes non-desired allocations.[6]

The strategy equilibrium extends the standard concept of equilibrium by opening up off-equilibrium paths at each date. The question of why non-desired allocations are not equilibria is answered by asking the agents what it is about the escape clause that encourages them not to leave the equilibrium path. Following Atkeson et al. (2010), we construct off-equilibrium paths by adopting the Dixit-Stiglitz production framework. Each intermediate good price setter chooses its price simultaneously and without coordinating. To decide what price to set, a price setter must form a conjecture about what prices the other agents choose and they must contemplate the associated continuation equilibrium for the economy. Thus, we can think of agents' choice of price as their best response to what

[6] Our equilibrium concept coincides with the 'sophisticated equilibrium' concept in Atkeson et al. (2010). We give ours a different name because the objects in our equilibrium are sequences rather than functions. Our choice of equilibrium concept is practical in our setting because we generally work with non-stochastic versions of our model. By working with sequences, we are able to minimize notation.

others do. In a competitive equilibrium, they select a belief about what others do that corresponds to the fixed point of this best response function. For this view about belief formation to be interesting, we require that for each conjecture about what others do, there is a well defined continuation equilibrium. In addition, we require that there exists a fixed point. If either of these two requirements are not satisfied, then the problem of forming a belief about inflation is not well defined and we say that a strategy equilibrium does not exist. Note that even though the individual agents are atomistic, the way they arrive at their belief about current inflation requires strategic thinking. Hence, the reason for the name of our equilibrium concept.

The escape clause rules out inflation above the monitoring range because it is not a fixed point of the best response function. A firm that conjectures high inflation understands that the government will respond by raising the nominal interest rate sharply (that reflects the Taylor Principle) and lowering future inflation (that reflects the switch to low money growth). The resulting high real interest rate produces a recession in the model, which reduces marginal cost. Firms' best response is to post low prices, so that high inflation does not occur. In short, with the escape clause the government asserts that it is ready to engineer a Volcker-type recession in case inflation is high, and the private economy responds by setting prices so that high inflation does not happen. No blowing up the world threats are involved.

Why do we reach a conclusion so different from Cochrane (2011)'s? The answer lies in his assumption of an endowment economy. That assumption cuts the heart out of the mechanism by which the escape clause does its work in our model. The fall in output and rise in the real interest rate that occurs in our model is impossible in an endowment economy. We reproduce Cochrane (2011)'s blow-up-the-world result in the endowment economy version of our model by showing that continuation equilibria do not exist for non-fixed point inflation conjectures (that is, that economy does not have a strategy equilibrium). So, we agree that the escape clause works by a blow-up-the-world threat in Cochrane (2011)'s model. But, his result does not generalize to a production economy. We conclude that Cochrane (2011)'s analysis is misleading for thinking about the issues addressed here.

We then consider Atkeson et al. (2010). Their primary economic conclusion is that a key tenet of the New Keynesian canon - that the Taylor principle is a necessary ingredient of good monetary policy - is false. They endorse the escape clause, but propose shrinking the monitoring range for inflation to a singleton that only includes the desired inflation rate. The desired equilibrium is indeed uniquely implemented by their policy and, as they emphasize, the size of the coefficient on inflation in the Taylor rule plays no role in ensuring that inflation remains at its desired level. We make two observations on Atkeson et al. (2010). First, the way that the escape clause works is very much in the spirit of the Taylor Principle. The idea behind the Taylor Principle is that with the big coefficient on inflation, a rise in inflation produces a rise in the real interest rate and, by slowing down the economy, that brings inflation back down to its desired level (see Taylor (1999, p.325)'s discussion of 'leaning against the wind'). That stabilizing force works reasonably well in a New Keynesian model in a neighborhood of the model steady state, but it does not exclude equilibria with inflation far from steady state. As explained above, the escape clause excludes those equilibria by a mechanism very similar to the way the

Taylor Principle works near steady state.[7] So, Atkeson et al. (2010)'s conclusion that the Taylor Principle is not necessary for unique implementation of the close-to-steady-state equilibrium is misleading, at least when viewed through the eyes of our model.[8]

Second, we show that the Atkeson et al. (2010) analysis is fragile. If a vanishingly small number of price setters make vanishingly small mistakes (i.e., trembles) the monitoring range would be violated by accident and not because an undesired equilibrium is forming. Yet, the Atkeson et al. (2010) policy would respond by shifting to the money growth regime. This regime-shift could induce a substantial drop in welfare if there are shocks to money demand.

Atkeson et al. (2010) report that the good performance of their monetary policy *is* robust to trembles, in contrast to the conclusion reached here. Their conclusion reflects that they linearize the map from individual intermediate good prices to the aggregate price index (see Atkeson et al. (2010, p. 53)). By the law of large numbers, zero-mean trembles

[7]How the Taylor Principle works near steady state is well understood. We briefly review that here for completeness. Consider the standard New Keynesian model without capital, linearized around the first-best equilibrium. The IS curve is $x_t = E_t x_{t+1} - [r_t - E_t \pi_{t+1} - r_t^*]$, where x_t denotes the log difference between output in the equilibrium with the Taylor rule and first-best output. The Taylor rule is $r_t = \phi \pi_t$, where $\phi > 1$. The Phillips curve is $\pi_t = \beta E_t \pi_{t+1} + \kappa x_t$, where $\kappa > 0$ and $\beta \in (0, 1)$. Finally, r_t^* denotes the natural rate of interest, $r_t^* = E_t a_{t+1} - a_t$ and a_t denotes a shock to technology. Suppose that the shock has the representation, $\Delta a_t = \rho \Delta a_{t-1} + \varepsilon_t$, where $\Delta a_t = a_t - a_{t-1}$, $\rho \in [0, 1)$ and ε_t is an *iid* shock. It is easy to verify that the locally unique equilibrium has the form:

$$r_t - E_t \pi_{t+1} = \psi \Delta a_t, \quad x_t = \frac{(1-\beta\rho)}{\kappa(\phi-\rho)} \psi \Delta a_t, \quad \pi_t = \frac{\psi}{\phi - \rho} \Delta a_t$$

where $\psi \equiv \rho \left[\frac{(1-\beta\rho)(1-\rho)}{\kappa(\phi-\rho)} + 1 \right]^{-1}$. For ϕ sufficiently large, ψ is close to unity and $r_t - E_t \pi_{t+1} \simeq r_t^*$, $\pi_t \simeq 0$ and $x_t \simeq 0$. So, a big value of ϕ stabilizes the equilibrium around first best and this is accomplished by the operation of the Taylor principle (i.e., the real rate of interest increases when inflation is high because $\psi > 0$ and $\phi > \rho$). It is easy to verify that this result also holds when the technology process is replaced by $a_t = \rho a_{t-1} + \varepsilon_t$ or when the shock is instead a stationary disturbance to labor supply.

[8]We do not mean to suggest that the Taylor principle is desirable in *all* possible models. For example, in Christiano et al. (2010b) and Christiano (2016) it is shown that if the working capital channel is strong enough, then the Taylor principle could be destabilizing. Christiano et al. (2010a) explain how the Taylor principle could inadvertently trigger a stock market boom in response to news about the future.

are wiped out in that linear representation. But, our analysis works with the actual nonlinear mapping, in which trembles matter. This is why we conclude that Atkeson et al. (2010)'s finding that their policy uniquely supports the desired equilibrium is not robust to trembles. Our policy, which adopts a wide monitoring range for inflation and the Taylor principle, uniquely implements the desired equilibrium and is robust to trembles.

Section 1.2 sets up our model economy and shows that a model with a Taylor rule has a continuum of equilibria. We include this well-known result here for convenience.[9] Section 1.3 shows how the introduction of a monitoring range for inflation and an escape clause renders the equilibrium unique. We explain Cochrane (2011)'s blowing-up-the-world critique of this uniqueness result. Section 1.4 defines a strategy equilibrium. We establish that our model has a strategy equilibrium, and that the desired equilibrium is uniquely implemented by the escape clause. We describe the key steps in the proof, but we move details to the Appendix. Section 1.5 reconciles our findings about the escape clause with Cochrane (2011)'s critique. Section 1.6 addresses Atkeson et al. (2010)'s conclusion that the Taylor Principle is not necessary to robustly and uniquely implement the desired equilibrium. We also explain the lack of robustness of their version of the escape clause to trembles. Finally, section 1.7 offers a brief conclusion.

1.2. Model With Taylor Rule and No Escape Clause

The model we work with is in some ways standard. We use the household preferences and Dixit-Stiglitz production structure used in the New Keynesian literature. For now, we assume flexible prices, though in later sections we adopt a form of price stickiness. For the purpose of our analysis, it is necessary to be explicit about the demand for money.

[9] See BSGU, as well as Woodford (2003).

We select our model for tractability and to maximize comparability with the models in Cochrane (2011) and Atkeson et al. (2010).

The government provides monetary transfers to households:

$$(\bar{\mu}_t - 1)\, \bar{M}_{t-1}, \quad \bar{\mu}_t = \bar{M}_t / \bar{M}_{t-1}, \tag{1.1}$$

where $\bar{M}_t$ denotes the end-of-period-stock of money and $\bar{\mu}_t$ denotes the money growth rate, a variable controlled directly by the government. The government levies sufficient lump sum taxes that, given the money growth rate, the government's budget is balanced in each period.[10]

Monetary policy policy selects a sequence, $\{\bar{\mu}_t\}_{t=0}^{\infty}$, so that, in equilibrium,

$$\bar{R}_t = \max\left\{1, \bar{R}^* \left(\frac{\bar{\pi}_t}{\bar{\pi}^*}\right)^{\phi}\right\}, \quad \bar{\pi}_{t+1} \equiv \frac{P_{t+1}}{P_t}, \quad \bar{R}^* \equiv \bar{\pi}^*/\beta, \tag{1.2}$$

where $\bar{\pi}^* \geq 1$ and $\bar{R}^*$ are the desired inflation and interest rate. Here, we assume that $\phi > 1$. Finally, $\beta \in (0, 1)$, denotes the representative household's discount rate.

The representative household's problem is:

$$\max_{\{c_t, l_t, m_t, b_t\}_{t=0}^{\infty}} \quad \sum_{t=0}^{\infty} \beta^t \left[\frac{c_t^{1-\gamma}}{1-\gamma} - \frac{l_t^{1+\psi}}{1+\psi}\right], \quad \gamma > 1, \psi > 0 \tag{1.3}$$

$$s.t. \quad m_t + b_t \leq W_t l_t + m_{t-1} - P_{t-1} c_{t-1} + \bar{R}_{t-1} b_{t-1} + T_t$$

$$P_t c_t \leq m_t$$

$$(m_{-1} - P_{-1} c_{-1} + R_{-1} b_{-1}) \text{ given.}$$

[10] A number of interesting issues concerning fiscal policy are left out of the analysis. For example, a property of our model is that non-negative money growth rate rules out a zero interest rate equilibrium. In the presence of government debt, this result is not necessarily true. For further discussion and a defense of the position taken here, see Christiano and Rostagno (2001, section 2.4).

Here, c_t and l_t denote consumption and employment; and m_t and b_t denote the household's end-of-period-t stock of money and one-period bonds. Note that the household has a cash constraint which requires that its end-of-period t cash balances be sufficient to cover its period t consumption expenditures. The term on the left of the equality in the first constraint on the household problem describes the allocation of the household's end-of-period financial resources between cash and bonds. The household sources of financial resources are: cash accumulated by working, $W_t l_t$, excess cash carried over from the previous period, $m_{t-1} - P_{t-1} c_{t-1}$, interest earned on the previous period's bond holdings, $\bar{R}_{t-1} b_{t-1}$, and lump sum transfers from taxes, money transfers and profits, T_t. The second constraint is the household cash constraint.

The first order necessary and sufficient conditions for household optimization are

$$\frac{W_t}{P_t} = c_t^{\gamma} l_t^{\psi}, \tag{1.4}$$

$$c_t^{-\gamma} = \beta c_{t+1}^{-\gamma} \frac{\bar{R}_t}{\bar{\pi}_{t+1}}, \tag{1.5}$$

$$0 = (R_t - 1)\left[m_t - P_t c_t\right], \quad m_t \geq P_t c_t, \tag{1.6}$$

$$m_t + b_t = W_t l_t + m_{t-1} - P_{t-1} c_{t-1} + \bar{R}_{t-1} b_{t-1} + T_t, \tag{1.7}$$

$$0 = \lim_{T \to \infty} \beta^T u'(c_T) \frac{m_T - P_T c_T + b_T}{P_T}. \tag{1.8}$$

Sufficiency and necessity of these conditions, as well as assumptions required to ensure boundedness of the household's intertemporal consumption opportunity set, are established in Christiano and Takahashi (2018).

A final output good is produced by a competitive, representative firm using the following production function:

$$Y_t = \left[\int_0^1 Y_{i,t}^{\frac{\varepsilon-1}{\varepsilon}} di \right]^{\frac{\varepsilon}{\varepsilon-1}}, \quad \varepsilon > 1 \tag{1.9}$$

The firm takes the price output, P_t, and the prices of the inputs, $p_{i,t}, \ i \in [0,1]$ as given. The first order conditions associated with its profit maximization is:

$$Y_{i,t} = Y_t \left(\frac{p_{i,t}}{P_t} \right)^{-\varepsilon}, \tag{1.10}$$

for $i \in [0,1]$. The first order conditions, together with the production function, impose a restriction across the aggregate price index and the price of intermediate goods:

$$P_t = \left[\int_0^1 p_{i,t}^{1-\varepsilon} di \right]^{\frac{1}{1-\varepsilon}}. \tag{1.11}$$

The i^{th} intermediate good, $Y_{i,t}$, is produced by a monopolist with the following production function:

$$Y_{i,t} = l_{i,t},$$

Here, $l_{i,t}$ denotes the labor input employed by the i^{th} firm. In the usual way, the i^{th} firm sets its price as a markup over its marginal cost, W_t :

$$p_{i,t} = \frac{\varepsilon}{\varepsilon - 1} (1 - \tau) W_t = W_t, \tag{1.12}$$

where τ denotes a government subsidy, which we assume cancels the markup. With all intermediate good firms setting the same price, we have that

$$P_t = W_t. \tag{1.13}$$

The goods, labor, money and bond market clearing conditions are:

$$c_t = Y_t, \quad \int_0^1 l_{i,t} = l_t, \quad \bar{M}_t = m_t, \quad b_t = 0. \tag{1.14}$$

Let

$$\bar{a}_t = \left(l_t, \{l_{i,t}\}, \{p_{i,t}\}, c_t, \bar{\pi}_t, \bar{R}_t, W_t, \bar{\mu}_t, \bar{M}_t, m_t, b_t\right). \tag{1.15}$$

Collecting the equilibrium conditions, we now define a monopolistically competitive equilibrium. We simplify the name to simply 'competitive equilibrium':

Definition 1. A *competitive equilibrium under the Taylor rule* is a sequence, $(\bar{a}_t)_{t=0}^{\infty}$, that satisfies, for $t \geq 0$, (i) intermediate good firm optimality, (1.12); (ii) final good firm optimality, (1.9)-(1.11); (iii) household optimization, (1.4)-(1.8), conditional on $m_{-1} - P_{-1}c_{-1} + R_{-1}b_{-1}$, P_{-1}; (iv) government policy, (1.1)-(1.2), and (v) market clearing, (1.14).

We define competitive equilibria under other monetary policy rules by suitable adjustment to condition (iv).

We now obtain a dynamic equation that can be used to identify all the equilibria in our model. We constructed our model so the equation that that equation is identical to the equation that characterizes the equilibria in Cochrane (2011)'s model. We will exploit this fact below. The similarity is completed by scaling and the logging the variables.

Combining (1.13) and (1.14), we obtain:

$$1 = c_t^{\gamma} l_t^{\psi} = c_t^{\psi+\gamma},$$

so that for all $t \geq 0$,

$$c_t = 1. \tag{1.16}$$

As a result, the intertemporal Euler equation reduces to the *Fisher equation*:

$$R_t = \pi_{t+1}, \tag{1.17}$$

where $R_t = \ln\left(\bar{R}_t/\bar{R}^*\right)$, $\pi_{t+1} = \ln\left(\bar{\pi}_t/\bar{\mu}^*\right)$.We can also express the monetary policy rule in scaled and logged form:

$$R_t = \max\left\{R^l, \phi\pi_t\right\}, \tag{1.18}$$

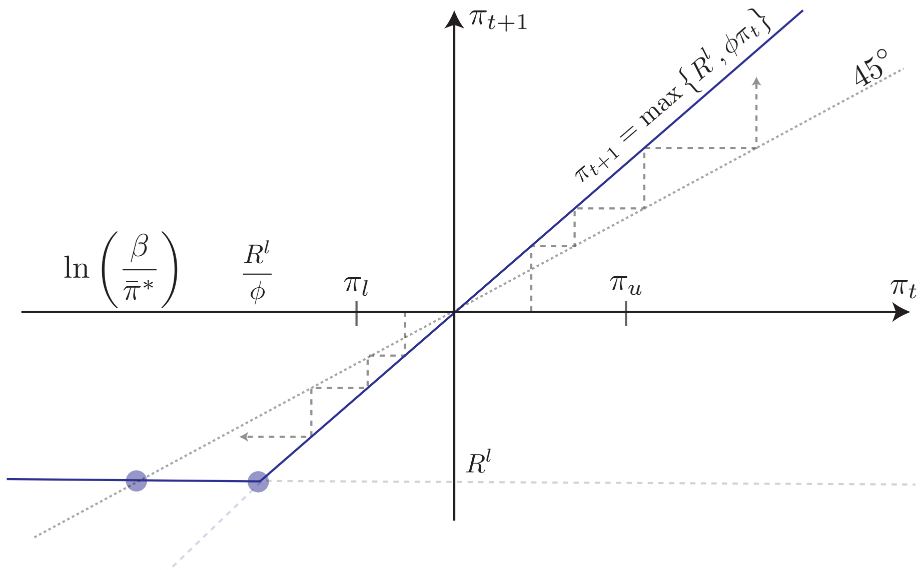
where $R^l \equiv \ln\left(1/\overline{R}^*\right)$. It is also useful to define the scaled and logged money growth rate:

$$\mu_t \equiv \ln\left(\frac{\bar{\mu}_t}{\bar{\mu}^*}\right). \tag{1.19}$$

Combining (1.17) and (1.18) we obtain:

$$\pi_{t+1} = \max\left\{R^l, \phi\pi_t\right\}. \tag{1.20}$$

Figure 1.2.1. Fisher Equation and Taylor Rule



Also, the transversality condition, taking into account (1.14) and (1.16), corresponds to:

$$\lim_{T\to\infty} \beta^T \frac{M_T}{P_T} = 0. \tag{1.21}$$

Equation (1.20) is useful for studying the equilibria in the model for the following reason:

Proposition 1. *For any sequence, $(\pi_t)_{t=0}^{\infty}$ that satisfies the difference equation, (1.20), it is possible to construct the other equilibrium variables in such a way that all the conditions for a competitive equilibrium are satisfied.*

For the proof, see Appendix 1.8.1.

Figure 1.2.1, a variant on the well-known figure in Benhabib et al. (2001b, Fig. 1), graphs (1.20).[11] From this figure it is easy to see that the model has many equilibria, each indexed by the value of π_0. The *desired equilibrium* corresponds to $\pi_t = 0$ for $t \geq 0$. Consider the two inflation rates marked in Figure 1.2.1 by π_l and π_u. We refer to this interval as the *inflation monitoring range*. Note that, due to the high value of ϕ, there is exactly one equilibrium in which inflation is always in the monitoring range, $\pi_t \in [\pi_l, \pi_u]$ for $t \geq 0$. The allocations in that equilibrium correspond to the desired equilibrium. This observation plays an important role in the analysis below.

The economic interpretation of (1.20) plays an important role in our analysis. According to Woodford (2003, p. 128), "...the equation indicates how the equilibrium inflation rate in period t is determined by expectations regarding inflation in the following period." Thus, an exploding inflation equilibrium is caused by high expected inflation. So, to exclude such an equilibrium policy must discourage agents from expecting high inflation.

1.3. Model With Escape Clause

The fact that the unique equilibrium with $\pi_t \in [\pi_l, \pi_u]$ is the desired equilibrium for all $t \geq 0$ is an important motivation for the following policy:

Definition 2. *Taylor rule with an escape clause*: if $\pi_s \in [\pi_l, \pi_u]$, for $s = 0, ..., t-1$ follow the Taylor rule, (1.18), with $\phi > 1$ in period t. Otherwise, set (scaled) money growth, μ_t, to a constant, $\mu \in [\mu_l, \mu_u]$. Here, $\mu_u = \pi_u$ and $\mu_l = \max\left\{\pi_l, \log \frac{1}{\bar{\mu}^*}\right\}$. Also,

$$\phi^{-1} R^l < \pi_l \leq 0 \leq \pi_u < \infty.$$

[11] For an extended discussion see, for example, Woodford (2003, Chapter 2, section 4, p. 123). This figure plays an important role throughout our analysis and so we include it here for completeness.

The lower bound on the range, $[\mu_l, \mu_u]$, is designed to guarantee that unscaled money growth under the escape clause, $\bar{\mu}$, is no less than unity. This helps to ensure uniqueness of equilibrium under constant money growth.[12] The lower bound on the monitoring range excludes inflation rates that imply the zero lower bound on the interest rate is binding. This assumption is simply made for analytic convenience.

A competitive equilibrium under a Taylor rule with an escape clause is unique and it is the desired equilibrium. This result is established in three steps. First, we establish that the constant money growth rate under the escape clause has a unique equilibrium:

Lemma 1. *Suppose $\bar{M}_{-1}$ is given and monetary policy sets $\bar{M}_t = \bar{\mu}\bar{M}_{t-1}$ for $t \geq 0$, where $\bar{\mu} \geq 1$. There exists a unique competitive equilibrium with the properties:*

$$\bar{R}_t = \beta^{-1}\bar{\mu} > 1, \quad c_t = 1, \quad \bar{\pi}_{t+1} = \bar{\mu}, \quad \textit{for} \quad t \geq 0, \quad \textit{and} \quad P_0 = \bar{M}_{-1}\bar{\mu}.$$

For the proof, see Appendix 1.8.2.

If Lemma 1 were not true and an equilibrium did not exist, it would be impossible to meaningfully ask what would happen if $\pi_t \notin [\pi_l, \pi_u]$, because agents would not know how to form expectations about $t+1$.[13] The second step shows that there is no competitive equilibrium in which $\pi_t \notin [\pi_l, \pi_u]$:

Lemma 2. *Consider the case in which monetary policy is the Taylor rule with the escape clause. An equilibrium has the following property: $\pi_t \in [\pi_l, \pi_u]$ for $t \geq 0$.*

[12] To see how multiplicity of equilibria can arise when money growth lies between β and unity, see Albanesi et al. (2003b, Proposition 4).

[13] For our analysis it is convenient that we have a unique equilibrium under constant money growth, but we have not investigated whether uniqueness is necessary for our conclusions.

This Lemma lies at the heart of our equilibrium uniqueness result. So, we include the proof here:

Proof. Suppose not, and that there exists an equilibrium with $\pi_T \notin [\pi_l, \pi_u]$ where T is the first date in which the monitoring range is violated. Consider the case, $\pi_T > \pi_u$. The Taylor rule implies $R_T = \phi\pi_T > \pi_u$. The escape clause and Lemma 1 imply $\pi_{T+1} = \mu \leq \pi_u$, so that

$$R_T - \pi_{T+1} > 0, \tag{1.22}$$

violating the Fisher equation, (1.17). This contradicts the assumption of equilibrium.

Next, consider the case $\pi_T < \pi_l$. Then, $R_T = \max\left\{R^l, \phi\pi_T\right\} \leq 0$. But, $R^l < \phi\pi_l < \pi_l$, so $R_T < \pi_l$. Also, $\pi_{T+1} = \mu \geq \pi_l$, so that

$$R_T - \pi_{T+1} < 0, \tag{1.23}$$

violating the Fisher equation, (1.17). This contradicts the assumption of equilibrium. This establishes the result sought. □

The third step is the main proposition:

Proposition 2. *Suppose monetary policy is governed by the Taylor rule with an escape clause. The only equilibrium is the desired equilibrium.*

The result is not surprising, given Lemma 2 and Figure 1.2.1. The former says that there is no equilibrium with $\pi_t \notin [\pi_l, \pi_u]$. The latter indicates that the only equilibrium with $\pi_t \in [\pi_l, \pi_u]$ is $\pi_t = 0$ for all t. For a formal proof, see Appendix 1.8.3.

Cochrane (2011) maintains that the uniqueness result in Proposition 2 is correct, but uninteresting in an economic sense. As we noted at the end of Section 1.2, the non-desired equilibria in the model are caused by high or low expectations about future inflation, so that to exclude those equilibria requires preventing such expectations from being formed. According to Cochrane (2011) the escape clause prevents such expectations by committing to do something infeasible (i.e., violate $R_T - \pi_{T+1} = 0$) in case those expectations are realized. We agree that a model in which policy works in this way is not interesting. Actual agents would presumably not believe government commitments to do infeasible things. Moreover, it is hard to imagine any government announcing such a policy in the first place.

1.4. Unique Implementation of the Desired Equilibrium Without Blowing up the World

In this section we describe an alternative concept of equilibrium which allows us to think about off-equilibrium events. The question of why non-desired allocations are ruled out by the escape clause is by definition a question about what happens off equilibrium. The conditions that hold in equilibrium shed no light on this question.

The alternative concept of equilibrium that we pursue here was first advocated in macroeconomics by Bassetto (2005). Bassetto (2005)'s framework applies in a situation like ours, in which the government has full commitment to implement its policy rule and we want to understand the economic reason that that policy rule excludes equilibria. Atkeson et al. (2010) also develop this approach by exploiting conceptual similarities between the framework used here and the framework developed in Chari and Kehoe (1990) to think about equilibrium when the government lacks the ability to commit.

To integrate off-equilibrium allocations into the analysis, we must, in effect, build off-ramps from the equilibrium path. In principle, that could be done in a variety of ways. For the purpose of comparison, we adopt the approach in Atkeson et al. (2010). This involves taking a closer look at the intermediate good firms' pricing decision. We do this in the first subsection below. That section also discusses the formation of firm beliefs, something that plays a central role in the analysis. Section 1.4.3 discusses the definition of a strategy equilibrium. The last two sections discuss how high and moderate inflation equilibria are ruled out in a strategy equilibrium of our model. Other cases are studied in the Appendix.

1.4.1. Sequence of Events During the Period

In our discussion of the model in Section 1.2 we assumed that the i^{th} firm knows the wage, W_t, at the time it sets its price, $p_{i,t}$ (see (1.12)). But, this assumption is problematic. The nominal rate rate, W_t, is determined in markets *simultaneously* with other variables like P_t. It is logically impossible for the i^{th} firm to actually observe P_t when it sets its own price, $p_{i,t}$. The reason is that by (1.11), P_t is the consequence of the prices set by intermediate good firms. Obviously, firms cannot see the consequences of their actions until *after* their actions have been taken.

Motivated by the preceding discussion, we assume that $p_{i,t}$ is set *before* W_t is realized. To capture this observation, we divide each period t into two parts: morning and afternoon. Intermediate good firms set their prices, $\{p_{i,t}\}_{i\in[0,1]}$, in the morning. They do so simultaneously and without knowing what the other intermediate good firms are doing. In the afternoon, the intermediate good firm prices are taken as given and are part of the state of the economy. Conditional on this state, a *period t continuation equilibrium* unfolds. We provide a formal definition of this equilibrium concept in the next subsection. This is a sequence of markets equilibrium that starts in the afternoon of period t and continues into all future periods.

1.4.2. Best Response Function, Fixed Point and Beliefs

In the morning of period t, the i^{th} intermediate good firm sets its price, $p_{i,t}$, as follows:

$$p_{i,t} = W_{i,t}^{e}. \tag{1.24}$$

Here, the superscript, e, indicates a firm's conjecture about the value that a variable (in this case, $W_{i,t}$) will take on in the afternoon, as part of the period t continuation equilibrium. As noted before, the economic state in this continuation equilibrium includes, among other things, the prices set by all the intermediate good firms, $\{p_{j,t}\}_{j\in[0,1]}$. Thus, to set its price, each firm must first form a conjecture about what other firms do.

We simplify the situation by making two assumptions about how the i^{th} firm forms a conjecture about each $\{p_{j,t}\}_{j\neq i}$. The first assumption is *symmetry:* the i^{th} firm assumes $p_{j,t} = p_{j',t} = p_{i,t}^{e}$ for all $j, j' \in [0,1]$. The second assumption is *consensus*: all firms form the same conjecture, so that $p_{i,t}^{e} = p_{j,t}^{e} = p_{t}^{e}$ for all i, j. Everyone knows that everyone else thinks in the same way (i.e., common knowledge).

Given the conjecture of the intermediate good prices, p_{t}^{e}, firms' conjecture about the aggregate price index is just p_{t}^{e} itself (see (1.11)). So, each firm's conjecture about the nominal wage, W_t^e, is given by

$$W_t^e = p_t^e \left(\frac{W_t^e}{p_t^e} \right) = p_t^e \left(c_t^e \right)^{\gamma} \left(l_t^e \right)^{\psi} = p_t^e \left(c_t^e \right)^{\gamma+\psi} . \tag{1.25}$$

In (1.25), we have taken into account the intra-temporal equilibrium condition for labor that must hold in the afternoon, (1.4), as well as the goods market clearing condition, (1.14), and the aggregate technology, which imply $c_t = l_t$.

Substituting (1.25) into (1.24), the expression for the i^{th} firm's price decision is:

$$p_t = p_t^e \left(c_t^e \right)^{\gamma+\psi} ,$$

where the subscript, i, on p_t is deleted because each firm sets the same price. After scaling by $P_{t-1}\bar{\mu}^*$, and then taking logs:

$$x_t = \pi_t^e + (\gamma + \psi) \ln c_t^e, \tag{1.26}$$

where x_t denotes the scaled and logged value of $p_{i,t}$ in (1.24). The fact that we drop the i index reflects our consensus assumption. Also, π_t^e denotes the conjectured inflation rate relative to its desired value:

$$\pi_t^e = \ln \left(\frac{p_t^e}{P_{t-1}\bar{\mu}^*} \right) .$$

Finally, c_t^e in (1.26) is the conjectured period t afternoon consumption in the continuation equilibrium.

We now provide a formal definition of a time t continuation equilibrium. We denote a time $t-1$ history, h_{t-1}, as follows:

$$h_{t-1} = \begin{cases} \left(m_{-1} - P_{-1}c_{-1} + R_{-1}b_{-1}, P_{-1}, \bar{M}_{-1}\right) & t = 0 \\ \left(h_{t-2}, a_{t-1}\right) & t \geq 1 \end{cases},$$

where a_t is defined in (1). Past history, h_{t-1}, matters in this model in part because of the nature of monetary policy. For example, if $\pi_j \in [\pi_l, \pi_u]$ for all $j \leq t-1$, then monetary policy in period t is the Taylor rule. Otherwise, it is a constant money growth rule. We define the history at the beginning of the afternoon of period t as:

$$h_{x,t} = \left(h_{t-1}, x_t\right),$$

where x_t is the (logged and scaled) price set by a representative intermediate good firm. We assume all firms set the same price, but otherwise x_t is unrestricted in our definition of $h_{x,t}$. The object, $h_{x,t}$ is the state of the economy at the beginning of the afternoon in period t. Let the scaled and logged version of the time t variables in (1.15) be denoted by

$$a_t = \left(l_t, \{l_{i,t}\}, x_t, c_t, \pi_t, R_t, W_t, \mu_t, \bar{M}_t, m_t, b_t\right),$$

where $\{p_{i,t}\}$ has been replaced by x_t. We define:

Definition 3. A *time t continuation equilibrium conditional on $h_{x,t}$* is a sequence, $\left(a_{t+s}\right)_{s=0}^{\infty}$, that satisfies, conditional on h_{t-1}, and x_t, all the date $t+s$, $s \geq 0$ competitive equilibrium conditions (see Definition 1), with the exception of the period t optimality condition for the intermediate good firm, (1.12).

Here, a_t is defined in (1.15).

We need the continuation equilibrium for given $h_{\pi^e,t}$ in order to compute c_t^e in (1.26). There are several different types of histories, $h_{\pi^e,t}$. There are two types of h_{t-1}: one in which the inflation monitoring range was never violated in $t-1$ and earlier; and the other in which it was violated at least once. The number of types of π_t^e depend on the type of h_{t-1}. If there has been no violation of the inflation monitoring range in the past, then, there are four types of π_t^e to consider: $\pi_t^e > \pi_u$, $\pi_t^e \in [\pi_l, \pi_u]$, $\frac{R^l}{\phi} < \pi_t^e \leq \pi_l$ and $\pi_t^e \leq \frac{R^l}{\phi}$. If h_{t-1} has the property that there has been a deviation in the past, then there are two types of π_t^e to consider: those for which the lower bound on the nominal rate of interest is binding and the others.

By Definition (3), c_t^e is a function of $h_{\pi^e,t}$, $c_t^e = c_t^e(h_{\pi^e,t})$. We refer to (1.26) as a 'best response function' from π_t^e to x_t, for given h_{t-1}:

$$x_t = \pi_t^e + (\gamma + \psi) \ln c_t^e (h_{\pi^e,t}) . \tag{1.27}$$

The object, x_t, in (1.27) is the (scaled and logged) price chosen by the i^{th} firm, given its conjecture, π_t^e, and given h_{t-1}. Because the i^{th} firm thinks (correctly) that everyone else behaves in the same way, the x_t chosen by the i^{th} firm is also its conjecture about what the others do and, hence, maps into a conjecture about the aggregate price index.[14] So a firm which conjectures that inflation will be π_t^e also conjectures that inflation will be the value of x_t implied by (1.26). We assume that a firm can only have one belief about a given variable. For this reason we suppose that a firm's belief is a conjecture that is a fixed point of (1.27).

[14]The object, x_t, is a scaled and logged price, $\log [p_{it}/(P_{t-1}\bar{\mu}^*)]$. So, $P_t = P_{t-1}\bar{\mu}^* e^{x_t}$.

Definition 4. Given history, h_{t-1}, the i^{th} firm's *belief* about period t inflation is a value of π_t^e with the property that $x_t = \pi_t^e$ in (1.27).

Thus, we distinguish between a conjecture and a special kind of conjecture, which we call a belief. Beliefs occur in equilibrium. We see from (1.16) that in a competitive equilibrium, $c_t = 1$, so that $x_t = \pi_t^e$, according to (1.27). Conjectures that are not beliefs allow us to contemplate off-equilibrium paths. Studying why such conjectures are not equilibria provides the economic answer to why a particular off-equilibrium path is not part of an equilibrium.

1.4.3. Strategy Equilibrium.

Consider the following definition.

Definition 5. A strategy equilibrium is a competitive equilibrium (see Definition (1)) with the property that for each possible history h_{t-1}: (i) there is a well defined continuation equilibrium corresponding to any value of π_t^e and (ii) there exists a π_t^e that is a fixed point of (1.26).

Because the equilibrium allows for arbitrary histories, h_{t-1}, we allow for multiple deviations from equilibrium. Condition (i) requires that a continuation equilibrium exist for any π_t^e so that the agent's choice of belief is a non-trivial one. To form a belief, agents must try out various conjectures until they settle on a belief assumed to exist by (ii). Condition (i) the important one for our purposes. It is what allows us to think in an organized way about why certain off-equilibrium paths are not equilibria.

We give our equilibrium concept a different name from the one in Atkeson et al. (2010) only because the objects in their equilibrium are functions while in our case they are sequences. We call ours a strategy equilibrium because intermediate good firms choose their belief about how others set prices based in part on how they think the economy will respond to that setting.

Finally, consider the following definition:

Definition 6. A policy uniquely implements an equilibrium if the equilibrium is unique and if the equilibrium is a strategy equilibrium.

The Taylor rule with the escape strategy uniquely implements the target equilibrium. Given that equilibrium is unique (see Proposition (2)) it remains to show that the target equilibrium satisfies (i) and (ii) in Definition (5). This requires evaluating the continuation equilibrium for all possible values of $h_{\pi^e,t}$, where all the possibilities are described after Definition (3). The two sections below consider the continuation equilibria and best response function for two interesting types of $h_{\pi^e,t}$. For the others, see the Appendix.[15]

1.4.4. How is it that High Inflation is Ruled Out in Equilibrium?

We first investigate why $\pi_t > \pi_u$ is impossible in a strategy equilibrium, when h_{t-1} is a history in which the monitoring range has never been violated. We need to show that $\pi_t^e > \pi_u$ cannot be a fixed point of (1.27) for this type of $h_{\pi^e,t}$. To do this, we first derive an expression for consumption in the period t continuation equilibrium. For our purposes,

[15] Atkeson et al. (2010) describe two other models for which they obtain unique implementation with the escape clause. We discuss those models below.

it is also convenient to have an expression for inflation in period $t+1$ given that $\pi_t^e > \pi_u$, which triggers the escape to constant money growth.

Period $t+1$ is the first period of the constant money growth regime. According to Definition 2 the money growth rate, $\bar{\mu}$, is greater than unity, so that Lemma 1 implies that $c_{t+1} = 1$ and $\bar{M}_{t+1} = P_{t+1}$. Dividing the latter by the binding (because $\pi_t^e > 0$) cash constraint in period t, $\bar{M}_t = c_t P_t$, we obtain

$$\bar{\mu} = \frac{\bar{\pi}_{t+1}}{c_t}. \tag{1.28}$$

Substituting into the household's intertemporal Euler equation, (1.5),

$$(c_t^e)^{-\gamma} = \beta c_{t+1}^{-\gamma} \frac{\bar{R}_t}{\bar{\pi}_{t+1}} = \beta \frac{\bar{R}_t}{c_t \bar{\mu}}.$$

Because of our assumption about h_{t-1}, the escape clause has not been violated in the past, and the Taylor rule, (1.2), is in operation in the current period. Thus,

$$(c_t^e)^{1-\gamma} = \beta \frac{\bar{R}_t}{\bar{\mu}} = \beta \frac{\bar{R}^*}{\bar{\mu}} \left(\frac{\bar{\pi}_t}{\bar{\pi}^*} \right)^{\phi} = \left(\frac{\bar{\pi}_t}{\bar{\pi}^*} \right)^{\phi} \frac{\bar{\mu}^*}{\bar{\mu}}$$

so that

$$c_t^e = \frac{\bar{\mu}^*}{\bar{\mu}} \left(\frac{\bar{\pi}_t}{\bar{\pi}^*} \right)^{\frac{\phi}{1-\gamma}}.$$

Taking logs of both sides,

$$\ln (c_t^e) = \frac{\phi}{1-\gamma} \pi_t^e - \frac{\mu}{1-\gamma}. \tag{1.29}$$

Substituting the last expression into (1.26), we obtain:

$$x_t = \left[1 + \frac{(\gamma + \psi)\,\phi}{1-\gamma}\right]\pi_t^e - \frac{(\gamma+\psi)\,\mu}{1-\gamma}. \tag{1.30}$$

The value of π_t^e such that $x_t = \pi_t^e$ is:

$$\pi_t^e = \frac{\mu}{\phi} < \pi_u.$$

Since the fixed point of the mapping, (1.30), is unique and lies below π_u it follows that there is no fixed point greater than π_u. After scaling and logging, (1.28) implies:

$$\begin{aligned}\pi_{t+1}^e &= \mu + \ln c_t^e \\ &= \frac{\phi}{1-\gamma}\pi_t^e - \frac{\gamma\mu}{1-\gamma}.\end{aligned} \tag{1.31}$$

We summarize these results in the form of a lemma:

Lemma 3. *Suppose* $\pi_t^e > \pi_u$. *Then,*

$$\ln\left(c_t^e\right) = \frac{\phi}{1-\gamma}\pi_t^e - \frac{\mu}{1-\gamma},\ \pi_{t+1}^e = \frac{\phi}{1-\gamma}\pi_t^e - \frac{\gamma\mu}{1-\gamma}, \tag{1.32}$$

and there is no fixed point of the best response function with $\pi_t > \pi_u$. *Also, the continuation consumption at date* t *and inflation at* $t+1$ *are the same as (1.32) when* $\pi_t^e \in (\phi^{-1}R_l, \pi_l)$

Proof. The first part of Lemma 3 is already done. For the second part, notice that exactly the same argument is applied for $\pi_t^e \in (\phi^{-1}R_l, \pi_l)$. □

Our analysis allows us to explain why $\pi_t > \pi_u$ is not a competitive equilibrium. When an intermediate good firm contemplates the possibility that inflation will be high because the other firms are setting high prices (i.e., $\pi_t^e > \pi_u$), it understands that the current interest rate will be high because of the Taylor rule and next period's inflation will be low because of (1.31). With the real rate interest high, the firm believes aggregate consumption will be low, so that the demand for labor and the real wage will be low too. With low anticipated marginal cost, the firm would raise price by less than the amount implicit in the rise in π_t^e (this is captured by the coefficient on π_t^e in (1.30) being less than unity)[16]. Understanding that other firms think in the same way, the firm would not form a belief, $\pi_t^e > \pi_u$ in the first place. This is why π_t cannot be greater than π_u in equilibrium.

Thus, we have an answer to the question, 'why can there be no hyperinflation?' If firms anticipated such a thing they would anticipate a Volcker-style recession with low marginal costs and they would choose not to set the high prices necessary for inflation to be high.

1.4.5. How is Moderate, but Not Desired, Inflation Ruled Out in Equilibrium?

We now consider the case, $\pi_t \in [\pi_l, \pi_u]$ and explain why it is that only $\pi_t = 0$ can be an equilibrium. Equivalently

Lemma 4. *The only fixed point of the best response function, (1.26), for* $\pi_t^e \in [\pi_l, \pi_u]$ *is* $\pi_t^e = 0$.

Proof. Consider $\pi_t^e \in [\pi_l, \pi_u]$. To evaluate the right side of the best response function, (1.26), we need to compute the period t afternoon continuation equilibrium. Consider the periods after t first. According to (2), the unique competitive equilibrium under the

[16]Recall from (1.3) that $\gamma > 1$

Taylor rule with escape clause has the property, $\pi_{t+1} = 0$ and $c_{t+1} = 1$. We then obtain period t consumption in the period t afternoon continuation equilibrium by substituting into the period t household intertemporal Euler equation, (1.5):

$$c_t^{-\gamma} = \beta c_{t+1}^{-\gamma} \frac{\bar{R}_t}{\bar{\pi}_{t+1}} = \beta \frac{\bar{R}_t}{\bar{\mu}^*} = \left(\frac{\bar{\pi}_t}{\bar{\pi}^*} \right)^{\phi},$$

where the last equality uses the Taylor rule, (1.2). The Taylor rule is in place in period t because of our construction of h_{t-1} which assumes that the escape clause has not been activated in the past. Substituting the log of the latter expression into (1.26) and collecting terms, we obtain:

$$x_t = \pi_t^e + (\gamma + \psi) \ln c_t^e = \left[1 - \frac{\phi}{\gamma} (\gamma + \psi) \right] \pi_t^e$$

The only fixed point for this expression is $\pi_t^e = 0$. Thus, $\pi_t^e \neq 0$ is not a fixed point, so that the desired result is established. □

The intuition here is similar to the intuition underlying Lemma 3. Suppose $\pi_u \geq \pi_t^e > 0$. Because inflation is above the desired level, the nominal interest rate is also high as well because of the Taylor rule. At the same time, the rate of inflation in the next period is expected to be at its desired level so that the real interest rate is high. This creates a fall in output, discouraging firms from raising prices. Knowing this, firms will not form a belief, $\pi_t^e > 0$. Now consider the case, $\pi_l \leq \pi_t^e < 0$. In this case, the government would generate a boom in output, giving firms an incentive to raise prices with the consequence that such a belief is not an equilibrium.

1.5. Where did Cochrane (2011) Go Wrong?

We have explored why it is that non-desired inflation paths fail to be equilibria when monetary policy is characterized by the Taylor rule with an escape clause. The reasons do not involve anyone doing something infeasible. In contrast, as explained in Section 1.3, Cochrane (2011) argues that the escape clause trims undesired equilibria by a commitment to do something infeasible in case an undesired equilibrium occurs.

Why does Cochrane (2011) reach such a different conclusion than we do? The different conclusions may at first seem surprising, since in both cases a competitive equilibrium is characterized by the same two equations: the Fisher equation, (1.17), and the Taylor rule, (1.18) (see 1). The reasons for the different conclusions lies in the different assumptions we make about the underlying economy.

Cochrane (2011, p. 574) assumes that the underlying economy is an endowment economy. To see the effect of this assumption, note that our model economy effectively reduces to an endowment economy if we assume that households supply a fixed amount of labor inelastically. In this case, consumption takes on the same value if we are on the equilibrium path, or we are on the off-equilibrium paths that are contemplated in a Strategy Equilibrium (see 5). So, the household's intertemporal Euler equation, (1.5), reduces to the Fisher equation, (1.17), both on and off-equilibrium. It follows that period t continuation equilibria simply do not exist when $\pi_t^e \neq 0$. Put differently, in the endowment economy version of our model, the government threatens to do something infeasible (raise the real rate of interest) in case inflation is higher than desired. In particular, deviations from the equilibrium path are enforced by a threat to blow up the economy, just as Cochrane (2011) suggests.

It follows that the unique competitive equilibrium of the endowment economy version of our model is not a Strategy Equilibrium. But, Cochrane (2011) is wrong to extrapolate from the properties of his model to those of monetary models in which output is endogenous. In particular, his analysis offers no reason to think that the escape clause is an uninteresting way to rule out non-desired equilibria in the kinds of monetary models used in practice, such as the New Keynesian model.

1.6. The Atkeson et al. (2010) Argument

Atkeson et al. (2010) direct attention to a policy with $,\pi_l = \pi_u = 0$, so that the monitoring range is a singleton composed just of the desired rate of inflation. Our model also supports their finding that the desired equilibrium is the unique equilibrium outcome under their policy. However, the first section below shows that this result is fragile and not robust to small trembles. In the second section below we explain why the fragility may have important welfare consequences.

Finally, in the introduction we expressed skepticism about Atkeson et al. (2010)'s suggestion that their policy reveals the Taylor principle is not a necessary part of good monetary policy. Under that policy, to keep the economy in the desired equilibrium, $\phi > 1$ is not necessary since the escape clause (assuming no trembles) can do that job all by itself. It is true that in the context of the Taylor rule, the parameter setting, $\phi > 1$, is often referred to as the Taylor Principle, and in that sense Atkeson et al. (2010) are right. But, in Section 1.5 we showed that in our model the escape clause is a commitment to raise the interest rate when inflation is high as a strategy for keeping inflation close to its desired level. As discussed in the introduction, this is basically a description of the Taylor

Principle. So, replacing a Taylor rule with $\phi > 1$ with an escape clause is tantamount to replacing one way of implementing the Taylor Principle with another.

1.6.1. Fragility of the Atkeson et al. (2010) Monetary Policy

We consider the possibility that a tiny subset (mass) of firms make a tiny mistake implementing their price decision. When this happens, then inflation drops out of the monitoring range, triggering a regime shift in money policy towards constant money growth.

We illustrate these points here with the use of trembles. Suppose that people form their beliefs as they do in previous sections, without imagining the possibility of trembles. They choose their belief as a solution to a fixed point problem, as discussed in Section 1.4.2. However, when the time comes for firms to post their price, a small number make a small mistake (the firm manager's hand 'trembles' as he/she writes the price on the door). In particular, let p_t^e denote the i^{th} firms expectation of how other firms post their price for all $i \in [0, 1]$:

$$p_{i,t} = \overbrace{p_t^e \left(c_t^e\right)^{\gamma+\psi}}^{\text{price in absence of tremble}} \upsilon_{i,t}. \tag{1.33}$$

This value of p_t^e has the fixed point property that, given the continuation equilibrium conditional on p_t^e, firms set the same price. Suppose there is a tremble in the form of the additional variable, $\upsilon_{i,t}$, on the right side of (1.33). That tremble is a unit mean random variable which is drawn independently by each firm. The distribution of υ_{it} has two parameters, $J_t, \delta_t \in [0, 1]$. With probability $1 - J_t$, the i^{th} firm does not tremble at all, so that $\upsilon_{i,t} \equiv 1$. With probability J_t the i^{th} firm draws $\upsilon_{i,t}$ from a uniform distribution with support, $[1 - \delta_t, 1 + \delta_t]$. We can allow both J_t and δ_t to be arbitrarily close to zero so that the subset of firms that tremble is very small and those that do tremble, do so by a small amount.

According to (1.11), the actual price index will be determined as follows :

$$P_t = \left[\int_0^1 \left(p_t^e \left(c_t^e\right)^{\gamma+\psi} \upsilon_{i,t}\right)^{1-\varepsilon} di\right]^{\frac{1}{1-\varepsilon}}$$

$$= p_t^e \left(c_t^e\right)^{\gamma+\psi} \exp\left(\kappa_t\right) \tag{1.34}$$

where

$$\exp\left(\kappa_t\right) = \left[1 - J_t + J_t \frac{\left(1+\delta_t\right)^{2-\varepsilon} - \left(1-\delta_t\right)^{2-\varepsilon}}{2\delta_t \left(2-\varepsilon\right)}\right]^{\frac{1}{1-\varepsilon}}.$$

As expected,

$$\kappa_t \to 0 \text{ as } J_t \to 0 \text{ for fixed } \delta_t, \quad \kappa_t \to 0 \text{ as } \delta_t \to 0 \text{ for fixed } J_t.$$

Divide both sides of (1.34) by $\bar{\mu}^* P_{t-1}$ and take logs, to obtain:

$$\pi_t = \pi_t^e + (\gamma + \psi) \ln\left(c_t^e\right) + \kappa_t. \tag{1.35}$$

Suppose the economy has, up to the present time, not experienced a tremble and that it has been in the desired equilibrium. As a result, firms set $\pi_t^e = 0,\ c_t^e = 1$ and they believe (they think $\kappa_t = 0$) $\pi_t = \pi_t^e = 0$. But, this is not what will actually occur, if $J_t, \delta_t > 0$. In this case, the tremble pushes the economy to switch to the money growth regime because $\pi_t = \kappa_t > 0$.

1.6.2. The Fragility has Real Consequences if there are High-Frequency Money Demand Shocks

In the model as it is set up now, the regime shift does not matter from a welfare standpoint. The same desired equilibrium outcomes occur whether the economy is following the Taylor rule or the money growth rule. But, if we assume money demand shocks are realized in the afternoon, then there is a substantial loss in jumping from the Taylor rule to the money growth rule. The assumption that the money demand shocks are realized in the afternoon is a way to capture the notion that money demand shocks operate at a higher frequency than price changes. The model is a variant of the sticky price model in Christiano et al. (1997), where time t prices are predetermined when time t shocks are realized.

We introduce money demand shocks, ν_t, by inserting them into the cash constraint:

$$P_t c_t \leq m_t \nu_t.$$

We assume these shocks are iid, have unit mean and are drawn from a uniform distribution with continuous support. It is easy to show that under the Taylor rule, l_t, c_t behave as they do in the desired equilibrium in the absence of money demand shocks. Under the Taylor rule, the money supply moves in such a way that consumption and employment are first-best and are insulated from the realizations of ν_t. In the money growth regime, things are different. It is easy to see that in that equilibrium,

$$l_t = c_t = \frac{\nu_t}{\left[E\nu_t^{\gamma+\psi}\right]^{\frac{1}{\gamma+\psi}}}.$$

A shift from the interest rate regime to the money growth regime, says due to trembles, would have a substantial negative welfare effect.

1.7. Conclusion

Our analysis reinforces the conclusions in earlier work that a Taylor rule with a coefficient on inflation that is bigger than one, which is coupled with an escape clause with a reasonably wide inflation monitoring range, has good stabilizing properties. In addition, it may rationalize the standard practice of focusing on the unique local-to-steady state equilibrium.

1.8. Proofs

1.8.1. Proposition 1

Proof. For each $\pi_0 > 0$, there is a sequence of inflation, $\{\pi_t\}_{t=1}^{\infty}$, that satisfies (1.18) in which inflation explodes to ∞. The interest rate associated with such a sequence is greater than unity for each t, so that (1.6) implies the cash constraint is binding and $m_T/P_T = c_T = 1$, or all T by (1.16). Trivially, the transversality condition, (1.21), is satisfied. Similarly, a sequence that satisfies (1.20) with $\pi_0 < 0$ converges to $\ln(\beta/\bar{\pi}^*)$, so that actual gross inflation converges to β. To see that this sequence satisfies (1.21) note that π_t converges in finite time to its fixed point, $\pi_t = \ln(\beta/\bar{\pi}^*)$. Suppose convergence occurs at $t = \bar{t} \geq 0$. Then, for $T > \bar{t}$, $R_T = 1$ so (1.5) implies $P_T = \beta^{T-\bar{t}} P_{\bar{t}}$. Setting $m_T = \beta^{T-\bar{t}} m_{\bar{t}}$ for all $T > \bar{t}$ so that $m_T/P_T = m_{\bar{t}}/P_{\bar{t}}$, a constant $\geq c_T$ for all $T > \bar{t}$, we have that the cash constraint, (1.6), is satisfied and

$$\frac{m_T}{P_T} = \frac{m_{\bar{t}}}{P_{\bar{t}}} \beta^T \rightarrow 0,$$

so that (1.21) is satisfied too. The uniqueness result is stated in Proposition 1. □

1.8.2. Lemma 1

Proof. First, we show that in any equilibrium, $\bar{R}_t > 1$ for all t. Recall the Fisher equation, (1.17), and the complementary slackness condition associated with the cash constraint, (1.6), $t \geq 0$:

$$\bar{\pi}_{t+1} = \beta \bar{R}_t \text{ (Fisher)}, \quad 0 = \overbrace{\left(\bar{R}_t - 1\right)}^{\geq 0,} \overbrace{\left(\bar{M}_t - P_t y\right)}^{\geq 0, \text{cash constraint}} .$$

Suppose, to the contrary, that $\bar{R}_t = 1$ for some t. It follows that $\bar{R}_{t+1} = 1$. To see this, note that $\bar{\pi}_{t+1} = \beta$ by the Fisher equation so that $P_{t+1} < P_t$. At the same time, $\bar{M}_{t+1} \geq \bar{M}_t$ because of our assumption, $\bar{\mu} \geq 1$. These two observations, together with $\bar{M}_t \geq P_t y$ imply that $\bar{M}_{t+1} > P_{t+1} y$. The complementary slackness condition then implies that $\bar{R}_{t+1} = 1$.

By induction, we conclude that $\bar{R}_t = 1$ implies $\bar{R}_{t+s} = 1$ and $\bar{\pi}_{t+s+1} = \beta$ for $s \geq 0$. But, this contradicts the hypothesis of equilibrium because the transversality condition is violated. To see this, note that for any fixed t,

$$\lim_{T \to \infty} \beta^T \frac{\bar{M}_T}{P_T} = \lim_{T \to \infty} \underbrace{\frac{\beta^{T-t}}{P_T / P_t}}_{=1} \beta^t \frac{\bar{M}_T}{P_t} = \frac{\beta^t}{P_t} \lim_{T \to \infty} \bar{M}_T > 0.$$

Thus, we conclude that in any equilibrium, $\bar{R}_t > 1$ for all $t \geq 0$. To suppose otherwise entails a contradiction.

Next, we show that with $\bar{R}_t > 1$, for all $t \geq 0$, the equilibrium conditions uniquely determine all variables. The cash constraint binds, $\bar{M}_t - P_t y = 0$, for each $t \geq 0$, so

$$\bar{\pi}_{t+1} = \bar{\mu}.$$

Also, the Fisher equation implies

$$\bar{R}_t = \beta^{-1}\bar{\pi}_{t+1} = \beta^{-1}\bar{\mu} > 1.$$

The cash constraint and $\bar{R}_0 > 1$ implies

$$P_0 y = \bar{M}_0 \Longrightarrow P_0 = \frac{\bar{M}_{-1}}{y}\frac{\bar{M}_0}{\bar{M}_{-1}} = \frac{\bar{M}_{-1}}{y}\bar{\mu}.$$

We have established: Suppose $\bar{M}_{-1}$ is given and monetary policy sets $\bar{M}_t = \bar{\mu}\bar{M}_{t-1}$ for $t \geq 0$, where $\bar{\mu} \geq 1$. There exists a unique equilibrium with the properties:

$$\bar{R}_t = \beta^{-1}\bar{\mu} > 1, \quad \bar{\pi}_{t+1} = \bar{\mu}, \quad \text{for} \quad t \geq 0, \quad \text{and} \quad P_0 = \bar{M}_{-1}\frac{\bar{\mu}}{y},$$

which is the result sought. □

1.8.3. Proposition 2

Proof. Suppose, to the contrary, that $\pi_0 \neq 0$. From Lemma 2 equilibrium has the property that the monitoring range is never violated, i.e., $\pi_t \in [\pi_l, \pi_u]$. The Taylor rule, (1.18), the Fisher equation, (1.17), and $\pi_l > \frac{R^l}{\phi}$, implies that in equilibrium:

$$\pi_{t+1} = \max\left\{R^l, \phi\pi_t\right\} = \phi\pi_t.$$

Evidently, $\pi_0 \neq 0$ implies $\pi_t \notin [\pi_l, \pi_u]$ for some t, given $\phi > 1$. This contradicts Lemma 2. We conclude that $\pi_0 = 0$, establishing the result. □

1.9. Arbitrary history

Now we consider an arbitrary history for the strategy equilibrium.

Definition 7. A strategy equilibrium is a sequence, a_t, for $t \geq 0$, with two properties: (i) it is a competitive equilibrium with the Taylor rule and escape clause and (ii) for any history h_{t-1}, there exists a continuation equilibrium, (iii) for any history (h_{t-1}, π_t^e) , there is a well defined time t afternoon continuation equilibrium conditional on $h_{\pi^e,t}$.

There are additional requirements if we consider a general history. Part (ii) requires that for any history h_{t-1}, a continuation equilibrium should exist. There are two kinds of history h_{t-1}; the monitoring range has never violated; the monitoring range has been violated. For each case, there is a continuation equilibrium trivially since there are no state variables. Therefore Part (ii) is satisfied trivially in our model. Part (iii) is almost the same as before. In the body of the paper, we do not consider the cases in which π_t^e is low enough so that interest rate hits the lower bound. We need to establish that even for such π_t^e, there exists a continuation equilibrium.

Lemma 5. *For any (h_{t-1}, π_t^e) , there exists a continuation equilibrium. In particular;*

(1) If the Taylor rule is operative at t *and* $\pi_t^e \leq \phi^{-1} R_l$,[17] *the continuation consumption, employment and money demand at date* t *are*

$$\ln c_t = \ln l_t = \frac{R_l - \mu}{1 - \gamma} \tag{1.36}$$

$$\mu_t = \pi_t^e + \ln c_t. \tag{1.37}$$

Notice that the cash constraint at date t *is binding.*

(2) If the money growth rule is operative and $\pi_t^e \in D(h_{t-1})$, *where*

$$D(h_{t-1}) = \left\{ \pi_t^e; (1-\gamma) \left[\mu + \ln \frac{M_{t-1}}{P_{t-1}} - \pi_t^e \right] + \mu > R_l \right\}, \tag{1.38}$$

then the continuation consumption, employment and interest rate at date t *are*

$$\ln c_t = \ln l_t = \mu + \ln \frac{M_{t-1}}{P_{t-1}} - \pi_t^e \tag{1.39}$$

$$R_t = (1-\gamma) \ln c_t + \mu. \tag{1.40}$$

(3) If the money growth rule is operative and $\pi_t \notin D(h_{t-1})$, *then*

$$\ln c_t = \ln l_t = -\frac{1}{\gamma} \left(R_l - \left[\mu + (\mu - \pi_t^e) + \ln \frac{M_{t-1}}{P_{t-1}} \right] \right) \tag{1.41}$$

$$R_t = R_l. \tag{1.42}$$

Proof. We construct a continuation equilibrium for each (h_{t-1}, π_t^e). We know that there exists a continuation equilibrium after $t+1$ from Lemma 1 and Proposition 2.

[17] The existence of a continuation equilibrium for $\pi_t^e > \phi^{-1} R_l$ is already established in Lemma 3 and 4.

So, we need to construct date-t variables so that all the date-t afternoon equilibrium conditions, (1.5), (1.6), and are satisfied.

First consider case (1); we consider a history in which the Taylor rule is operative at date t and $\pi_t^e \leq \phi^{-1} R_l$. Suppose that $\pi_t^e \leq \phi^{-1} R_l$. The monitoring range is violated so that the continuation equilibrium from $t+1$ is unique, and consumption and price are

$$c_{t+1} = 1, \quad \pi_{t+1} = \mu + \ln \frac{M_t}{P_t}. \tag{1.43}$$

By construction of (1.37), the cash constraint (1.6) is satisfied as equality. binding cash constraint, $m_t = P_t c_t$. Also, the Euler equation (1.5) is also satisfied. To show it,

$$\begin{aligned} & -\gamma \underbrace{\ln c_{t+1}}_{=0} + \underbrace{R_t}_{=R_l} - \underbrace{\pi_{t+1}}_{=\mu + \ln c_t} + \gamma \ln c_t \\ =& R_l - (\mu + \ln c_t) + \gamma \ln c_t \\ =& R_l - \mu + \underbrace{(\gamma - 1) \ln c_t}_{=\mu - R_l} = 0. \end{aligned}$$

Therefore all the afternoon equilibrium conditions are satisfied by (1.36) and (1.37).

Second, consider case (2). Again we need to show that the Euler equation (1.5) and cash constraint (1.6) are satisfied with (1.39) and (1.40) conditional on the continuation equilibrium from $t+1$. In particular, consumption and inflation rate is

$$c_{t+1} = 1, \quad \pi_{t+1} = \mu + \ln \frac{M_t}{P_t}. \tag{1.44}$$

Cash constraint is trivially satisfied by construction. Put differently, the real balance M_t/P_t is equal to consumption c_t. The Euler equation is satisfied since

$$-\gamma \underbrace{\ln c_{t+1}}_{=0} + \underbrace{R_t}_{=(1-\gamma)\ln c_t+\mu} - \underbrace{\pi_{t+1}}_{=\mu+\ln c_t} + \gamma \ln c_t$$
$$= (1-\gamma)\ln c_t + \mu - \mu - \ln c_t + \gamma \ln c_t = 0.$$

Also, notice that the nominal rate is greater than the lower bound R_l :

$$R_t = (1-\gamma)\left[\mu + \ln \frac{M_{t-1}}{P_{t-1}} - \pi_t^e\right] + \mu > R_l \tag{1.45}$$

since $\pi_t^e \in D(h_{t-1})$.

Finally consider case (3). Note that in this case the interest rate implied by (1.45) is weakly less than R_l. Again from $t+1$, there exists a continuation equilibrium with

$$c_{t+1} = 1, \quad \pi_{t+1} = \mu + \ln \frac{M_t}{P_t} = \mu + (\mu - \pi_t^e) + \ln \frac{M_{t-1}}{P_{t-1}}. \tag{1.46}$$

For the last equation, we use the fact that the monetary policy at date h is a constant money growth policy. Using (1.41), (1.42), and (1.46), we can show that the Euler equation is satisfied:

$$-\gamma \underbrace{\ln c_{t+1}}_{=0} + \underbrace{R_t}_{=R_l} - \underbrace{\pi_{t+1}}_{=\mu+(\mu-\pi_t^e)+\ln \frac{M_{t-1}}{P_{t-1}}} + \gamma \ln c_t$$
$$= R_l - \left[\mu + (\mu - \pi_t^e) + \ln \frac{M_{t-1}}{P_{t-1}}\right] - \left(R_l - \left[\mu + (\mu - \pi_t^e) + \ln \frac{M_{t-1}}{P_{t-1}}\right]\right) = 0.$$

The cash constraint is also satisfied. First notice that

$$M_t \geq P_t c_t \iff \mu - \pi_t^e + \ln \frac{M_{t-1}}{P_{t-1}} \geq \ln c_t.$$

It is easy to show that

$$\begin{aligned} & \mu - \pi_t^e + \ln \frac{M_{t-1}}{P_{t-1}} - \ln c_t \\ =& (\mu - \pi_t^e) + \ln \frac{M_{t-1}}{P_{t-1}} + \frac{1}{\gamma}\left(R_l - \left[\mu + (\mu - \pi_t^e) + \ln \frac{M_{t-1}}{P_{t-1}}\right]\right) \\ =& \frac{1}{\gamma}\left\{(R_l - \mu) - (1-\gamma)\left[(\mu - \pi_t^e) + \ln \frac{M_{t-1}}{P_{t-1}}\right]\right\}. \end{aligned} \tag{1.47}$$

Since $\pi_t^e \notin D(h_{t-1})$, we have

$$\begin{aligned} & R_l \geq (1-\gamma)\left[\mu + \ln \frac{M_{t-1}}{P_{t-1}} - \pi_t^e\right] + \mu \\ \iff & R_l - \mu - (1-\gamma)\left[\mu + \ln \frac{M_{t-1}}{P_{t-1}} - \pi_t^e\right] \geq 0. \end{aligned} \tag{1.48}$$

Combining (1.48) with (1.47), we get

$$\mu - \pi_t^e + \ln \frac{M_{t-1}}{P_{t-1}} - \ln c_t \geq 0,$$

which is equivalent to cash constraint at date t. Thus (1.41) and (1.42) satisfy the date-t afternoon equilibrium conditions from (h_{t-1}, π_t^e). □

CHAPTER 2

A Mathematical Background and A Sophisticated Equilibrium Approach

with Lawrence Christiano

2.1. Introduction

This chapter is a supplemental material for the previous chapter. We study rigorously the household's maximization problem, and derive the necessary and sufficient conditions which characterize the optimality. Then in Section 2.3, we extend the deterministic model used in the previous chapter to a stochastic version. This extension is crucial to discuss importance of the Taylor principle. In the deterministic model, there are no real consequences of using the money growth rule. However, in the stochastic version of the model in which the economy has velocity shocks, the welfare under the constant money growth rule is strictly worse than the efficient level. Therefore the government prefers to use the interest rate based rule, and the failure of doing it has a real consequence.

In Section 2.4 and 2.5, we formally introduce a sophisticated equilibrium concept, which is proposed by Atkeson et al. (2010). In the strategy equilibrium in the previous section, the equilibrium is a sequence of endogenous variables. In the sophisticated equilibrium, the equilibrium is a collection of mapping from history to a number. While the previous treatment might be more accessible for a wider range of audience, the sophisticated equilibrium might be clearer if the economy has a shock. Note that conceptually they are identical.

In Section 2.6, we show that the Taylor rule with the escape clause uniquely implements the desired allocation. Moreover we replicate the main result by Atkeson et al. (2010) which argues that the Taylor principle is not necessary for the unique implementation. While the result is mathematically true, we show that the result is fragile to introduction of *trembles*, and we conclude that the Taylor principle is in the end necessary for unique robust implementation.

2.2. Households

This section discusses the maximization problem by households in the previous chapter deeply. In particular, we derive a sufficient condition to make sure that the maximization problem is well-defined. Then we discuss necessary and sufficient conditions for the optimality of the problem.

2.2.1. Maximization problem

There is a unit mass of identical households. The date 0 problem of the representative household is as follows:

$$V = \max_{\{c_t, l_t, m_t, b_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t U(c_t, l_t) \tag{2.1}$$

$$\text{s.t.} \quad m_t + b_t = W_t l_t + m_{t-1} - P_{t-1}c_{t-1} + \bar{R}_{t-1}b_{t-1} + T_t \tag{2.2}$$

$$P_t c_t \leq m_t \tag{2.3}$$

$$(m_{-1} - P_{-1}c_{-1} + R_{-1}b_{-1}) \text{ given,}$$

where U is separable w.r.t. c_t and l_t :

$$U(c_t, l_t) = u(c_t) + \bar{v}(N - l_t).$$

Here, c_t and l_t denote time t household consumption and employment, respectively, while N denotes the household's time endowment. Define $v(l_t)$ as $\bar{v}(N - l_t)$. Also, W_t, P_t and R_t denote the nominal wage rate, the price level and the gross nominal rate of interest on a bond, b_t, purchased at end of time t. Equation (2.1) is the household's asset accumulation equation. The household begins period t with its end of period $t-1$ cash balances, m_{t-1}, net of the cash it owes on its period $t-1$ consumption expenditures, $P_{t-1}c_{t-1}$. During period t, the household accumulates more cash by working, from interest on b_{t-1} and from government transfers and profits, T_t. All this cash is then split into end of period t money, m_t, and bonds, b_t. This split is determined subject to m_t being large enough to cover the household's period t consumption expenditures according to the cash constraint, (2.3). For the household problem to be well defined it is also necessary that it respect the natural debt constraint, which is described in section 2.2.2 below. There are additional restrictions on prices which are required for the household problem to be well defined. For example, we must have $P_t, W_t > 0$ and $R_t \geq 1$ for $t \geq 0$. These and other restrictions on prices are also discussed in Section 2.2.2. Sections below also discuss the necessity and sufficiency of the first order and transversality conditions for household optimization.

The household first order conditions are:

$$\frac{W_t}{P_t} = -\frac{v'(l_t)}{u'(c_t)} \tag{2.4}$$

$$u'(c_t) = \beta u'(c_{t+1}) \frac{\bar{R}_t}{\bar{\pi}_{t+1}},$$

$$(R_t - 1)[m_t - P_t c_t] = 0 \tag{2.5}$$

where $\bar{\pi}_t$ denotes the gross inflation rate at date t, P_t/P_{t-1}. The transversality condition is:

$$\lim_{T\to\infty} \beta^T u'(c_T) \frac{m_T - P_T c_T + b_T}{P_T} = 0.$$

Note that the transversality condition is expressed in terms of end-of-period money, net of that period's expenditures on consumption goods. Necessity and sufficiency of the transversality condition for solving the household problem is discussed below.

2.2.2. Conditions for the Household Problem to be Well Defined

The household takes as given the sequences, $(P_t, R_t, T_t)_{t=-1}^{\infty}$. We suppose that $P_t > 0, \bar{R}_t \geq 1$, for otherwise there would be no equilibrium. The period utility function, u, is strictly increasing in consumption and leisure, $N - l_t$, and we assume the function is concave.

We use five more assumptions in our analysis of the household problem. First, the present value of the household's income must be finite:

$$\sum_{t=0}^{\infty} q_t \left[W_t N + T_t \right] < \infty, \tag{2.6}$$

where N denotes the time endowment of the household and

$$q_t = \begin{cases} 1 & t = 0 \\ \frac{1}{\bar{R}_0 \bar{R}_1 \cdots \bar{R}_{t-1}} & t \geq 1 \end{cases}. \tag{2.7}$$

Second, we impose the natural debt limit. To describe this, it is useful to reformulate the problem following the approach in Woodford (1994). In particular, we define the household's "asset" a_t by

$$a_t = b_t + \left(m_t - P_t c_t \right).$$

This is the sum of the household's bonds and money at the end of the period. The money held at the end of the period by households is net of the amount used for period t consumption expenditures, $P_t c_t$. Rewriting the flow budget equation, (2.2):

$$\begin{aligned} m_t + b_t - P_t c_t &= W_t \left(l_t - N \right) + W_t N + T_t + m_{t-1} - P_t c_t - P_{t-1} c_{t-1} + \bar{R}_{t-1} b_{t-1} \\ &= \underbrace{W_t N + T_t}_{=i_t} - \left[\underbrace{P_t c_t + W_t \left(N - l_t \right) + \left(R_{t-1} - 1 \right) \left(m_{t-1} - P_{t-1} c_{t-1} \right)}_{=s_t} \right] \\ &+ \bar{R}_{t-1} \left(\underbrace{b_{t-1} + m_{t-1} - P_{t-1} c_{t-1}}_{=a_{t-1}} \right), \end{aligned}$$

so that

$$a_t = (i_t - s_t) + \bar{R}_{t-1} a_{t-1}. \tag{2.8}$$

When assets, a_t, is negative, the household is in debt. The natural debt limit is the amount of debt that the household could just barely pay off by setting spending to zero forever after, $s_{t+j} = 0$ for $j \geq 1$. Note that by our construction of s_{t+1}, setting that to zero requires not holding excess cash balances, $m_t - P_t c_t = 0$, in period t. The limit is:

$$a_t \geq -\frac{1}{q_t} \sum_{s=1}^{\infty} q_{t+s} i_{t+s}, \tag{2.9}$$

We say $\{c_t, l_t, m_t, b_t\}_{t=0}^{\infty}$ is *a feasible allocation* if (2.2), (2.3), and (2.9) are satisfied. Given (2.6), then (2.9) is equivalent to the following condition:

$$\lim_{T \to \infty} q_T a_T \geq 0. \tag{2.10}$$

Below, we use this equivalence to work with (2.10) rather than (2.9).

Our third assumption is that discounted utility is well defined at the solution of the household problem:

$$|V| < \infty. \tag{2.11}$$

Prices at which this condition is not satisfied are not interesting for our analysis. First, we know that $V < \infty$ in any equilibrium, since leisure is bounded above and the resource constraint imposes $c_t \leq N$. Second, $V > -\infty$ because this can only occur when all feasible paths generate present discounted utility equal to $-\infty$. This is not an economically interest case.

Our fourth assumption is that for feasible allocations, $\{x_t\}_{t=0}^{\infty}$, the following sequence

$$\lim_{T\rightarrow\infty}\sum_{t=0}^{T}\beta F\left(x_{t-1},x_t\right), \tag{2.12}$$

converges to a finite number, or minus infinity. Finally, we define an *interior feasible allocation* as a feasible allocation in which consumption is strictly positive at all dates. Our fifth assumption is that a feasible interior sequence, $\{x_t\}_{t=0}^{\infty}$, has the property that (2.12) converges to a finite number. In our model, F has well defined derivatives at all interior feasible allocations. We refer to our fourth and fifth assumptions as *regularity conditions.*

2.2.3. Restatement of the Household Problem

In the next section, we derive the transversality condition associated with the household problem. To this end, it is convenient to reformulate the household problem stated in the previous section in two ways. First, we impose (obviously without loss of generality) the flow budget constraint, (2.2), as a strict equality. Second, we use that equation to substitute out for l_t. Third, we express the problem in a vector notation that allows us to establish the sufficiency of the transversality condition using an adaption of the strategy used in Stokey et al. (1989) and necessity using Kamihigashi (2002).

Solving (2.2) for l_t after replacing the weak inequality with a strict inequality:

$$l_t=\frac{m_t+b_t-\left(m_{t-1}-P_{t-1}c_{t-1}+\bar{R}_{t-1}b_{t-1}+T_t\right)}{W_t},$$

for all $t \geq 0$. We denote the date t choice variables by the 3×1 column vector x_t :

$$x_t = (c_t, m_t, b_t)^{'},$$

We denote the period utility function in the reformulated system by the function, F :

$$F(x_{t-1}, x_t) = u\left(c_t, \frac{m_t + b_t - (m_{t-1} - P_{t-1}c_{t-1} + R_{t-1}b_{t-1} + T_t)}{W_t}\right).$$

Note that F is concave (x_{t-1}, x_t) since u is concave. Obviously, the function, $F(x_{t-1}, x_t)$, is not only determined by household choice variables (x_{t-1}, x_t), but also by other variables, such as $P_t, \bar{R}_t, W_t, T_t$, beyond the control of the household. We do not make the dependence of F on the latter variables explicit in order to keep the notation simple.

The household's cash constraint is expressed as

$$A_t^{'} x_t \geq 0,$$

where A_t is a sequence taken as given by the household:

$$A_t = (-P_t, 1, 0)^{'}.$$

We express the natural debt limit as:

$$\lim_{T \to \infty} q_T \tilde{A}_T^{'} x_T \geq 0, \; \tilde{A}_T^{'} = \begin{bmatrix} -P_t & 1 & 1 \end{bmatrix}^T. \tag{2.13}$$

As noted above, the latter is equivalent to the natural debt limit, (2.9).

We say $\{x_t\}_{t=0}^{\infty}$ is *a feasible* sequence if $A_t' x_t \geq 0$ for all $t \geq 0$ and (2.13). The reformulated problem is as follows:

$$v(x_{-1}) = \max_{\text{feasible } \{x_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t F(x_{t-1}, x_t) \tag{2.14}$$

2.2.4. Necessary and Sufficient Conditions for Household Optimality

In this section we discuss the necessary and sufficient conditions for household optimization. That the first order conditions are part of the necessary and sufficient conditions is easy to verify and we do not discuss that here. Instead, we focus on the sufficiency and necessity of a transversality condition.

The first order conditions are:

$$\underbrace{F_2(x_{t-1}, x_t)}_{1\times 3} + \underbrace{\mu_t}_{1\times 1} \times \underbrace{A_t'}_{1\times 3} + \beta \underbrace{F_1(x_t, x_{t+1})}_{1\times 3} = 0, \tag{2.15}$$

$$\underbrace{\mu_t A_t' x_t}_{1\times 1} = 0, \tag{2.16}$$

$$\mu_t \geq 0, \tag{2.17}$$

where F_1 and F_2 denote 1×3 row vectors of derivatives with respect to the first and second arguments, respectively. Also, μ_t is the multiplier on the cash constraint, $A_t' x_t \geq 0$, in the Lagrangian representation of the household problem. The transversality condition is:

$$\lim_{T\to\infty} \beta^T F_2(x_{T-1}, x_T)\, x_T = 0. \tag{2.18}$$

To interpret 2.18 in terms of terms of our underlying model, note

$$-\frac{u_{c,0}}{P_0}\beta^T F_2\left(x_{T-1},x_T\right)x_T = -\frac{u_{c,0}}{P_0}\beta^T \left[\begin{array}{ccc} u_{c,T}, \frac{u_{l,T}}{W_T} & , & \frac{u_{l,T}}{W_T}\end{array}\right]\left[\begin{array}{c} c_T \\ m_T \\ b_T \end{array}\right]$$

$$= -\frac{u_{c,0}}{P_0}\beta^T \left[\begin{array}{ccc} u_{c,T}, -\frac{u_{c,T}}{P_T} & , & -\frac{u_{c,T}}{P_T}\end{array}\right]\left[\begin{array}{c} c_T \\ m_T \\ b_T \end{array}\right]$$

$$= q_T a_T \rightarrow_{T\rightarrow\infty} 0. \tag{2.19}$$

The second equality uses the household's labor first order condition. The third equality uses the intertemporal Euler equation for debt, as well as the definition of q_T in (2.7). Also, a_T denotes time T assets, $b_T + m_T - P_T c_T$ (see (2.8)). The convergence at the end shows that, in terms of our underlying model, the transversality condition, (2.18), is equivalent to 2.10.

2.2.4.1. Sufficiency of Transversality Condition. According to the following proposition, if a feasible interior sequence, $\{x_t\}_{t=0}^{\infty}$, satisfies (2.15), (2.16) and (2.17) for $t \geq 0$ and the transversality condition, (2.18), then that sequence solves the household problem in that there is no other feasible sequence that generates higher utility.

Proposition 3. *Suppose (i) the regularity conditions in section (2.2.2) hold and (ii) an interior feasible $\{x_t^*\}_{t=0}^{\infty}$ satisfies (2.15), (2.16), (2.17), and (2.18). Then, the sequence, $\{x_t^*\}_{t=0}^{\infty}$, solves the household problem, (2.14).*

Proof. The proof style follows that of Theorem 4.15 in Stokey et al. (1989). Let $\{x_t\}_{t=0}^{\infty}$ denote an arbitrary feasible sequence. Define

$$D_T = \sum_{t=0}^{T} \beta^t \left[F\left(x_{t-1}^*, x_t^*\right) - F\left(x_{t-1}, x_t\right) \right].$$

We wish to show that $\lim_{T\to\infty} D_T \geq 0$. The fact that F is concave implies:

$$F\left(x_{t-1}, x_t\right) \leq F_1\left(x_{t-1}^*, x_t^*\right)\left(x_{t-1} - x_{t-1}^*\right) + F_2\left(x_{t-1}^*, x_t^*\right)\left(x_t - x_t^*\right).$$

Then,

$$D_T \geq \sum_{t=0}^{T} \beta^t \left[F_1\left(x_{t-1}^*, x_t^*\right)\left(x_{t-1}^* - x_{t-1}\right) + F_2\left(x_{t-1}^*, x_t^*\right)\left(x_t^* - x_t\right) \right].$$

Collecting terms

$$\begin{aligned} D_T \geq\ & F_1\left(x_{-1}^*, x_0^*\right)\left(x_{-1}^* - x_{-1}\right) \\ & + \left[F_2\left(x_{-1}^*, x_0^*\right) + \beta F_1\left(x_0^*, x_1^*\right)\right]\left(x_0^* - x_0\right) \\ & + \beta\left[F_2\left(x_0^*, x_1^*\right) + \beta F_1\left(x_1^*, x_2^*\right)\right]\left(x_1^* - x_1\right) \\ & + \cdots \\ & + \beta^{T-1}\left[F_2\left(x_{T-2}^*, x_{T-1}^*\right) + \beta F_1\left(x_{T-1}^*, x_T^*\right)\right]\left(x_{T-1}^* - x_{T-1}\right) \\ & + \beta^T F_2\left(x_{T-1}^*, x_T^*\right)\left(x_T^* - x_T\right), \end{aligned}$$

or, after using (2.15):

$$D_T \geq F_1\left(x_{-1}^*, x_0^*\right)\left(x_{-1}^* - x_{-1}\right) + \sum_{t=0}^{T-1} \beta^t \left[-\mu_t A_t\left(x_t^* - x_t\right)\right] + \beta^T F_2\left(x_{T-1}^*, x_T^*\right)\left(x_T^* - x_T\right).$$

Now, consider

$$\sum_{t=0}^{T-1} \beta^t \left[-\mu_t A_t \left(x_t^* - x_t\right)\right] = \sum_{t=0}^{T-1} \beta^t \left[\left(-\mu_t A_t x_t^* + \mu_t A_t x_t\right)\right].$$

By the complementary slackness condition, $\mu_t A_t x_t^* = 0$ for all t. Thus,

$$D_T \geq F_1\left(x_{-1}^*, x_0^*\right)\left(x_{-1}^* - x_{-1}\right) + \sum_{t=0}^{T-1} \beta^t \mu_t A_t x_t + \beta^T F_2\left(x_{T-1}^*, x_T^*\right)\left(x_T^* - x_T\right).$$

By (2.17), $\mu_t \geq 0$. Feasibility requires $A_t x_t \geq 0$ and that the initial conditions be respected, so $x_{-1}^* - x_{-1} = 0$. We conclude

$$\begin{aligned} D_T &\geq \beta^T F_2\left(x_{T-1}^*, x_T^*\right)\left(x_T^* - x_T\right) \\ &= \beta^T F_2\left(x_{T-1}^*, x_T^*\right) x_T^* + \underbrace{\frac{P_0}{u_c(0)}}_{\geq 0} q_T \tilde{A}_T^T x_T. \end{aligned}$$

The transversality condition, (2.18), and feasibility of x_t implies that

$$\lim_{T\to\infty} \beta^T F_2\left(x_{T-1}^*, x_T^*\right) x_T^* = 0, \quad \lim_{T\to\infty} q_T \tilde{A}_T^T x_T \geq 0.$$

We conclude that

$$D_T \geq \psi_T,$$

where ψ_T is a sequence which converges, as $T \to \infty$, to a non-negative number. But, we do not know whether D_T itself converges. For example, D_T is the difference between two partial sums and we have not made any assumption that those partial sums converge. The regularity conditions in (i) guarantee that those sums do converge. This establishes our result. □

2.2.4.2. Necessity of Transversality Condition. We turn to the necessity of 2.18. We establish the result using the argument in Kamihigashi (2002). We show that his argument applies almost without change, even though he works with a version of the Stokey et al. (1989) model, while ours is a monetary model with a cash constraint.

Proposition 4. *Suppose (i)* $\{x_t^*\}_{t=0}^{\infty}$ *is a interior solution to the household problem, (2.14), and (ii) the value of the household problem is finite, i.e., condition (2.11) is satisfied. Then,* $\{x_t^*\}_{t=0}^{\infty}$ *satisfies (2.18).*

Proof. Consider a class of perturbations, $\{x_t(\lambda, T)\}_{t=0}^{\infty}$, on the optimal path, $\{x_t^*\}$:

$$x_t(\lambda, T) = \begin{cases} x_t^* & t \leq T \\ \lambda x_t^* & t > T \end{cases},$$

where $\gamma \leq \lambda < 1$. Also, γ is a scalar, $0 < \gamma < 1$, having the property that $\{x_t(\lambda, T)\}$ is a feasible interior sequence with finite discounted utility value for each $\lambda \in [\gamma, 1)$. Feasibility is trivial and imposes no restriction on γ, since $A_t^T x_t^* \geq 0$ implies $A_t^T x_t(\lambda, T) \geq 0$ for all $t \geq 0$, and each $\lambda \in [\gamma, 1)$. Similarly, the natural debt limit, (2.13), also places no restriction on γ. To see this, note that

$$\lim_{T \to \infty} q_t \tilde{A}_t' x_t(\lambda, T) = \lambda \lim_{T \to \infty} q_t \tilde{A}_t' x_t^* \geq 0.$$

That we can always choose a value of γ (perhaps very near to unity) so that $\{x_t(\lambda, T)\}_{t=0}^{\infty}$ has finite discounted utility follows from continuity of F.

Since $\{x_t^*\}$ is optimal, it follows that:

$$\infty > \sum_{t=0}^{\infty} \beta^t F\left(x_{t-1}^*, x_t^*\right) \geq \sum_{t=0}^{\infty} \beta^t F\left(x_{t-1}(\lambda, T), x_t(\lambda, T)\right). \tag{2.20}$$

Kamihigashi (2002)'s argument exploits a property of concavity. In particular, the fact that the slope of a concave function, $f : [\gamma, 1] \to \mathbb{R}$, is declining implies

$$\frac{f(1) - f(\lambda)}{1-\lambda} \leq \frac{f(1) - f(\gamma)}{1-\gamma}, \tag{2.21}$$

for $\lambda \in [\gamma, 1)$ (see Kamihigashi (2002), Lemma 3). Rearranging 2.20, we obtain

$$\begin{aligned} \beta^{T+1} \frac{F\left(x_T^*, \lambda x_{T+1}^*\right) - F\left(x_T^*, x_{T+1}^*\right)}{1-\lambda} &\leq \sum_{t=T+1}^{\infty} \beta^{t+1} \frac{F\left(x_t^*, x_{t+1}^*\right) - F\left(\lambda x_t^*, \lambda x_{t+1}^*\right)}{1-\lambda} \\ &\leq \sum_{t=T+1}^{\infty} \beta^{t+1} \frac{F\left(x_t^*, x_{t+1}^*\right) - F\left(\gamma x_t^*, \gamma x_{t+1}^*\right)}{1-\gamma}, \end{aligned} \tag{2.22}$$

where the second inequality is an application of (2.21). Note

$$\begin{aligned} \beta^{T+1} \frac{F\left(x_T^*, x_{T+1}^*\right) - F\left(x_T^*, \lambda x_{T+1}^*\right)}{1-\lambda} &\overset{\lambda \uparrow 1}{\to} \beta^{T+1} F_2\left(x_T^*, x_{T+1}^*\right) x_{T+1}^* \\ &= -\frac{P_0}{u_{c,0}} q_{T+1} a_{T+1}^* \leq 0. \end{aligned}$$

Here, the limit result follows from differentiability of F, the equality uses (2.19) and the weak inequality reflects feasibility (specifically, the natural debt constraint, 2.9).

Driving $\lambda \to 1$ in (2.22) and using the latter result, we obtain,

$$0 \leq -\beta^{T+1} F_2\left(x_T^*, x_{T+1}^*\right) x_{T+1}^* \leq \sum_{t=T+1}^{\infty} \beta^{t+1} \frac{F\left(x_t^*, x_{t+1}^*\right) - F\left(\gamma x_t^*, \gamma x_{t+1}^*\right)}{1-\gamma}$$

As $T \to \infty$, the term on the right converges to zero since the both infinite sums are finite when summed over all $t \geq 0$. This establishes (2.18). $\square$

2.3. Stochastic Standard Equilibrium

In this section we extend the deterministic monetary model introduced in the previous section to incorporate velocity shocks. We show two main results in this section. First, under the Taylor rule with the Taylor principle, the efficient allocations can be supported as an equilibrium. Second, the efficient allocations cannot be an equilibrium under the constant money growth rule. These results play a role when we discuss equilibrium robustness later in Section 2.6.2.

2.3.1. Model with velocity shock

We modify the deterministic model by adding an iid velocity shock, ν_t, to cash constraint (2.3):

$$P_t c_t \leq M_t \exp\left(\nu_t\right),$$

where $\nu_t \in \left\{\nu^1, \cdots, \nu^S\right\}$ and $\nu^i \leq \nu^j$ if $i \leq j$ for all $t \geq 1$, and $\nu_0 = 0$. The associated probability probability is denoted as $p \in \Delta^S \equiv \left\{p \in \mathbb{R}^S_+; \sum_{i=1}^S p_i = 1\right\}$. We assume that

$$\frac{\exp\left(-\gamma \nu^S\right)\bar{\mu}}{\beta E\left[\exp\left(-\gamma \nu_{t+1}\right)\right]} > 1. \tag{2.23}$$

The price setters choose their price before the realization of ν_t. So, the i^{th} intermediate good producer sets its price as follows:

$$p_{i,t} = E_{t-1}W_t. \tag{2.24}$$

The price index is the same as before:

$$P_t = \left(\int p_{i,t}^{1-\varepsilon} di\right)^{\frac{1}{1-\varepsilon}}. \tag{2.25}$$

Since $p_{i,t}$ is not a function of ν_t, then P_t is also not a function of ν_t. Households make their decision after observing ν_t. As a result, the first order necessary and sufficient conditions for optimization are modified as follows:

$$c_t^{-\gamma} = \beta \frac{\bar{R}_t}{\bar{\pi}_{t+1}} E_t c_{t+1}^{-\gamma} \tag{2.26}$$

$$1 = E_{t-1} c_t^{\gamma+\psi} \tag{2.27}$$

$$0 = [R_t - 1] [\exp(\nu_t) M_t - P_t c_t] \tag{2.28}$$

$$0 = \lim_{t\uparrow\infty} \beta^t E_0 \frac{u'(c_t)}{P_t} (M_t \exp(\nu_t) - P_t c_t). \tag{2.29}$$

Notice that the efficient allocation is the same as the deterministic model. To show it formally, consider the following Planning problem in order to define an efficient allocation.

$$\max_{c_t(\nu_t), l_t(\nu_t)} \quad E_0 \sum_{t=0}^{\infty} \beta^t u(c_t(\nu_t), l_t(\nu_t))$$
$$s.t. \quad c_t(\nu_t) = l_t(\nu_t).$$

$\{c_t^*(\nu_t), l_t^*(\nu_t)\}$ denotes the solution to above problem. The problem is separable across time and state so that the solution should satisfy $c_t^*(\nu_t) = c^*$ and $l_t^*(\nu_t) = l^*$. (c^*, l^*) is a solution to

$$\max_{c,l} \quad u(c,l)$$
$$s.t. \quad c = l.$$

The first order condition is

$$\frac{-u_l(c,c)}{u_c(c,c)} = 1 \iff c^{\psi+\gamma} = 1 \iff c = 1.$$

Therefore the efficient allocation is $(c_t, l_t) = (1,1)$. Obviously the welfare associated with the efficient allocation is strictly higher than other equilibrium.

Proposition 5. *Suppose that there exists a non-efficient equilibrium. The welfare associated with* $(c_t, l_t)_{t=0}^{\infty}$ *is strictly lower than the welfare associated with the first best allocation.*

Proof. Since the period utility is strictly concave and $(c_t, l_t)_{t=0}^{\infty}$ is not an efficient allocation, we obtain

$$u(c_t, l_t) \leq u(c^*, l^*) + u_1(c^*, l^*)(c_t - c^*) + u_2(c^*, l^*)(l_t - l^*)$$

for all t and the inequality becomes strict for some $t \geq 0$. Then taking the expectation and sum over t, we obtain

$$\begin{aligned}\sum_{t=0}^{\infty} \beta^t Eu(c_t, l_t) &< \sum_{t=0}^{\infty} \beta^t \left[Eu(c^*, l^*) + u_1(c^*, l^*)(Ec_t - c^*) + u_2(c^*, l^*)(El_t - l^*)\right] \\ &= \sum_{t=0}^{\infty} \beta^t \left[Eu(c^*, l^*) + u_1(c^*, l^*) \underbrace{\left[1 + \frac{u_2(c^*, l^*)}{u_1(c^*, l^*)}\right]}_{=0} (Ec_t - c^*)\right] \\ &= \sum_{t=0}^{\infty} \beta^t Eu(c^*, l^*),\end{aligned}$$

which is desired. □

2.3.2. Equilibrium Properties

Now we study equilibrium properties.

2.3.2.1. Under Taylor rule. We show that the efficient allocations can be supported as an equilibrium under the Taylor rule with the Taylor principle. Note that in the efficient allocations,

$$c_t = l_t = 1 \tag{2.30}$$

for all t and ν_t. The efficiency requires that the real variables are independent of the shocks since the velocity shocks do not affect the households' intra-temporal conditions and the production frontier. We construct an equilibrium in which the levels of consumption and labor are (2.30) as follows:

$$\left(c_t, l_t, \bar{\pi}_t, \bar{M}_t, \bar{R}_t\right) = \left(1, 1, \bar{\pi}^*, \frac{\left(\bar{\pi}^*\right)^t P_{-1}}{\exp\left(\nu_t\right)}, \bar{R}^*\right). \tag{2.31}$$

Proposition 6. *(2.31) is an equilibrium, and the equilibrium is determinate.*

Proof. It is easy to verify that (2.31) satisfy the equilibrium conditions, (2.26), (2.27), (2.29). To show (2.28), note that the interest rate is strictly positive and

$$\bar{M}_t = \frac{\left(\bar{\pi}^*\right)^t P_{-1}}{\exp\left(\nu_t\right)} \iff M_t \exp\left(\nu_t\right) = P_t c_t.$$

The equilibrium is determinate since the Taylor principle is satisfied. Thus we establish Proposition (2.31). □

In the stochastic model, (2.31) is the desired or target allocations.

2.3.2.2. Money growth rule. We establish two results under the constant money growth rule. First, there exists a unique equilibrium in which the nominal rate is strictly positive. Second, we show that (2.30) is not supported as equilibrium outcomes.

We assume (and then verify) that $\bar{R}_t > 1$ for all $t \geq 0$. In that case, the cash constraint is satisfied as a strict equality in all dates and states, so that

$$c_t = \frac{M_t \exp\left(\nu_t\right)}{P_t}.$$

Multiplying both sides to the power of $\gamma + \psi$, we obtain:

$$c_t^{\gamma+\psi} = \left(\frac{M_t}{P_t}\right)^{\gamma+\psi} \exp\left(\left(\gamma + \psi\right) \nu_t\right) \tag{2.32}$$

Taking the expectation at time $t-1$ and using (2.27), we obtain

$$E_{t-1}c_t^{\gamma+\psi} = \left(\frac{M_t}{P_t}\right)^{\gamma+\psi} E_{t-1}\exp\left(\left(\gamma+\psi\right)\nu_t\right) = 1,$$

for $t \geq 0$. Then, the price index is expressed in terms of M_t and exogenous variables:

$$P_t = M_t\left[E_{t-1}\exp\left(\left(\gamma+\psi\right)\nu_t\right)\right]^{\frac{1}{\gamma+\psi}}. \tag{2.33}$$

Since the government follows constant money growth rule with rate $\bar{\mu}$, the inflation rates from 1 is $\bar{\mu}$:

$$\bar{\pi}_{t+1} = \bar{\mu} \quad t \geq 0.$$

From 2.32 and 2.33 consumption and employment are also expressed in terms of exogenous variables:

$$c_t = l_t = \left[E\exp\left(\left(\gamma+\psi\right)\nu_t\right)\right]^{\frac{-1}{\gamma+\psi}}\exp\left(\nu_t\right),$$

where E reflects our assumption that ν_t are iid. We have

$$c_t^{-\gamma} = \left[E\exp\left(\left(\gamma+\psi\right)\nu_t\right)\right]^{\frac{\gamma}{\gamma+\psi}}\exp\left(-\gamma\nu_t\right)$$

The Euler equation residually determine the nominal interest rate:

$$\bar{R}_t = \frac{c_t^{-\gamma}\bar{\mu}}{\beta E_t c_{t+1}^{-\gamma}} = \frac{\exp\left(-\gamma\nu_t\right)\bar{\mu}}{\beta E_t\exp\left(-\gamma\nu_{t+1}\right)}. > 1$$

From 2.23, $\bar{R}_t$ is strictly greater than 1. Condition (2.29) is satisfied trivially because the cash in advance constraint holds as a strict equality in all periods.

We summarize this result in the form of a proposition:

Proposition 7. *Suppose the velocity shock,* ν_t*, satisfies (2.23). Under the constant money growth policy,* $\bar{\mu}$*,there exists a unique equilibrium in which* $\bar{R}_t > 1$ *for all* $t \geq 0$*. In particular, the allocations and the prices are given as follows:*

$$P_t = M_t \left(E \left[\exp \left(\left(\gamma + \psi \right) \nu_t \right) \right] \right)^{\frac{1}{\gamma+\psi}}$$

$$c_t = \left[E \exp \left(\left(\gamma + \psi \right) \nu_t \right) \right]^{\frac{-1}{\gamma+\psi}} \exp \left(\nu_t \right)$$

$$\bar{R}_t = \frac{\exp \left(-\gamma \nu_t \right) \bar{\mu}}{\beta E \left[\exp \left(-\gamma \nu_{t+1} \right) \right]}$$

$$W_t = P_t c_t^{\gamma+\psi}.$$

We do not have proof that the equilibrium in Proposition 7 is the only equilibrium of the constant money growth model. Instead, we now show that equilibrium consumption cannot be constant when there are velocity shocks. This is obviously true in the equilibrium described in 7. But, it is also true for any other equilibria, if they exist.

Proposition 8. *Suppose* $\bar{\mu} > 1$ *and*

$$\frac{\bar{\mu}}{\beta} \exp \left(\nu_1 - \nu_S \right) > 1. \tag{2.34}$$

Then in any equilibrium, consumption c_t *cannot be constant.*

Proof. Suppose the proposition is false, so that $c_t = c$ for all t. Then, by (2.27), $c = 1$. First, we show that if $\bar{R}_t = 1$ for some t, then $\bar{R}_{t+s} = 1$ for all $s \geq 0$. To see this, suppose that $\bar{R}_t = 1$. Then the Euler equation implies

$$\bar{\pi}_{t+1} = \beta.$$

Then

$$\frac{M_{t+1}\exp(\nu_{t+1})}{P_{t+1}c_{t+1}} = \frac{\bar{\mu}}{\beta}\underbrace{\frac{M_t\exp(\nu_t)}{P_t}}_{\geq 1}\exp(\nu_{t+1}-\nu_t) \geq \frac{\bar{\mu}}{\beta}\exp(\nu_{t+1}-\nu_t) > 1,$$

by assumption (2.34). We conclude that $\bar{R}_{t+1} = 1$. Repeating this argument for $s \geq 2$, we obtain the result sought. That is, $\bar{R}_{t+s} = 1$ for all $s \geq 0$.

With inflation equal to β in each period, we have $P_t = \beta^t P_0$. Also, $M_t = \bar{\mu}^t M_0$, by constant money growth. We then have

$$\beta^t E_0\left(\frac{M_t\exp(\nu_t)}{P_t} - c_t\right) = \left[\frac{M_0}{P_0}\bar{\mu}^t - \beta^t\right] \uparrow \infty,$$

since $\bar{\mu} > 1$ and $\beta < 1$. We now have a contradiction, since the transversality condition, (2.29), is violated. We conclude that $\bar{R}_t > 1$ for all $t \geq 0$. From Proposition

Second, since $\bar{R}_t > 1$ for all $t \geq 0$, the cash constraint is binding for all $t \geq 0$.

$$P_t = M_t\exp(\nu_t) = \bar{\mu}^t M_0\exp(\nu_t).$$

But, from (2.24) and (2.25) we see that P_t is not a function of ν_t. Contradiction. □

2.4. Setting for Sophisticated Equilibrium

We describe a sophisticated equilibrium for the model in Section 2.3. We divide the period into two parts. In the first sub-period, (i), intermediate good firms set their prices, $p_{i,t}$, $i \in I$. In the second sup-period, (ii), the prices set by intermediate firms are a state variable. A velocity shock ν_t realizes and is observed by households and the government. At this point goods, labor and financial market decisions are made by households, final good firms and government. Intermediate good firms are simply required to satisfy whatever demand occurs at the price they set in morning. Monetary policy can be governed by an interest rate rule or a money growth rule. We discuss strategy equilibria under these two scenarios in separate subsections below.

2.4.1. Timing

Events occur in a particular order in a period in the sophisticated equilibrium. This section explains that ordering.

2.4.1.1. Morning. Consider the i^{th} intermediate good firm, $i \in I$. That firm understands that it faces a constant elasticity demand curve. Given a government subsidy that cancels the effects of monopoly power and the fact that intermediate good firms have the following constant returns to scale production function:

$$Y_{t,i} = l_{t,i},$$

the i^{th} firm sets price as follows:

$$p_{i,t} = E_{t-1} W_t^e, \tag{2.35}$$

where W_t^e denotes the wage rate that the i^{th} firm believes will prevail in the second subperiod. The expectation operator, $E_{t-1}(\cdot)$ represents the expectation operator conditional on the information available at the beginning of the period (i). The operator is needed since a shock hits an economy in the afternoon and the i^{th} intermediate firm needs to set its price before the realization of the shock. (see the discussion in section 2.4.1.2.) In what follows we first explain why the i^{th} firm makes its price decision based on a believed wage, rather than the actual realized wage. Then, we explain why the believed wage, W_t^e, is not indexed by i.

The reason the realized period t wage rate is not observed in morning is that the labor market does not clear until the afternoon. The existence of this delay in the realization of W_t reflects that the labor market cannot clear until the aggregate price index, P_t, is realized (households need to know P_t to determine their labor supply response to the wage rate, W_t). But, it is no possible for P_t to be known at the time firm i sets $p_{i,t}$ since (a)P_t is the aggregate over all $p_{j,t}$ for all $j \in I \backslash \{i\}$ (see below) and (b) all intermediate good firms set their price simultaneously. At the time that the firm is contemplating what to set $p_{i,t}$ to, all the other firms are doing the same thing, so P_t does not yet exist.

The i^{th} intermediate good firm forms its belief about the nominal wage as the product of its belief about the real wage and its belief about the price level:

$$W_t^e = \left(\frac{W_t}{P_t}\right)^e P_t^e. \tag{2.36}$$

Because the real wage will be determined in a labor market which will clear in the afternoon and that market is competitive, the i^{th} firm understands that the above expression can be written as

$$W_t^e = (MRS_t^e)\, P_t^e,$$

where MRS_t^e is the marginal rate of substitution of the representative household in the labor market that will occur in the afternoon. Again, the 'e' reflects that that market will occur later, and so it is the i^{th} intermediate good firm's *belief* about MRS_t that appears here, rather than the actual realized value of MRS_t. As explained above, it is the belief of the i^{th} intermediate good firm about the aggregate price level, P_t^e, that appears here.

Later, we will show that MRS_t^e can be computed as part of a continuation equilibrium conditional on P_t^e. We begin with a discussion of P_t^e.

The i^{th} intermediate good firm forms its belief, P_t^e, knowing that P_t will be determined by the decisions of all the other intermediate good price setters, i.e., the j^{th} price setters, $j \neq i$, $j \in I$. So a belief by the i^{th} price setter about P_t must begin with a belief about $p_{j,t}$ for $j \neq i$, $j \in I$. We denote the i^{th} price setter's belief about $p_{j,t}$ by $p_{j,t}^e(i)$. We make the following assumption about $p_{j,t}^e(i)$:

Assumption 1. *Symmetry:* $p_{j,t}^e(i) = p_{j',t}^e(i)$ *for all* $j, j' \in I - i$.

Under this assumption the i^{th} price setter believes that all the other price setters will choose the same price. This perhaps seems natural because of the symmetry across intermediate good firms. We denote the i^{th} firm's belief about the price set by other firms by $p_t^e(i)$, for $i \in I$.

We also adopt a second assumption:

Assumption 2. *Consensus*: $p_t^e(i) = p_t^e(i')$ *for all* $i, i' \in I$.

According to the consensus assumption, each intermediate good producer has the same belief about the price that the others will set. This assumption is not particularly 'natural' since the intermediate price setters do not communicate. In equilibrium, we will adopt the standard rational expectations assumption that intermediate good producers set W_t^e, and therefore price, as a given function of publicly available history. We do not address the important question of how this comes to be. However, to gain a deeper understanding into the economics of the equilibrium, we explore what would happen 'out of equilibrium' if price setters departed from their rules. For the most part, we will explore departures which maintain the consensus assumption, Assumption (2). However, we will also consider 'trembles' when the consensus assumption is violated.

We denote a firm's belief about what other firms will do under Assumption (2) by p_t^e. In this case, all firms believe the aggregate price index will be:

$$P_t^e = p_t^e,$$

given the linear homogeneity of P_t^e as a function of intermediate good prices.

2.4.1.2. Afternoon. In the afternoon a velocity shock is realized, and all markets meet and clear conditional on the intermediate goods prices, $p_{i,t}$, set in morning. There is a large number of identical and competitive final good producers. The representative producer's production is:

$$Y_t = \left[\int_0^1 Y_{i,t}^{\frac{\varepsilon-1}{\varepsilon}} di \right]^{\frac{\varepsilon}{\varepsilon-1}}, \quad \varepsilon > 1.$$

Final good firms also take the price of final goods, P_t, as given. The first order conditions associated with profit maximization are given by:

$$Y_{i,t} = Y_t \left(\frac{p_{i,t}}{P_t} \right)^{-\varepsilon}, i \in I.$$

Intermediate good firms are required to supply whatever demand materializes at the price that they set in morning. Combining the demand curves with this production function, we obtain:

$$P_t = \left[\int_0^1 p_{i,t}^{1-\varepsilon} di \right]^{\frac{1}{1-\varepsilon}},$$

where $p_{i,t}$ denotes the price set by the i^{th} intermediate good producer in morning. This price is a state variable in the afternoon.

Household optimality in the labor market is summarized by the following:[1]

$$\frac{W_t}{P_t} = MRS_t = l_t^{\psi} c_t^{\gamma}. \tag{2.37}$$

Similarly, household participation in the bond market is associated with the following intertemporal first order condition:

$$c_t^{-\gamma} = \beta E_t \left[c_{t+1}^{-\gamma} \frac{\bar{R}_t}{\bar{\pi}_{t+1}} \right], \tag{2.38}$$

where $\bar{\pi}_t$ denotes the gross inflation rate at time t, P_t/P_{t-1}. Optimality in the money market, given the household's cash constraint, $\exp(\nu_t) M_t \geq P_t c_t$, implies the following complementary slackness condition:

$$(R_t - 1)\left[M_t \exp(\nu_t) - P_t c_t \right] = 0.$$

The transversality condition is:

$$\lim_{T \to \infty} E_0 \beta^T c_T^{-\gamma} \frac{M_T \exp(\nu_T) - P_T c_T + b_T}{P_T} = 0.$$

[1]We assume the household period utility function is $\ln c_t - l_t^{1+\psi} / (1+\psi)$, where $\psi \geq 0$.

The government chooses the gross growth rate, $\bar{\mu}_t$, of the aggregate money stock, M_t :

$$\bar{\mu}_t = \frac{M_t}{M_{t-1}}.$$

The money is injected by transfers to households:

$$T_t = M_t - M_{t-1} - \tau_t,$$

where τ_t denote the nominal value of subsidies provided to intermediate good firms.

We consider monetary policy with an escape clause. In particular, if inflation has been in the monitoring range in each past period, then the government follows the Taylor rule in the current period:

$$\bar{R}_t = \max\left\{1, \bar{R}^*\left(\frac{\bar{\pi}_t}{\bar{\pi}^*}\right)^{\phi}\right\}, \quad \bar{\pi}_{t+1} \equiv \frac{P_{t+1}}{P_t}, \quad \bar{R}^* \equiv \bar{\pi}^*/\beta, \tag{2.39}$$

where $\bar{\pi}^* \geq 1$ and $\bar{R}^*$ are the desired inflation and interest rate, respectively. Here, $\phi > 1$ corresponds to the Taylor principle. If, in the past, inflation has ever fallen outside the monitoring range, then the government switches to a constant money growth rate rule in which the scaled money growth rate is $0 \leq \mu \leq \pi_u$.

Goods market clearing in the afternoon, as well as the technology and the fact that intermediate good prices are all the same, corresponds to:

$$c_t = Y_t = l_t. \tag{2.40}$$

Bond market clearing:

$$b_t = 0.$$

2.4.2. Scaling and logging the variables

We find it convenient to express the variables in log deviation from the desired equilibrium, in which $c_t = 1$, $\bar{\pi}_t = \bar{\pi}^*$, $\bar{R}_t = \bar{R}^*$, $\bar{\mu}_t = \bar{\mu}^*$, where

$$\beta \bar{R}^* = \bar{\pi}^* = \bar{\mu}^*.$$

It is easy to verify that the level of consumption and employment in the desired equilibrium is first best.

Also,

$$R_t \equiv \ln\left(\frac{\bar{R}_t}{\bar{R}^*}\right), \quad \pi_{t+1} \equiv \ln\left(\frac{\bar{\pi}_t}{\bar{\pi}^*}\right).$$

The Taylor rule can be expressed as follows:

$$R_t = \max\left\{R^l, \phi\pi_t\right\}, \quad R^l \equiv \ln\left(\frac{1}{\bar{R}^*}\right). \tag{2.41}$$

The logged and scaled money growth rate is

$$\mu_t = \ln\left(\frac{\bar{\mu}_t}{\bar{\pi}^*}\right). \tag{2.42}$$

The Euler equation, 2.38, in scaled terms, is:

$$R_t - \pi_{t+1} = -\ln\left\{E_t\left[\left(\frac{c_{t+1}}{c_t}\right)^{-\gamma}\right]\right\}, \tag{2.43}$$

where E_t means the condition expectation based on information available in the afternoon of time t.

Combining (2.35) with (2.36), (2.37) and (2.40), we obtain

$$\ln p_{i,t} = \ln E_{t-1}\left[P_t^e\left(c_t^e\right)^{\gamma+\psi}\right] = \ln P_t^e + \ln E_{t-1}\left(c_t^e\right)^{\gamma+\psi},$$

where P_t^e is the conjecture that the i^{th} firm has about P_t at the time it sets its price. Now, subtract $\ln\left(\bar{\pi}^* P_{t-1}\right)$ from both sides to obtain:

$$x_{i,t} = \pi_t^e + \ln E_{t-1}\left(c_t^e\right)^{\gamma+\psi}, \tag{2.44}$$

where

$$x_{i,t} \equiv \ln p_{i,t} - \ln P_{t-1} - \ln \bar{\pi}^*,\ \pi_t^e \equiv \ln P_t^e - \ln P_{t-1} - \ln \bar{\pi}^*.$$

In the model without trembles,

$$x_{i,t} = x_t \text{ for all } \quad i \in I. \tag{2.45}$$

That is to say, x_t denotes the scaled price chosen by the typical firm.

2.4.3. Continuation Equilibrium Conditions

In Section 2.5 below, we define a sophisticated equilibrium. For this, we must define the model equilibrium conditions, conditional on any past history. There are two types of past history that are of interest. In what we later call a continuation equilibrium given h_{t-1}, we require all the equilibrium conditions which apply in date t and also the equilibrium conditions that apply in later dates. In what we call a continuation equilibrium given $h_{x,t}$, we suppose that morning date t price choices have been determined, so that $\{p_{i,t}\}_{i\in I}$ (hence, x_t) and are state variables. For this type of continuation equilibrium, with takes

x_t as give, we obviously drop the associated optimality condition. All the equilibrium conditions in the afternoon of period t, as well as all equilibrium conditions in $t+1$ and later are included.

The continuation equilibrium conditions given h_{t-1} are as follows. Combining (2.44) with (2.45), we have the price setting equation of the typical intermediate good producer:

$$x_t = \pi_t + \ln E_{t-1} c_t^{\gamma+\psi}. \tag{2.46}$$

This latter condition was discussed in the previous section. It convolves the intermediate good price setter condition with the household intra-temporal Euler equation. Given that all intermediate good price setters set the same price, we have:

$$\pi_t = x_t.$$

We have the household intertemporal Euler equation, (2.43):

$$R_t - \pi_{t+1} = -\ln E_t \left[\left(\frac{c_{t+1}}{c_t} \right)^{-\gamma} \right].$$

The household Kuhn-Tucker condition is:

$$\left[\bar{R}_t - 1\right] \left[\bar{M}_t \exp\left(\nu_t\right) - P_t c_t\right] = 0, \quad \bar{R}_t \geq 1,\ \bar{M}_t \exp\left(\nu_t\right) \geq P_t c_t. \tag{2.47}$$

The household transversality condition is:

$$\lim_{T \to \infty} \beta^T E_0 c_T^{-\gamma} \frac{\bar{M}_T \exp\left(\nu_T\right) - P_T c_T}{P_T} = 0.$$

Finally, monetary policy in period t is a constant money growth rate, $\mu \in [\pi_l, \pi_u]$ in the event that the monitoring range was violated in any past period. Otherwise, time t monetary policy is the Taylor rule, (2.41).

The continuation equilibrium conditions given $h_{x,t}$ are all of the above, except that (2.46) is dropped.

Here we illustrate how the continuation equilibrium is uniquely determined at $h_{x,t}$ if the government uses the constant money growth rule as its monetary policy. For simplicity, consider a deterministic version of the model. From Lemma 1, we know that the equilibrium is unique from $t = 1$: for all $t \geq 0$,

$$c_{t+1} = l_{t+1} = 1$$

$$\bar{\pi}_{t+2} = \bar{R}_{t+1} = 0,$$

and the initial inflation rate is

$$\bar{\pi}_1 = \bar{\mu}\frac{M_0}{P_0}.$$

Assume (and verify later) that the interest rate at date 0 is positive; $\bar{R}_0 > 1$. Then

$$c_0 = \frac{M_0}{P_0} = \frac{\bar{\mu}}{x_0}\frac{M_{-1}}{P_{-1}}.$$

Then the equilibrium variables are all determined. However, the marginal rate of substitution between the consumption and labor is not equalized to the marginal transformation between them

$$MRT = 1 \neq c_0^{\gamma+\psi} = MRS,$$

unless x_0 is $\bar{\mu}\frac{M_{-1}}{P_{-1}}$, which is an equilibrium condition at the beginning of period (i).

2.4.4. Best Responses, Trembles and Strategically Interesting Agents

2.4.4.1. Out of Equilibrium Versus Equilibrium. In the analysis of economic models we obviously like to understand the properties of their equilibria. But we also often like to understand why it is that alternative allocations are *not* equilibria. For the latter question, we need to make precise concepts of 'out of equilibrium' versus 'in equilibrium'. In macroeconomics there is a lot of experience with this type of thinking, dating back at least to the 1980s. At that time, the interest was in modeling the decision of a large agent (the government) who did not have the ability to commit to anything (i.e., not to actions or strategies). To make its decision, the large agent had to choose each period between the alternative actions available to it. In order for the choice to be well defined, the large agent needed to know what the continuation equilibrium was after each candidate action. In particular, the agent had to understand and take into account the mapping from its actions to the equilibrium that would unfold as a result. Because the agent takes into account the economy's response to its actions, the agent is said to be *strategic*. Early work that formalized these ideas include, for example, Kydland and Prescott (1977), Stokey (1991) and V.V. and Kehoe Patrick (1993). More recent applications that apply this approach to monetary policy include Chari et al. (1998) and Albanesi et al. (2003a).

The environment we consider here is in substance very different. Here, the large agent - the government - is able to commit at date 0. However, we follow Bassetto (2005) in insisting that the government can only commit to things that are ex post feasible. For the reasons explained in Bassetto (2005), ex post feasibility in general implies that the government must commit to strategies (i.e., reactions to the private economy outcomes) and cannot commit to date and state contingent actions. Bassetto (2005) discusses the *implementation problem*, the design of government strategies for the purpose of achieving particular desired equilibrium outcomes. Bassetto (2005) shows that the implementation problem is surprisingly challenging. There are cases in which there is no solution. Sometimes there can be cases there is a solution, but the desired outcome is not the only outcome possible.

In a more recent paper, Atkeson et al. (2010) show how minor adjustments to the conceptual tools developed in the earlier literature apply to the environment considered here. In our analysis we follow the formalism in Atkeson et al. (2010).

In our environment, each intermediate good producer (like the government in the earlier literature) has to make a choice among a series of alternatives and each alternative is evaluated according to the continuation equilibrium associated with it. This choice arises for the i^{th} intermediate good firm from its need to set the price, $p_{i,t}$, for its good. As we explain below, to set $p_{i,t}$ the intermediate good producer must form a belief, P_t^e, about what value the aggregate price level will take on. This is because the intermediate good producer cannot actually see the aggregate price, P_t, at the time that it sets p_{it}. This is for the simple reason that the aggregate price is a function of the prices set by all intermediate good producers and so its actual value cannot possibly be realized until after the intermediate good firms have set their price.

To form its belief, P_t^e, the i^{th} intermediate good producer considers a range of potential values of P_t^e. Corresponding to each potential value of P_t^e there is a belief about the continuation equilibrium, which we denote by $A\left(P_t^e\right)$. Given $A\left(P_t^e\right)$, the intermediate good producer's problem is well defined and it chooses $p_{i,t}$. We refer to the mapping from P_t^e to $p_{i,t}$ as the intermediate good firm's 'best response', $p_{i,t} = f\left(P_t^e\right)$.

To select P_t^e, the i^{th} intermediate good producer proceeds as follows. In our environment, the consensus assumption discussed in section 2.4.1.1 is that all intermediate good producers have the same belief, P_t^e. As a result, the i^{th} producer can extrapolate from its own best response to P_t, which aggregates the best responses of all the firms. To see this, recall that the aggregate price index, P_t, is given by

$$P_t = \left[\int_0^1 p_{j,t}^{1-\varepsilon} dj \right]^{\frac{1}{1-\varepsilon}} .$$

In this way, the i^{th} producer can construct a mapping from P_t^e to P_t, which we denote by F:

$$P_t = F\left(P_t^e\right) \equiv \left[\int_0^1 \left(f\left(P_t^e\right)\right)^{1-\varepsilon} di \right]^{\frac{1}{1-\varepsilon}} = f\left(P_t^e\right).$$

So, $F\left(P_t^e\right)$ is the i^{th} firm's belief about P_t conditional on P_t^e. Evidently, unless $P_t^e = F\left(P_t^e\right)$ there is an internal inconsistency in the i^{th} firm's beliefs. This is why we assume that the i^{th} producer selects a value for P_t^e that is a fixed point of F.

So, the firm selects about the aggregate price index are internally In selecting a value for P_t^e, the i^{th} firm would never choose one that fails to be a fixed point of F. This gives us to talk about on and off equilibrium allocations. Off equilibrium allocations are $A\left(P_t^e\right)$ for values of P_t^e that are not fixed points of F.

The situation of the intermediate good producer is analogous to that of the large agent in the earlier literature without commitment. There, the choice of P_t^e corresponds to the government's choice of an action and $A\left(P_t^e\right)$ corresponds to the consequence of that action. Of course, there is an important distinction between our environment and that in the earlier literature. Because the intermediate good agent is atomistic, there is

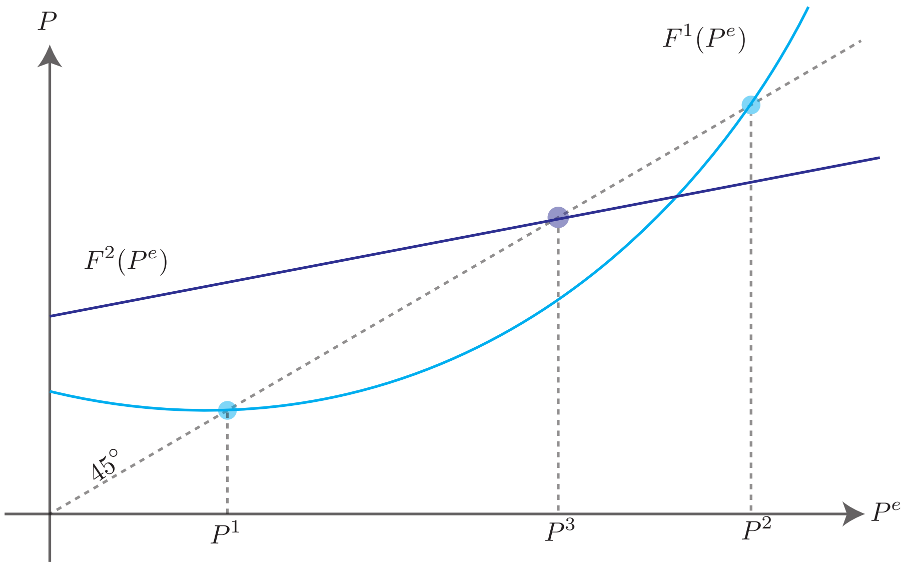


Figure 2.4.1. Illustration of fixed point

no sense in which the selection of P_t^e by an intermediate firm actually *causes* $A\left(P_t^e\right)$ in the sense that a government action causes $A\left(P_t^e\right)$ in the earlier literature. So the sense in which the intermediate good firm is strategic is not exactly the same as the sense in which the government in the earlier literature is strategic. This is why we follow Atkeson et al. (2010) in referring to the intermediate good agents as *strategically interesting*.

A simple way to illustrate the above ideas uses Figure 2.4.1. That displays two F functions, F^1 and F^2. Note that F^1 has two fixed points, P_1^e and P_2^e. Suppose that government in period -1 is contemplation its monetary policy strategy. The properties of the policy strategy influence the shape of F. So, F^1 is associated with one particular policy and this policy has two equilibria. Suppose that the government does not like those equilibria (that is, it does not like the allocations, $A\left(P_1^e\right)$ and $A\left(P_2^e\right)$). Suppose that a different policy produces the function, F^2, that crosses the 45 degree line once at P_3^e, which is associated with an equilibrium, $A(P_3^e)$, that the government desires. In this case, the second policy discourages firms from choosing the beliefs, P_2^e and P_1^e and encourages them to instead choose P_3^e. From the point of view of the second policy, P_2^e is not an equilibrium because it is not an fixed point.

2.4.4.2. Trembles and Observations on Atkeson et al. (2010). The discussion in the previous subsection allows us to illustrate the use of trembles to establish robustness, as well as the heart of our discussion of the Atkeson et al. (2010). Let P^* denote the price in the desired equilibrium, $A(P^*)$. Atkeson et al. (2010) construct a monetary policy in which $A(P^e)$ is discontinuous at $P^e = P^*$. That is,

$$\lim_{P^e \to P^*} A(P^e) \neq A(P^*) . \tag{2.48}$$

The discontinuity at the desired equilibrium arises because the escape clause is activated as soon as there is any deviation of any magnitude from the desired equilibrium. In the deviation, monetary policy switches to a constant money growth rule. The equilibrium under this rule is very different from the equilibrium under the interest rate rule if the economy is subject to velocity shocks.

We now consider a tremble. Suppose that firms know what price they would like to set, but they get it wrong (their hand 'trembles' when they write down the price). Thus, we modify the best response as follows:

$$p_{i,t} = f(P_t^*)\, \upsilon_i,$$

where υ_i is a uniform random variable with support $[1-\delta, 1+\delta]$. Each intermediate good firm, $i \in I$ draws independently from the uniform distribution. In this case, the price index is also "trembled". P_t^δ denote the associated price index under the trembles.

$$\begin{aligned} P_t^\delta = F\left(P_t^*;\delta\right) &\equiv \left[\int_0^1 \left(f\left(P_t^*\right)\upsilon_i\right)^{1-\varepsilon} di\right]^{\frac{1}{1-\varepsilon}} \\ &= f\left(P_t^*\right)\left[\int_0^1 \upsilon_i^{1-\varepsilon} di\right]^{\frac{1}{1-\varepsilon}} \\ &= f\left(P_t^*\right)\kappa^\delta, \end{aligned}$$

where

$$\begin{aligned} \kappa^\delta &\equiv \left[\int_0^1 \upsilon_i^{1-\varepsilon} di\right]^{\frac{1}{1-\varepsilon}} \\ &= \left[\frac{1}{2-\varepsilon}\left\{\frac{(1+\delta)^{2-\varepsilon} - (1-\delta)^{2-\varepsilon}}{2\delta}\right\}\right]^{\frac{1}{1-\varepsilon}}. \end{aligned} \tag{2.49}$$

Note that κ^δ is positive and goes to 1 as δ goes to 0. The F we had before corresponds to $F(\cdot;0)$. Suppose that P_t^* is a fixed point of $F(\cdot;0)$, so that

$$P_t^* = F\left(P_t^*;0\right).$$

Then, $F\left(P_t^*;\delta\right) = F\left(P_t^*;0\right)\kappa^\delta$, so that P_t^* is obviously not a fixed point for $F(\cdot;\delta)$:

$$P_t^* \neq F\left(P_t^*;0\right)\kappa^\delta.$$

We can see that $\lim_{\delta \to 0} P_t^{\delta} = P_t^*$ since $\kappa^{\delta} \to 1$ as $\delta \downarrow 0$. But, of course $\lim_{\delta \downarrow 0} A\left(P^{\delta}\right) \neq A\left(P^*\right)$ because of the discontinuity of A, (2.48). In this case, we say that the equilibrium is not robust to trembles in actions. Even with an extremely small tremble, there is a discrete shift in the equilibrium allocations.

Atkeson et al. (2010) argue that their equilibrium is robust to trembles in actions. However, that is because they assume that the price index, P_t, is the arithmetic average of the individual prices. As a consequence, $P^{\delta} = P^*$ for $\delta \geq 0$. In this case, trivially,

$$A\left(P_t^{\delta}\right) = A\left(P_t^*\right),$$

for all $\delta \geq 0$. Arguably, their robustness result is not interesting, because models where the aggregate price level is an equally weighted average of individual prices are not used in New Keynesian models, which are the focus of their analysis.

2.4.4.3. Strategically Interesting Agents. The intermediate good firms are said to be strategically interesting because they select a conjecture about the aggregate price level from a range of possibilities. Each possibility implies a conjecture for the continuing equilibrium and, given our consensus assumption, for the aggregate price level. Consistency of the conjecture requires that it be a fixed point. Because they evaluate different conjectures based on their implication for the continuing equilibrium we call them strategically interesting.

In principle we could think of other agents as being strategically interesting too. Our choice reflects a desire to preserve comparability between our paper and Atkeson et al. (2010).

2.5. Sophisticated Equilibrium

This section formally describes the sophisticated equilibrium concept byAtkeson et al. (2010).

We first define the strategies and then we state the equilibrium. The variables determined at time t are summarized in the vector,

$$q_t = (\mathbf{x}_t, \nu_t, z_t, g_t) .$$

Here, $\mathbf{x}_t = (x_{i,t})_{i \in I}$ is the price set in morning of period t by the typical intermediate good producers and the other variables are determined in the afternoon (see subsection 2.4.1 for a discussion of the sub-periods). The (state) variable, $\mathbf{x}_t$, is discussed in detail in the previous section. With the assumption of symmetry and consensus, $x_{i,t} = x_t$ for all $i \in I$ in any equilibrium if the economy is not trembled. Therefore in this case, $\mathbf{x}_t$ is replaced by a number x_t without loss of generality. The variable, ν_t, the velocity shock, which is realized after x_t is determined. The variables determined in the afternoon are:

$$z_t = (\pi_t, \ln c_t, \ln l_t) ,$$

$$g_t = (\mu_t, R_t) .$$

Let h_{t-1} denote the history of events up to time $t-1$. We define h_t recursively as follows:

$$h_t = (h_{t-1}, q_t)$$

for $t \geq 0$, with $h_{-1} = M_{-1}/P_{-1}$ given.

$x_t(h_{t-1})$ denotes the typical intermediate good strategy if there are no trembles. Let $h_{x,t}$ denote the history at the end of the afternoon in period $t : h_{x,t} = (h_{t-1}, x_t, \nu_t)$. Again if there are no trembles, then $\mathbf{x}_t$ is replaced by x_t in $h_{x,t}$ for simplicity. Let $z_t(h_{x,t})$ denote rules for setting the variables in z_t. Let $g_t(h_{x,t})$ denote the government rule. These functions are defined for $t \geq 0$ and for all $h_{x,t}$. We denote the collection of functions, $x_t(h_{t-1})$, $z_t(h_{x,t})$, and $g_t(h_{x,t})$ by σ_x ,σ_z, and σ_g respectively. Let σ denotes the set, $\sigma = (\sigma_x, \sigma_z, \sigma_g)$.

By a recursion using σ for any given h_{t-1}, we can generate a sequence of continuation outcomes from h_{t-1}, $\left\{a_{t+j}\left(\nu_{t+j}^t; h_{t-1}, \sigma\right)\right\}_{j=0}^{\infty}$, where

$$\nu_s^t = \begin{cases} (\nu_t, \cdots, \nu_s) & s \geq t \\ \emptyset & s < t \end{cases}.$$

For example, for $j = 0$:

$$a_t\left(\nu_t^t; h_{t-1}, \sigma\right) = (x_t(h_{t-1}), \nu_t, z_t(h_{t-1}, x_t(h_{t-1}), \nu_t), g_t(h_{t-1}, x_t(h_{t-1}), \nu_t)).$$

Now consider $j = 1$. Let

$$h_t = (h_{t-1}, x_t(h_{t-1}), \nu_t, z_t(h_{t-1}, x_t(h_{t-1}), \nu_t), g_t(h_{t-1}, x_t(h_{t-1}), \nu_t))$$

and

$$a_{t+1}\left(\nu_{t+1}^t; h_{t-1}, \sigma\right) = (x_{t+1}(h_t), \nu_{t+1}, z_{t+1}(h_t, x_{t+1}(h_t), \nu_{t+1}), g_{t+1}(h_t, x_{t+1}(h_t), \nu_{t+1})),$$

and so on for $j \geq 2$. The sequence, $\left\{a_{t+j}\left(\nu_{t+j}^{t}; h_{t-1}, \sigma\right)\right\}_{j=0}^{\infty}$, is a *continuation equilibrium given* h_{t-1}, if all date t and later equilibrium conditions are satisfied.

Note that the previous sequence of outcomes is a function of h_{t-1}. We now define a slightly different sequence of outcomes, which is a function of h_{t-1}, x_t and ν_t. Let

$$h_{x,t} = \left(h_{t-1}, x_t, \nu_t\right),$$

for any h_{t-1}, x_t, and ν_t. For any given $h_{x,t}$ a sequence of continuation outcomes from $h_{x,t}$, $\left\{a_{x,t+j}\left(\nu_{t+j}^{t+1}; h_{x,t}, \sigma\right)\right\}_{j=0}^{\infty}$, can be generated recursively in a similar way. Thus, for $j = 0$:

$$a_{x,t}\left(\nu_{t}^{t+1}; h_{x,t}, \sigma\right) = \left(x_t, z_t\left(h_{x,t}\right), g_t\left(h_{x,t}\right)\right).$$

Now consider $j = 1$. Set $h_t = \left(h_{x,t}, z_t\left(h_{x,t}\right), g_t\left(h_{x,t}\right)\right)$ and $h_{x,t+1} = \left(h_t, x_{t+1}\left(h_t\right), \nu_{t+1}\right)$, and

$$a_{x,t+1}\left(\nu_{t+1}^{t+1}; h_{x,t}, \sigma\right) = \left(x_{t+1}\left(h_t\right), \nu_{t+1}, z_{t+1}\left(h_{x,t+1}\right), g_{t+1}\left(h_{x,t+1}\right)\right),$$

and so on for $j \geq 2$. The sequence, $\left\{a_{x,t+j}\left(\nu_{t+j}^{t+1}; h_{x,t}, \sigma\right)\right\}_{j=0}^{\infty}$, is a *continuation equilibrium given* $h_{x,t}$ if all relevant date t and later equilibrium conditions are satisfied. Since we take x_t as given, the relevant date t and later equilibrium conditions do not include the equilibrium condition associated with x_t. That condition corresponds to 2.44, which we express in the notation of this section as follows:

$$x_t = \pi_t + \ln \sum_{i=1}^{N} p\left(\nu^i\right)\left(c_t\left(h_{t-1}, x_t, \nu^i\right)\right)^{\gamma+\psi}. \tag{2.50}$$

In the continuation equilibrium given h_{t-1}, $x_t = x_t\left(h_{t-1}\right)$. The expression on the right of the equality is a function of x_t and it can be thought of as a mapping from x_t into itself.

Thus, $x_t(h_{t-1})$ is a fixed point of that mapping. In a continuation equilibrium given $h_{x,t}$ the expression (2.49) does not appear.

Now we formally define our equilibrium concept.

Definition 8. A *sophisticated equilibrium, given the policy rule* σ_g, is a strategy, σ_x, and an allocation rule, σ_z, such that

(1) for any h_{t-1}, $\left\{a_{t+j}\left(\nu_{t+j}^{t}; h_{t-1}, \sigma\right)\right\}_{j=0}^{\infty}$, constitute a continuation equilibrium;

(2) for any $h_{x,t}$, $\left\{a_{x,t+j}\left(\nu_{t+j}^{t+1}; h_{x,t}, \sigma\right)\right\}_{j=0}^{\infty}$, constitute a continuation equilibrium.

To understand this concept of equilibrium, consider first condition 1. The state variable at the start of a period is h_{t-1}. When $t = 0$, this says that the allocations induced by σ represent the date zero competitive equilibrium considered in Lemma 1. When $t > 0$ then the allocations describe the competitive equilibrium for any $t > 0$, and any possible h_{t-1}. Part 2 of the definition considers values of x_t that need not be equal to $x_t(h_{t-1})$.

The continuation outcomes allow us to define best response function from x_t^e to what intermediate good producers in sup-period (i) would do, and from there to compute actual x_t. In particular, time h_{t-1} consumption and employment in $a_t(h_{x,t}; \sigma)$, for $h_{x,t} = (h_{t-1}, x_t^e)$ can be substituted into (2.44) to obtain the individual best response, $x_{i,t}$. Trivial aggregation over all i then delivers the aggregate response, x_t. Only in case $x_t^e = x_t(h_{t-1})$ for some function $x_t(h_{t-1})$ that satisfies (2.50) is it the case that actual x_t and x_t^e are the same. When we compute the continuation equilibrium from (h_{t-1}, x_t^e) we in effect pretend that x_t^e actually happens and this is why in the definition above we use x_t. In sum, Part 2 allows us to construct the best response function, and Part 1 requires that $x_t(h_{t-1})$ is a fixed point of the best response function.

Also for a later purpose, it is useful to define the utility level associated with a sophisticated equilibrium σ. $U(\sigma)$ denotes the utility level associated with the outcomes of the sophisticated equilibrium σ. For example, if the outcome associated with the consumption of a sophisticated equilibrium σ is $\{c_t(\nu_t)\}_{\nu_t}$, then the utility level $U(\sigma)$ is

$$U(\sigma) = E_0 \sum_{t=0}^{\infty} \beta^t \left[\frac{c_t(\nu_t)^{1-\sigma}}{1-\sigma} - \frac{c_t(\nu_t)^{1+\psi}}{1+\psi} \right]. \tag{2.51}$$

U^* denotes the welfare level associated with the desired allocation. In particular,

$$U^* = \frac{1}{1-\beta} \left[\frac{1}{1-\sigma} - \frac{1}{1+\psi} \right]. \tag{2.52}$$

Now we formally define *unique implementation.*

Definition 9. The government's policy, σ_g^*, uniquely implements the desired allocation if the continuation outcomes from h_{-1} of any sophisticated equilibrium (σ_x, σ_z) given σ_g^* coincides with the desired allocation.

We illustrate Definition (9) in Figure (2.5.1). The latter figure illustrates two possible histories, the two nodes labeled 1. The large and small arrows indicate two strategy equilibria and note that they both implement the same outcome. They do differ, but only on off equilibrium paths, the continuation associated with going down from 0 to 1. So, here we have unique implementation of the outcome associated with the horizontal line at the top. This example can be converted into an example where unique implementation does not occur by supposing that the large arrow goes down from the upper node in period

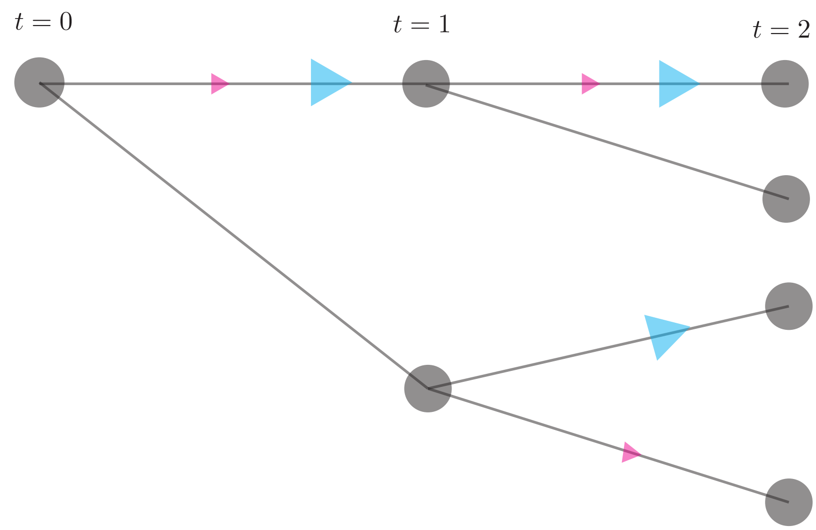


Figure 2.5.1. Unique Implementation

1. What the two equilibria in Figure (9) have in common is that the agent is discouraged from going down in both periods 0 and 1.

2.5.1. Trembles and Fragility

How robust is a result? This question motivates us to consider *trembles* to a sophisticated equilibrium σ. By trembles in actions we imagine that the i^{th} intermediate good producer knows what the 'right' value of $p_{i,t}$ is , but when it is time to actually post that price, it posts instead $p_{i,t}\upsilon_{i,t}$. Here the right value of $p_{i,t}$ is given by the sophisticated equilibrium outcome. In particular, randomly selected J firms experience a idiosyncratic tremble so $\upsilon_{i,t}$ is independently from an uniform distribution $U\left[1-\delta_t, 1+\delta_t\right]$. [2]For others, $\upsilon_{i,t}$ is 1. Note that the individual best response is the same as before, but the actual posted price

[2] J_t can be time-varying too. For simplicity, we assume that $J_t = J$ for all $t \geq 0$.

is

$$x_{i,t}^{\delta_t}\left(h_{t-1}\right)=x_t\left(h_{t-1}\right)+\ln \upsilon_{i,t}. \tag{2.53}$$

Note that the firms do not recognize such trembles in the first place, so that the actual aggregate price index is not a fixed point for the aggregate best response function. We denote the collection of functions, $x_t^{\delta_t}\left(h_{t-1}\right)$ by $\sigma_x^{\delta_t}$. Let σ^{δ_t} denotes the set, $\sigma^{\delta_t}=\left(\sigma_x^{\delta_t},\sigma_z,\sigma_g\right)$. To see how continuation outcomes are determined, for example, the aggregate price index under tremble δ_t is derived as follows:

$$\pi_t\left(h_{t-1}\right)=\ln\left[\int_i\left(\exp x_{i,t}^{\delta_t}\left(h_{t-1}\right)\right)^{1-\varepsilon}di\right]^{\frac{1}{1-\varepsilon}}=\kappa^{\delta_t}+x_t\left(h_{t-1}\right), \tag{2.54}$$

where $\kappa^{\delta_t}=\ln\left[\int_0^1\upsilon_{i,t}^{1-\varepsilon}di\right]^{\frac{1}{1-\varepsilon}}$. Note that κ^{δ_t} is

$$\kappa^{\delta_t}=\ln\left[1-J+J\frac{\left(1+\delta_t\right)^{2-\varepsilon}-\left(1-\delta_t\right)^{2-\varepsilon}}{\left(2-\varepsilon\right)2\delta_t}\right]^{\frac{1}{1-\varepsilon}}<0.$$

for all $\delta_t>0$. The other continuation outcomes are similarly computed based on $\left(h_{t-1},\left(x_{i,t}^{\delta}\right)_{i\in I}\right)$, where $x_{i,t}^{\delta}$ is given by (2.53). $\left\{a_{t+j}^{\delta_t}\left(\nu_{t+j}^t;h_{t-1},\sigma\right)\right\}_{j=0}^{\infty}$ and $\left\{a_{t+j}^{\delta_t}\left(\nu_{t+j}^t;h_{x,t},\sigma\right)\right\}_{j=0}^{\infty}$ denotes the continuation outcome at date h_{t-1} and $h_{x,t}$ under tremble δ^t respectively.

Definition 10. Fix a σ_g which uniquely implements the desired allocation and σ denotes the unique sophisticated equilibrium. σ_g is said to be *fragile to* δ_t if the equilibrium outcome associated with the trembles

$$\lim_{\delta_t\downarrow 0}U\left(\left(\sigma_x^{\delta_t},\sigma_z,\sigma_g\right)\right)\neq U^*.$$

σ_g is said to be *fragile* if σ_g is fragile to some δ_t. σ_g is *robust* if σ_g is not fragile.

2.6. Implementation by the Taylor rule with Escape Clause

In this section, we show that the Taylor rule with an escape clause uniquely implements the desired allocation in the sense of Definition 9. The first subsection below considers the deterministic model, and the second section proceeds to

2.6.1. Deterministic case

This section is organized as follows. First we formalize the strategies intuitively mentioned above. Second we show that the strategy is indeed a sophisticated equilibrium, and uniquely implements the desired allocation. Finally, we discuss whether the implementation is robust or not.

2.6.1.1. Without trembles. For convenience, we introduce the following notations. $\mathcal{H}_{t-1}$ denotes the set of all possible histories at the beginning of the date t. $\mathcal{H}_{t-1}^N$ is a subset of $\mathcal{H}_{t-1}$ in which the inflation rates until date $t-1$ never violates the monitoring range:

$$\mathcal{H}_{t-1}^N = \{h_{t-1}; \pi_s \in [\pi_l, \pi_u] \quad 0 \leq s \leq t-1\}.$$

$\mathcal{H}_{t-1}^D$ is a collection of histories in which the inflation rate for some date $s \leq t-1$ violates the monitoring range:

$$\mathcal{H}_{t-1}^D = \mathcal{H}_{t-1} \setminus \mathcal{H}_{t-1}^N.$$

Now we construct the strategies, σ. First the government strategy σ_g^* is formulated as follows. Fix any history (h_{t-1}, x_t), where $h_{t-1} \in \mathcal{H}_{t-1}^N$. Then the monetary policy follows the interest rate rule:

$$R_t\left(h_{t-1},x_t\right)=\max\left\{\phi\pi_t,R_l\right\}.$$

If $h_{t-1}\in\mathcal{H}^D_{t-1}$, then the government sets its money growth rate $\mu_t\left(h_{t-1},x_t\right)=\mu$. σ^*_g denotes the collection of the government policies.

The intermediate firms set the price as follows:

$$x_t\left(h_{t-1}\right)=\begin{cases}0 & h_{t-1}\in\mathcal{H}^N_{t-1}\\ \mu+\ln\frac{M_{t-1}}{P_{t-1}} & h_{t-1}\in\mathcal{H}^D_{t-1}\end{cases},$$

where $\ln\frac{M_{t-1}}{P_{t-1}}$ is measurable at history h_{t-1}

$$\ln\frac{M_{t-1}}{P_{t-1}}=\sum_{s=0}^{t-1}\left(\mu_s-\pi_s\right)+\ln\frac{M_{-1}}{P_{-1}}.$$

The inflation rate is computed based on the actual prices posted by the intermediate firms:

$$\pi_t\left(h_{t-1},\mathbf{x}_t\right)=\ln\left[\int\exp\left(\left(1-\varepsilon\right)x_{i,t}\right)di\right]^{\frac{1}{1-\varepsilon}}.$$

Note that we do *not* impose symmetry here.

The consumption strategy is more involved:

$$\ln c_t\left(h_{t-1},x_t\right)=\begin{cases}-\frac{1}{\gamma}R_t\left(h_{t-1},x_t\right) & \left(h_{t-1},x_t\right)\in\mathcal{H}^N_{t-1}\times\left[\pi_l,\pi_u\right]\\ \frac{1}{1-\gamma}\left[R_t\left(h_{t-1},x_t\right)-\mu\right] & \left(h_{t-1},x_t\right)\in\mathcal{H}^N_{t-1}\times\left(\mathbb{R}\setminus\left[\pi_l,\pi_u\right]\right)\\ \mu+\ln\frac{M_{t-1}}{P_{t-1}}-\pi_t\left(x_t\right) & \left(h_{t-1},x_t\right)\in\mathcal{H}^D_{t-1}\times D\left(h_{t-1}\right)\\ -\frac{1}{\gamma}\left(R_l-\left[\mu+\left(\mu-x_t\right)+\ln\frac{M_{t-1}}{P_{t-1}}\right]\right) & \left(h_{t-1},x_t\right)\in\mathcal{H}^D_{t-1}\times D^c\left(h_{t-1}\right)\end{cases},$$

where the set $D(h_{t-1})$ is defined as

$$D(h_{t-1}) = \left\{x_t; (1-\gamma)\left[\mu + \ln\frac{M_{t-1}}{P_{t-1}} - x_t\right] + \mu > R_l\right\}. \tag{2.55}$$

There are four cases to consider to define a consumption rule. First, if the escape clause is not activated and the price level chosen by the typical intermediate price setter belong to $[\pi_l, \pi_u]$, then we know that the (unique) continuation outcomes from $t+1$ are unique, and consumption and inflation rate at date $t+1$ are

$$(c_{t+1}, \pi_{t+1}) = (1, 0).$$

The Euler equation at $h_{x,t}$ implies that the consumption level is

$$\ln c_t = \ln c_{t+1} - \frac{1}{\gamma}(R_t - \pi_{t+1}) = -\frac{1}{\gamma}R_t(h_{t-1}, x_t).$$

Second case deals with the case in which the price set by the typical intermediate firm does not belong to $[\pi_l, \pi_u]$. In this case, the constant money growth rule will be used from $t+1$, which affects the continuation outcomes. In particular, the unique competitive equilibrium levels of consumption and inflation rate at date $t+1$ under the constant money growth rule are

$$(c_{t+1}, \pi_{t+1}) = \left(1, \mu + \ln\frac{M_t}{P_t}\right). \tag{2.56}$$

Note that the real balance Again from the Euler equation at history $h_{x,t}$, the consumption at date t is $\frac{1}{1-\gamma}[R_t(h_{t-1}, x_t) - \mu]$ by assuming the cash constraint at date t binds. Such assumption is justified because the opportunity cost of holding money is zero if the interest rate is zero. Thus, it is indifferent for the households to hold more or less money.

Third and forth cases deal with the case in which the monitoring range is already violated in the past. If the constant money growth rule is operative and cash constraint at date t is satisfied as equality, then the interest rate is residually determined by the Euler equation.

$$\ln c_t = \mu + \ln \frac{M_{t-1}}{P_{t-1}} - \pi_t\left(x_t\right) \tag{2.57}$$

$$R_t = (1-\gamma) \ln c_t\left(h_{t-1}, x_t\right) + \mu \tag{2.58}$$

To obtain (2.58), we use Euler equation, cash constraint at date t, and the fact that consumption and inflation rate at $t+1$ are given by (2.56). These two equations tell that if the chosen prices, x_t, become higher, then the households can buy less consumption goods, which pushes up the nominal interest rate. Notice that this level of interest rate, (2.58), will be negative if the intermediate firms (irrationally) chose a substantially low price level. Third case deals with the case in which the implied interest rate, (2.58), is weakly larger than R_l. Forth case deals with the case in which not. For notational convenience, x_l denotes the inflation level in which the implied interest rate R_t is exactly R_l:

$$\mu + \ln \frac{M_{t-1}}{P_{t-1}} - x_l = -\frac{1}{\gamma}\left(R_l - \mu - (\mu - x_l) - \ln \frac{M_{t-1}}{P_{t-1}}\right). \tag{2.59}$$

If x_t is lower than x_l, then the interest rate determined by (2.58) is lower than R_l. Because of the zero lower bound, R_t cannot be below R_l. In such cases, $x_t < x_l$, the consumption and the interest rate are given by

$$\ln c_t = -\frac{1}{\gamma}\left(R_l - \left[\mu + (\mu - x_t) + \ln \frac{M_{t-1}}{P_{t-1}}\right]\right)$$
$$R_t = R_l.$$

Note that by construction, the Euler equation at date t is satisfied:

$$\ln c_t = \underbrace{0}_{=\ln c_{t+1}} - \frac{1}{\gamma}\left(\underbrace{R_l}_{=R_t} - \left[\underbrace{\mu + (\mu - x_t) + \ln \frac{M_{t-1}}{P_{t-1}}}_{=\pi_{t+1}}\right]\right).$$

We need to show additionally cash constraint at date t is satisfied. This is carefully done in the proof below. Finally, the labor supply strategy, l_t, is chosen to be consistent with the aggregate constraint:

$$l_t\left(h_{x,t}\right) = \ln c_t\left(h_{x,t}\right) - \ln p^{\delta}\left(h_{x,t}\right),$$

where

$$p^{\delta_t}\left(h_{x,t}\right) \equiv \left(\frac{\left(\int_0^1 \left(\exp\left(x_{i,t}\right)\right)^{-\varepsilon} di\right)^{-\frac{1}{\varepsilon}}}{\exp\left(\pi_t\left(h_{x,t}\right)\right)}\right)^{\varepsilon}.$$

Notice that when $x_{i,t} = x_t$ for all i, then $p^{\delta_t}\left(h_{x,t}\right) = 1$.

Now we specify the level of the money growth rate μ_t, (interest rate, R_t) at $(h_{t-1}, x_t) \in \mathcal{H}_{t-1}^N \times \mathbb{R}$ ($(h_{t-1}, x_t) \in \mathcal{H}_{t-1}^D \times \mathbb{R}$, respectively). If the the Taylor rule is operative at date $(h_{t-1}, x_t) \in \mathcal{H}_{t-1}^N \times \mathbb{R}$, then the money growth rate is residually determined by by

$$\mu_t\left(h_{t-1}, x_t\right) = \pi_t\left(x_t\right) + \ln c_t\left(h_{t-1}, x_t\right) - \ln \frac{M_{t-1}}{P_{t-1}}.$$

Notice with this level of money demand, cash constraint at time t is always satisfied. This property of μ_t is frequently used in the proof for Proposition 9.

On the other hand, if the money growth rule is operative at history $(h_{t-1}, x_t) \in \mathcal{H}_{t-1}^D \times \mathbb{R}$, then the interest rate is residually determined by

$$R_t(h_{t-1}, x_t) = \max\{(1-\gamma)\ln c_t(h_{t-1}, x_t) + \mu_t(h_{t-1}, x_t), R_l\}. \tag{2.60}$$

By definition, the interest rate (2.60) is always weakly greater than R_l.σ_x^* denotes the collection of the intermediate firms strategy, and the σ_z^* denotes the collections of other functions including (c_t, l_t).

Now we show that the constructed strategy σ^* constitutes a sophisticated equilibrium.

Proposition 9. $\sigma^* = (\sigma_x^*, \sigma_z^*, \sigma_z^*)$ *constitutes a sophisticated equilibrium.*

Proof. The proof proceeds as follows; **(1)** we show that the continuation outcome from history h_{t-1} is an equilibrium; **(2)** we show that the continuation outcome from history (h_{t-1}, x_t) is an equilibrium too.

Part (1) First fix any $h_{t-1} \in \mathcal{H}_{t-1}^N$. It is easy to show that the continuation outcomes associated with h_{t-1} is the desired allocation. Thus the continuation outcomes satisfy the equilibrium conditions from h_{t-1}.

Second fix any $h_{t-1} \in \mathcal{H}_{t-1}^D$. The government uses a money growth rule as its instrument. Then the continuation outcome is computed as follows:

$$\ln c_{t+s} = \ln l_{t+s} = 0 \quad \pi_{t+s+1} = R_{t+s} = \mu \quad \text{for all } s \geq 0.$$

It might be useful to derive the continuation outcome for $s = 0$. Note that the inflation rate chosen by the intermediate firms is

$$x_t\left(h_{t-1}\right) = \mu + \ln \frac{M_{t-1}}{P_{t-1}},$$

and the monetary policy is

$$\mu_t\left(h_{t-1}, x_t\left(h_{t-1}\right)\right) = \mu$$

since $\left(2-\gamma\right)\mu + \left(1-\gamma\right)\left[\ln \frac{M_{t-1}}{P_{t-1}} - x_t\right] = \mu$. Then the consumption level is

$$\begin{aligned}
\ln c_t\left(h_{t-1}, x_t\left(h_{t-1}\right)\right) &= \mu_t\left(h_{t-1}, x_t\right) + \ln \frac{M_{t-1}}{P_{t-1}} - \pi_t\left(x_t\left(h_{t-1}\right)\right) \\
&= \mu + \ln \frac{M_{t-1}}{P_{t-1}} - \mu - \ln \frac{M_{t-1}}{P_{t-1}} \\
&= 0.
\end{aligned}$$

The interest rate and the inflation rate are given by

$$\begin{aligned}
\pi_t\left(x_t\left(h_{t-1}\right)\right) &= x_t\left(h_{t-1}\right) = \mu, \\
R_t\left(h_{t-1}, x_t\left(h_{t-1}\right)\right) &= \left(1-\gamma\right)\ln c_t\left(h_{t-1}, x_t\left(h_{t-1}\right)\right) + \mu_t\left(h_{t-1}, x_t\right) = \mu.
\end{aligned}$$

We can recursively compute the rest of the allocations. Lemma 1 tells that the allocations satisfy the equilibrium conditions.

Part (2) Now we show that the continuation outcomes from any history $\left(h_{t-1}, x_t\right)$ satisfy the equilibrium conditions. In this case, there are four cases to consider; **(i)** $\left(h_{t-1}, x_t\right) \in \mathcal{H}_{t-1}^{N} \times \left[\pi_l, \pi_u\right]$; **(ii)** $\left(h_{t-1}, x_t\right) \in \mathcal{H}_{t-1}^{N} \times \mathbb{R}\backslash\left[\pi_l, \pi_u\right]$; **(iii)** $\left(h_{t-1}, x_t\right) \in \mathcal{H}_{t-1}^{D} \times$

$D(h_{t-1})$**;** **(iv)** $(h_{t-1}, x_t) \in \mathcal{H}^D_{t-1} \times D^c(h_{t-1})$. We show separately that the continuation outcomes associated with these four cases satisfy the equilibrium conditions.

It is worthwhile to mention that the continuation outcomes from h_t for each case constitute an equilibrium due to Part (1). Therefore to show that the continuation outcomes satisfy the equilibrium conditions, we need to show that the date-t equilibrium conditions (i.e., cash constraint, Euler equation, and the monetary policy) are satisfied given the continuation outcomes, e.g. c_{t+1}, π_{t+1} and so on.

Case (i) Fix any history $h_{x,t} = (h_{t-1}, x_t) \in \mathcal{H}^N_{t-1} \times [\pi_l, \pi_u]$. Then the date-$t$ continuation outcomes are

$$\begin{aligned} R_t &= \phi x_t \\ \ln c_t &= -\frac{1}{\gamma} R_t \\ \mu_t &= \pi_t(x_t) + \ln c_t(h_{t-1}, x_t) - \ln \frac{M_{t-1}}{P_{t-1}}, \end{aligned}$$

and from $t+1$, the economy goes back to the desired allocations, $\ln c_{t+s} = 0$ and $R_{t+s} = \pi_{t+s+1} = 0$ for all $s \geq 1$. As mentioned above, the allocation from $t+1$ satisfies the equilibrium conditions. So, it suffices to check whether the date-t equilibrium conditions are satisfied.

The cash constraint at date t is satisfied because of the construction of $\mu_t(h_{x,t})$. The Euler equation at date t is also satisfied since

$$\ln c_t = -\frac{1}{\gamma} R_t = \underbrace{\ln c_{t+1}}_{=0} - \frac{1}{\gamma}\left(R_t - \underbrace{\pi_{t+1}}_{=0}\right).$$

Therefore the equilibrium conditions from date t are all satisfied by the continuation outcomes. Therefore the continuation outcomes constitutes an equilibrium.

Case (ii) Fix any history $h_{x,t} = (h_{t-1}, x_t) \in \mathcal{H}_{t-1}^N \times \mathbb{R} \backslash [\pi_l, \pi_u]$. In this case, from h_t, the constant money growth rule will be operative. The date-t continuation outcomes are

$$\begin{aligned}
\pi_t &= x_t \\
R_t &= \max\{\phi x_t, R_l\} \\
\ln c_t &= \frac{1}{1-\gamma} [R_t (h_{t-1}, x_t) - \mu] \\
\mu_t &= \pi_t (x_t) + \ln c_t (h_{t-1}, x_t) - \ln \frac{M_{t-1}}{P_{t-1}},
\end{aligned}$$

and the continuation outcomes from $(h_{x,t}, (c_t, \pi_t, R_t, \mu_t)) \in \mathcal{H}_t^D$ constitutes an equilibrium.

Because of construction of μ_t, the cash constraints are satisfied:

$$\mu_t = \pi_t (x_t) + \ln c_t (h_{t-1}, x_t) - \ln \frac{M_{t-1}}{P_{t-1}} \quad \Longleftrightarrow \quad P_t c_t = M_t.$$

To show the Euler equation at date t first notice that the inflation rate at $t+1$ is $\pi_{t+1} = \mu + \ln c_t$. Then

$$\begin{aligned}
\ln c_t &= \frac{1}{1-\gamma} [R_t (h_{t-1}, x_t) - \mu] \iff \\
(1-\gamma) \ln c_t &= R_t (h_{t-1}, x_t) - \pi_{t+1} - \ln c_t \iff \\
\ln c_t &= \underbrace{\ln c_{t+1}}_{=0} - \frac{1}{\gamma} [R_t (h_{t-1}, x_t) - \pi_{t+1}],
\end{aligned}$$

which is the Euler equation at date t. Therefore the continuation outcome from $h_{x,t}$ constitutes an equilibrium.

Case (iii) Fix any history $(h_{t-1}, x_t) \in \mathcal{H}_{t-1}^D \times D(h_{t-1})$. The consumption level at date t are given by

$$\ln c_t(h_{x,t}) = \mu + \ln \frac{M_{t-1}}{P_{t-1}} - \pi_t(h_{x,t})$$

so that the cash constraint at date t is satisfied. Notice that the interest rate is positive

$$R_t(h_{x,t}) = (1-\gamma)\ln c_t(h_{x,t}) + \mu > R_l$$

since $x_t \in D(h_{t-1})$. Also, the above equation is equivalent to the Euler equation:

$$R_t(h_{x,t}) = (1-\gamma)\ln c_t(h_{x,t}) + \mu \iff \ln c_t = \underbrace{\ln c_{t+1}}_{=0} - \frac{1}{\gamma}\left[R_t - \pi_{t+1}\right].$$

Therefore the continuation outcomes from $h_{x,t} \in \mathcal{H}_{t-1}^D \times D(h_{t-1})$ satisfy the equilibrium conditions.

Case (iv) Fix any history $(h_{t-1}, x_t) \in \mathcal{H}_{t-1}^D \times D^c(h_{t-1})$. As in Case (iii), the continuation outcome from h_t will be an equilibrium. The inflation rate at $t+1$ is

$$\pi_{t+1} = \mu + (\mu - x_t) + \ln \frac{M_{t-1}}{P_{t-1}}.$$

The consumption level at history $\mathcal{H}_{t-1}^D \times D^c(h_{t-1})$ is now

$$\ln c_t(h_{x,t}) = -\frac{1}{\gamma}\left(R_l - \left[\mu + (\mu - x_t) + \ln \frac{M_{t-1}}{P_{t-1}}\right]\right). \tag{2.61}$$

As mentioned above, Euler equation is satisfied since $\ln c_{t+1} = 0$ and the square bracket term corresponds to π_{t+1}.

However, unlike Case (iii), the cash constraint at date t is not trivially satisfied. To show it, notice that

$$\begin{aligned}\ln\frac{M_t}{P_t}-\ln c_t\left(h_{x,t}\right)&=\left(\mu-x_t\right)+\ln\frac{M_{t-1}}{P_{t-1}}+\frac{1}{\gamma}\left(R_l-\left[\mu+\left(\mu-x_t\right)+\ln\frac{M_{t-1}}{P_{t-1}}\right]\right)\\&=\left(1-\frac{1}{\gamma}\right)\left[\left(\mu-x_t\right)+\ln\frac{M_{t-1}}{P_{t-1}}\right]+\frac{1}{\gamma}\left(R_l-\mu\right).\end{aligned}$$

Combining above equation with the definition of x_l (2.59),

$$\begin{aligned}\ln\frac{M_t}{P_t}-\ln c_t\left(h_{x,t}\right)&=\left(\frac{\gamma-1}{\gamma}\right)\left[\mu-x_t+\ln\frac{M_{t-1}}{P_{t-1}}\right]+\frac{1-\gamma}{\gamma}\left[\mu-x_l+\ln\frac{M_{t-1}}{P_{t-1}}\right]\\&=\underbrace{\frac{\gamma-1}{\gamma}}_{\geq 0}\underbrace{\left[x_l-x_t\right]}_{\geq 0}\geq 0,\end{aligned}$$

which is desired. □

Proposition 10 (Atkeson et al. (2010))**.** *The Taylor rule with escape clause, σ_g^*, uniquely implements the desired allocation if one of the following conditions is satisfied;*

(1) $\pi_l<\pi_u$ and the Taylor principle is satisfied, $\phi>1$;

(2) $\pi_l=\pi_u$.

Proof. First suppose that $\pi_l<\pi_u$ and $\phi>1$. Also suppose that there exists a sophisticated equilibrium (σ_x,σ_z) such that the continuation outcomes from h_{-1} differs from the desired allocation. Since the continuation outcomes differ from the desired allocation, there exists a date t such that such that

$$x_t\neq 0, x_s=0\quad 0\leq s\leq t-1.$$

$(\pi_{t+s}, R_{t+s}, c_{t+s}, l_{t+s})_{s=0}^{\infty}$ denotes the continuation equilibrium at history h_{t-1}. There are two scenarios to consider; **(1)** $x_t(h_{t-1}) \in [\pi_l, \pi_u] \setminus \{0\}$; **(2)** $x_t(h_{t-1}) \notin [\pi_l, \pi_u]$.

Part (1). The continuation equilibrium should satisfy the following equilibrium difference equations for all $s \geq 0$. Therefore especially, the following equilibrium difference equation needs to be satisfied if $\pi_{t+s-1} \in [\pi_l, \pi_u]$ for all $0 \leq s \leq t-1$:

$$\pi_{t+1+s} = \phi \pi_{t+s}.$$

Then the sequence generated by this difference equation implies there exists T such that $\pi_T \notin [\pi_l, \pi_u] \quad \pi_{T-s} \in [\pi_l, \pi_u]$ for all $0 \leq s \leq T-t$ since $\pi_t \neq 0$. Then if $\pi_T > \pi_u$, then $\pi_{T+1} = \mu \in [\pi_l, \pi_u]$,and the value of the interest rate does not belong to the monitoring range too since the Taylor principle is not satisfied:

$$R_T = \phi \pi_T \notin [\pi_l, \pi_u].$$

However, the Euler equation implies that the interest rate is $\pi_{T+1} \in [\pi_l, \pi_u]$, which is a contradiction.

If $\pi_T < \pi_l$, then $\pi_{T+1} = \mu \in [\pi_l, \pi_u]$. Then the interest rate does not belong to the monitoring range

$$R_T = \phi \pi_T \notin [\pi_l, \pi_u].$$

Contradiction..

Part (2). Lemma 1 implies that the continuation outcomes from $h_t = (h_{t-1}, (\pi_t, R_t, \mu_t, \ln c_t))$ is unique and given by

$$\pi_{t+s+1} = R_{t+s} = \mu \in [\pi_l, \pi_u]$$

$$\ln c_{t+s} = 0$$

for all $s \geq 1$. Since R_{t+1} is positive, cash constraint at date $t+1$ binds:

$$P_{t+1} = M_{t+1} \iff \pi_{t+1} = \mu + \ln \frac{M_t}{P_t}. \tag{2.62}$$

The date-t variables $(\pi_t, R_t, \mu_t, \ln c_t)$ should satisfy date-t equilibrium conditions:

$$\begin{aligned} 0 &= \ln c_t, \\ P_t c_t &\leq M_t, \\ \ln c_t &= \ln c_{t+1} - \frac{1}{\gamma}(R_t - \pi_{t+1}) \\ R_t &= \max\{\phi x_t, R_l\}. \end{aligned} \tag{2.63}$$

Suppose that $x_t \in D(h_{t-1})$, then the interest rate is positive, i.e. $R_t = \phi x_t$, which implies that the cash constraint at date t binds, $c_t = M_t/P_t$. So, (2.62) is reduced to

$$\pi_{t+1} = \mu \in [\pi_l, \pi_u],$$

and

$$R_t = \phi x_t \notin [\pi_l, \pi_u],$$

which contradicts with (2.63).

Now suppose that $x_t \notin D(h_{t-1})$, then the zero-lower bound binds, i.e. $R_t = R_l < \mu$. Cash constraint might be slack, $P_t \leq M_t$, so $\pi_{t+1} \geq \mu$. In this case, Euler equation at date t (2.63) is

$$\mu > R_l = \pi_{t+1} \geq \mu,$$

which is a contradiction.

Now we prove that σ^* uniquely implements the desired allocations without the Taylor principle when $\pi_l = \pi_u$. This claim can be shown by repeating the previous argument with modifications. First notice that only in **Part (1)** of the previous argument, the Taylor principle is used. Also since $\pi_l = \pi_u$, $[\pi_l, \pi_u] \setminus \{0\}$ is an empty set. Therefore we only need to show **Part (2)** of the previous proof. Since the same argument is applied here, we have a unique implementation. □

It is worthwhile to emphasize that the Taylor principle $\phi > 1$ does not need to hold when $\pi_l = \pi_u$. Since the government immediately intervenes if undesired outcomes realize, the government can force the intermediate price setters to coordinate on the desired allocation directly. Notice here that the Taylor rule is operative on the equilibrium path. If a stochastic version of Proposition 10 holds, then we cannot really tell from the relation between the interest rate and inflation rate whether the Taylor principle holds or not. This is a critique by Atkeson et al. (2010) on Clarida et al. (2000). This result is very provocative since the literature finds that the Taylor principle is actually necessary.

2.6.1.2. With trembles. Now we extend our analysis by incorporating trembles in action. In particular, we show that if $\pi_l = \pi_u$, then the government is forced to use the money growth rule under the trembles. Under the trembles, the intermediate firms make (uncorrelated) small mistakes, and the aggregated price index does not correspond to what the government wants. Therefore, the Taylor rule stops operative immediately.

Lemma 6. *Suppose that $\pi_l = \pi_u$. For any sophisticated equilibrium σ which uniquely implements the desired equilibrium, the outcomes associated with tremble δ_t entails the regime switching.*

Proof. Fix any tremble δ_t. At date t, the inflation rate is

$$\pi_t\left(h_{t-1}\right) = \kappa^{\delta_t} + \pi_t\left(x_t\left(h_{t-1}\right)\right) = \kappa^{\delta_t} \neq 0.$$

Therefore from $t+1$, the government start using the constant money growth rule. □

2.6.2. Stochastic case

In Lemma 6, we show that the government start using the constant money growth rule at date t under tremble δ_t. However, the sophisticated equilibrium σ is superficially robust to this tremble, since the economy does not have a real consequence by using the constant money growth rule. If the interest rate rule outperforms a constant money growth rule, then such regime-shift has a real consequence, and σ becomes fragile. We materialize this logic in the stochastic version of the model. In particular, we argue that to robustly implement the desired allocation, we need to have a wide monitoring range and Taylor principle, which is not required for robust implementation in the deterministic model.

Proposition 11. *(Necessity of the Taylor principle) Suppose that there exists an strategy σ_g uniquely implementing the desired allocation in the stochastic model. Then if σ_g is robust to the trembles, then the Taylor principle is satisfied and $\pi_l < \pi_u$.*

Proof. The proof proceeds with two steps; **(i)** if $\pi_l = \pi_u$, then the σ_g is fragile; **(ii)** if $\pi_l < \pi_u$, then the Taylor principle is necessary for unique implementation. The contraposition of step **(i)** is that if σ_g is robust, then $\pi_l < \pi_u$. Step **(ii)** implies that in that case, the Taylor principle needs to be satisfied, which is desired. Notice that step (ii) is trivial since the desired allocation is locally determinate if the Taylor principle is not satisfied. Thus we only show step **(i)**.

Step (i) Suppose that $\pi_l = \pi_u$. Then consider a tremble at date $t = 0$, δ_0 and the associated sophisticated equilibrium σ^{δ_0}. The initial period prices chosen by the intermediate firms are

$$\mathbf{x}_0^{\delta}(h_{-1}) = x_0(h_{-1}) + \upsilon_i,$$

where $x_0(h_{-1})$ is zero since σ_g uniquely implements the desired allocation, $\pi_0 = 0$. Therefore the inflation rate at date-t afternoon is

$$\pi_t(h_{t-1}, \mathbf{x}_t) = \ln\left[\int \exp\left((1-\varepsilon)\, x_{i,t}\right) di\right]^{\frac{1}{1-\varepsilon}} = \kappa^{\delta_0} \notin [\pi_l, \pi_u] = \{0\}\,.$$

Therefore the monitoring range is immediately violated. Then from Proposition 8, consumption and employment start fluctuating due to the velocity shock ν_t. $\left\{c_t^{\delta_0}(\nu_t), l_t^{\delta_0}(\nu_t)\right\}$ denotes consumption and employment outcome given σ^{δ_0}. The outcomes from date $t \geq 1$ are independent of the trembles since the economy gets hit by a tremble only at date 0.

The welfare associated with σ^{δ_0} is

$$U\left(\sigma^{\delta_0}\right) = u\left(c_0^{\delta_0}, l_0^{\delta_0}\right) + E_0 \sum_{t=1}^{\infty} \beta^t u \left(\underbrace{c_t^{\delta_0}\left(\nu_t\right), l_t^{\delta_0}\left(\nu_t\right)}_{\text{independent of } \delta_0} \right)$$

$$\overset{\delta_0 \downarrow 0}{\to} u\left(c_0^*, l_0^*\right) + E_0 \sum_{t=1}^{\infty} \beta^t u\left(c_t^{\delta_0}\left(\nu_t\right), l_t^{\delta_0}\left(\nu_t\right)\right).$$

Notice that from Proposition 8, $\left\{c_t^{\delta_0}\left(\nu_t\right), l_t^{\delta_0}\left(\nu_t\right)\right\}_{t=1}^{\infty}$ is not an efficient allocation from date $t = 1$. So, Proposition 5 implies that the welfare associated with $\left\{c_t^{\delta_0}\left(\nu_t\right), l_t^{\delta_0}\left(\nu_t\right)\right\}_{t=1}^{\infty}$ is strictly lower than the welfare associated with the efficient allocation. Therefore,

$$\lim_{\delta_0 \downarrow 0} U\left(\sigma^{\delta_0}\right) < U^*,$$

which completes the proof. □

2.7. Conclusion

In this chapter, we provide a supplement material for the previous chapter. Especially we define a sophisticated equilibrium and analyze the same model through the lens of that equilibrium concept. While the notation used in the previous chapter is more accessible to a wider range of audience, the notation used in this section gives us rigorousness, and perhaps clarity. Also when the economy has a shock, the notation used in this section would be easier than the one used in the previous section.

The rigorousness easily allows us to connect applied macroeconomics with microeconomic theory. In microeconomics, there has been a concern whether Nash equilibrium concept or similar equilibrium concepts would be reasonable concepts or not for human behaviors, and microeconomists have propose several new concepts to analyze games and try to give a better prediction about outcomes of their games. See, for example, **?** and **?**. These concerns are, in principle, applied in our model too. We assume that all the agents in our model correctly need to understand what would happen on off-equilibrium-paths and the agents can rationally react to such deviations. Also the agents do need to understand what the government would do for all possible histories. But, one can argue that it makes less sense to assume such a high level of rationality. We do not discuss at all these concerns since it is beyond the scope of this paper. It might be interesting to use the model developed here to investigate how the results in this paper change under the newly developed concepts in microeconomics which assumes a lower level of rationality. For example, it is interesting the determinacy .

CHAPTER 3

Universal Gravity

with Treb Allen and Costas Arkolakis

This paper studies the theoretical properties and counterfactual predictions of a large class of general equilibrium trade and economic geography models. We begin by presenting a framework that combines aggregate factor supply and demand functions with market clearing conditions. We prove that existence, uniqueness and – given observed trade flows – the counterfactual predictions of any model within this framework depend only on the demand and supply elasticities (the "gravity constants"). We propose a new strategy to estimate these gravity constants using an instrumental variables approach that relies on the general equilibrium structure of the model. Finally, we use these estimates to compute the impact of a trade war between US and China.

3.1. Introduction

Over the past fifteen years, there has been a quantitative revolution in spatial economics. The proliferation of general equilibrium gravity models incorporating flexible linkages across many locations now gives researchers the ability to conduct a rich set of real world analyses. However, the complex general equilibrium interactions and the variegated assumptions underpinning different models has resulted in our understanding of the models' properties to lag behind. As a result, many important questions remain either partially or fully unresolved, including: When does an equilibrium exists and when is it unique? Do different models have different counterfactual implications?

In this paper, we characterize the theoretical and empirical properties common to a large class of gravity models spanning the fields of international trade and economic geography. We first provide a "universal gravity" framework combining aggregate demand and supply equations with standard market clearing conditions that incorporates many workhorse trade and economic geography models.[1] We show that existence and uniqueness of the equilibria of all models under the auspices of our framework can be characterized solely based on their aggregate demand and supply elasticities (the "gravity constants"). Moreover, the counterfactual predictions for trade flows, incomes, and real output prices of these models can be expressed solely as a function of the gravity constants and observed data. Hence, the key theoretical properties and positive counterfactual predictions of all gravity models depend ultimately on the value of two parameters – the elasticities of supply and demand. We show how these gravity constants can be estimated using an instrumental variables approach that relies on the general equilibrium structure of the model. Finally, we use these estimates to compute the impact of a trade war between US and China.

[1]Examples of gravity trade models included in our framework are perfect competition models such as Anderson (1979), Anderson and Van Wincoop (2003), Eaton and Kortum (2002),Dekle et al. (2008), Caliendo and Parro (2010) monopolistic competition models such as Krugman (1980), Melitz (2003) as specified by Chaney (2008), Arkolakis et al. (2008), Di Giovanni and Levchenko (2008), , and the Bertrand competition model of Bernard et al. (2003). Economic geography models incorporated in our framework include Allen and Arkolakis (2014) and Redding (2016). See Table 3.8.1 for the mapping from work-horse trade and economic geography models into the universal gravity framework.

To construct our framework, we consider a representative economy in which an aggregate good is traded across locations subject to the following six economic conditions: 1) "iceberg" type bilateral trade frictions; 2) a constant elasticity of substitution (CES) aggregate demand function; 3) a CES aggregate supply function; 4) market clearing; 5) balanced trade; and 6) a choice of the numeraire. Any model in which the equilibrium can be represented in a way that satisfies these conditions is said to be contained within the universal gravity framework. Moreover, these conditions impose sufficient structure to completely characterize all general equilibrium interactions of trade flows, incomes, and real output prices. It turns out that the aggregate demand elasticity from condition 2 and the aggregate supply elasticity from condition 3 play a particularly important role in this characterization.

We first provide sufficient conditions for the existence, uniqueness, and interiority of the equilibrium of the model that depend solely on the gravity constants. Existence occurs everywhere except for a knife-edge constellation of parameters (corresponding e.g. to Leontief preferences in an Armington trade model or when agglomeration forces are just strong enough to create a "black hole" equilibrium in an economic geography model). An equilibrium is unique as long as the demand elasticity is (weakly) negative and the supply elasticity is (weakly) positive (or vice versa and both elasticities are greater than one in magnitude); moreover, if the inequalities are strict, an iterative algorithm is guaranteed to converge to the the unique equilibrium from any interior starting point. Multiplicity may occur if demand and supply elasticities are both negative (for example, in an economic geography model if agglomeration forces are sufficiently strong) or if demand and supply elasticities are both positive (for example, in a trade model if goods are complementary).

We also show that these sufficient conditions can be extended further if trade frictions are "quasi" symmetric – a common assumption in the literature and provide conditions under which an equilibrium exists and an iterative algorithm is guaranteed to converge to the equilibrium.

We then examine how a shock to bilateral trade frictions affects equilibrium trade flows, incomes, and real output prices. To do so, we derive an analytical expression for the counterfactual elasticities of these endogenous variables to changes in all bilateral trade frictions that elucidates the networks effects of trade. In particular, we show how can this expression be written as series of terms expressing how a shock propagates through the trading network, e.g. the direct effect of a shock, the effect of the shock on all locations' trading partners, the effect on all locations' trading partners' trading partners, etc. Importantly, we show that this expression depends only on observed trade flows and the gravity constants, demonstrating that conditional on these two model parameters, the positive macro-economic implications for all gravity models are the same.[2] Moreover, we analytically prove that when trade frictions are "quasi" symmetric, the impact of a trade friction shock on the real output prices and real expenditure in directly affected locations will always exceed the impact on other indirectly-affected locations.

We proceed by estimating the gravity constants using a novel procedure that can be applied to any model contained within the universal gravity framework. We show that the supply and demand elasticities can be estimated by regressing a location's fixed effect (recovered from a gravity equation) on its own expenditure share (the coefficient of which

[2]While the implications for real output prices are the same for all gravity models, the mapping from real output prices to welfare will in general depend on the particular model. As a result, the normative (welfare) implications will vary across different models, as we discuss in detail below.

is the supply elasticity) and its income (the coefficient of which is the demand elasticity). Identifying the elasticities requires a set of instruments that are correlated with own expenditure share and income, but uncorrelated with unobserved supply shifters (such as productivity) in the residual. We construct such instruments using the general equilibrium structure of the model by calculating the equilibrium own expenditure shares and incomes of a hypothetical world where no such unobserved supply shifters exist and bilateral trade frictions are only a function of distance. Using this procedure, we estimate a demand elasticity in line with previous estimates from the trade literature (e.g. Simonovska and Waugh (2014)) and a supply elasticity that is larger than is typically (implicitly) calibrated to in trade models but appears reasonable given estimates from the economic geography literature.

Finally, we use the estimated gravity constants along with the expression for comparative statics to evaluate the effect of a trade war between the U.S. and China on the real expenditure of all countries in the world. Given our large estimated supply elasticity, we find modest declines in real output prices but large declines in real expenditure. Third country effects are also substantial, with important trading partners of China (e.g. Vietnam and Japan) and the U.S. (e.g. Canada and Mexico) being especially adversely affected.

This paper is related to a number of strands of literature in the fields of international trade, economic geography, and general equilibrium theory. There is a small but growing literature examining the structure of general equilibrium models of trade and economic geography. In particular, Arkolakis et al. (2012a) provide conditions under which a model yields a closed form expression for changes in welfare as a function of changes in openness, while in a recent paper Adao et al. (2017) show how to conduct counterfactual predictions in neo-classical trade models without imposing gravity. In contrast, our paper incorporates models with elastic aggregate supply curves, thereby allowing analysis of both economic geography models and trade models with intermediate "round-about" production. A key characteristic of the class of models we study is that the "gravity constants" are the same across all locations; while strong, this assumption imposes sufficient structure to completely characterize all general equilibrium interactions while retaining tractability even in the presence of a large number of locations.[3]

In terms of the theoretical properties of the equilibrium, Alvarez and Lucas (2007) use the gross substitutes property to establish sufficient conditions for uniqueness for gravity trade models. We instead generalize results from the study of nonlinear integral equations (see e.g. Karlin and Nirenberg (1967); Zabreyko et al. (1975); Polyanin and Manzhirov (2008)) to systems of nonlinear integral equations. As a result, the sufficient conditions we provide are strictly weaker than those derived by Alvarez and Lucas (2007). In particular, our conditions allows the supply elasticity to be larger in magnitude than the demand elasticity (in which case gross substitutes may not hold), which is what we find when we estimate the elasticities. In previous work, Allen and Arkolakis (2014) provide sufficient conditions for existence and uniqueness for economic geography models. Unlike those results, our conditions do not require symmetric trade frictions nor do we require finite trade frictions between all locations. Unlike both Alvarez and Lucas (2007) and Allen

[3]In contrast, the literature on Computable General Equilibrium models typically focuses on models with a large number of elasticities (e.g. location or region specific) but only a small number of regions; for a review of these models see Menezes et al. (2006). Although outside the purview of this paper, it would be perhaps be interesting future work to determine whether some of the tools developed below could be applied to those models.

and Arkolakis (2014), our theoretical results cover both trade and economic geography models simultaneously.

Our analytical characterization of the counterfactual predictions is related to the "exact hat algebra" methodology pioneered by Dekle et al. (2008) and extended in Costinot and Rodriguez-Clare (2013) (and many others). Unlike that approach, we characterize the elasticity of endogenous variables to trade shocks (i.e. we examine local shocks instead of global shocks). There are several advantages of our local approach: first, all possible counterfactuals can be calculated simultaneously through a single matrix inversion. Second, our analytical characterization holds for local shocks around the observed equilibria even if there are other possible equilibria (in which case we are unaware of a procedure that ensures the solution to the "exact hat" approach that corresponds to the observed equilibria). Third, the local analytical expression admits a simple economic interpretation as a shock propagating through the trading network. In this regard, our paper is related to the recent working paper by Bosker and Westbrock (2016) which examines how shocks propagate through global production networks. Fourth, our analytical derivation allows us to characterize the relative size of the elasticity of real output prices and real output in different locations from a trade friction shock, providing (to our knowledge) one of the first analytical results about the relative size of the direct and indirect impacts of a trade friction shock in a model with many locations and arbitrary bilateral frictions.[4]

Our estimation strategy uses equilibrium income and own expenditure shares from a hypothetical economy as instruments to identify the demand and supply elasticities. Following Eaton and Kortum (2002), we use the fixed effects of a gravity equation as the

[4]Mossay and Tabuchi (2015) prove a similar result in a three country world.

dependent variable in an instrumental variables regression (although we use the regression to estimate the supply elasticity along with the demand elasticity). One advantage of our approach is the simplicity of calculating our instruments using bilateral distances and observed geographic variables; in this regard, we owe credit to Frankel and Romer (1999) who instrument for trade with geography (albeit not in a general equilibrium context).

The idea of using the general equilibrium structure of the gravity model to recover key parameters is originally due to Anderson and Van Wincoop (2003). Following this, several papers have sought to improve the typical gravity equation estimation by accounting for equilibrium conditions. For example, Anderson and Yotov (2010) pursues an estimation strategy imposing that the equilibrium "adding up constraints" of the multilateral resistance terms are satisfied, whereas Fally (2015) proposes the use of a Poisson Pseudo-Maximum-Likelihood estimator whose fixed effects ensure that such constraints are satisfied, and Egger and Nigai (2015) develops a two-step model consistent approach that overcomes bias arising from general equilibrium forces and unobserved trade frictions. Unlike these papers, here our focus is on recovering the demand and supply elasticities rather than estimating trade friction coefficients in a model consistent manner.

Recent work by Anderson et al. (2016) explores the relationship between trade and growth examined by Frankel and Romer (1999) in a structural context. They recover the demand (trade) elasticity from a regression of income on a multilateral resistance term, where endogeneity concerns are addressed by calculating multilateral resistance based on international linkages only. Our estimation strategy, in contrast, recovers both the demand and supply elasticities from a gravity regression and overcomes endogeneity concerns using an instrumental variables approach based on the general equilibrium structure of the model.

Finally, we should note that the brief literature review above is by no means complete and refer the interested reader to the excellent review articles by Baldwin and Taglioni (2006), Head and Mayer (2013), Costinot and Rodriguez-Clare (2013) and Redding and Rossi-Hansberg (2017), where the latter two focus especially on quantitative spatial models.

The remainder of the paper is organized as follows. In the next section, we present the universal framework and discuss how it nests existing general equilibrium gravity models. In Section 3.3, we present the theoretical results for existence and uniqueness. In Section 3.4, we present the results concerning the counterfactual predictions of the model. In Section 3.5, we estimate the gravity constants. In Section 3.6 we calculate the effects of a U.S. - China trade war. Section 3.7 concludes.

3.2. A universal gravity framework

Before turning to the universal gravity framework, we present two variants of the simple Armington gravity model to provide a concrete example of the type of models that fall within our framework. Suppose there are N locations each producing a a differentiated good and in what follows we define the set $S \equiv \{1, ..., N\}$. The only factor of production is labor, where we denote the allocation of labor in location $i \in S$ as L_i and assume the total world labor endowment is $\sum_{i \in S} L_i = \bar{L}$. Shipping the good from $i \in S$ to final destination j incurs an iceberg *trade friction*, where $\tau_{ij} \geq 1$ units must be shipped in order for one unit to arrive. Consumers have CES preferences with elasticity of substitution $\sigma \geq 0$.

In the first variant, which we call the "trade" model, suppose that the labor endowed to a location is exogenous and perfectly inelastic, as in Anderson (1979) and Anderson and Van Wincoop (2003). Suppose too that there is roundabout production, as in Eaton and Kortum (2002), that combines labor and an intermediate input in a Cobb-Douglas fashion. Thus, the quantity of output produced in location i is $Q_i = (A_i L_i)^{\zeta} I_i^{1-\zeta}$, with $\zeta \in (0,1]$ the labor share, A_i is the labor productivity in location $i \in S$ and I_i is an intermediate input equal to a CES aggregate of the differentiated varieties in all locations with the same elasticity of substitution σ as final demand. In this case, the output price in location i is $p_i = (w_i/A_i)^{\zeta} P_i^{1-\zeta}$, where w_i is the wage and $P_j \equiv \left(\sum_{k\in S} (p_j \tau_{kj})^{1-\sigma}\right)^{\frac{1}{1-\sigma}}$ is both the CES price index for the consumer and the price per unit of intermediate input.

In the second variant, the "economic geography" model, we suppose instead that the labor supplied to a location is perfectly elastic so that welfare is equalized across locations, as in Allen and Arkolakis (2014).[5] Welfare in this model is the product of the real expenditure of labor and the amenity value of living in a location, denoted by , and . welfare equalization implies $\frac{w_i}{P_i} u_i = \frac{w_j}{P_j} u_j$ for all $i, j \in S$. We further assume that productivities and amenities are subject to spillovers: $A_i = \bar{A}_i L_i^a$ and $u_i = \bar{u}_i L_i^b$. In this variant of the model, the quantity of output produced in location i is $Q_i = \bar{A}_i L_i^{1+a}$ and the output price is $p_i = w_i / \left(\bar{A}_i L_i^a\right)$.[6]

[5]In addition, this formulation incorporates many prominent economic geography models, e.g. Helpman (1998); Donaldson and Hornbeck (2012); Bartelme (2014); Redding (2016).
[6]It is straightforward to add round-about production into the economic geography variant of the model (see Table 3.8.1); we omit to do so here to keep our illustrative examples as simple as possible.

In both variants of the model, CES consumer preferences for the goods from each location yields a gravity equation that characterizes the aggregate demand in location j for the differentiated variety from location i:

$$X_{ij} = \frac{(p_i \tau_{ij})^{1-\sigma}}{\sum_{k \in S} (p_j \tau_{kj})^{1-\sigma}} E_j, \quad \text{for all } j, \tag{3.1}$$

where $E_j = \sum_{j \in S} X_{ji}$ is the expenditure in location j.

More subtly, both variants of the model also feature an aggregate supply for the quantity of output produced in each location. In the trade variant of the model – despite the labor supply being perfectly inelastic – we can use the fact that a constant share of revenue is paid to both workers and intermediates to write the output of location i as:

$$Q_i = A_i L_i \left(\frac{p_i}{P_i} \right)^{\frac{1-\zeta}{\zeta}}. \tag{3.2}$$

Similarly, in the economic geography variant of the model we can use the welfare equalization condition to write:

$$Q_i = \kappa \bar{A}_i^{\frac{b-1}{a+b}} \bar{u}_i^{-\frac{1+a}{a+b}} \left(\frac{p_i}{P_i} \right)^{-\frac{1+a}{a+b}}, \tag{3.3}$$

where $\kappa \equiv \left(\bar{L} / \left(\sum_{i \in S} \left(\bar{A}_i \bar{u}_i \right)^{-\frac{1}{a+b}} \left(\frac{p_i}{P_i} \right)^{-\frac{1}{a+b}} \right) \right)^{1+a}$ is an (endogenous) scalar that depends on the aggregate labor endowment $\bar{L}$ and we refer to $\frac{p_i}{P_i}$ as the *real output price* in location $i \in S$.[7] Finally, in both variants, we close the model by requiring that the value of total

[7]In these two examples – as in most of the analysis that follows – we focus on interior equilibria where production is positive in all locations. In the Online Appendix 3.10.2 we generalize our setup to allow for the possibility of non-interior solutions where production is zero in some locations, which allows e.g. for the case that welfare in unpopulated locations may be lower than populated locations. In Theorem 1 below, we provide sufficient conditions under which all equilibria are guaranteed to be interior.

output equals total sales (market clearing), i.e.

$$Y_i \equiv p_i Q_i = \sum_{j \in S} X_{ij}, \tag{3.4}$$

and that total expenditure equals total output (balanced trade), i.e.:

$$E_i = p_i Q_i. \tag{3.5}$$

Substituting the CES demand (equation 3.1) and supply equations (equations 3.2 or 3.3) into the market clearing and balanced trade conditions yields the following identical system of equilibrium equations for both variants of the model. In particular,

$$p_i^{1+\phi} C_i \left(\frac{p_i}{P_i}\right)^{\psi} = \sum_{j \in S} \tau_{ij}^{-\phi} P_j^{\phi} p_j C_j \left(\frac{p_j}{P_j}\right)^{\psi} \forall i \in S \tag{3.6}$$

$$P_i^{-\phi} = \sum_{j \in S} \tau_{ji}^{-\phi} p_j^{-\phi} \; \forall i \in S, \tag{3.7}$$

where in the trade variant of the model $\psi \equiv \frac{1-\zeta}{\zeta}$ and $C_i \equiv A_i L_i$, in the economic geography variant of the model $\psi \equiv -\frac{1+a}{a+b}$ and $C_i \equiv \bar{A}_i^{\frac{b-1}{a+b}} \bar{u}_i^{-\frac{1+a}{a+b}}$, and in both models $\phi \equiv \sigma - 1$. Note in both models the constants $\{C_i\}_{i \in S}$ are exogenous model location-specific fundamentals, which we refer to as *supply shifters* in what follows, and ϕ, ψ are global parameters. Given supply shifters, trade frictions, and the two parameters, one can use equations (3.6) and (3.7) to solve for output prices p_i and prices indices P_i (up-to-scale). One can then use a normalization that total world income is equal to one, i.e. $\sum_{i \in S} Y_i = 1$ and the gravity equation (equation 3.1) to calculate trade flows X_{ij}. Given trade flows, income

Y_i can then be recovered from market clearing (equation 3.4). Note that although the endogenous scalar κ from the economic geography model does not enter the equilibrium system of equations (and hence does not affect trade flows or incomes), it does affect the level of output, a point we return to below.

This example highlights the close relationship between trade and geography models and suggests the possibility for a unified analysis of the properties of such spatial gravity models. In what follows, we present a framework comprising six simple economic conditions about aggregate trade flows of a representative good between many locations. We show that the equilibrium of any model that satisfies these conditions can be represented by the solution to equations (3.6) and (3.7).

To proceed with out universal gravity framework, it is helpful to first introduce some terminology. Define the *output* $Q_i \geq 0$ to be the quantity of the representative good produced in location $i \in S$; the *quantity traded* $Q_{ij} \geq 0$ be the quantity of the representative good in location $i \in S$ that is consumed in location $j \in S$; the *output price* $p_i \geq 0$ to be the (factory gate) price per unit of the representative good in location $i \in S$; the *bilateral price* $p_{ij} \geq 0$ to be the cost of the representative good from location $i \in S$ in location $j \in S$; the *income* $Y_i \equiv p_i Q_i$ to be the total value of the representative good in location $i \in S$; the *trade flows* $X_{ij} \equiv p_{ij} Q_{ij}$ to be the value of the good in $i \in S$ sold to $j \in S$; the *expenditure* $E_i \equiv \sum_{j \in S} X_{ji}$ to be the total value of imports in $i \in S$; the *real expenditure* $W_i \equiv E_i / P_i$ is a measure of expenditure in location $i \in S$, where P_i is a *price index* defined below; and the *real output price* to be p_i / P_i.

We say that an equilibrium is *interior* if output and output prices are strictly positive in all locations, i.e. $Q_i > 0$ and $p_i > 0$ for all $i \in S$. In what follows, we focus our attention to interior equilibria and disregard the trivial equilibrium where $Q_i = 0$ for all $i \in S$. We provide sufficient conditions to ensure all equilibria are interior below and examine non-interior solutions in depth in Online Appendix 3.10.2. Clearly, because of the presence of complementarities there is a possibility of multiple interior equilibria. This is true in the economic geography model because of labor mobility and agglomeration externalities or even in the trade model when complementarities in consumption are large (low σ).

We first start with a condition that describes the relationship between the output price in location i and the bilateral price:

Condition 1. The bilateral price is equal to the product of the output price and a bilateral scalar:

$$p_{ij} = p_i \tau_{ij}, \tag{3.8}$$

where, as above, $\{\tau_{ij}\}_{i,j \in S} \in \overline{\mathbb{R}}_{++}$ are referred to as *trade frictions*.[8]

Given prices, the next condition can be used to derive aggregate demand.

Condition 2. (CES Aggregate Demand). There exists an exogenous (negative of the) *demand elasticity* $\phi \in \mathbb{R}$ such that the expenditure in location $j \in S$ can be written as:

$$E_j = \left(\sum_{i \in S} p_{ij}^{-\phi} \right)^{-\frac{1}{\phi}} W_j, \tag{3.9}$$

where W_j is the real expenditure and the associated price index is $P_j \equiv \left(\sum_{i \in S} p_{ij}^{-\phi} \right)^{-\frac{1}{\phi}}$. By Shephard's lemma, condition 2 (or, for short, C.2 thereafter) implies that the trade

[8] $\overline{\mathbb{R}}_{++}$ is defined as $\mathbb{R}_{++} \cup \{\infty\}$. If $\tau_{ij} = \infty$, then there is no trade between i and j.

flows from $i \in S$ to $j \in S$ can be written as::

$$X_{ij} = \frac{p_{ij}^{-\phi}}{\sum_{k \in S} p_{kj}^{-\phi}} E_j. \tag{3.10}$$

We refer to equation (3.10) as the *aggregate demand* of the universal gravity model. The aggregate demand equation (3.10) combined with C.1 yields a *gravity equation* equivalent to equation (2) in Anderson and Van Wincoop (2004), Condition R3' in Arkolakis et al. (2012a) and the CES factor demand specification considered in Adao et al. (2017). Accordingly, we note that the demand elasticity ϕ is often referred to as the "trade elasticity" in the literature.

It is important to emphasize that real expenditure $W_i = \frac{E_i}{P_i}$ and real output prices $\frac{p_i}{P_i}$ are distinct concepts from welfare, as neither necessarily correspond to the welfare of the underlying factor of production (such as labor) of a particular model. In the models above, for example, the welfare of a worker corresponds to her real wage, which is equal to the marginal product of a worker divided by the price index. Because of the presence of roundabout production (in the trade model) or externalities (in the economic geography model), a workers marginal product is not equal to the price per unit (gross) output. [9]

We furthermore assume that output in a location is potentially endogenous and specify the following supply-side equation:

Condition 3. (CES Aggregate Supply) There exists exogenous *supply shifters* $\{C_i\} \in \mathbb{R}_{++}^N$, an exogenous *aggregate supply elasticity* $\psi \in \mathbb{R}$, and an endogenous scalar $\kappa > 0$

[9] The relationship between real output prices and welfare for a number of seminal models are summarized in the last column of Table 3.8.1 and discussed in detail in Online Appendix 3.10.10.

such that output in each location $i \in S$ can be written as: (3.11)

$$Q_i = \kappa C_i \left(\frac{p_i}{P_i}\right)^{\psi}. \tag{3.11}$$

In what follows, we refer to equation (3.11) as the *aggregate supply* of the universal gravity model and the pair of demand and supply elasticities $(-\phi, \psi)$ as the *gravity constants*.

In general, the value of the endogenous scalar κ will depend on the particular model; for example, as we saw above, in the trade model $\kappa = 1$, whereas in the economic geography model κ is endogenously determined. Without taking a particular stance on the underlying model (and the implied value of κ), the scale of output is unspecified. However, we show below that we can still identify the equilibrium trade flows, incomes, and real output prices – including their level – without knowledge of κ.

Finally, to close the model, we impose two standard conditions and choose our numeraire:

Condition 4. (Output market clearing). For all $i \in S$, $Q_i = \sum_{j \in S} \tau_{ij} Q_{ij}$.

Note that by multiplying both sides of C.4 by the output price we have that income is equal to total sales as in equation (3.4) in our example economy.[10]

Condition 5. (Balanced trade). For all $i \in S$, $E_i = p_i Q_i$.

[10] As Anderson and Van Wincoop (2004) show, one can combine C.1, C.2, and C.4 to derive a gravity equation of the form $X_{ij} = \left(\frac{\tau_{ij}}{\Pi_i P_j}\right)^{-\phi} Y_i E_j$, where $\Pi_i^{-\phi} \equiv \sum_{j \in S} \left(\frac{\tau_{ij}}{P_j}\right)^{-\phi} E_j$ and $P_j^{-\phi} \equiv \sum_{i \in S} \left(\frac{\tau_{ij}}{\Pi_i}\right)^{-\phi} Y_i$ are outward and inward multilateral resistance terms, respectively.

Balanced trade is a standard assumption in (static) gravity models, despite trade imbalances being a common occurrence empirically. When we combine the general equilibrium structure of the model with data to characterize the counterfactual implications of gravity models, we relax C.5 to allow for exogenous trade deficits.

Our final condition is a normalization:

Condition 6. World income equals to one:

$$\sum_i Y_i = 1. \tag{3.12}$$

In the absence of a normalization, the level of prices are undetermined because equations (3.6) and (3.7) are homogeneous of degree 0 in $\{p_i, P_i\}_{i\in S}$. Moreover, without specifying κ in equation (3.11), the level of output is also unknown. The choice of normalizing world income to one in C.6 addresses both these issues simultaneously by pining down the product of the level of these two unknown scalars. As a result, we can determine the equilibrium level (i.e. including scale) of nominal incomes and trade flows. However, the cost of doing is that both the level of output (in quantities) and prices remain unknown. As a result, the primary focus in the following analysis is on three endogenous model outcomes for which we can pin down the levels: incomes, trade flows, and real output prices $\{p_i/P_i\}_{i\in S}$ (which are invariant to the both κ and the scale of prices and hence determined including scale).

Given any gravity constants $\{\phi, \psi\}$, supply shifters, $\{C_i\}_{i\in S}$, and bilateral trade frictions $\{\tau_{ij}\}_{i,j\in S}$, we define an *equilibrium of the universal gravity framework* to be a set of endogenous outcomes determined up-to-scale, namely: outputs $\{Q_i\}_{i\in S}$, quantities traded

$\{Q_{ij}\}_{i,j\in S}$, output prices $\{p_i\}_{i\in S}$, bilateral prices $\{p_{ij}\}_{i,j\in S}$, price indices $\{P_i\}_{i\in S}$, and real expenditures, as well as a set of endogenous outcomes for which the scale is known, namely: incomes $\{Y_i\}_{i\in S}$, expenditures $\{E_i\}_{i\in S}$, trade flows $\{X_{ij}\}_{i,j\in S}$ and real output prices $\{p_i/P_i\}_{i\in S}$ that together satisfy C.2-C.6.

As Table 3.8.1 summarizes, many well-known trade and economic geography models are contained within the universal gravity framework. On the demand side, it is well known (see e.g. Arkolakis et al. (2012b) and Adao et al. (2017)) that many trade models imply an aggregate CES demand system as specified in C.2.[11] For example, in the Armington perfect competition model, a CES demand combined with linear production functions implies $\phi = \sigma - 1$, in the Eaton and Kortum (2002) model, a Ricardian model with endogenous comparative advantage across goods and Frechet distributed productivities across sectors with elasticity θ implies that $\phi = \theta$. Similarly, a class of monopolistic models with CES or non-CES demand, linear production function, and Pareto distributed productivities with elasticity θ, summarized in Arkolakis et al. (2012b), also implies $\phi = \theta$. Economic geography models delivering gravity equations for trade flows such as Allen and Arkolakis (2014) and Redding (2016) also satisfy C.2.

As discussed in the example above, labor mobility across locations generates a CES aggregate supply satisfying C.3, with a supply elasticity of $\psi = -\frac{1+a}{a+b}$. In this case, the supply elasticity depends on the strength of the agglomeration / dispersion forces summarized by $a + b$. Assuming $a > -1$, if dispersion forces dominate ($a + b < 0$), the

[11]The class of trade models considered by Arkolakis et al. (2012a) (under their CES demand assumption R3') are a strict subset of the models which fall within the universal gravity framework, corresponding to the case of $\psi = 0$.

supply elasticity is positive, whereas when agglomeration forces dominate ($a+b>0$), the supply elasticity is negative.

Perhaps more surprising, trade models incorporating "round-about" trade with intermediates goods also exhibit an aggregate CES supply, even though workers are immobile across locations. As discussed in the example above, the supply elasticity is $\psi = \frac{1-\zeta}{\zeta}$ and hence positive and increasing in the share of intermediates in the production. In the next two sections, we show that any trade and economic geography models sharing the same gravity constants will also share the same theoretical properties and counterfactual implications.

What types of models are not contained within the universal gravity framework? C.2 and C.3 are violated by models that do not exhibit constant demand and supply elasticities, which include Novy (2010), Head et al. (2014), Melitz and Redding (2014), Fajgelbaum and Khandelwal (2013) and Adao et al. (2017). Models with multiple factors of production with non-constant factor intensities will generally not admit a single aggregate good representation and hence are also not contained within the universal gravity framework (although the tools developed below can often be extended to analyze such models depending on the particular functional forms). C.5 is violated both by dynamic models in which the trade deficits are endogenously determined and by models incorporating additional sources of revenue (like tariffs); hence these models are not contained within the universal gravity framework. However, we show in Online Appendix 3.10.8 how the results below can be applied to a simple Armington trade model with tariffs.[12]

[12] It is important to note that while the universal gravity framework can admit tariffs, how tariffs affect the model implications will in general depend on the micro-economic foundations of a model. In particular, the Armington model presented in Online Appendix 3.10.8 abstracts from two additional complications that may arise with the introduction of tariffs. First, the elasticity of trade to tariffs may be different

3.3. Existence, uniqueness, and interiority of equilibria

We proceed by deriving a number of theoretical properties of the equilibria of all models contained within the universal gravity framework.

To begin, we note that we can combine C.1 through C.5 to write the equilibrium output prices and price indices (to-scale) as the solution to equations (3.6) and (3.7). These equations are sufficient to recover the equilibrium level of real output prices and – given the normalization in C.6 – the equilibrium level of incomes, expenditures, and trade flows as well as all other endogenous variables up-to-scale.[13] As a result, equations (3.6) and (3.7) (together with the normalization in C.6) are sufficient to characterize the equilibrium of the universal gravity framework.

Before proceeding, we impose two mild conditions on bilateral trade frictions $\{\tau_{ij}\}_{i,j\in S}$:

Assumption 3. *The following parameter restrictions hold:*

i) $\tau_{ii} < \infty$ *for all* $i \in S$.

ii) The graph of the matrix of trade frictions $\{\tau_{ij}\}_{i,j\in S}$ *is strongly connected.*

The first part of the assumption imposes strictly positive diagonal elements of the matrix of bilateral trade frictions. The second part of the assumption – strong connectivity – requires that there is a sequential path of finite bilateral trade frictions that can link any two locations i and j for any $i \neq j$. This condition has been applied previously in general

than the elasticity of trade to trade frictions depending on the model; second, if one does not impose that tariffs are uniform for all trade flows between country pairs, the construction of (good-varying) optimal tariffs will depend on the particular micro-economic structure of the model; see Costinot et al. (2016) for a detailed discussion of these issues.

[13]See Online Appendix 3.10.1 for these derivations.

equilibrium analysis as a condition for existence in McKenzie (1959, 1961), Arrow et al. (1971), invertibility by Cheng (1985); Berry et al. (2013), and uniqueness by Arrow et al. (1971), Allen (2012). In our case these two assumptions are the weakest assumptions on the matrix of trade frictions we can accommodate in order to analyze existence and uniqueness of interior equilibrium.

We mention briefly (but do not need to assume) a third condition. We say that trade frictions are *quasi-symmetric* if there exist a pair of strictly positive vectors $\left(\tau_i^A, \tau_i^B\right) \in \mathbb{R}_{++}^{2N}$ such that for any $i, j \in S$, we can write $\tau_{ij} = \tilde{\tau}_{ij}\tau_i^A\tau_j^B$, where $\tilde{\tau}_{ij} = \tilde{\tau}_{ji}$. Quasi-symmetry is a common assumption in the literature (see for example Anderson and Van Wincoop (2003), Eaton and Kortum (2002), Waugh (2010), Allen and Arkolakis (2014)), and we prove in Online Appendix 3.10.3 that C.1, C.2, C.4, and C.5 taken together imply that the origin and destination-specific terms in the bilateral trade flow expression are equal up to scale, i.e. $p_i^{-\phi} \propto p_i^{1+\psi}P_i^{\phi-\psi}C_i$, which in turn implies that equilibrium trade flows will be symmetric, i.e. $X_{ij} = X_{ji}$ for all $i, j \in S$. The only way the trade can be balanced when trade frictions are quasi-symmetric is to make trade flows bilaterally balanced. As a result, equations (3.6) and (3.7) simplify to a single set of equilibrium equations, which allows allows us to relax the conditions on the following theorem regarding existence and uniqueness:

Theorem 1. *Consider any model contained within the universal gravity framework satisfying Assumption 3. Then:*

(i) If $1 + \psi + \phi \neq 0$*, then there exists an interior equilibrium.*

(ii) If $\phi \geq -1$*, and* $\psi \geq 0$ *then all equilibria are interior.*

(iii) If $\{\phi \geq 0, \psi \geq 0\}$ *or* $\{\phi \leq -1, \psi \leq -1\}$ *(or, if trade frictions are quasi-symmetric and either* $\left\{\phi \geq -\frac{1}{2}, \psi \geq -\frac{1}{2}\right\}$ *or* $\left\{\phi \leq -\frac{1}{2}, \psi \leq -\frac{1}{2}\right\}$*) then there is a unique interior equilibrium.*

(iv) If $\{\phi > 0, \psi > 0\}$ *or* $\{\phi < -1, \psi < -1\}$ *(or, if trade frictions are quasi-symmetric and either* $\left\{\phi > -\frac{1}{2}, \psi > -\frac{1}{2}\right\}$ *or* $\left\{\phi < -\frac{1}{2}, \psi < -\frac{1}{2}\right\}$*).*

Proof. See Appendix 3.9.1 for parts (i) and (iii) and Online Appendix 3.10.2 for part (ii). □

A key advantage of Theorem 1 is that despite the large dimensionality of the parameter space (N supply shifters $\{C_i\}_{i \in S}$ and N^2 trade frictions $\{\tau_{ij}\}_{i,j \in S}$), the conditions are only stated in terms of the two gravity constants. Of course, since we provide sufficient conditions, there may be certain parameter constellations such as particular geographies of trade frictions where uniqueness may still occur even if the conditions of Theorem 1 are not satisfied.[14,15]

The sufficient conditions for existence, interiority, and uniqueness from Theorem 1 are illustrated in Figure 3.8.1. In the case of existence, standard existence theorems (see e.g. Mas-Colell et al. (1995)) guarantee existence for endowment economies when preferences

[14]Alvarez and Lucas (2007) provide an alternative approach based on the gross substitute property to provide conditions for uniqueness of the Eaton and Kortum (2002) model. In Online Appendix 3.10.6, we show that the gross substitutes property directly applied to our system may fail if the supply elasticity ψ is larger in magnitude than the demand elasticity ϕ, i.e. in ranges $\psi > \phi \geq 0$ or $\psi < \phi \leq -1$. Theorem 1 provides strictly weaker sufficient conditions in that regard. Such parameter constellations are consistent with economic geography models with weak dispersion forces or trade models with large intermediate goods shares. Importantly, in Section 3.5, we estimate that $\psi > \phi > 0$ empirically.

[15]Theorem 1 generalizes Theorem 2 of Allen and Arkolakis (2014) in three ways: 1) it allows for asymmetric trade frictions; 2) it allows for infinite trade frictions between certain locations; and 3) it applies to a larger class of general equilibrium spatial model, including notably trade models with inelastic labor supplies (i.e. models in which $\psi = 0$). Theorem 1 also provides a theoretical innovation, as it shows how to extend the mathematical argument of Karlin and Nirenberg (1967) to multi-equation systems of non-linear integral equations.

are strictly convex. This is also true in the universal gravity framework: existence of an interior equilibrium may fail only when $1+\psi+\phi=0$, which corresponds to the Armington trade model (without intermediate goods) where $\sigma=0$, i.e. with Leontief preferences that are not strictly convex. Moreover, in the economic geography example above, an interior equilibrium does not exist in the knife-edge case where $\sigma=\frac{1+a}{a+b}$, as agglomeration forces lead to the concentration of all economic activity in one location (see Allen and Arkolakis (2014)).

As long as the partial elasticity of aggregate demand with respect to own output price is greater than negative 1 and the partial elasticity of supply with respect to the real output price is positive, all equilibria are interior. For example, in the economic geography model above, if these conditions are satisfied, one can show that the welfare of an uninhabited location approaches infinity as its population approaches zero, ensuring that all locations will be populated in equilibrium.

An equilibrium is unique as long as the partial elasticity of aggregate demand to output prices is negative (i.e. $\phi \geq 0$) and the partial elasticity of aggregate supply is positive (i.e. $\psi \geq 0$). There is also a unique interior equilibrium the demand elasticity is positive and the supply elasticity is negative and both elasticities have magnitudes greater than one, although such parameter constellations are less economically meaningful (and there may also exist non-interior equilibria). Multiplicity of interior equilibria may arise in cases when supply and demand elasticities are both positive (which occurs e.g. in trade models when goods are complements) or when supply and demand elasticities are both negative (which occurs e.g. in economic geography models when agglomeration forces are stronger than dispersion forces). Such examples of multiplicity are easy to construct - Appendix

3.10.7 provides examples of multiplicity in a two location world where either the demand elasticity is negative (in which case the relative demand and supply curves are both upward sloping) or the supply elasticity is negative (in which case the relative demand and supply curves are both downward sloping). Finally, quasi-symmetric trade frictions allow us to extend the range of gravity constants for which uniqueness is guaranteed, but do not qualitatively change the intuition for the results.

3.4. The network effects of a trade shock

We now turn to how the universal gravity framework can be used to make predictions of how a change in trade frictions alter equilibrium trade flows, incomes, and real output prices in each location.[16]

To begin, we define two $N \times 1$ vectors (which, with some abuse of language, we will call "curves"): define the supply curve $\mathbf{Q}^s$ to be the set of supply equations (3.11) from C.3 (multiplied by output prices and divided by κ); and define the demand curve $\mathbf{Q}^d$ to be the set of market clearing (demand) equations combining C.1, C.2, C.4, and C.5, i.e.:

$$\mathbf{Q}^s\left(\mathbf{p},\mathbf{P}\right) \equiv \left(p_i \times C_i \left(\frac{p_i}{P_i}\right)^{\psi}\right)_{i\in S} \tag{3.13}$$

$$\mathbf{Q}^d\left(\mathbf{p},\mathbf{P};\boldsymbol{\tau}\right) \equiv \left(\sum_{j\in S} \tau_{ij}^{-\phi} p_i^{-\phi} P_j^{\phi} p_j C_j \left(\frac{p_j}{P_j}\right)^{\psi}\right)_{i\in S}, \tag{3.14}$$

where $\mathbf{p} \equiv (p_i)_{i\in S}$ and $\mathbf{P} \equiv \left(\left[\sum_j \tau_{ji}^{-\phi} p_j^{-\phi}\right]^{-\frac{1}{\phi}}\right)_{i\in S}$ are $N \times 1$ vectors and $\tau \equiv (\tau_{ij})_{i,j\in S}$ is an $N^2 \times 1$ vector. Note that we express both the supply and demand curves in value

[16] In what follows, we focus on the policy shocks that alter bilateral trade frictions $\{\tau_{ij}\}_{i,j\in S}$. In Online Appendix 3.10.8, we show how one can apply similar tools to characterize the theoretical properties and conduct counterfactuals in an Armington trade model with tariffs.

terms, which will prove helpful in deriving the comparative statics in terms of observed trade flows.

In equilibrium, supply is equal to demand, i.e. $\mathbf{Q}^s(\mathbf{p},\mathbf{P}) = \mathbf{Q}^d(\mathbf{p},\mathbf{P};\boldsymbol{\tau})$. We fully differentiate this equation, along with the definition of the price index, to yield the following system of $2N$ linear equations relating a small change in trade costs, $D\ln\boldsymbol{\tau}$, to a small change in output prices and price indices, $D\ln\mathbf{p}$ and $D\ln\mathbf{P}$, respectively:

$$\left(\underbrace{\begin{pmatrix} D_{\ln\mathbf{p}}\mathbf{Q}^s & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{pmatrix}}_{\equiv\mathbf{S}} - \underbrace{\begin{pmatrix} D_{\ln\mathbf{p}}\mathbf{Q}^d & D_{\ln\mathbf{P}}\mathbf{Q}^d - D_{\ln\mathbf{P}}\mathbf{Q}^s \\ D_{\ln\mathbf{p}}\ln\mathbf{P} & \mathbf{0} \end{pmatrix}}_{\equiv\mathbf{D}}\right)\begin{pmatrix} D\ln\mathbf{p} \\ D\ln\mathbf{P} \end{pmatrix} = \underbrace{\begin{pmatrix} D_{\ln\boldsymbol{\tau}}\mathbf{Q}^d \\ D_{\ln\boldsymbol{\tau}}\ln\mathbf{P} \end{pmatrix}}_{\equiv\mathbf{T}} D\ln\boldsymbol{\tau},$$

where $\mathbf{S}$ (*the supply matrix*) and $\mathbf{D}$ (*the demand matrix*) are $2N \times 2N$ matrices capturing the marginal effects of a change in the output price on the supply and demand curves (where the demand matrix also captures the net effect of a change in the price index), respectively, and $\mathbf{T}$ is a $2N \times N^2$ matrix capturing the marginal effects of a change in trade costs on the demand curve and price index.

Given expressions (3.13) and (3.14), we can write all three matrices solely as a function of the gravity constants and observables as follows:

(3.15)
$$\mathbf{S} = \begin{pmatrix} (1+\psi)\,\mathbf{Y} & ,\mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{pmatrix},\ \mathbf{D} = \begin{pmatrix} -\phi\mathbf{Y} + (1+\psi)\,\mathbf{X} & (\phi-\psi)\,\mathbf{X} + \psi\mathbf{Y} \\ \mathbf{E}^{-1}\mathbf{X}^T & \mathbf{0} \end{pmatrix},$$

(3.16)
$$\mathbf{T} = \begin{pmatrix} -\phi\,(\mathbf{X}\otimes\mathbf{1})\circ(\mathbf{I}\otimes\mathbf{1}) \\ \left(\mathbf{E}^{-1}\mathbf{X}^T\otimes\mathbf{1}\right)\circ(\mathbf{1}\otimes\mathbf{I}) \end{pmatrix},$$

where $\mathbf{X}$ is the (observed) $N \times N$ trade flow matrix whose $\langle i,j\rangle^{th}$ element is X_{ij}, $\mathbf{Y}$ is the $N \times N$ diagonal income matrix whose i^{th} diagonal element is Y_i, $\mathbf{E}$ is the $N \times N$ diagonal income matrix whose i^{th} diagonal element is E_i, $\mathbf{I}$ is the $N \times N$ identity matrix and $\mathbf{1}$ is an $1 \times N$ matrix of ones,$\mathbf{I}_i$ is the standard i-th basis for $\mathbb{R}^N$, and where $\otimes$ represents the Kronecker product and $\circ$ represents the element-wise multiplication (i.e. Hadamard product). [17]

A simple application of the implicit function theorem allows us to characterize the elasticity of prices and price indices to any trade cost shock. Define the $2N \times 2N$ matrix $\mathbf{A} \equiv \mathbf{S} - \mathbf{D}$ and, with a slight abuse of notation, let $A^{-1}_{k,l}$ denote the $\langle k,l\rangle^{th}$ element of the (pseudo) inverse of $\mathbf{A}$. Then:

Theorem 2. *Consider any model contained in the universal gravity framework. Suppose that* $\mathbf{X}$ *satisfies strong connectivity. If* $\mathbf{A}$ *has rank* $2N-1$*, then:*

(i) The elasticities of output prices and output price indices are given by:

$$\frac{\partial \ln p_l}{\partial \ln \tau_{ij}} = -\phi X_{ij} A^{-1}_{l,i} + \frac{X_{ij}}{E_j} A^{-1}_{l,N+j} \quad \textit{and} \quad \frac{\partial \ln P_l}{\partial \ln \tau_{ij}} = -\phi X_{ij} A^{-1}_{N+l,i} + \frac{X_{ij}}{E_j} A^{-1}_{N+l,N+j}. \tag{3.17}$$

(ii) If the largest absolute value of eigenvalues of $\mathbf{S}^{-1}\mathbf{D}$ *is less than one, then* $\mathbf{A}^{-1}$ *has the following series expansion:*

$$\mathbf{A}^{-1} = \sum_{k=0}^{\infty} \left(\mathbf{S}^{-1}\mathbf{D}\right)^k \mathbf{S}^{-1},$$

[17] In what follows (apart from part (iii) of Theorem 2), we do not assume that C.5 holds in the data, i.e. that income is necessarily equal to expenditure; rather, we allow for income and expenditure to differ by a location-specific scalar, i.e. we allow for (exogenous) deficits.

(iii) If trade frictions are quasi-symmetric and $\phi, \psi \geq 0$ *then for all* $i, l \in S$ *and* $j \neq i, l$,

$$\frac{\partial \ln (p_i/P_i)}{\partial \ln \tau_{il}}, \frac{\partial \ln (p_l/P_l)}{\partial \ln \tau_{li}} < \frac{\partial \ln (p_j/P_j)}{\partial \ln \tau_{il}}$$
$$\frac{\partial \ln (p_i Q_i/P_i)}{\partial \ln \tau_{il}}, \frac{\partial \ln (p_l Q_l/P_l)}{\partial \ln \tau_{li}} < \frac{\partial \ln (p_j Q_j/P_j)}{\partial \ln \tau_{il}}.$$

and the inequalities have the opposite sign $(>)$ *if* $(\phi, \psi \leq -1)$.

Proof. See Appendix 3.9.2. □

Recall from Section 3.3 that knowledge of the output prices and price indices up-to-scale is sufficient to recover real output prices and – along with the normalization C.6 – is sufficient to recover equilibrium trade flows, expenditures, and incomes.[18] As a result, part (i) of Theorem 2 states that given gravity constants and observed data, the (local) counterfactuals of these variables for all models contained in the university gravity framework are the same.[19]

[18] Because of homogeneity of degree 0, we can without loss of generality normalize one price; moreover, from Walras' law, if $2N-1$ equilibrium conditions hold, then the last equation holds as well. As a result, $\mathbf{A}$ will have at most $2N-1$ rank and $\mathbf{A}^{-1}$ can be calculated by simply eliminating one row and column of $\mathbf{A}$ and then calculating its inverse. The values of the eliminated row can then be determined using the normalization C.6. For example, if one removes the first row and column, $\frac{\partial \ln p_1}{\partial \ln \tau_{ij}}$ can be chosen to ensure that $\sum_{i \in S} \frac{\partial \ln Y_i}{\partial \ln \tau_{ij}} = 0$ so that C.6 is satisfied.

[19] In Online Appendix 3.10.9, we show how the "exact hat algebra" (Dekle et al. (2008), Costinot and Rodriguez-Clare (2013)) can be applied to any model in the universal gravity framework to calculate the effect of any (possibly large) trade shock. The key takeaway – that counterfactual predictions depend only on observed data and the value of the gravity constants – remains true globally. However, if the uniqueness conditions of Theorem 1 do not hold, we are unaware of any procedure that guarantees that the solution found using the "exact hat algebra" approach corresponds to the counterfactual of the observed equilibrium. Indeed, it is straightforward to construct a simple example where in the presence of multiple equilibria, iterative algorithms used to solve the "exact hat algebra" system of equations will converge to qualitatively different equilibria than what is observed in the data even for arbitrarily small shocks, implying arbitrarily large counterfactual elasticities. In contrast, the elasticities in Theorem 2 will provide the correct local counterfactual elasticities around the observed equilibrium even in the presence of multiple equilibria.

The second part of Theorem 2 provides a simple interpretation of the counterfactuals as a shock propagating through the trade network. Consider a shock that decreases the trade cost between i and j by a small amount $\partial \ln \tau_{ij}$ and define $\left(\mathbf{S}^{-1}\mathbf{D}\right)^k \mathbf{S}^{-1}$ as the k^{th} *degree effect* of the shock. It turns out the k^{th} degree effect is simply the effect of the $k-1^{th}$ degree shock on the output prices and price indices of all locations' trading partners, holding constant their trading partners' prices and price indices. To see this, consider first the 0^{th} degree effect. Holding constant the prices and price indices in all other locations, the direct effect of a decrease in $\partial \ln \tau_{ij}$ is a shift of the demand curve upward in i by $\phi X_{ij} \times \partial \ln \tau_{ij}$ and a decrease in the price index in j by $\frac{X_{ij}}{E_j} \times \partial \ln \tau_{ij}$. To re-equilibriate supply and demand (holding constant prices and price indices in all other locations), we then trace along the supply curve to where supply equals demand by scaling the effect by $\mathbf{S}^{-1}$, for a total effect of $\mathbf{S}^{-1}\partial \ln \tau$. Consider now the 1^{st} degree effect. We first take the resulting changes in the price and price index from the 0^{th} degree effect and calculate how they shift the demand curve (and alter the price index) in all i and j trading partners by multiplying the 0^{th} degree effect by the demand matrix, i.e. $\mathbf{D}\left(\mathbf{S}^{-1}\partial \ln \tau\right)$. To find how this changes the price and price index in each trading partner, (holding constant the prices and price indices in the trading partners' trading partners), we then trace along the supply curve by again scaling the shock by $\mathbf{S}^{-1}$, for a combined effect of $\mathbf{S}^{-1}\mathbf{D}\mathbf{S}^{-1}\partial \ln \tau$. The process continues iteratively, with the k^{th} degree effect shifting the demand curve and price index according to the $k-1$ shock and then re-equilibriated supply and demand by tracing along the supply curve (holding constant the prices and price indices in all trading partners), for an effect of $(\mathbf{S}^{-1}\mathbf{D})^k \mathbf{S}^{-1}\partial \ln \tau$, as claimed.[20] The total change in prices and price indices is the infinite sum of all k^{th} degree shocks.

The third part of Theorem 2 says that the direct impact of a symmetric decline in trade frictions $\partial \ln \tau_{il}$ and $\partial \ln \tau_{li}$ on real output prices (and real expenditure) in the directly affected locations i and l will be larger than the impact of that shock in any other indirectly affected location $j \neq i, l$. If the demand and supply elasticities are positive, then a decline in trade frictions will cause the real output prices in the directly affected locations to rise more than any indirectly affected location (the ordering is reversed if the demand and supply elasticities are negative). This analytical result characterizes the relative impact of a trade friction shock on different locations in a model with many locations and arbitrary bilateral frictions.[21]

3.5. Estimating the gravity constants

In the previous section, we saw that the impact of a trade friction shock on trade flows, incomes, expenditures, and real output prices in any gravity model can be determined solely from observed trade flow data and the value the demand and supply elasticities. In this section, we show how these gravity constants can be estimated. We use data on international trade flows, so for the remainder of the paper we refer to a location as a country.

[20] One can also derive the alternative representation $\mathbf{A}^{-1} = -\sum_{k=0}^{\infty} \mathbf{D}^{-1} \left(\mathbf{S}\mathbf{D}^{-1}\right)^k$, in which the ordering is reversed: the k^{th} degree effect is calculated by first shifting the supply curve by the $k-1$ degree shock and then tracing along the demand curve to re-equilibriate supply and demand.

[21] Mossay and Tabuchi (2015) prove a similar result in a three country world.

3.5.1. Methodology

We first derive an equation that shows that the relationship between three observables – relative trade shares, relative incomes, and relative own expenditure shares – are governed by the two gravity constants. We then show how this relationship under minor assumptions can be used as an estimating equation to recover the gravity constants. We begin by combining C.1 and C.2 to express the expenditure share of country j on trade from i relative to its expenditure on its own goods as a function of the trade frictions, the output prices in i and j, and the aggregate demand elasticity:

$$\frac{X_{ij}}{X_{jj}} = \left(\frac{\tau_{jj}p_j}{\tau_{ij}p_i}\right)^{\phi}.$$

We then use the relationship $p_i = Y_i/Q_i$ to re-write this expression in terms of incomes and aggregate quantities and rely on C.3 to write the equilibrium output as a function of output prices and the output price index:

$$\frac{X_{ij}}{X_{jj}} = \left(\frac{\tau_{jj}\left(\frac{Y_j}{C_j}\right)\left(\frac{p_i}{P_i}\right)^{\psi}}{\tau_{ij}\left(\frac{Y_i}{C_i}\right)\left(\frac{p_j}{P_j}\right)^{\psi}}\right)^{\phi}. \tag{3.18}$$

We now define $\lambda_{jj} \equiv X_{jj}/E_j$ to be the fraction of income country j spends on its own goods (j's "own expenditure share"). By combining C.1 and C.2, we note j's own expenditure share can be written as $\lambda_{jj} = \left(\tau_{jj}\frac{p_j}{P_j}\right)^{-\phi}$, which allows us to write equation (3.18) (in log form) as:

$$\ln\frac{X_{ij}}{X_{jj}} = -\phi\ln\frac{\tau_{ij}}{\tau_{jj}} + \phi\ln\frac{Y_j}{Y_i} + \psi\ln\frac{\lambda_{jj}}{\lambda_{ii}} - \phi\ln\frac{C_j}{C_i} + \phi\psi\ln\frac{\tau_{jj}}{\tau_{ii}}. \tag{3.19}$$

Equation (3.19) shows that the demand elasticity ϕ is equal to the partial elasticity of trade flows to relative incomes, whereas the supply elasticity ψ is equal to the partial elasticity of trade flows to the relative own expenditure shares. Intuitively, the greater j's income relative to i (holding all else equal, especially the relative supply shifters $\ln \frac{C_j}{C_i}$), the greater the price in j relative to i and hence the more it would demand from i relative to j; the greater the demand elasticity ϕ, the greater the effect of the price difference on expenditure. Conversely, because the real output price is inversely related to a country's own expenditure share, the greater j's own expenditure share relative to i, the lower the relative aggregate supply to j and hence the more j will consume from i relative to j; the larger the supply elasticity ψ, the more responsive supply will be to differences in own expenditure share.

Equation (3.19) forms the basis of our strategy for estimating the gravity elasticities ϕ and ψ. However, it also highlights two important challenges in estimation. First, equation (3.19) suggests that for any observed set of trade flows $\{X_{ij}\}$ and any assumed set of gravity elasticities $\{\phi, \psi\}$, own trade frictions $\{\tau_{ii}\}$, and supply shifters $\{C_i\}$, there will exist a unique set of trade frictions $\{\tau_{ij}\}_{i \neq j}$ for which the observed trade flows are the equilibrium trade flows of the model.[22] As a result, trade flow data alone will not provide sufficient information to estimate the gravity elasticities. Second, equation (3.19) highlights that the gravity elasticities are partial elasticities holding the (unobserved) relative supply shifters $\{C_i\}$ fixed. Because both income and own expenditure shares are correlated with supply shifters through the equilibrium structure of the model, any estimation procedure must contend with this correlation between observables and unobservables.

[22] See Online Appendix 3.10.10 for a formal proof of this result.

In order to address both concerns, we combine plausibly exogenous observed geographic variation with the general equilibrium structure of the model to estimate the gravity elasticities. We proceed in a two-stage procedure.[23] First, we re-write equation (3.19) as:

$$\ln \frac{X_{ij}}{X_{jj}} = -\phi \ln \frac{\tau_{ij}}{\tau_{jj}} - \ln \pi_i + \ln \pi_j,$$

where $\ln \pi_i \equiv \phi \ln Y_i + \psi \ln \lambda_{ii} - \phi \ln C_i + \phi\psi \ln \tau_{ii}$ is a country-specific fixed effect. We assume relative trade frictions scaled by the trade elasticity can be written as a function of their continent of origin c, continent of destination d, and the decile of distance between the origin and destination countries, l:

$$-\phi \ln \frac{\tau_{ij}}{\tau_{jj}} = \beta_{cd}^l + \varepsilon_{ij},$$

where ε_{ij} is a residual assumed to be independent across origin-destination pairs. The country-specific fixed effect can then be recovered from the following the following equation:

$$\ln \frac{X_{ij}}{X_{jj}} = \beta_{cd}^l - \ln \pi_i + \ln \pi_j + \varepsilon_{ij}, \tag{3.20}$$

[23]While the two step procedure we follow resembles the procedure used in Eaton and Kortum (2002) to recover the trade elasticity from observed wages, there are two important differences. First, our procedure applies to a large class of trade and economic geography models and allows us to simultaneously estimate both the demand (trade) elasticity and the supply elasticity (rather than assuming e.g. that the population of a country is exogenous and calibrating the model to a particular intermediate good share). Second, our procedure relies on the general equilibrium structure of the model to generate the identifying variation (rather than e.g. instrumenting for wages with the local labor supply, which would be inappropriate for economic geography models).

where we estimate β^l_{cd} non-parametrically using a set of 360 dummy variables (10 distances deciles $\times$ 6 origin continents $\times$ 6 destination continents). Let $\ln \hat{\pi}_i$ denote the estimated fixed effect and define $\hat{\nu}_i \equiv \ln \hat{\pi}_i - \ln \pi_i$ to be its estimation error.

In the second stage, we write the estimated fixed effect as a function of income and own expenditure share:

$$\ln \hat{\pi}_i = \phi \ln Y_i + \psi \ln \lambda_{ii} + \nu_i, \tag{3.21}$$

where $\nu_i \equiv -\phi \ln C_i + \phi\psi \ln \tau_{ii} + \hat{\nu}_i$ is a residual that combines the unobserved supply shifter, the unobserved own trade friction, and the estimation error from the first stage. As mentioned above, it is not appropriate to estimate equation (3.21) via ordinary least squares, as variation in the supply shifter will affect income and the own expenditure share through the equilibrium structure of the model, creating a correlation between the residual and the observed covariates. Intuitively, the larger the supply shifter of a country, the greater its output and hence the greater the trade flows for a given observed income; since the country-specific fixed effect $\ln \pi_i$ is decreasing in relative trade flows, the OLS estimate of ϕ will be biased downwards.

To overcome this bias, we pursue an instrumental variables (IV) strategy, where we use the general equilibrium structure of the model to construct a valid instrument. To do so, we calculate the equilibrium trade flows of a hypothetical world where the bilateral trade frictions and supply shifters depend only on observables. We then use the incomes and relative own expenditure shares of this hypothetical world as instruments for the observed incomes and own expenditure shares. These counterfactual variables are valid instruments as long as (1) they are correlated with their observed counterparts (which we can verify); and (2) the observable components of the bilateral trade frictions and supply shifters are uncorrelated with unobserved supply shifters.

Because the first-stage estimation of (3.20) provides an unbiased estimate of $-\phi \ln \frac{\tau_{ij}}{\tau_{jj}}$, we use the estimated origin-continent-destination-continent-decile coefficients $\hat{\beta}^l_{cd}$ to create our counterfactual measure of bilateral trade frictions (normalizing own trade frictions $\tau_{jj} = 1$). In the simplest version of our procedure, we then calculate the equilibrium income and own expenditure share given these bilateral trade frictions, assuming that the supply shifter C_i is equal in all countries. In this version of the procedure, the instrument is valid as long as the the general equilibrium effects of distance on the origin fixed effects of a gravity equation are uncorrelated with unobserved heterogeneity in supply shifters (or own trade frictions). Because we calculate the equilibrium of the model in a counterfactual world where there is no heterogeneity in supply shifters, it seems reasonable to assume that the resulting equilibrium income and own expenditure shares that we use as instruments are uncorrelated with any real world heterogeneity. However, our instrument would be invalid if there were a correlation between unobserved supply shifters and the observed geography of a country (e.g. if countries more remotely located were also less productive or less attractive places to reside).

To mitigate such a concern (and to allow for more realistic variation across countries in supply), we extend the approach to allow the supply shifter to vary across countries depending on a vector of (exogenous) observables X_i^c, e.g. land controls like the amount of fertile land, geographic controls like the distance to nearest coast, institutional controls like the rule of law, historical controls like the population in 1400, and schooling and R&D controls like average years of schooling. Given a set of supply shifters $\{C_i\}$

that depend only these observables and the set of trade frictions that depend only on our non-parametric estimates from above, we re-calculate the equilibrium income and own expenditure share in each country. We then use the equilibrium values from this hypothetical world as our instruments, while and control directly for the observables X_i^c in equation (3.21). As a result, the identifying variation from the instruments only arises through the general equilibrium structure of the model.[24] Intuitively, differences in observables like land area in neighboring countries generates variation in the demand that a country faces for its production, as well as variation in the price it faces for its consumption, even conditional on its own observables.

There are two things to note about the above procedure. First, to construct the hypothetical equilibrium incomes and own expenditure shares requires assuming values of the gravity constants ϕ and ψ for the hypothetical world. In what follows, we choose a demand elasticity $\phi = 8.28$ and a supply elasticity $\psi = 3.76$, which correspond to the (estimated) demand elasticity estimated and (implicitly calibrated) supply elasticity in Eaton and Kortum (2002). We should note that while the particular choice of the these parameters will affect the strength of the constructed instruments, they will not affect the consistency of our estimates of the gravity constants under the maintained assumption

[24]Calculating the counterfactual equilibrium income and own expenditure share in each country when the supply shifters depend on observables requires assuming a particular mapping between the observables X_i^c and the supply shifter C_i. We assume that $C_i = X_i^c \beta^c$ and note that the theory implies the following equilibrium condition:

$$\ln Y_i = \frac{\phi}{\phi - \psi} \ln C_i + \frac{1+\psi}{\psi - \phi} \ln \gamma_i + \frac{\psi}{\psi - \phi} \ln \delta_i.$$

As a result, we choose the β^c that arise from the OLS regression $\ln Y_i = \frac{\phi}{\phi - \psi} X_i^c \beta^c + \epsilon_i$. Although our estimates of β^c may be biased due to the correlation between X_i^c and ϵ_i, this bias only affects the strength of the instrument, because if each X_i^c is uncorrelated with the residual ν_i in equation (3.21) (i.e. X_i^c is exogenous), then any linear combination of X_i^c will also be uncorrelated with the residual.

that bilateral distances are uncorrelated with the unobserved supply shifters conditional on observables.[25]

The second thing to note about the estimation procedure is more subtle. As mentioned in Section 3.3 and discussed in detail in Online Appendix 3.10.3, when bilateral trade frictions are "quasi-symmetric" the equilibrium origin and destination shifters will be equal up to scale. In this case, there will be a perfect log linear relationship between the income of a country, its own expenditure share and its supply shifter.[26] As a result, if we were to impose quasi-symmetric bilateral trade frictions in the hypothetical world, the equilibrium income and expenditure shares generated would be perfectly collinear, preventing us from simultaneously identifying the demand and supply elasticities in the second stage. Intuitively, identification of the demand elasticity requires variation in a country's supply curve (its destination fixed effect), whereas identification of the supply elasticity requires variation in a country's demand curve (its origin fixed effect); when trade frictions are quasi-symmetric, however, the two co-vary perfectly. Our choice to allow distance to affect trade frictions differently depending on the continent of origin and continent of destination introduces the necessary asymmetries in the trade frictions to allow the model constructed instruments to vary separately, allowing for identification of both the supply and demand elasticities simultaneously. To address concerns about the extent to which these asymmetries are sufficient to separately identify the two, we report the Sanderson-Windmeijer F-test (see Sanderson and Windmeijer (2016)) in the results that follow.

[25] In principle, we could search over different values of the gravity constants to find the constellation that maximizes the power of our instruments. In practice, however, our estimates vary only a small amount across different values of the gravity constants.

[26] In particular, $(1+2\phi)\ln E_i = (2\phi)\ln C_i + (1-2\psi)\ln \lambda_{ii} + C$.

3.5.2. Data

We now briefly describe the data we use to estimate the gravity constants.

Our trade data comes from the Global Trade Analysis Project (GTAP) Version 7 (Narayanan, 2008). This data provides bilateral trade flows between 94 countries for the year 2004. To construct own trade flows, we subtract total exports from the total sales of domestic product, i.e. $X_{ii} = X_i - \sum_{j \neq i} X_{ij}$. We use the bilateral distances between countries from the CEPII gravity data set of Head et al. (2010) to construct deciles of distance between two countries. We rely on the data set of Nunn and Puga (2012) to provide a number of country level characteristics that plausibly affect supply shifters, including “land controls” (land area interacted with the fraction of fertile soil, desert, and tropical areas), “geographic controls” (distance to the nearest coast and the fraction of country within 100 kilometers of an ice free coast), “historical controls” (log population in 1400 and the percentage of the population of European descent), “institutional controls” (the quality of the rule of law). Finally, following Eaton and Kortum (2002), we also consider “schooling and R&D controls” including the average years of schooling from UNESCO (2015) and the R&D stocks from Coe et al. (2009), where a dummy variable is included if the country is not in each respective data set.

3.5.3. Estimation results

Table 3.8.2 presents the results of our estimation of equation (3.19). The first column presents the ordinary least squares regression; we estimate a positive supply elasticity and negative demand elasticity, consistent with the discussion above that the OLS estimate of the demand elasticity is biased downward. Column 2 presents the instrumental variable estimation where the counterfactual income and own expenditure shares comprising our instrument are constructed assuming equal supply shifters. After correcting for the bias arising from the correlation between the unobserved supply shifters and observed incomes and own expenditure shares, we find positive supply and demand elasticities, although the demand elasticity is not statistically significant. Columns 3 through 7 sequentially allows the supply shifter in the construction of the instrument to vary across countries depending on an increasing number of observables (while including these same observables as controls in both the first and second stages of the IV estimation of equation (3.19)). Including these observables both increases the strength of the instruments and reduces the concern that the instruments are correlated with unobserved supply shifters. Reassuringly, our estimated demand and supply elasticities vary only slightly with the inclusion of additional controls.[27]

In our preferred specification (column 7), we estimate a demand elasticity of $\phi = 3.72$ (95% confidence interval [1.14,6.29] and a supply elasticity $\psi = 68.49$ (95% confidence interval [5.38,131.60]).[28] Hence, our demand elasticity estimate is somewhat lower than the preferred estimate of Eaton and Kortum (2002) of 8.28 (although similar to their estimate using variation in wages of 3.6), as well as similar to estimates of trade elasticity around 4 in Anderson and Van Wincoop (2004), Simonovska and Waugh (2014), and Donaldson (forthcoming). Unlike these papers, however, we also estimate the supply elasticity. Our point estimate, while noisily estimated, is substantially larger than and

[27]Figure 3.10.2 in the online appendix shows that our instrumental variables of counterfactual income and own expenditure shares are positively correlated with their observed counterparts, even after differencing out the observables in the supply shifters.

[28]While the p-value of the Sanderson-Windmeijer F-test is statistically significant in the first stage for income, it is only marginally statistically significant for expenditure shares, suggesting that the wide confidence interval for the supply elasticity may be due in part to a weak instrument.

statistically different (at the 5% level) from the supply elasticity to which Eaton and Kortum (2002) implicitly calibrate. Moreover, our estimated value is consistent with recent estimates of labor mobility from the migration literature. To see this, consider an economic geography framework with intermediate goods, agglomeration forces, and Frechet distributed preferences over location (see the last row of Table 3.8.1). If we match the labor share in production of 0.21 in Eaton and Kortum (2002) and the agglomeration force of $\alpha = 0.10$ in Rosenthal and Strange (2004), then our point estimate of ψ is consistent with a migration elasticity (Fréchet shape parameter) of 1.4. This is similar to estimates from the migration literature using observed labor flows and about one-third to one-half the size of within-country estimates.[29]

3.6. The impact of a U.S.-China trade war

We now apply the estimates from Section 3.5 to evaluate the impact of a trade war between the U.S. and China. We model the trade war as an increase in the trade frictions between the U.S. and China (holding constant all other trade frictions). We then characterize how such a trade war propagates through the trade network using the methodology developed in Section 3.4.[30]

[29] Ortega and Peri (2013) estimates an migration elasticity to destination country income of 0.6 using international migration flows and an estimate of 1.8 for the sub-sample of migration flows within the European Union, albeit not using a log-linear gravity specification. Within countries (aud with log-linear gravity specifications), Monte et al. (2015) estimate a migration elasticity of 4.4 in the U.S.; Tombe et al. (2015) estimate a migration elasticity of 2.54 in China, and Morten and Oliveira (2014) estimate a migration elasticity of 3.4 in Brazil.

[30] In the counterfactuals that follow, we accommodate the deficits observed in the data by assuming that the observed ratio of expenditure to income for each country remains remains constant and impose an aggregate market clearing condition that total income is equal to total expenditure. The results are qualitatively similar if we instead solve for the (unique) set of balanced trade flows that match the observed import shares and treat these balanced trade flows as the data.

There are two 0^{th} degree effects of the trade war: first, the U.S. and China export less to each other, causing the output prices in both countries to fall; second, the the cost of importing increases, causing the price index in both countries to rise. Both effects cause the real output price to decline, with a greater decline in China because both its export and import shares with the U.S. are relatively larger.

The top panel of Figure 3.8.2 depicts the 1^{st} degree effect on the real output price in all countries. The effect in the U.S. and China is positive, as the degree 0 decline in output price reduces the cost of own expenditure (causing the price index to fall in both countries). In other countries, however, the degree 1 effect is negative, as the U.S. and China demand less of their goods, causing their trading partner's output prices to fall. The most negatively affected countries are those who export the most to the U.S. and China.

Summing across all degree shocks yields the total elasticity of real output prices in each country to the trade war shock, which the bottom panel of Figure 3.8.2 depicts.[31] Not surprisingly, the two countries hurt most by a trade war are the U.S. and China. Moreover, while all countries are made worse off, the countries who are closely linked through the trading network with the U.S. and China (e.g. Canada, Mexico, Vietnam, and Japan) are hurt more than those countries that are less connected (e.g. India). All told, we estimate that a 10% increase in bilateral trade frictions is associated with a decline in real output price of 0.04% in the U.S. and 0.14% in China. These modest changes in the real output price are due to the large supply elasticity, causing the aggregate output to

[31]Figures 3.10.3 through 3.10.7 in the Online Appendix depict the impact of the degrees 0, 1, 2, and higher on the relative prices, relative output, income, the relative price index, and real output prices in each country.

reallocate away from the U.S. and China in response to the trade war. The converse of this result, however, is that the reallocation of the aggregate output results in large changes to *total* real expenditure: for example, in the Armington trade model interpretation, a 10% increase in bilateral trade frictions causes the total real expenditure to fall by 2.7% in the U.S. and by 9.8% in China.[32]

There are two potential concerns about these estimated effects. First, because the elasticities correspond to an infinitesimal shock, one may worry that the effects of a large trade war may differ. To address this concern, we calculate the effect of a 50% increase in bilateral trade frictions using the methodology discussed in Online Appendix 3.10.9. The correlation between the local elasticities and global changes exceeds 0.99, indicating that the local *relative* effect of the trade war is virtually the same as the global effect.[33] However, the local effect does overstate the global effect of such a shock, as we find that log first differences implied by the global shock are roughly 80% the size of those implied by the local elasticities. Second, the effects of the trade war above were calculated given the gravity constants estimated in Section 3.5; one may be concerned that the effects of the trade wars may differ substantially across alternative values of these elasticities. To address this concern, we calculate the effects of a trade war for a large number of different combinations of supply and demand elasticities.[34] Across all constellations in the 95% confidence interval of the two estimated gravity constants, the calculated elasticities are quite similar, with a 10% increase in bilateral trade frictions associated with a decline in

[32]Recall from Section 3.2 that while the changes in real output prices are identified from the value of trade flows alone, without specifying κ in equation (3.11), the change in total real expenditure is only identified up to scale. In Armington trade models with intermediates, however, this is not a problem, as $\kappa = 1$.

[33]See Figure 3.10.8 in the Online Appendix.

[34]See Figure 3.10.9 in the Online Appendix.

real output price between 0.03% and 0.05% in the U.S. and 0.07% and 0.26% in China. Of course, as Section 4 emphasizes, the particular value of the gravity constants may substantially affect the impact of counterfactuals more generally.

3.7. Conclusion

In this paper, we provide a framework that unifies a large set of trade and geography models. We show that the properties of models within this framework depend crucially on the value of two gravity constants: the aggregate supply and demand elasticities. Sufficient conditions for the existence and uniqueness of the equilibria depend solely on the gravity constants. Moreover, given observed trade flows, these gravity constants are sufficient to determine the effect of a trade friction shock on trade flows, incomes, and real output price without needing to specify a particular underlying model.

We then develop a novel instrumental variables approach for estimating the gravity constants using the general equilibrium structure of the framework. Using our estimates, we find potentially large losses may arise due to a trade war between U.S. and China occur.

By providing a universal framework for understanding the general equilibrium forces in trade and geography models, we hope that this paper provides a step toward unifying the quantitative general equilibrium approach with the gravity regression analysis common in the empirical trade and geography literature. Toward this end, we have developed a toolkit that operationalizes all the theoretical results presented in this paper.[35] We also hope the tools developed here can be extended to understand other general equilibrium

[35]The toolkit is available for download on Allen's website.

spatial systems, such as those incorporating additional types of spatial linkages beyond trade frictions.

3.8. Tables and Figures

Table 3.8.1. Examples of models in the universal gravity framework

Model	ϕ	ψ	Model parameters
Armington (1969), Anderson (1979), Anderson and Van Wincoop (2003)	$\sigma - 1$	$\frac{1-\zeta}{\zeta}$	σ subs. param. ζ labor share
(with intermediates) Krugman (1980) (with intermediates)	$\sigma - 1$	$\frac{1-\zeta}{\zeta}$	σ subs. param. ζ labor share
Eaton and Kortum (2002) (with intermediates)	θ	$\frac{1-\zeta}{\zeta}$	θ heterogeneity ζ labor share
Melitz (2003), Di Giovanni and Levchenko (2013)	$\theta\left(1+\frac{\theta-(\sigma-1)}{\theta(\sigma-1)}\right)$	$\frac{1-\zeta}{\zeta}$	σ subs. param. θ heterogeneity.
Redding (2016)	θ	$\frac{\alpha\varepsilon}{1+\varepsilon(1-\alpha)}$	θ good hetero. ε laobr hetero. α expenditure share

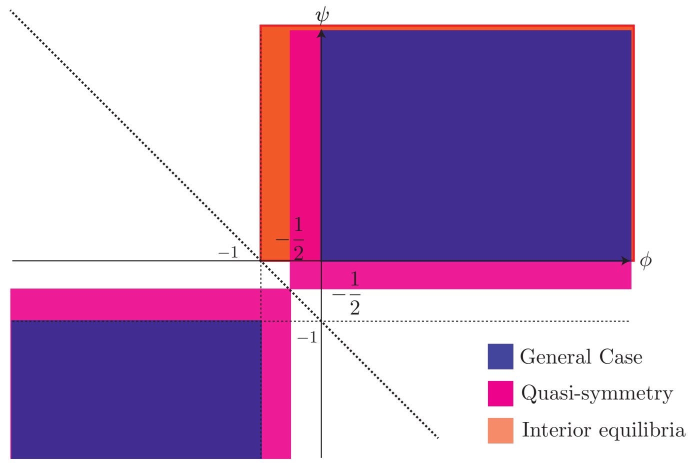
Notes: This table includes a (non-exhaustive) list of trade and economic geography models that can be examined within the universal gravity framework, the mapping of their structural parameters to the gravity constants. B_i is an exogenous location specific parameter whose interpretation depends on the particular model. λ is an endogenous variable which affects every country simultaneously.

Table 3.8.2. Estimating the gravity constants

	(1) OLS	(2) IV	(3) IV	(4) IV	(5) IV	(6) IV	(7) IV
Log Income	-0.403**	1.484	3.278	4.364*	3.882**	3.539***	3.715***
	(0.171)	(1.157)	(2.674)	(2.371)	(1.838)	(1.356)	(1.312)
Log Own Expenditure	3.381**	92.889***	108.592**	116.649**	71.859**	64.968**	68.488**
Share	(1.600)	(13.417)	(48.104)	(47.944)	(35.883)	(33.014)	(32.198)
Controls							
Land	No	No	Yes	Yes	Yes	Yes	Yes
Geographic	No	No	No	Yes	Yes	Yes	Yes
History	No	No	No	No	Yes	Yes	Yes
Institution	No	No	No	No	No	Yes	Yes
Schooling and R&D	No	No	No	No	No	No	Yes
First stage F-test:							
Income		25.909	3.994	6.349	20.095	34.198	25.763
(p-value)		0.004	0.102	0.053	0.007	0.002	0.004
Own expenditure share		72.702	4.388	4.923	3.561	4.577	5.205
(p-value)		0.000	0.090	0.077	0.118	0.085	0.071
Observations	94	94	94	94	94	94	94

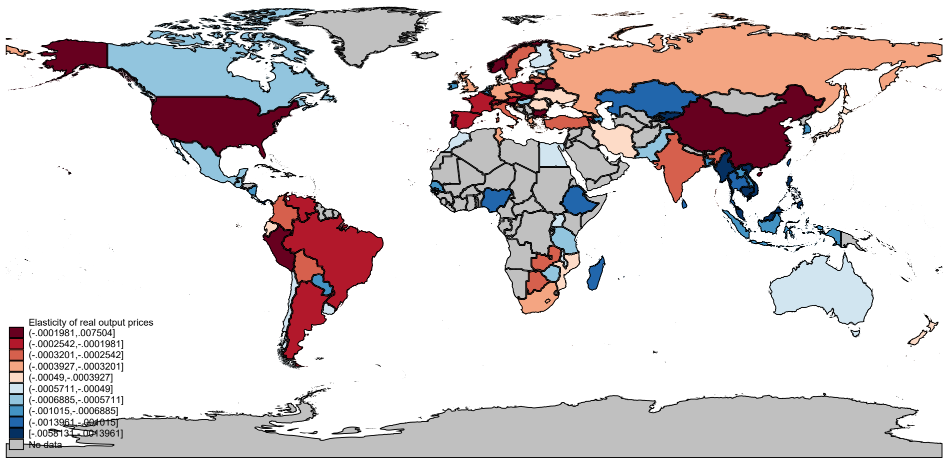
Notes: The dependent variable is the estimated country fixed effect of a gravity regression of the log ratio of bilateral trade flows to destination own trade flows on categorical deciles of distance variables, where the coefficient is allowed to vary by continent of origin and destination. Hence, each observation in the regressions above is a country. Instruments for income and own expenditure share are the equilibrium values from a trade model where the bilateral trade frictions are those predicted from the same gravity equation and countries are either identical in their supply shifters (column 2) or their supply shifters are estimated from a regression of observed income on observables (columns 3 through 7). In the latter case, the observables determining the supply shifters are controlled for directly in the first and second stage regressions, so identification of the demand and supply elasticities arise only from the general equilibrium effect on income and own expenditure shares. Land controls include land area interacted with fraction fertile soil, desert, and tropical areas. Geographic controls include the distance to nearest coast and the fraction of country within 100 km of an ice free coast. Historical controls include the log population in 1400 and the percentage of the population of European descent. Institutional controls include the quality of the rule of law. Schooling and R&D controls are average years of schooling (from UNESCO) and the R&D stocks (from Coe et al. (2009)), where a dummy variable is included if the country is not in each respective data set. Land, geographic, and historical control are from Nunn and Puga (2012). Standard errors clustered at the continent level are reported in parentheses. Stars indicate statistical significance: * $p<.10$ ** $p<.05$ *** $p<.01$.

Figure 3.8.1. Existence and uniqueness

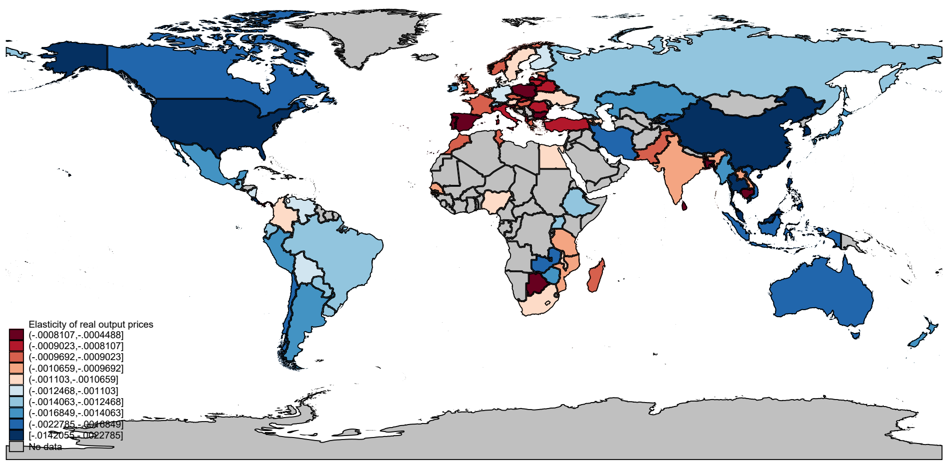


Notes: This figure shows the regions in (ϕ, ψ) space for which the gravity equilibrium is unique and interior. Existence can be guaranteed throughout the entire region except for the case $1 + \phi + \psi = 0$.

Figure 3.8.2. The network effect of a U.S.-China trade war



(a) Degree 1 Effect



(b) Total Effect

Notes: This figure depicts the elasticity of real output prices to an increase in the bilateral trade frictions between the U.S. and China (a "trade war") in all countries. The top panel depicts the "Degree 1" effect, which is the effect of the direct shock on the U.S. and China on all countries through the trade network, holding constant the output prices and quantities of their trading partners fixed. The bottom panel shows the total effect of the trade war on the real output price in each country.

3.9. Proofs

3.9.1. Proof of Theorem 1

Proof. Part i) The proof proceeds as follows. First we transform the equilibrium conditions to the associated non-linear integral equations form. However, we cannot directly apply the fixed point theorem for the non-linear integral equations since the system does not map to a compact space. Therefore we need to "scale" the system so that we can apply the fixed point, which implies that there exists a fixed point for the scaled system. Finally we construct a fixed point for the original non-linear integral equations. In this subsection, we show how to set up in the associated integral equation form, and apply the fixed point theorem. The other technical parts are proven in Online Appendix 3.10.4. Note that our result proposition is a natural generalization of Karlin and Nirenberg (1967) to a system of non-linear integral equations.

Define z as follows:

$$z \equiv \begin{pmatrix} (x_i)_i \\ (y_i)_i \end{pmatrix} \equiv \begin{pmatrix} \left(p_i^{1+\psi+\phi} P_i^{-\psi}\right)_i \\ \left(P_i^{-\phi}\right)_i \end{pmatrix}.$$

Then the system of equations (3.6) and (3.7) of the general equilibrium gravity model is re-written in vector form:

$$\begin{pmatrix} (x_i)_i \\ (y_i)_i \end{pmatrix} = \begin{pmatrix} \sum_j K_{ij} C_i^{-1} C_j x_j^{a_{11}} y_j^{a_{12}} \\ \sum_j K_{ji} x_j^{a_{21}} y_j^{a_{22}} \end{pmatrix}, \tag{3.22}$$

where $A = (a_{ij})_{i,j}$ is given by

$$A = \begin{pmatrix} \frac{1+\psi}{1+\psi+\phi} & -\frac{1+\phi}{1+\psi+\phi} \\ -\frac{\phi}{1+\psi+\phi} & \frac{\psi}{1+\psi+\phi} \end{pmatrix}.$$

Also the kernel, K_{ij}, is given by $K_{ij} = \tau_{ij}^{-\phi}$. Notice that we cannot directly apply Browser's fixed point theorem for equation (3.22) since there are no trivial compact domain for equation (3.22). Therefore consider the following "scaled" version of equation (3.22).

$$z = \begin{pmatrix} (x_i)_i \\ (y_i)_i \end{pmatrix} = \begin{pmatrix} \frac{\sum_j K_{ij} C_i^{-1} C_j x_j^{a_{11}} y_j^{a_{12}}}{\sum_{i,j} K_{ij} C_i^{-1} C_j x_j^{a_{11}} y_j^{a_{12}}} \\ \frac{\sum_j K_{ji} x_j^{a_{21}} y_j^{a_{22}}}{\sum_{i,j} K_{ji} x_j^{a_{21}} y_j^{a_{22}}} \end{pmatrix} \equiv F(z), \tag{3.23}$$

and F is defined over the following compact set$C$:

$$C = \left\{x \in \Delta\left(R_+^N\right); x_i \in [\underline{x}, \overline{x}]\ \forall i\right\} \times \left\{y \in \Delta\left(R_+^N\right); y_i \in \left[\underline{y}, \overline{y}\right]\ \forall i\right\}, \tag{3.24}$$

where the bounds for x and y are respectively given as follow:

$$\bar{x} \equiv \max_{i,j} \frac{K_{ij} C_i^{-1} C_j}{\sum_{i,j} K_{ij} C_i^{-1} C_j} \quad \underline{x} \equiv \min_{i,j} \frac{K_{ij} C_i^{-1} C_j}{\sum_{i,j} K_{ij} C_i^{-1} C_j}$$
$$\overline{y} \equiv \max_{i,j} \frac{K_{ji}}{\sum_{i,j} K_{ji}} \quad \underline{y} = \min_{i,j} \frac{K_{ji}}{\sum_{i,j} K_{ji}}.$$

It is trivial to show that F maps from C to C and continuous over the following compact set C, so that we can apply Brouwer's fixed point and there exists an fixed point $z^* \in C$.

There are two technical points needed to be proven; first, there exists a fixed point for the original (un-scaled) system (3.22); second, the equilibrium z^* is strictly positive. These two claims are proven in Lemma 7 and Lemma 8 in Online Appendix 3.10.4, respectively.

Part (iii) It suffices to show that there exists a unique interior solution for equation (3.22). Suppose that there are two strictly positive solutions (x_i, y_i) and $(\widehat{x}_i, \widehat{y}_i)$ such that there does not exist $t, s > 0$ satisfying

$$(x_i, y_i) = (t\widehat{x}_i, s\widehat{y}_i)\,.$$

Namely, two solutions are "linearly independent." First note that for any $i \in S$, we can evaluate one of equation (3.22).

$$\frac{x_i}{\widehat{x}_i} = \frac{1}{\widehat{x}_i}\sum_{j\in S} K_{ij}C_i^{-1}C_j\left(\frac{x_j}{\widehat{x}_j}\right)^{\alpha_{11}}\left(\frac{y_j}{\widehat{y}_j}\right)^{\alpha_{12}}(\widehat{x}_j)^{\alpha_{11}}(\widehat{y}_j)^{\alpha_{12}} \tag{3.25}$$

$$\leq \max_{j\in S}\left(\frac{x_j}{\widehat{x}_j}\right)^{\alpha_{11}}\max_{j\in S}\left(\frac{y_j}{\widehat{y}_j}\right)^{\alpha_{12}}. \tag{3.26}$$

Taking the maximum of the left hand side,

$$\max_{i\in S}\frac{x_i}{\widehat{x}_i} \leq \max_{j\in S}\left(\frac{x_j}{\widehat{x}_j}\right)^{\alpha_{11}}\max_{j\in S}\left(\frac{y_j}{\widehat{y}_j}\right)^{\alpha_{12}}. \tag{3.27}$$

Lemma 9 in Online Appendix 3.10.4 shows that the inequality is actually strict. Analogously, we obtain

$$\min_{i\in S}\frac{x_i}{\widehat{x}_i} \geq \min_{j\in S}\left(\frac{x_j}{\widehat{x}_j}\right)^{\alpha_{11}}\min_{j\in S}\left(\frac{y_j}{\widehat{y}_j}\right)^{\alpha_{12}}. \tag{3.28}$$

Dividing equation (3.27) by equation (3.28), it is shown that

$$1 \leq \mu_x \equiv \frac{\max_{i\in S}\frac{x_i}{\widehat{x}_i}}{\min_{i\in S}\frac{x_i}{\widehat{x}_i}} < \frac{\max_{j\in S}\left(\frac{x_j}{\widehat{x}_j}\right)^{\alpha_{11}}}{\min_{j\in S}\left(\frac{x_j}{\widehat{x}_j}\right)^{\alpha_{11}}} \times \frac{\max_{j\in S}\left(\frac{y_j}{\widehat{y}_j}\right)^{\alpha_{12}}}{\min_{j\in S}\left(\frac{y_j}{\widehat{y}_j}\right)^{a_{12}}} = \mu_x^{|\alpha_{11}|} \times \mu_y^{|\alpha_{12}|},$$

where

$$\mu_y \equiv \frac{\max_{i\in S}\frac{y_i}{\widehat{y}_i}}{\min_{i\in S}\frac{y_i}{\widehat{y}_i}}.$$

The same argument is applied to obtain the following inequality

$$1 \leq \mu_y \equiv \frac{\max_{i\in S}\frac{y_i}{\widehat{y}_i}}{\min_{i\in S}\frac{y_i}{\widehat{y}_i}} < \frac{\max_{j\in S}\left(\frac{x_j}{\widehat{x}_j}\right)^{\alpha_{21}}}{\min_{j\in S}\left(\frac{x_j}{\widehat{x}_j}\right)^{\alpha_{21}}} \times \frac{\max_{j\in S}\left(\frac{y_j}{\widehat{y}_j}\right)^{\alpha_{22}}}{\min_{j\in S}\left(\frac{y_j}{\widehat{y}_j}\right)^{\alpha_{22}}} = \mu_x^{|\alpha_{21}|} \times \mu_y^{|\alpha_{22}|}.$$

Taking logs in the two inequalities and exploiting the restriction we can write

$$\begin{pmatrix} \ln \mu_x \\ \ln \mu_y \end{pmatrix} < \underbrace{\begin{pmatrix} |\alpha_{11}| & |\alpha_{12}| \\ |\alpha_{21}| & |\alpha_{22}| \end{pmatrix}}_{=|A|} \begin{pmatrix} \ln \mu_x \\ \ln \mu_y \end{pmatrix}, \tag{3.29}$$

which from the Collatz–Wielandt formula, equation (3.29) implies that the largest eigenvalue of $|A|$ is greater than one:

$$\rho\left(|A|\right) > 1.$$

However, we prove in Lemma 10 in Online Appendix 3.10.4 that the sufficient condition in part (ii) of Theorem 1 guarantees that the largest absolute eigenvalue is 1. As a result, this is a contradiction.

Quasi-symmetry) When the bilateral trade frictions satisfy quasi-symmetry, then we can reduce the system to$N$ dimensional integral system (see Online Appendix 3.10.3). Then the same logic used above can be applied to show there exists a unique strictly positive solution. As mentioned above, this result follows directly from Karlin and Nirenberg (1967) and is summarized in Theorem 2.19 of Zabreyko et al. (1975). The same argument for (iv) is used for convergence. □

3.9.2. Proof of Theorem 2

Proof. Part (i) Equation (3.17) is a direct application of the implicit function theorem. Define a function $F : R^{2N} \to R^{2N}$ as follows.

$$F_i\left((\ln p_i)_{i=1}^N, (\ln P_i)_{i=1}^N\right) = \kappa C_i p_i^{1+\psi} P_i^{-\psi} - \kappa \sum_k \tau_{ik}^{-\phi} p_i^{-\phi} C_k P_k^{\phi-\psi} p_k^{1+\psi}$$

$$F_{N-1+i}\left((\ln p_i)_{i=1}^N, (\ln P_i)_{i=1}^N\right) = P_i^{-\phi} - \sum_k \tau_{ki}^{-\phi} p_k^{-\phi}$$

Applying the implicit function theorem for F, we obtain the comparative static (3.17). As in Dekle et al. (2008), the matrix $\mathbf{A}$ and $\mathbf{T}$ can be expressed in terms of observables.

Part (ii) Notice that $\mathbf{A}$ is written as follows:

$$\mathbf{A} = \mathbf{S}\left(\mathbf{I} - \mathbf{S}^{-1}\mathbf{D}\right),$$

where $\mathbf{S}$ and $\mathbf{D}$ are defined by equation (3.15). If the largest absolute eigenvalue for $\mathbf{S}^{-1}\mathbf{D}$ is less than 1, then $\mathbf{A}^{-1}$ is expressed as $\sum_{k=0}^{\infty}\left(\mathbf{S}^{-1}\mathbf{D}\right)^k \mathbf{S}^{-1}$. Note that we could have similarly written $\mathbf{A} = -\left(\mathbf{I} - \mathbf{S}\mathbf{D}^{-1}\right)\mathbf{D}$, so that if the largest eigenvalue for $\mathbf{S}\mathbf{D}^{-1}$ is less than 1, $\mathbf{A}^{-1}$ can be expressed as $-\sum_{k=0}^{\infty}\mathbf{D}^{-1}\left(\mathbf{S}\mathbf{D}^{-1}\right)^k$, as noted in footnote 20.

Part (iii) When quasi-symmetric assumption is imposed, destination effects are proportional to the associated origin effects. Therefore as shown in Online Appendix 3.10.3, the equilibrium is characterized by the following single non-linear system of equations:

$$\underbrace{p_i^{1+\psi-\psi\frac{1+\psi+\phi}{\psi-\phi}}\left(\frac{\tau_i^A}{\tau_i^B}\right)^{-\psi\frac{\phi}{\psi-\phi}}\left(C_i\right)^{\frac{\phi}{\psi-\phi}}}_{=Y_i/\kappa} = \sum_{j\in S}\underbrace{\tilde{\tau}_{ij}^{-\phi} p_i^{-\phi}\left(\tau_i^A\right)^{-\phi}\left(\tau_j^A\right)^{-\phi} p_j^{-\phi}}_{=X_{ij}/\kappa} \tag{3.30}$$

As before, define z_i for all $i \in S$ as follows:

$$z_i\left(p;\tau\right) = \kappa p_i^{1+\psi-\psi\frac{1+\psi+\phi}{\psi-\phi}} \left(\frac{\tau_i^A}{\tau_i^B}\right)^{-\psi\frac{\phi}{\psi-\phi}} \left(C_i\right)^{\frac{\phi}{\psi-\phi}} - \kappa \sum_{j\in S} \tilde{\tau}_{ij}^{-\phi} p_i^{-\phi} \left(\tau_i^A\right)^{-\phi} \left(\tau_j^A\right)^{-\phi} p_j^{-\phi}.$$

Then apply the implicit function theorem to (3.30),

$$\frac{\partial \ln p}{\partial \ln \tau_{il}} = -2 \left(\underbrace{\frac{\partial z}{\partial \ln p}}_{N\times N}\right)^{-1} \underbrace{\frac{\partial z}{\partial \ln \tau_{il}}}_{N\times 1}. \tag{3.31}$$

Note that numerical number 2 shows up to preserve quasi-symmetry of trade frictions. As in the general trade friction case, $\frac{\partial z}{\partial \ln p}$ is expressed as observables:

$$\frac{\partial z}{\partial \ln p} = \left[\phi\frac{1+\psi+\phi}{\phi-\psi}\right]\left[\mathbf{Y} + \frac{\phi-\psi}{1+\psi+\phi}\mathbf{X}\right],$$

where $\mathbf{Y} = diag\left(Y_i\right)$ and $\mathbf{X} = \left(X_{ij}\right)_{i,j\in S}$. Define $\mathbf{A}$ as follows:

$$\mathbf{A} = \mathbf{Y} + \frac{\phi-\psi}{1+\psi+\phi}\mathbf{X}.$$

From Lemma 11, $\mathbf{A}$ has positive diagonal elements and is dominant of its rows. Equation (3.31) is

$$\frac{\partial \ln p_i}{\partial \ln \tau_{il}} = -2\frac{\phi-\psi}{1+\psi+\phi}A_{ii}^{-1}X_{il}, \quad \frac{\partial \ln p_j}{\partial \ln \tau_{il}} = -2\frac{\phi-\psi}{1+\psi+\phi}A_{ji}^{-1}X_{il}.$$

Since the price index is log-linear w.r.t. the associated output price, we have

$$\frac{\partial \ln P_i}{\partial \ln \tau_{il}} = \frac{1+\psi+\phi}{\psi-\phi}\frac{\partial \ln p_i}{\partial \ln \tau_{il}}.$$

Therefore, the real output price is

$$\frac{\partial \ln (p_i/P_i)}{\partial \ln \tau_{il}} = \left(\frac{2\phi+1}{\phi-\psi}\right)\frac{\partial \ln p_i}{\partial \ln \tau_{il}} = -2\frac{2\phi+1}{1+\psi+\phi}A_{ii}^{-1}X_{il}.$$

Then the ordering of the real output price follows from part (iii) of Theorem 2 , $A_{ii}^{-1} > A_{ji}^{-1}$ for $j \in S - i$. The result for real expenditure then follows immediately from C.5 and equation (3.11), as $E_i/P_i \propto C_i\,(p_i/P_i)^{1+\psi}$:

$$\frac{\partial \ln (p_i Q_i/P_i)}{\partial \ln \tau_{il}} = -2\frac{2\phi+1}{1+\psi+\phi}\,(1+\psi)\,A_{ii}^{-1}X_{il} + \underbrace{\frac{\partial \ln \kappa}{\partial \ln \tau_{il}}}_{\text{common}}.$$

By the same argument, the ordering of $\left(\frac{\partial \ln(p_i Q_i/P_i)}{\partial \ln \tau_{il}}\right)$ follows. □

3.10. Additional materials

3.10.1. Recovering the equilibrium variables from the Universal Gravity conditions

In this subsection, we show how the universal gravity conditions C.1-C.5 can be combined to derive equations (3.6) and (3.7), which can be used to solve for equilibrium prices and price indices up to scale. We then show how information of these prices and price indices up-to-scale can be used to solve for the level of real output prices $\{p_i/P_i\}_{i\in S}$ and, combined with the numeraire in C.6, to determine the equilibrium level of income $\{Y_i\}_{i\in S}$, expenditure $\{E_i\}_{i\in S}$, and trade flows $\{X_{ij}\}_{i,j\in S}$. Finally, we show how all other endogenous variables can be recovered up-to-scale if the equilibrium prices and price indices are known up to scale.

3.10.1.1. From Universal Gravity C.1-C.5 to Equations (3.6) and (3.7). We first show Universal Gravity C.1-C.5 imply equations (3.6) and (3.7).

Combing C.1 and C.2 (in particular the gravity equation (3.10)):

$$X_{ij} = \tau_{ij}^{-\phi} p_i^{-\phi} P_j^{\phi} E_j, \tag{3.32}$$

where recall from C.2 that the price index can be written as:

$$P_i^{-\phi} \equiv \sum_{j \in S} \tau_{ji}^{-\phi} p_j^{-\phi} \tag{3.33}$$

Combining equation (3.32) with C.(4) and C.(5) yields:

$$p_i Q_i = \sum_{j \in S} \tau_{ij}^{-\phi} p_i^{-\phi} P_j^{\phi} p_j Q_j \tag{3.34}$$

Finally, we substitute C.3 into equation (3.34) to yield:

$$p_i \left(C_i \left(\frac{p_i}{P_i} \right)^{\psi} \right) = \sum_{j \in S} \tau_{ij}^{-\phi} p_i^{-\phi} P_j^{\phi} p_j \left(C_j \left(\frac{p_j}{P_j} \right)^{\psi} \right) \tag{3.35}$$

Note that equations (3.33) and (3.35) are equivalent to equations (3.6) and (3.7). Hence, C.1-C. 5 imply equations (3.6) and (3.7), as claimed. There are two things to note about equilibrium equations (3.33) and (3.35): first, they depend only on output prices $\{p_i\}$, the price indices $\{P_i\}$, and exogenous model fundamentals (in particular, they do not depend on the endogenous scalar κ); second, they are homogeneous of degree zero with respect to $\{p_i, P_i\}$, so the scale of prices (and price indices) are undetermined.

3.10.1.2. From Equations (3.6) and (3.7) to endogenous variables. We now show that given a solution to equations (3.6) and (3.7), we can construct all endogenous variables in the models. We divide the derivations into endogenous variables determined up to scale and endogenous variables for which the scale is known (given the choice of numeraire in C.6. Suppose that we have a set of prices $\{p_i\}_{i \in S}$ and price indices $\{P_i\}_{i \in S}$ that solve equations (3.6) and (3.7). Note that because equations (3.6) and (3.7) are homogeneous of degree zero with respect to $\{p_i, P_i\}_{i \in S}$, for any scalar α, the normalized prices $\tilde{p}_i \equiv \frac{1}{\alpha} p_i$ and price indices $\tilde{P}_i \equiv \frac{1}{\alpha} P_i$ continue to satisfy equations (3.6) and (3.7).

We first solve for the real output price. Note that for any choice of α, the real output price $\{p_i/P_i\}_{i \in S}$ remains unchanged, so its level is unaffected by the unknown scalar.

We now solve for quantities. From equation (3.11), the quantity in location i does not depend on α, but it does depend on the unknown scalar κ as follows:

$$Q_i = \kappa C_i \left(\frac{p_i}{P_i} \right)^{\psi}.$$

Hence, equilibrium quantities are only determine up-to-scale.

We now solve for income and expenditure. From C.4 and C.5 we have:

$$E_i = Y_i = p_i Q_i.$$

Applying the numeraire in C.6 then yields:

$$\sum_{i \in S} Y_i = 1 \iff$$
$$\sum_{i \in S} p_i Q_i = 1 \iff$$
$$\kappa\alpha \sum_{i \in S} \tilde{p}_i C_i \left(\frac{\tilde{p}_i}{\tilde{P}_i}\right)^{\psi} = 1 \iff$$
$$\kappa\alpha = \left(\sum_{i \in S} \tilde{p}_i C_i \left(\frac{\tilde{p}_i}{\tilde{P}_i}\right)^{\psi}\right)^{-1},$$

which, as claimed, pins down the product of the unknown quantity scalar and unknown price scalar. Given $\kappa\alpha$, we can now determine the level of income and expenditure as follows:

$$E_i = Y_i = p_i Q_i \iff$$
$$E_i = Y_i = \frac{\tilde{p}_i C_i \left(\frac{\tilde{p}_i}{\tilde{P}_i}\right)^{\psi}}{\left(\sum_{j \in S} \tilde{p}_j C_j \left(\frac{\tilde{p}_j}{\tilde{P}_j}\right)^{\psi}\right)},$$

as claimed.

We now determine the level of trade flows using equation (3.32):

$$X_{ij} = \tau_{ij}^{-\phi} p_i^{-\phi} P_j^{\phi} E_j \iff$$
$$X_{ij} = \frac{\tau_{ij}^{-\phi} \tilde{p}_i^{-\phi}}{\sum_{k \in S} \tau_{kj}^{-\phi} \tilde{p}_k^{-\phi}} \left(\frac{\tilde{p}_j C_j \left(\frac{\tilde{p}_j}{\tilde{P}_j}\right)^{\psi}}{\left(\sum_{k \in S} \tilde{p}_k C_k \left(\frac{\tilde{p}_k}{\tilde{P}_k}\right)^{\psi}\right)}\right).$$

Other than real output prices $\{p_i/P_i\}_{i\in S}$, income $\{Y_i\}_{i\in S}$, expenditure $\{E_i\}_{i\in S}$, and trade flows $\{X_{ij}\}_{i,j\in S}$, all other endogenous variables are determined only up-to-scale, as they depend either on the price scalar α (i.e. output prices $\tilde{p}_i$, price indices $\tilde{P}_i$, bilateral prices $p_{ij} = \tau_{ij}\tilde{p}_i$, and the quantity traded $Q_{ij} = X_{ij}/\tau_{ij}p_i$) or the quantity scalar κ (i.e. quantities Q_i).

3.10.2. Proof of Theorem 1 part (ii)

We first provide a general mathematical formulation to incorporate non-interior solutions. Let the equilibrium be a duple $(p_i, Q_i) \in \overline{\mathbb{R}}_+^N \times \mathbb{R}_+^N$ such that for all $i \in S$,

$$Q_i = \sum_j \frac{\tau_{ij}^{-\phi-1} p_i^{-\phi-1}}{\sum_{k\in S} \tau_{kj}^{-\phi} p_k^{-\phi}} p_j Q_j \tag{3.36}$$

$$(p_i, Q_i) \in F_i(p, Q) \tag{3.37}$$

where F is a supply condition, which might be a correspondence. (The fact that F might be correspondence allows us to extend the framework to allow for non-interior solutions). In particular, we define F as follows: We say $(p_i, Q_i) \in F_i(p, Q)$ if and only if

$$sign(\psi)\left[Q_i - \kappa\left(\frac{p_i}{P_i(p)}\right)^{\psi}\right] \geq 0 \tag{3.38}$$

$$Q_i = \kappa\left(\frac{p_i}{P_i(p)}\right)^{\psi} \quad \text{if } Q_i > 0, \tag{3.39}$$

and where $\left(\frac{0}{0}\right)$ is defined as 0. That is, if $Q_i = 0$, then we replace C.3 with an inequality. For example, in an economic geography model, inequality constraint (3.38) corresponds to welfare equalization. If there are people living in location i, then Q_i is given by equality

(3.39). If not, then the welfare living in location i should be lower than one obtained in other places, which is represented as the inequality (3.38).

As we mentioned in Section 3.3, we restrict our attention to non-trivial equilibria where there is positive production in at least one location. To show that all (non-trivial) equilibria are interior, it then suffices to show that if some locations produces nothing, then all other locations must also produce nothing.

Suppose that $Q_l = 0$ for some $l \in S$. Then from equation (3.36) for l:

$$0 = \sum_j \underbrace{\frac{\tau_{lj}^{-\phi} p_l^{-\phi-1}}{\sum_{k\in S} \tau_{kj}^{-\phi} p_k^{-\phi}} p_j Q_j}_{\geq 0}, \tag{3.40}$$

which in turn implies that for all $j \in S$,

$$\frac{\tau_{lj}^{-\phi} p_l^{-\phi-1}}{g_j} p_j Q_j = 0, \tag{3.41}$$

where $g_j = \sum_{k\in S} \tau_{kj}^{-\phi} p_k^{-\phi}$.

Note that there are two reasons why equation (3.41) is zero for all j; either (1) ; or (2) for all $j \in S$, $\tau_{lj}^{-\phi} \frac{p_j Q_j}{g_j} = 0$. We will prove a contradiction in both cases.

First assume that (1) $p_l^{-\phi-1} = 0$, which if $\phi > -1$ implies that $p_l = \infty$. While $(p_l, Q_l) = (\infty, 0)$ satisfies equation (3.40), it does not satisfy equation (3.37). To see this, note:

$$0 = Q_i < \kappa \left(\frac{p_i}{g_i^{-\frac{1}{\phi}}} \right)^{\psi} = \infty,$$

which contradicts with equation (3.38) since $\psi \geq 0$. Therefore p_l needs to be finite, $p_l < \infty$.

Now assume that (2) for all $j \in S$, $\tau_{lj}^{-\phi} \frac{p_j Q_j}{g_j} = 0$. Since the price for country l is finite, equation (3.41) is reduced to

$$\tau_{lj}^{-\phi} \frac{p_j Q_j}{g_j} = 0$$

for all $j \in S$. An equivalent expression is that for all countries connected with l, $j \in S_l = \{k \in S; \tau_{lk} < \infty\}$,

$$p_j Q_j = 0 \quad \text{or} \quad g_j = \infty. \tag{3.42}$$

Fix any country $j \in S_l$. Suppose that $p_j, Q_j > 0$ Then equation (3.42), $g_j = \infty$. Then for all $(p_j, Q_j) \in \overline{\mathbb{R}}_+ \times \mathbb{R}$ if $\psi \geq 0$ we have

$$\infty = \kappa \left(\frac{p_j}{g_j^{-\frac{1}{\phi}}} \right)^{\psi} \leq Q_j = 0,$$

which is a contradiction. Therefore in order to satisfy equation (3.42), p_j or Q_j needs to be zero. Suppose that $p_j = 0$. Then we have

$$0 = \kappa \left(\frac{p_j}{g_j^{-\frac{1}{\phi}}} \right)^{\psi} \leq Q_j.$$

If $Q_j > 0$, then C. (3). Therefore, $Q_j = 0$. Therefore Q_j needs to be zero for all $j \in S_l$.

So far, we have shown that if $Q_l = 0$ then the connected countries $j \in S_l$ produce nothing, $Q_j = 0$. Because of strong connectedness, any country n is connected with l through third countries. Therefore, by repeating the argument along with the chain, we have $Q_n = 0$ for all $n \in S$.

As a result, if $\phi \geq -1$, and $\psi \geq 0$ then all equilibria are interior, as claimed.

3.10.3. Quasi-symmetric trade frictions

In this subsection, we show that when trade frictions are quasi-symmetric, then balanced trade implies that the origin and destination fixed effects of the gravity trade flow expression are equal up to scale.

We first formally define "quasi-symmetry." We say that the set of trade frictions $\{\tau_{ij}\}_{i,j\in S}$ are *quasi-symmetric* if there exists a set of origin scalars $\left\{\tau_i^A\right\}_{i\in S} \in \mathbb{R}^N_{++}$, destination scalars $\left\{\tau_i^B\right\}_{i\in S} \in \mathbb{R}^N_{++}$, and a symmetric matrix $\{\tilde{\tau}_{ij}\}_{i,j\in S}$ where $\tilde{\tau}_{ij} = \tilde{\tau}_{ji}$ for all $i, j \in S$ such that we can write:

$$\tau_{ij} = \tau_i^A \tau_i^B \tilde{\tau}_{ij} \ \forall i, j \in S.$$

Loosely speaking, quasi-symmetric trade frictions are those that are reducible to a symmetric component and exporter- and importer-specific components. While restrictive, it is important to note that the vast majority of papers which estimate gravity equations assume that trade frictions are quasi-symmetric; for example Eaton and Kortum (2002) and Waugh (2010) assume that trade frictions are composed by a symmetric component that depends on bilateral distance and on a destination or origin fixed effect.

Combining the universal gravity conditions C. 1 and C. 2 allows us to write the value of bilateral trade flows from i to j as:

$$X_{ij} = \tau_{ij}^{-\phi} p_i^{-\phi} P_j^{\phi} E_j,$$

which we now re-write as:

$$X_{ij} = \kappa \tau_{ij}^{-\phi} \gamma_i \delta_j, \tag{3.43}$$

where we call $\gamma_i \equiv p_i^{-\phi}$ the *origin fixed effect* and $\delta_i \equiv P_i^{\phi} E_i = C_i P_i^{\phi-\psi} p_i^{1+\psi}$ the *destination fixed effect*.

Proposition 12. *If trade frictions are quasi-symmetric, then in any model within the universal gravity framework, the product of the equilibrium origin fixed effect and the origin scalar will be equal to the product of the equilibrium destination fixed effects and the destination fixed effect up to scale, i.e.: for some scalar $\lambda \geq 0$,*

$$\left(\tau_i^A\right)^{-\phi} \gamma_i = \lambda \left(\tau_i^B\right)^{-\phi} \delta_i \ \forall i \in S.$$

Proof. We first note that market clearing condition C.4 and balanced trade condition C.5 together imply that: $\sum_{j \in S} X_{ij} = \sum_{j \in S} X_{ji} \ \forall i \in S$. Combining this with the gravity expression (3.43) and quasi-symmetry implies:

$$\sum_j \underbrace{\kappa \tau_{ij}^{-\phi} \gamma_i \delta_j}_{=X_{ij}} = \sum_j \underbrace{\kappa \tau_{ji}^{-\phi} \gamma_j \delta_i}_{X_{ji}} \iff$$

$$\frac{\left(\tau_i^A\right)^{-\phi} \gamma_i}{\left(\tau_i^B\right)^{-\phi} \delta_i} = \frac{\sum_{j \in S} \tilde{\tau}_{ij}^{-\phi} \left(\tau_j^A\right)^{-\phi} \gamma_j}{\sum_{j \in S} \tilde{\tau}_{ij}^{-\phi} \left(\tau_j^B\right)^{-\phi} \delta_j} = \sum_{j \in S} \frac{\tilde{\tau}_{ij}^{-\phi} \left(\tau_j^B\right)^{-\phi} \delta_j}{\sum_{k \in S} \tilde{\tau}_{ik}^{-\phi} \left(\tau_k^B\right)^{-\phi} \delta_k} \times \frac{\left(\tau_j^A\right)^{-\phi} \gamma_j}{\left(\tau_j^B\right)^{-\phi} \delta_j}.$$

It is easy to show that $\frac{\left(\tau_i^A\right)^{-\phi} \gamma_i}{\left(\tau_i^B\right)^{-\phi} \delta_i} = 1$ is a solution to this problem for any kernel. From the Perron-Frobenius theorem, the solution is unique up to scale. Therefore we have:

$$\left(\tau_i^A\right)^{-\phi} = \lambda \left(\tau_i^B\right)^{-\phi} \delta_i \ \forall i \in S, \tag{3.44}$$

as required. □

Proposition 12 has a number of important implications. First, Proposition 12 allows one to simplify the equilibrium system of equations 3.6 and 3.7 into a single non-linear equation when $\phi \neq \psi$:

$$\left(p_i^{\frac{1+\psi+\phi}{\psi-\phi}}\right)^{-\phi} = (\lambda)^{\frac{\phi}{\psi-\phi}} (C_i)^{\frac{\phi}{\psi-\phi}} \sum_{j \in S} \tilde{\tau}_{ij}^{-\phi} \left(\tau_i^A\right)^{\frac{\phi^2}{\psi-\phi}} \left(\tau_i^B\right)^{-\frac{\phi\psi}{\psi-\phi}} \left(\tau_j^A\right)^{-\phi} p_j^{-\phi}, \quad i \in S, \tag{3.45}$$

which simplifies the characterization of the theoretical and empirical properties of the equilibrium. Notice that λ is an endogenous scalar. Since (3.45) holds for any location $i \in S$, λ is expressed as

$$\lambda^{\frac{\phi}{\psi-\phi}} = \frac{\sum_i \left(p_i^{-\phi}\right)^{\frac{1+\psi+\phi}{\psi-\phi}}}{\sum_i \sum_{j \in S} \tau_{ij}^{-\phi} \left(\frac{\tau_i^A}{\tau_i^B}\right)^{\frac{\phi^2}{\psi-\phi}} C_i^{\frac{\phi}{\psi-\phi}} p_j^{-\phi}}.$$

Substituting above expression, we obtain:

$$\frac{\left(p_i^{-\phi}\right)^{\frac{1+\psi+\phi}{\psi-\phi}}}{\sum_i \left(p_i^{-\phi}\right)^{\frac{1+\psi+\phi}{\psi-\phi}}} = \sum_{j \in S} \frac{\tau_{ij}^{-\phi} \left(\frac{\tau_i^A}{\tau_i^B}\right)^{\frac{\phi^2}{\psi-\phi}} C_i^{\frac{\phi}{\psi-\phi}} p_j^{-\phi}}{\sum_i \sum_{j \in S} \tau_{ij}^{-\phi} \left(\frac{\tau_i^A}{\tau_i^B}\right)^{\frac{\phi^2}{\psi-\phi}} C_i^{\frac{\phi}{\psi-\phi}} p_j^{-\phi}}.$$

Notice that the system is now homogeneous degree 0. Therefore, if $\phi \notin \left\{-\frac{1}{2}, \psi, 0\right\}$, then we can normalize $\lambda = 1$ without loss of generality.

Second, by showing that the origin and destination fixed effects are equal up to scale, Proposition 12 provides offers an analytical characterization of the equilibrium. For example, given the definition of the origin and destination fixed effects, Proposition 12 can equivalently be expressed as:

$$p_i P_i \propto \frac{\tau_i^B}{\tau_i^A} E_i^{-\frac{1}{\phi}}, \tag{3.46}$$

i.e. there is a log-linear relationship between output prices, the price index and total expenditure in a location.

Third, it is straightforward to show that quasi-symmetry implies that equilibrium trade flows will be bilaterally symmetric, i.e. $X_{ij} = X_{ji}$ for all $i, j \in S$, allowing one to test whether trade frictions are quasi-symmetric directly from observed trade flow data.

Finally, we should note that the results of Proposition 12 have already been used in the literature for particular models, albeit implicitly. The most prominent example is Anderson and Van Wincoop (2003), who use the result to show the bilateral resistance is equal to the price index.[36] To our knowledge, Head and Mayer (2013) are the first to recognize the importance of balanced trade and market clearing in generating the result for the Armington model; however, Proposition 12 shows that the result applies more generally to any model with quasi-symmetrical trade frictions in the universal gravity framework.

3.10.4. Proofs of the lemmas used in Theorem (1)

There are 4 lemmas which are not proven in the paper. In this section, we discuss them carefully. Before proving these lemmas, we discuss how we use them in the proof. In the proof, we show a fixed point for the "scaled" system, not the actual system. Therefore it needs to be shown that there exists a fixed point for the actual system, which is shown in Lemma 7. Then we argue that the solution we obtain is strictly positive, which is guaranteed by Assumption 3. We emphasize the connectivity assumption is crucial here. These two lemmas are used in **Part i)** Theorem 1.

[36]The result is also used in economic geography byAllen and Arkolakis (2014) to simplify a set on non-linear integral equations into a single integral equation.

Part ii) shows that there exists an unique solution. During the proof, we argue that 3.27 should hold with strict inequality. Again the connectivity allows us to show this result (Lemma 9). After establishing this strict inequality, we follow the argument by Allen et al. (2014), which requires that the largest absolute eigenvalues for $|A|$ are less than 1. Since A is a 2-by-2 matrix, we can compute the eigenvalues by hand and show that one of them is exactly 1, and the other is less than 1 if the conditions in **Part ii)** are satisfied.

Lemma 7. *Suppose that z solves (3.23). Then there exists $\widehat{z}$ solving (3.22).*

Proof. First it is easy to show[37]

$$\sum_{i,j\in S} K_{ij}C_i^{-1}C_j x_j^{a_{11}} y_j^{a_{12}} = \sum_{i,j\in S} K_{ji} x_j^{a_{21}} y_j^{a_{22}}. \tag{3.49}$$

Guess a solution

$$\widehat{z} = \begin{pmatrix} (\widehat{x}_i)_i \\ \\ (\widehat{y}_i)_i \end{pmatrix} = \begin{pmatrix} t^{-1}(x_i)_i \\ \\ t^{-1}(y_i)_i \end{pmatrix}, \tag{3.50}$$

[37]To see this, multiply $C_i x_i^{a_{21}} y_i^{a_{22}} = C_i p_i^{-\phi}$, to the first equations of (3.23) and sum over i;

$$\sum_i C_i p_i^{1+\psi} P_i^{-\psi} = \frac{\sum_i \sum_j K_{ij} C_j x_i^{a_{21}} y_i^{a_{22}} x_j^{a_{11}} y_j^{a_{12}}}{\sum_{i,j} K_{ij} C_i^{-1} C_j x_j^{a_{11}} y_j^{a_{12}}}. \tag{3.47}$$

Also multiply $C_i x_i^{a_{11}} y_i^{a_{12}} = C_i P_i^{\phi-\psi} p_i^{1+\psi}$ to the second equations (3.23) and sum over i;

$$\sum_{i\in S} C_i p_i^{1+\psi} P_i^{-\psi} = \frac{\sum_{i\in S} \sum_{j\in S} K_{ij} C_j x_i^{a_{21}} y_i^{a_{22}} x_j^{a_{11}} y_j^{a_{12}}}{\sum_{i\in S, j\in S} K_{ji} x_j^{a_{21}} y_j^{a_{22}}}. \tag{3.48}$$

Notice that the LHS is the same as one in (3.47). Also the numerator of the RHS in (3.47) is the same as one in (3.48). Therefore the following double sum terms should be the same:

$$\sum_{i,j} K_{ij} C_i^{-1} C_j x_j^{a_{11}} y_j^{a_{12}} = \sum_{i\in S, j\in S} K_{ji} x_j^{a_{21}} y_j^{a_{22}}.$$

where $t = \left(\sum_{i,j\in S} K_{ij} C_i^{-1} C_j x_j^{a_{21}} y_j^{a_{22}}\right)^{\frac{1}{1-a_{11}-a_{12}}} = \left(\sum_{i,j\in S} K_{ji} x_j^{a_{21}} y_j^{a_{22}}\right)^{\frac{1}{1-a_{21}-a_{22}}}$.[38] Then it is easy to verify that (3.50) solves (3.22); in particular, note that

$$\begin{aligned}\widehat{x}_i &= t^{-1}\frac{\sum_{j\in S} K_{ij} C_i^{-1} C_j x_j^{a_{11}} y_j^{a_{12}}}{\sum_{i,j\in S} K_{ij} C_i^{-1} C_j x_j^{a_{11}} y_j^{a_{12}}} = t^{1-a_{11}-a_{12}}\frac{\sum_{j\in S} K_{ij} C_i^{-1} C_j \left(\widehat{x}_j\right)^{a_{11}} \widehat{y}_j^{a_{12}}}{\sum_{i,j\in S} K_{ij} C_i^{-1} C_j x_j^{a_{11}} y_j^{a_{12}}} \\ &= \sum_{j\in S} K_{ij} C_i^{-1} C_j \widehat{x}_j^{a_{11}} \widehat{y}_j^{a_{12}}.\end{aligned}$$

We can also show that the second equations in (3.22) are also solved in the same vein:

$$\begin{aligned}\widehat{y}_i &= t\frac{\sum_{j\in S} K_{ji} x_j^{a_{21}} y_j^{a_{22}}}{\sum_{i,j\in S} K_{ji} x_j^{a_{21}} y_j^{a_{22}}} = t^{1-a_{21}-a_{22}}\frac{\sum_{j\in S} K_{ji} \widehat{x}_j^{a_{21}} \widehat{y}_j^{a_{22}}}{\sum_{i,j\in S} K_{ji} x_j^{a_{21}} y_j^{a_{22}}} \\ &= \sum_{j\in S} K_{ji} \widehat{x}_j^{a_{21}} \widehat{y}_j^{a_{22}}.\end{aligned}$$

The above two equations confirm that $\widehat{x}_i$ and $\widehat{y}_i$ is a solution to (3.22). □

Lemma 8. *If $\{\tau_{ij}\}_{i,j}$ satisfies Assumption 3, then the fixed point for (3.23) is strictly positive.*

Proof. We need to consider four different cases for the combinations of a_{11}, a_{12} satisfying different inequalities. We will consider the case $a_{11}, a_{12} > 0$ since the logic in the other cases is the same. We proceed by contradiction. Suppose that there is a solution x to equation (3.23) such that for some $i \in S$ $x_i = 0$. Consider an arbitrary location $n \neq i$ and consider a connected path, $K_{in}^c \equiv K_{i\pi_1} \times ... \times K_{\pi_m n} > 0$ for some $m(*)$. Then, from

[38]Notice that $a_{11} + a_{12} = a_{21} + a_{22}$.

the first of equations in (3.22) notice that

$$x_i = \sum_{j \in S} K_{ij} x_j^{a_{11}} y_j^{a_{12}} \geq \underbrace{K_{i\pi_1}}_{\neq 0} x_{\pi_1}^{a_{11}} y_{\pi_1}^{a_{12}}.$$

Note that $K_{i\pi_1}$ is strictly positive due to $(*)$. Then either x_n or y_n or both are zero if a_{11} and $a_{11} > 0$. If$x_n = 0$ this argument holds for any$n$ so this is a contradiction with the non-zero equilibrium proved above. Else if $y_n = 0$ we can repeat the argument the second of the equations in (3.22) to establish another contradiction. Notice that if either of $\alpha_{11}, \alpha_{12} = 0$ a contradiction is also easy to establish. □

Lemma 9. *Equation 3.27 holds with strict inequality.*

To that end, define the set of directly connected countries to each location $i \in S$ as $S_i^c \equiv \{j \in S : K_{ij} > 0\}$. Then notice that equation (3.25) combined with our equality assumption on equation (3.27) yields

$$\frac{x_i}{\widehat{x}_i} = \frac{1}{\widehat{x}_i} \sum_{j \in S_i^c} K_{ij} C_i^{-1} C_j \left(\frac{x_j}{\widehat{x}_j}\right)^{\alpha_{11}} \left(\frac{y_j}{\widehat{y}_j}\right)^{\alpha_{12}} (\widehat{x}_j)^{\alpha_{11}} (\widehat{y}_j)^{\alpha_{12}} = \max_{j \in S} \left(\frac{x_j}{\widehat{x}_j}\right)^{\alpha_{11}} \max_{j \in S} \left(\frac{y_j}{\widehat{y}_j}\right)^{\alpha_{12}}.$$

Notice that given that $\hat{x}_i$ is a solution, this implies that the following has to be true for all $j \in S_i^c$

$$\left(\frac{x_j}{\widehat{x}_j}\right)^{\alpha_{11}} = \max_{j \in S} \left(\frac{x_j}{\widehat{x}_j}\right)^{\alpha_{11}} \quad \left(\frac{y_j}{\widehat{y}_j}\right)^{\alpha_{12}} = \max_{j \in S} \left(\frac{y_j}{\widehat{y}_j}\right)^{\alpha_{12}}.$$

Now notice that if $\alpha_{11} \neq 0$ then for all $n \in S_i^c$,$x_j/\widehat{x}_j = x_n/\widehat{x}_n$. However, because of C. 3, we assume that there exists an indirectly connected path from any location to any other location, so that repeating this argument for all j and using the indirect connectivity we can prove that $x_j/\widehat{x}_j = x_n/\widehat{x}_n$ for all $j, n \in S$ i.e. the solutions are the same up-to-scale, a contradiction.

Lemma 10. *If $\phi, \psi \geq 0$ or $\phi, \psi \leq -1$, the eigenvalue for $|A|$ is*

$$\lambda = \frac{\phi - \psi}{1 + \phi + \psi}, 1,$$

and

$$\left| \frac{\phi - \psi}{1 + \phi + \psi} \right| < 1.$$

Proof. Notice that

$$|A| = \begin{pmatrix} \left|\frac{1+\psi}{1+\psi+\phi}\right| & \left|\frac{1+\phi}{1+\psi+\phi}\right| \\ \left|\frac{\phi}{1+\psi+\phi}\right| & \left|\frac{\psi}{1+\psi+\phi}\right| \end{pmatrix} = \begin{pmatrix} \frac{1+\psi}{1+\psi+\phi} & \frac{1+\phi}{1+\psi+\phi} \\ \frac{\phi}{1+\psi+\phi} & \frac{\psi}{1+\psi+\phi} \end{pmatrix}.$$

Then we can solve the following characteristic functions

$$\lambda^2 - \left(\frac{1+\psi}{1+\psi+\phi} + \frac{\psi}{1+\psi+\phi} \right) \lambda + \frac{1+\psi}{1+\psi+\phi} \frac{\psi}{1+\psi+\phi} - \frac{1+\phi}{1+\psi+\phi} \frac{\phi}{1+\psi+\phi} = 0.$$

Then

$$\lambda = \frac{\phi - \psi}{1 + \phi + \psi}, 1.$$

We need to show that $\left| \frac{\phi-\psi}{1+\phi+\psi} \right| < 1$. To show it, it suffices to show

$$g = |1 + \phi + \psi| - |\phi - \psi| > 0$$

Suppose that $\phi, \psi \geq 0$. Then g is strictly positive as follows:

$$\begin{aligned} g &= 1 + \phi + \psi - |\phi - \psi| \\ &\geq 1 + \phi + \psi - (|\phi| + |\psi|) = 1 > 0. \end{aligned}$$

Suppose that $\phi, \psi \leq -1$. Then g is given by

$$g = -1 - \phi - \psi - |\phi - \psi| .$$

If $\phi \leq \psi$, then

$$\begin{aligned} g &= -1 - \phi - \psi + \phi - \psi \\ &= -1 - 2\psi \geq 1. \end{aligned}$$

If $\phi \geq \psi$, then

$$\begin{aligned} g &= -1 - \phi - \psi - \phi + \psi \\ &= -1 - 2\psi \geq 1, \end{aligned}$$

which completes the proof. □

3.10.5. Lemmas and Proposition used in Theorem 2 (iii)[39]

In this section, we prove the lemma and proposition used in Theorem 2 (iii).

[39] A similar argument is found in Johnson and Smith (2011).

Lemma 11. *If $\phi, \psi \geq 0$ or $\phi, \psi \leq -1$, then A has strictly positive diagonal elements and is diagonal dominant in its rows; namely, for all $i \in S$*

$$A_{ii} > 0, \tag{3.51}$$

$$|A_{ii}| > \sum_{j \in S-i} |A_{ij}|\,. \tag{3.52}$$

Proof. Recall that A matrix is

$$A = \mathbf{Y} + \frac{\phi - \psi}{1 + \psi + \phi}\mathbf{X},$$

and from Lemma 10,

$$\left|\frac{\phi - \psi}{1 + \phi + \psi}\right| < 1.$$

Then the diagonal elements for A are positive; for all $i \in S$,

$$\begin{aligned} A_{ii} &= Y_{ii} + \frac{\phi - \psi}{1 + \psi + \phi}X_{ii} \\ &= Y_{ii} - \left|\frac{\phi - \psi}{1 + \psi + \phi}\right| X_{ii} \\ &> Y_{ii} - X_{ii} \geq 0. \end{aligned}$$

Also, for all $i \in S$,

$$
\begin{aligned}
&|A_{ii}| - \sum_{l \in S-i} |A_{il}| \\
=&\left|\underbrace{Y_{ii} + \frac{\phi-\psi}{1+\psi+\phi} X_{ii}}_{>0}\right| - \left|\frac{\phi-\psi}{1+\psi+\phi}\right| \sum_{l \in S-i} X_{il} \\
=&Y_{ii} + \frac{\phi-\psi}{1+\psi+\phi} X_{ii} - \left|\frac{\phi-\psi}{1+\psi+\phi}\right| (Y_i - X_{ii}) \\
=&\left(\underbrace{1 - \left|\frac{\phi-\psi}{1+\psi+\phi}\right|}_{>0}\right) Y_{ii} + \left[\underbrace{\frac{\phi-\psi}{1+\psi+\phi} + \left|\frac{\phi-\psi}{1+\psi+\phi}\right|}_{\geq 0}\right] X_{ii} > 0,
\end{aligned}
$$

which is equation (3.52). □

The next proposition plays a crucial role in the proof for Theorem 2 (iii).

Proposition 13. *If A has strictly positive diagonal elements and is dominant of its rows, then for all $i \neq j$,*

$$A_{ii}^{-1} > A_{ji}^{-1} > 0.$$

Proof. The co-factor expansion of A^{-1} is[40]

$$
\begin{aligned}
A_{ii}^{-1} - A_{ji}^{-1} &= \frac{det\left(A\left[S-i\right]\right) - (-1)^{i+j} det\left(A\left[S-i, S-j\right]\right)}{det\left(A\right)} \\
&= \frac{det\left(T\right)}{det\left(A\right)},
\end{aligned}
$$

[40]Remember

$$A_{ij}^{-1} = (-1)^{i+j} \frac{det\left(A\left[N-j, N-i\right]\right)}{det\left(A\right)}.$$

where T is defined as follows:

$$\widetilde{T} = A + \left(\underbrace{\mathbf{0}, \cdots, \mathbf{0}}_{N \times (j-1)}, \underbrace{A_i}_{N \times 1}, I \underbrace{\mathbf{0}, \cdots, \mathbf{0}}_{N \times (N-j)} \right).$$

T is a principal component of $\widetilde{T}$:

$$T = \widetilde{T} \left[S - i, S - i \right].$$

If a matrix C has positive diagonal elements, and is diagonally dominant of its rows, then $det\,(C) > 0$. [41] Then if T has such properties, then

$$\frac{det\,(T)}{det\,(A)} > 0$$

since A is assumed to have these properties. Thus it suffices to show that T has positive diagonal elements and is dominant of its rows.

By construction of T, it suffices to show

$$A_{kk} > 0 \quad k \in S - i - j \tag{3.53}$$

$$A_{kk} + A_{ki} > 0 \quad k = j \tag{3.54}$$

$$|A_{kk}| > \sum_{l \in S-i-k} |A_{kl} + 1_{l=j} A_{ki}| \quad k \in S - i - j \tag{3.55}$$

$$|A_{kk} + A_{ki}| > \sum_{l \in S-i-k} |A_{kl}| \quad k = j. \tag{3.56}$$

[41]See also Theorem 3 of Evmorfopoulos (2012).

First we show equation (3.53) and equation (3.54). since A has a strictly positive diagonal, for all $k \in S$,

$$A_{kk} > 0,$$

which is equation (3.53) . Also since A is diagonal dominant,

$$A_{jj} + A_{ji} > \sum_{l \neq j} |A_{jl}| + A_{ji} \geq |A_{ji}| + A_{ji} \geq 0,$$

which is equation (3.54).

Second, we show equation (3.55) and equation (3.56). Fix $k \in N - i - j$. Since A is diagonally dominant,

$$\begin{aligned}
|A_{kk}| &> \sum_{l \in S-k} |A_{kl}| \\
&= \sum_{l \in S-k-i-j} |A_{kl}| + |A_{ki}| + |A_{kj}| \\
&\geq \sum_{l \in S-i-k-j} |A_{kl}| + |A_{ki} + A_{kj}| \\
&= \sum_{l \in S-i-k} \left| A_{kl} + 1_{l=j} A_{ki} \right|,
\end{aligned}$$

which is equation (3.55). Fix $k = j$. Since A has positive diagonal elements, and is diagonally dominant,

$$
\begin{aligned}
|A_{kk} + A_{ki}| &\geq ||A_{kk}| - |A_{ki}|| \\
&= |A_{kk}| - |A_{ki}| \\
&= \sum_{l \in S-k-i} |A_{kl}| + |A_{ki}| - |A_{ki}| \\
&= \sum_{l \in S-k-i} |A_{kl}| ,
\end{aligned}
$$

which is equation (3.56). □

3.10.6. Existence and Uniqueness using Gross Substitutes Methodology (a la Alvarez and Lucas (2007))

In this subsection, we prove the existence and uniqueness of an equilibrium in our universal gravity framework using the gross substitutes methodology employed by Alvarez and Lucas (2007). As we show below, the sufficient conditions here are stronger than we provide in Theorem 1 above.

Proposition 14. *Consider any model within the universal gravity framework. If $\phi > \psi > 0$ and $\tau_{ij} \in (0, \infty)$ for all $i, j \in S$, then the excess demand system of the model satisfies gross substitutes and, as a result, the equilibrium exists and is unique.*

Proof. Recall the equilibrium conditions of the universal gravity framework from equations (3.6) and

$$p_i C_i \left(\frac{p_i}{P_i}\right)^{\psi} = \sum_{j\in S} \tau_{ij}^{-\phi} p_i^{-\phi} P_j^{\phi} p_j C_j \left(\frac{p_j}{P_j}\right)^{\psi} \forall i \in S \tag{3.57}$$

$$P_i^{-\phi} = \sum_{j\in S} \tau_{ji}^{-\phi} p_j^{-\phi} \ \forall i \in S \tag{3.58}$$

Substituting equation (3.58) into (3.57) yields a single equilibrium system of equations that depends only on the output prices in every location:

$$p_i^{1+\phi+\psi} \left(\sum_{j\in S} \tau_{ji}^{-\phi} p_j^{-\phi}\right)^{\frac{\psi}{\phi}} C_i = \sum_{j\in S} \tau_{ij}^{-\phi} C_j p_j^{1+\psi} \left(\sum_{k\in S} \tau_{kj}^{-\phi} p_k^{-\phi}\right)^{\frac{\psi-\phi}{\phi}} \quad \forall i \in S$$

We define the corresponding excess demand function as:

$$Z_i(\mathbf{p}) = \frac{1}{p_i}\left(\frac{1}{\sum_{k\in S} C_k \left(\sum_{l\in S} \tau_{lk}^{-\phi} (\beta p_l)^{-\phi}\right)^{\frac{\psi}{\phi}} (\beta p_k)^{\psi}}\right) \times$$

$$\left[\sum_{j\in S} \tau_{ij}^{-\phi} C_j p_i^{-\phi} p_j^{1+\psi} \left(\sum_{k\in S} \tau_{kj}^{-\phi} p_k^{-\phi}\right)^{\frac{\psi-\phi}{\phi}} - p_i^{1+\psi} \left(\sum_{j\in S} \tau_{ji}^{-\phi} p_j^{-\phi}\right)^{\frac{\psi}{\phi}} C_i\right], \tag{3.59}$$

whereP_i is defined by equation (3.58).This system written as such needs to satisfy 6 properties to be an excess demand system and the gross substitute property to establish existence and uniqueness. The six conditions are:

1.$Z(\mathbf{p})$ is continuous for$\mathbf{p} \in \Delta\left(R_+^N\right)$

2.$Z(\mathbf{p})$ is homogeneous of degree zero.

3.$Z(\mathbf{p}) \cdot \mathbf{p} = 0$ (Walras' Law).

4. There exists a$k > 0$ such that$Z_j(\mathbf{p}) > -k$ for all$j$.

5. If there exists a sequence $p^m \to p^0$, where $p^0 \neq 0$ and $p_i^0 = 0$ for some i, then it must be that:

$$max_j \left\{ Z_j \left(p^m \right) \right\} \to \infty \tag{3.60}$$

and the gross-substitute property:

6. Gross substitutes property:$\frac{\partial Z(p_j)}{\partial p_k} > 0$ for all$j \neq k$.

We verify each of these properties in turn. Property 1 is trivial given equation (3.59) for excess demand. To see property 2, consider multiplying output prices by a scalar $\beta > 0$, which immediately yields $Z_i \left(\beta \mathbf{p} \right) = Z_i \left(\mathbf{p} \right)$.as required. Property 3 can be seen as follows:

$$\begin{aligned}
Z \left(\mathbf{p} \right) \cdot \mathbf{p} = &= \sum_{i \in S} Z_i \left(\mathbf{p} \right) p_i \iff \\
&= \left(\frac{1}{\sum_{k \in S} C_k \left(\sum_{l \in S} \tau_{lk}^{-\phi} p_l^{-\phi} \right)^{\frac{\psi}{\phi}} p_k^{\psi}} \right) \times \\
&= \sum_{i \in S} \left(\sum_{j \in S} \tau_{ij}^{-\phi} C_j p_i^{-\phi} p_j^{1+\psi} \left(\sum_{k \in S} \tau_{kj}^{-\phi} p_k^{-\phi} \right)^{\frac{\psi - \phi}{\phi}} - p_i^{1+\psi} \left(\sum_{j \in S} \tau_{ji}^{-\phi} p_j^{-\phi} \right)^{\frac{\psi}{\phi}} C_i \right) \iff \\
&= 0,
\end{aligned}$$

as required. Property 4 can be seen as follows:

$$Z_i(\mathbf{p}) = \frac{1}{p_i}\frac{\sum_{j\in S}\tau_{ij}^{-\phi}C_j p_i^{-\phi}p_j^{1+\psi}\left(\sum_{k\in S}\tau_{kj}^{-\phi}p_k^{-\phi}\right)^{\frac{\psi-\phi}{\phi}}}{\sum_{k\in S}C_k\left(\sum_{l\in S}\tau_{lk}^{-\phi}(\beta p_l)^{-\phi}\right)^{\frac{\psi}{\phi}}(\beta p_k)^{\psi}} - Q_i \implies$$

$$Z_i(\mathbf{p}) > -Q_i > \bar{Q}$$

since $\frac{1}{p_i}\left(\frac{1}{\sum_{k\in S}C_k\left(\sum_{l\in S}\tau_{lk}^{-\phi}(\beta p_l)^{-\phi}\right)^{\frac{\psi}{\phi}}(\beta p_k)^{\psi}}\right)\sum_{j\in S}\tau_{ij}^{-\phi}C_j p_i^{-\phi}p_j^{1+\psi}\left(\sum_{k\in S}\tau_{kj}^{-\phi}p_k^{-\phi}\right)^{\frac{\psi-\phi}{\phi}} > 0$ for all $\mathbf{p} \gg 0$ and $Q_i \leq \bar{Q}$ from C. 3. Property 5 can be seen as follows: consider any $\mathbf{p} \in \Delta\left(R_+^N\right)$ such that there exists an$l \in S$ where$p_l = 0$ and an$l' \in S$ where$p_{l'} > 0$. Consider any sequence of output prices such that $\mathbf{p}^n \to \mathbf{p}$ as$n \to \infty$. Then we need to show that:

$$\max_{i\in S} Z_i(\mathbf{p}) \to \infty.$$

To see this note that:

$$\max_{i\in S} Z_i(\mathbf{p}^n) = \max_{i\in S}\frac{\frac{1}{p_i}\sum_{j\in S}(\tau_{ij}p_i)^{-\phi}C_j p_j^{1+\psi}\left(\sum_{k\in S}(\tau_{kj}p_k)^{-\phi}\right)^{\frac{\psi-\phi}{\phi}}}{\sum_{k\in S}C_k\left(\sum_{l\in S}\tau_{lk}^{-\phi}p_l^{-\phi}\right)^{\frac{\psi}{\phi}}p_k^{\psi}} - Q_i \implies$$

$$\max_{i\in S} Z_i(\mathbf{p}^n) > \max_{i,j\in S}\frac{p_j}{p_i}\tau_{ij}^{-\phi}\frac{C_j p_i^{-\phi}p_j^{\psi}\left(\sum_{k\in S}\tau_{kj}^{-\phi}p_k^{-\phi}\right)^{\frac{\psi-\phi}{\phi}}}{\sum_{k\in S}C_k\left(\sum_{l\in S}\tau_{lk}^{-\phi}p_l^{-\phi}\right)^{\frac{\psi}{\phi}}p_k^{\psi}} - \bar{Q}.$$

Hence, if it is the case that $\max_{i,j\in S}\frac{p_j}{p_i}\tau_{ij}^{-\phi}\frac{C_j p_i^{-\phi}p_j^{\psi}\left(\sum_{k\in S}\tau_{kj}^{-\phi}p_k^{-\phi}\right)^{\frac{\psi-\phi}{\phi}}}{\sum_{k\in S}C_k\left(\sum_{l\in S}\tau_{lk}^{-\phi}p_l^{-\phi}\right)^{\frac{\psi}{\phi}}p_k^{\psi}} \to \infty$, then because $\max_{i\in S} Z_i(\mathbf{p}^n)$ is bounded below it, it must be that $\max_{i\in S} Z_i(\mathbf{p}^n) \to \infty$ as well. Note that:

$$\begin{aligned}
&\max_{i,j\in S}\frac{p_j}{p_i}\tau_{ij}^{-\phi}\frac{C_j p_i^{-\phi}p_j^{\psi}\left(\sum_{k\in S}\tau_{kj}^{-\phi}p_k^{-\phi}\right)^{\frac{\psi-\phi}{\phi}}}{\sum_{k\in S}C_k\left(\sum_{l\in S}\tau_{lk}^{-\phi}p_l^{-\phi}\right)^{\frac{\psi}{\phi}}p_k^{\psi}}\\
>&\max_{i,j\in S}\frac{p_j}{p_i}\tau_{ij}^{-\phi}\frac{C_j p_i^{-\phi}p_j^{\psi}\left(\sum_{k\in S}\tau_{kj}^{-\phi}\left(p^{\min}\right)^{-\phi}\right)^{\frac{\psi-\phi}{\phi}}}{\sum_{k\in S}C_k\left(\sum_{l\in S}\tau_{lk}^{-\phi}\left(p^{\min}\right)^{-\phi}\right)^{\frac{\psi}{\phi}}\left(p^{\max}\right)^{\psi}}\implies\\
>&C_{ij}\min_{l\in S}p_l^{-(\phi-\psi)},
\end{aligned}$$

where $p^{\min}\equiv\min_{l\in S}p_l$,$p^{\max}\equiv\max_{l\in S}$, and $C_{ij}\equiv\tau_{ij}^{-\phi}\frac{C_j\left(\sum_{k\in S}\tau_{kj}^{-\phi}\left(p^{\min}\right)^{-\phi}\right)^{\frac{\psi-\phi}{\phi}}}{\sum_{k\in S}C_k\left(\sum_{l\in S}\tau_{lk}^{-\phi}\left(p^{\min}\right)^{-\phi}\right)^{\frac{\psi}{\phi}}\left(p^{\max}\right)^{\psi}}$. Since $\phi>\psi>0$ and there exists an$l\in S$ such that $p_l^n\to\infty$ as $n\to\infty$, then we have $\max_{i\in S}Z_i\left(\mathbf{p}^n\right)\to\infty$ as well.

Finally, we verify gross-substitutes. Without loss of generality, we differentiate only the bracketed term (as the term outside the bracket will be multiplied by zero since the bracket term is equal to zero in the equilibrium). We have:

$$\begin{aligned}
\frac{\partial Z_i\left(\mathbf{p}\right)}{\partial p_j}=&\frac{\partial}{\partial p_j}\left[\sum_{j\in S}\tau_{ij}^{-\phi}C_j p_i^{-\phi}p_j^{1+\psi}\left(\sum_{k\in S}\tau_{kj}^{-\phi}p_k^{-\phi}\right)^{\frac{\psi-\phi}{\phi}}-p_i^{1+\psi}\left(\sum_{j\in S}\tau_{ji}^{-\phi}p_j^{-\phi}\right)^{\frac{\psi}{\phi}}C_i\right]\\
=&\left(1+\psi\right)\tau_{ij}^{-\phi}C_j p_i^{-\phi}p_j^{\psi}\left(\sum_{k\in S}\tau_{kj}^{-\phi}p_k^{-\phi}\right)^{\frac{\psi-\phi}{\phi}}+\\
&\left(\phi-\psi\right)p_j^{-\phi-1}\sum_{l\in S}\tau_{il}^{-\phi}C_l p_i^{-\phi}p_l^{\psi}\left(\sum_{k\in S}\tau_{kl}^{-\phi}p_k^{-\phi}\right)^{\frac{\psi-\phi}{\phi}-1}+\\
&\psi p_j^{-\phi-1}p_i^{1+\psi}\left(\sum_{j\in S}\tau_{ji}^{-\phi}p_j^{-\phi}\right)^{\frac{\psi}{\phi}-1}>0
\end{aligned}$$

because $\phi > \psi > 0$ and prices, trade frictions, and supply shifters C_l are strictly positive. Because properties 1-6 hold, by Propositions 17.B.2, 17.C.1 and 17.F.3 of Mas-Colell et al. (1995), the equilibrium exists and unique. □

Note that in the case where $\psi > \phi > 0$ – which is the ordering we find when we estimate the gravity constants in Section 3.5 – Theorem 1 still proves existence and uniqueness of the equilibrium. The following example shows that gross substitutes may not be satisfied in this case.

Example 1. (Gross substitution) Consider the three location economy. Take p_3 as the numeraire The gross substitute is violated if there exists $\overline{p}_1$ such that $Z_1(\overline{p}_1, p_2, 1)$ is not monotonic w.r.t. p_2. Consider the following parameter values:

$$(\phi, \psi) = (2, 5)$$

$$\tau_{ij} = 1 \quad \text{for } i, j \in \{1, 2, 3\}$$

$$C_i = (.9, .6, .1)^{\mathbf{T}}.$$

Figure 3.10.10 shows that with these parameter values, $Z_1(\overline{p}_1, p_2, 1)$ is not monotonic w.r.t. p_2 when $\overline{p}_1 = .5$.

3.10.7. Examples of multiplicity in two location world

In this subsection, we derive the equilibrium conditions of a two location world and provide examples for different combinations of the gravity constants (i.e. the demand elasticity ϕ and supply elasticity ψ).

We first derive equations for the demand and supply of the representative good in each location as a function of parameters and prices in all other locations. Combining C. 2 (aggregate demand) and C. 3 (market clearing) yields the following aggregate demand equation:

$$Q_i^d = p_i^{-(1+\phi)} \times \left(\sum_{j \in S} \frac{\tau_{ij}^{-\phi}}{\sum_k \tau_{kj}^{-\phi} p_k^{-\phi}} p_j Q_j^d \right), \tag{3.61}$$

where we denote the quantity of the representative good demanded in location i as Q_i^d. Similarly, C. 3 (aggregate supply) yields the following aggregate supply equation:

$$Q_i^s = \kappa C_i \left(\frac{p_i}{\sum_{j \in S} \tau_{ji}^{-\phi} p_j^{-\phi}} \right)^{\psi}, \tag{3.62}$$

where we denote the quantity of the representative good supplied in location i as Q_i^d.

Now consider the two-location case (i.e. $S \equiv \{1,2\}$) where $\tau_{12} = \tau_{21} = \tau \geq 1$ and $C_1 = C_2 = 1$. Dividing Q_1^d by Q_2^d using equation (3.61) delivers the following relative demand equation:

$$\frac{Q_1^d}{Q_2^d} = \left(\frac{p_1}{p_2}\right)^{-(1+\phi)} \times \frac{\left(\frac{\tau^{-\phi}\left(\frac{p_1}{p_2}\right)^{-\phi} + 1}{\left(\frac{p_1}{p_2}\right)^{-\phi} + \tau^{-\phi}} \right) \times \frac{p_1}{p_2} \times \frac{Q_1^d}{Q_2^d} + \tau^{-\phi}}{\tau^{-\phi} \left(\left(\frac{\tau^{-\phi}\left(\frac{p_1}{p_2}\right)^{-\phi} + 1}{\left(\frac{p_1}{p_2}\right)^{-\phi} + \tau^{-\phi}} \right) \times \frac{p_1}{p_2} \times \frac{Q_1^d}{Q_2^d} \right) + 1} \tag{3.63}$$

Similarly, dividing Q_1^s by Q_2^s delivers the following relative supply equation:

$$\frac{Q_1^s}{Q_2^s} = \left(\frac{p_1}{p_2}\right)^{\psi} \times \left(\frac{\tau^{-\phi}\left(\frac{p_1}{p_2}\right)^{-\phi} + 1}{\left(\frac{p_1}{p_2}\right)^{-\phi} + \tau^{-\phi}} \right)^{-\frac{\psi}{\phi}} \tag{3.64}$$

Note that given the trade friction τ and gravity constants, the relative demand and relative supply can be solved solely as a function of relative output price $\frac{p_1}{p_2}$ using equations (3.63) and (3.64), allowing us to analytically characterize the equilibria using standard (relative) supply and demand curves.

Figure 3.10.1 depicts example equilibria possible for different combinations of gravity constants; the points where the two curves intersect are possible equilibria. The top left figure shows that when the supply and demand elasticities are both positive (corresponding to a case where the relative aggregate supply is increasing and the relative aggregate demand is decreasing), there is a unique equilibrium. The top right figure shows that when the supply elasticity is positive but the demand elasticity is negative, both the relative aggregate supply and demands are increasing, potentially resulting in multiple equilibria. Similarly, the bottom left figure shows that when the supply elasticity is negative and the demand elasticity is positive, both the relative aggregate supply and demand curves are decreasing, also potentially resulting in multiple equilibria. Finally, the bottom right figure shows that when both the supply and demand elasticities are negative and suitably large in magnitude, the relative aggregate supply curve is downward sloping and the relative aggregate demand curve is upward sloping, allowing for a unique equilibria (albeit one without much economic relevance). These examples are consistent with the sufficient conditions for uniqueness presented in Theorem 1.

3.10.8. Tariffs in the universal gravity framework

In this subsection, we show how one can use the tools developed above to analyze the effect of tariffs in a simple Armington trade model.

Because tariffs introduce an additional source of revenue, they are are not strictly contained within the universal gravity framework. However, it turns out that the equilibrium structure of an Armington trade model with tariffs is mathematically equivalent to the equilibrium structure of the universal gravity framework. As a result, we can apply Theorems 1 and 2 almost immediately to the case of tariffs in this model.

To see this, consider a simple Armington trade model with N locations.[42] Each location $i \in S$ is endowed with its own differentiated variety and L_i workers who supply their unit labor inelastically and consume varieties from all locations with CES preferences and an elasticity of substitution σ. Suppose that trade is subject to technological iceberg trade frictions $\tau_{ij} \geq 1$ and ad-valorem tariffs $\tilde{t}_{ij} \geq 0$. Define $t_{ij} \equiv 1 + \tilde{t}_{ij}$. Then we can write the value of trade flows from i to j (excluding the tariffs) as:

$$X_{ij} = \tau_{ij}^{1-\sigma} t_{ij}^{-\sigma} A_i^{\sigma-1} w_i^{1-\sigma} P_j^{\sigma-1} E_j, \tag{3.65}$$

where A_i is the productivity in location $i \in S$, w_i is the wage, P_j is the ideal Dixit-Stiglitz price index, and E_j is expenditure.

Income in location i from trade is equal to its total sales (excluding tariffs):

$$Y_i = \sum_{j \in S} X_{ij}. \tag{3.66}$$

Total income (and hence expenditure) also includes the revenue earned from tariffs T_i:

$$E_i = Y_i + T_i, \tag{3.67}$$

[42]We consider an Armington model in order to have an explicit welfare function, the results that follow will hold for any general equilibrium model where the aggregate supply elasticity$\psi = 0$.

where tariff revenue is equal to the bilateral tariff charged on all trade being sent[43]:

$$T_i = \sum_{j \in S} \tilde{t}_{ji} X_{ji}. \tag{3.68}$$

The total expenditure by consumers in location i is also equal to its total imports plus the tariffs incurred:

$$E_i = \sum_{j \in S} \left(1 + \tilde{t}_{ji}\right) X_{ji}. \tag{3.69}$$

Combining equations (3.67), (3.68), (3.69), we can demonstrate that trade flows are balanced:

$$\begin{aligned} E_i &= \sum_{j \in S} \left(1 + \tilde{t}_{ji}\right) X_{ji} \iff \\ Y_i + \sum_{j \in S} \tilde{t}_{ji} X_{ji} &= \sum_{j \in S} \left(1 + \tilde{t}_{ji}\right) X_{ji} \iff \\ Y_i &= \sum_{j \in S} X_{ji} \end{aligned} \tag{3.70}$$

Finally, total expenditure is equal to the payment to workers plus tariff revenue:

$$\begin{aligned} E_i &= w_i L_i + T_i \iff \\ Y_i &= w_i L_i \end{aligned} \tag{3.71}$$

Define $K_{ij} \equiv \tau_{ij}^{1-\sigma} t_{ij}^{-\sigma}$ as the bilateral "kernel", $B_i \equiv A_i L_i$ as the "income shifter", $\gamma_i \equiv A_i^{\sigma-1} w_i^{1-\sigma}$ as the origin fixed effect, $\delta_j \equiv P_j^{\sigma-1} E_j$ as the destination fixed effect, and

[43]If we had instead supposed that tariffs are only levied on goods that actually arrive, we would have$T_i = \sum_j \frac{\tilde{t}_{ji}}{\tau_{ji}} X_{ji}$, which does not change the following analysis in any substantive way.

$\alpha \equiv \frac{1}{1-\sigma}$. Combining equations (3.66), (3.70), and (3.71) yields the following system of equilibrium equations:

$$w_i L_i = \sum_{j \in S} X_{ij} \iff$$

$$B_i \gamma_i^{\alpha} = \sum_{j \in S} K_{ij} \gamma_i \delta_j \tag{3.72}$$

$$w_i L_i = \sum_{j \in S} X_{ji} \iff$$

$$B_i \gamma_i^{\alpha} = \sum_{j \in S} K_{ji} \gamma_j \delta_i. \tag{3.73}$$

Equations (3.72) and (3.73) can be jointly solved to recover the equilibrium $\{\gamma_i\}_{i \in S}$ and $\{\delta_i\}_{i \in S}$; given $\{\gamma_i\}_{i \in S}$ and $\{\delta_i\}_{i \in S}$, in turn, we can solve for all endogenous variables, as wages can be written as $w_i = \gamma_i^{\frac{1}{1-\sigma}} A_i$, the price index can be written as $P_i = \left(\sum_{j \in S} \tau_{ji}^{1-\sigma} t_{ji}^{1-\sigma} \gamma_j\right)^{\frac{1}{1-\sigma}}$, expenditure can be written as $E_i = \delta_i \left(\sum_{j \in S} \tau_{ji}^{1-\sigma} t_{ji}^{1-\sigma} \gamma_j\right)$, and real expenditure can be written as $W_i \equiv \frac{E_i}{P_i} = \delta_i \left(\sum_{j \in S} \tau_{ji}^{1-\sigma} t_{ji}^{1-\sigma} \gamma_j\right)^{\frac{\sigma}{\sigma-1}}$. As we note at the beginning of Section 3.3, this equilibrium system is identical in mathematical structure to the universal gravity equilibrium equations 3.6 and 3.7. Hence, Theorem 1 applies directly (with existence as long as $\sigma \neq 0$ and uniqueness as long as $\sigma \geq 1$). Moreover, a similar methodology as employed in Theorem 2 can be used to determine how the equilibrium variables γ_i and δ_i respond to shocks that alter the kernel K_{ij} (be they due to changes in iceberg trade frictions or tariffs). In particular:

$$\frac{\partial \ln \gamma_l}{\partial \ln K_{ij}} = X_{ij} \times \left(A^{+}_{l,i} + A^{+}_{N+l,j} - c\right) \tag{3.74}$$

$$\frac{\partial \ln \delta_l}{\partial \ln K_{ij}} = X_{ij} \times \left(A^{+}_{N+l,i} + A^{+}_{l,j} - c\right), \tag{3.75}$$

where $\tilde{A}^{-1}_{i,j}$ is the $\langle i, j \rangle$ element of the $2N \times 2N$ matrix the (pseudo) inverse $\tilde{\mathbf{A}}^{-1}$:[44]

$$\tilde{\mathbf{A}}^{-1} = \begin{pmatrix} \frac{\sigma}{1-\sigma}\mathbf{Y} & -\mathbf{X} \\ \frac{1}{1-\sigma}\mathbf{Y} - \mathbf{X}^T & -\mathbf{Y} \end{pmatrix}^{-1}, \tag{3.76}$$

Because all endogenous variables in the model are simple functions $\{\gamma_i\}_{i\in S}$ and $\{\delta_i\}_{i\in S}$, one can apply equations (3.74) and (3.75) to immediately derive any elasticity of interest, e.g. the effect of welfare in location l from changing the tariffs j impose on goods coming from i.

3.10.9. Global shocks

In this subsection we show that the "exact hat algebra" pioneered by Dekle et al. (2008) and extended by Costinot and Rodriguez-Clare (2013) can be applied to any model in the universal gravity framework to calculate the effect of any (possibly large) trade shock. (Note that Section 3.4 instead showed how to calculate the *elasticity* of endogenous variables to any trade friction shock). We show that the key takeaway from Section 3.4 holds

[44] The psuedo-inverse can be calculated simply by removing the first row and column and taking the inverse; see footnote 18.

for all trade shocks: Given observed data, all the gravity models with the same gravity constants imply the same counterfactual predictions for all endogenous variables (i.e. output prices, price indices, nominal incomes, real expenditures, and trade flows).

Consider an arbitrary change in the trade friction matrix $\{\tau_{ij}\}_{S\times S}$. In what follows, we denote with a hat the ratio of the counterfactual to initial value of the variable, i.e. $\hat{x}_i \equiv \frac{x_i^{counterfactual}}{x_i^{initial}}$. The following proposition provides an analytical expression relating the change in the output price and the associated price index to the change in trade frictions and the initial observed trade flows:

Proposition 15. *Consider any given set of observed trade flows* $\mathbf{X}$*, gravity constants* ϕ *and* ψ*, and change in the trade friction matrix* $\widehat{\tau}$*. Then the percentage change in the exporter and importer shifters,* $\{\hat{p}_i\}$ *and* $\left\{\hat{P}_i\right\}$*, if it exists, will solve the following system of equations:*

(3.77)

$$\widehat{p}_i^{1+\phi+\psi}\widehat{P}_i^{-\psi} = \sum_{j\in S}\frac{X_{ij}}{Y_i}\widehat{\tau}_{ij}^{-\phi}\widehat{P}_j^{\phi}\widehat{p}_j\left(\frac{\widehat{p}_j}{\widehat{P}_j}\right)^{\psi} \quad and \quad \widehat{P}_i^{-\phi} = \sum_{j\in S}\left(\frac{X_{ji}}{E_j}\right)\hat{\tau}_{ji}^{-\phi}\hat{p}_j^{-\phi},\ \forall i\in S$$

Proof. We first note that equilibrium equations (3.10) and (3.7) must hold for both the initial and counterfactual equilibria. Taking ratios of the counterfactual to initial values yields:

$$\widehat{p}_i^{1+\phi+\psi}\widehat{P}_i^{-\psi} = \frac{\sum_{j\in S}\left(\tau'_{ij}\right)^{-\phi}\left(P'_j\right)^{\phi}p'_jC_j\left(\frac{p'_j}{P'_j}\right)^{\psi}}{\sum_{j\in S}\tau_{ij}^{-\phi}P_j^{\phi}p_jC_j\left(\frac{p_j}{P_j}\right)^{\psi}}\forall i\in S$$

$$\widehat{P}_i^{-\phi} = \frac{\sum_{j\in S}\left(\tau'_{ji}\right)^{-\phi}\left(p'_j\right)^{-\phi}}{\sum_{j\in S}\tau_{ji}^{-\phi}p_j^{-\phi}},\ \forall i\in S$$

where we denote the counterfactual equilibrium variables with a prime and the initial equilibrium variables as unadorned. Note that from the gravity equation (3.10) (and C. 3 - C. 5) we have $X_{ij} = \tau_{ij}^{-\phi} p_i^{-\phi} P_j^{\phi} p_j C_j \left(\frac{p_j}{P_j}\right)^{\psi}$, where $p_j C_j \left(\frac{p_j}{P_j}\right)^{\psi} = E_j$, so that the above equations become:

$$\widehat{p}_i^{1+\phi+\psi} \widehat{P}_i^{-\psi} = \frac{\sum_{j\in S} \left(\tau'_{ij}\right)^{-\phi} \left(P'_j\right)^{\phi} p'_j C_j \left(\frac{p'_j}{P'_j}\right)^{\psi}}{p_i^{\phi} \sum_{j\in S} X_{ij}} \forall i \in S$$

$$\widehat{P}_i^{-\phi} = \frac{\sum_{j\in S} \left(\tau'_{ji}\right)^{-\phi} \left(p'_j\right)^{-\phi}}{P_i^{-\phi} \frac{1}{E_i} \sum_{j\in S} X_{ji}}, \ \forall i \in S$$

Finally, note that from C. 2 and C. 4 we have $E_i = \sum_{j\in S} X_{ij}$ and $Y_i = \sum_{j\in S} X_{ij}$, respectively. Then using our definition $\hat{x}_i \equiv \frac{x_i^{counterfactual}}{x_i^{initial}} \iff x_i^{counterfactual} = \hat{x}_i x_i^{initial}$ we have:

$$\widehat{p}_i^{1+\phi+\psi} \widehat{P}_i^{-\psi} = \sum_{j\in S} \left(\frac{X_{ij}}{Y_i}\right) \hat{\tau}_{ij}^{-\phi} \hat{P}_j^{\phi} \hat{p}_j \left(\frac{\widehat{p}_j}{\widehat{P}_j}\right)^{\psi} \forall i \in S$$

$$\widehat{P}_i^{-\phi} = \sum_{j\in S} \left(\frac{X_{ji}}{E_j}\right) \hat{\tau}_{ji}^{-\phi} \hat{p}_j^{-\phi} \forall i \in S,$$

as required. □

Note that equation (3.77) inherits the same mathematical structure as equations (3.6) and (3.7). As a result, part (i) of Theorem 1 proves that there will exist a solution to equation (3.77) and part (ii) of Theorem 1 provides conditions for its uniqueness.

3.10.10. Identification

In this subsection, we show how one can always choose a set of bilateral trade frictions to match observed trade flows for any choice of gravity constants, own trade frictions, and supply shifters. We first state the result as a proposition before providing a proof.

Proposition 16. *Take as given the set of observed trade flows $\{X_{ij}\}$, an assumed set of supply shifters $\{C_i\}$, an aggregate scalar κ, and own trade frictions $\{\tau_{ii}\}$, and the gravity constants ϕ and ψ. Then there exists a unique set of trade frictions $\{\tau_{ij}\}_{i\neq j}$, output prices $\{p_i\}$, price indices $\{P_i\}$, and output $\{Q_i\}$ such that the following equilibrium conditions hold:*

(1) For all locations $i \in S$, income is equal to the product of the output price and theoutput:

$$Y_i = p_i Q_i$$

(2) For all location pairs $i, j \in S$, the value of trade flows from i to j can be written in the following gravity equation form:

$$X_{ij} = \tau_{ij}^{-\phi} p_i^{-\phi} P_j^{\phi} E_j$$

(3) For all locations $i \in S$, output satisfies the following supply condition:

$$Q_i = \kappa C_i \left(\frac{p_i}{P_i}\right)^{\psi}$$

Proof. First, note that the income $Y_i = \sum_{j\in S} X_{ij}$, expenditure $E_i = \sum_{j\in S} X_{ji}$, and own expenditure share $\lambda_{jj} \equiv \frac{X_{jj}}{E_j}$, are all immediately derived from the observed trade flow data.

Second, let us define our unknown parameters and endogenous variables as functions of data and known parameters. The trade frictions are defined follows:

$$\tau_{ij} = \tau_{jj} \left(\frac{Y_j}{Y_i}\right) \left(\frac{\lambda_{jj}}{\lambda_{ii}}\right)^{\frac{\psi}{\phi}} \left(\frac{C_i}{C_j}\right) \left(\frac{\tau_{jj}}{\tau_{ii}}\right)^{\psi} \left(\frac{X_{jj}}{X_{ij}}\right)^{\frac{1}{\phi}}$$

for all $i, j \in S$ such that $i \neq j$.

The output prices are defined as

$$p_i = Y_i \left(\lambda_{ii} \tau_{ii}^{\phi}\right)^{\frac{\psi}{\phi}} / \kappa C_i$$

for all $i \in S$.

Given the output prices and trade frictions, the price index is defined as: for all $i \in S$,

$$P_i = \left(\sum_{j \in S} \tau_{ji}^{-\phi} p_j^{-\phi}\right)^{-\frac{1}{\phi}}.$$

Finally, the output in each location is defined as: for all $i \in S$,

$$Q_i = \kappa C_i \left(\frac{p_i}{P_i}\right)^{\psi}.$$

It is first helpful to note that given the above definitions of the trade frictions and output price indices, we have the following convenient relationship between own expenditure shares and prices:

$$\lambda_{jj} = \left(\tau_{jj} \frac{p_j}{P_j}\right)^{-\phi}$$

To see this, note that we can write:

$$\begin{aligned}
\lambda_{jj} &= \left(\tau_{jj}\frac{p_j}{P_j}\right)^{-\phi} \iff \\
\frac{X_{jj}}{E_j} &= \frac{\tau_{jj}^{-\phi}p_j^{-\phi}}{\sum_{i\in S}\tau_{ij}^{-\phi}p_i^{-\phi}} \iff \\
\tau_{jj}^{-\phi}p_j^{-\phi} &= \left(\frac{X_{jj}}{E_j}\right)\sum_{i\in S}\tau_{ij}^{-\phi}p_i^{-\phi} \iff \\
\tau_{jj}^{-\phi}p_j^{-\phi} &= \sum_{i\in S}\left(\frac{X_{ij}}{E_j}\right)\left(\frac{(Y_i/C_i)^{\phi}\left(\lambda_{ii}\tau_{ii}^{\phi}\right)^{\psi}}{(Y_j/C_j)^{\phi}\left(\lambda_{jj}\tau_{jj}^{\phi}\right)^{\psi}}\right)\tau_{jj}^{-\phi}p_i^{-\phi} \iff \\
p_j^{\phi-\phi} &= \sum_{i\in S}\left(\frac{X_{ij}}{E_j}\right)p_i^{\phi-\phi} \iff \\
E_j &= \sum_{i\in S}X_{ij},
\end{aligned}$$

which is the definition of E_j.

We now confirm each of the three equilibrium conditions. To see that income is equal to the product of the output price and the output, we write:

$$\begin{aligned}
p_i \times Q_i &= Y_i \times \left(\left(\lambda_{ii}\tau_{ii}^{\phi}\right)^{\frac{\psi}{\phi}}/\kappa C_i\right) \times Q_i \iff \\
p_i \times Q_i &= Y_i \times \left(\kappa C_i\left(\frac{p_i}{P_i}\right)^{\psi}\right)^{-1} \times Q_i \iff \\
p_i \times Q_i &= Y_i \times \frac{Q_i}{Q_i} \iff \\
p_i \times Q_i &= Y_i,
\end{aligned}$$

as required.

To see that the value of trade flows can be written in the gravity equation form, we write the gravity equation as follows:

$$\tau_{ij}^{-\phi} p_i^{-\phi} P_j^{\phi} E_j = \left(\tau_{jj} \left(\frac{Y_j}{Y_i} \right) \left(\frac{\lambda_{jj}}{\lambda_{ii}} \right)^{\frac{\psi}{\phi}} \left(\frac{C_i}{C_j} \right) \left(\frac{\tau_{jj}}{\tau_{ii}} \right)^{\psi} \left(\frac{X_{jj}}{X_{ij}} \right)^{\frac{1}{\phi}} \right)^{-\phi} p_i^{-\phi} P_j^{\phi} E_j \iff$$

$$\tau_{ij}^{-\phi} p_i^{-\phi} P_j^{\phi} E_j = X_{ij} \left(\frac{(Y_i/C_i)^{\phi} \lambda_{ii}^{\psi} \tau_{ii}^{\phi\psi}}{(Y_j/C_j)^{\phi} \lambda_{jj}^{\psi} \tau_{jj}^{\phi\psi}} \right) \left(\frac{p_i}{p_j} \right)^{-\phi} \frac{\tau_{jj}^{-\phi} p_j^{-\phi} P_j^{\phi} E_j}{X_{jj}}$$

Recall from above that we have the following relationship between prices and own expenditure shares:

$$\lambda_{ii} = \left(\tau_{ii} \frac{p_i}{P_i} \right)^{-\phi}$$

so that:

$$\tau_{ij}^{-\phi} p_i^{-\phi} P_j^{\phi} E_j = X_{ij} \left(\frac{(Y_i)^{\phi} \left(\left(\frac{p_i}{P_i} \right)^{\psi} C_i \right)^{-\phi}}{(Y_j)^{\phi} \left(\left(\frac{p_j}{P_j} \right)^{\psi} C_j \right)^{-\phi}} \right) \left(\frac{p_i}{p_j} \right)^{-\phi} \frac{\tau_{jj}^{-\phi} p_j^{-\phi} P_j^{\phi} E_j}{X_{jj}}$$

Furthermore, recall that we have defined our quantities as follows:

$$Q_i = \kappa C_i \left(\frac{p_i}{P_i} \right)^{\psi},$$

which implies that:

$$\tau_{ij}^{-\phi} p_i^{-\phi} P_j^{\phi} E_j = X_{ij} \left(\frac{(Y_i/Q_i)^{\phi}}{(Y_j/Q_j)^{\phi}} \right) \left(\frac{p_i}{p_j} \right)^{-\phi} \frac{\tau_{jj}^{-\phi} p_j^{-\phi} P_j^{\phi} E_j}{X_{jj}}$$

We have shown above that $p_i Q_i = Y_i$, so that we have:

$$\tau_{ij}^{-\phi} p_i^{-\phi} P_j^{\phi} E_j = X_{ij} \frac{\tau_{jj}^{-\phi} p_j^{-\phi} P_j^{\phi} E_j}{X_{jj}}$$

We claim that this implies that observed trade flows are explained by the gravity equation, i.e.:

$$X_{ij} = \tau_{ij}^{-\phi} p_i^{-\phi} P_j^{\phi} E_j$$

To see this, suppose not. Then we have

$$\tau_{ij}^{-\phi} p_i^{-\phi} P_j^{\phi} E_j = X_{ij} \frac{\tau_{jj}^{-\phi} p_j^{-\phi} P_j^{\phi} E_j}{X_{jj}}$$

but $X_{ij} \neq \tau_{ij}^{-\phi} p_i^{-\phi} P_j^{\phi} E_j$. Then without loss of generality we can write $X_{ij} = \tau_{ij}^{-\phi} p_i^{-\phi} P_j^{\phi} E_j \varepsilon_{ij}$, where $\varepsilon_{ij} \neq 1$.

$$\begin{aligned}
\tau_{ij}^{-\phi} p_i^{-\phi} P_j^{\phi} E_j &= \left(\tau_{ij}^{-\phi} p_i^{-\phi} P_j^{\phi} E_j \varepsilon_{ij}\right) \frac{\tau_{jj}^{-\phi} p_j^{-\phi} P_j^{\phi} E_j}{\left(\tau_{jj}^{-\phi} p_j^{-\phi} P_j^{\phi} E_j \varepsilon_{jj}\right)} \iff \\
1 &= \frac{\varepsilon_{ij}}{\varepsilon_{jj}} \iff \\
\varepsilon_{ij} &= \varepsilon_{jj} \equiv \varepsilon_j \ \forall i \in S
\end{aligned}$$

which then implies that we have:

$$X_{ij} = \tau_{ij}^{-\phi} p_i^{-\phi} P_j^{\phi} E_j \varepsilon_j$$

however, we know that:

$$\sum_{i \in S} X_{ij} = E_j \iff$$

$$\sum_{i \in S} \tau_{ij}^{-\phi} p_i^{-\phi} P_j^{\phi} E_j \varepsilon_j = E_j \iff$$

$$\frac{\sum_{i \in S} \tau_{ij}^{-\phi} p_i^{-\phi}}{\sum_{i \in S} \tau_{ij}^{-\phi} p_i^{-\phi}} = \frac{1}{\varepsilon_j} \iff$$

$$\varepsilon_j = 1,$$

which is a contradiction. Hence, the observed trade flows are explained by the gravity equation.

Finally, we note that the third equilibrium condition trivially holds by the definition of Q_i:

$$Q_i = \kappa C_i \left(\frac{p_i}{P_i} \right)^{\psi}.$$

Hence, given our definitions, we have found a unique set of trade frictions $\{\tau_{ij}\}_{j \neq i}$, output prices $\{p_i\}_{i \in S}$, price indices $\{P_i\}_{i \in S}$, and output $\{Q_i\}_{i \in S}$ such that the equilibrium conditions hold for any set of observed trade flows $\{X_{ij}\}_{i,j \in S}$, an assumed set of supply shifters $\{C_i\}_{i \in S}$ and own trade frictions $\{\tau_{ii}\}_{i \in S}$, and the gravity constants (ϕ, ψ).Real output prices, welfare, and the openness to trade $\square$

In this section, we explore the relationship between the real output E_i/P_i and real output price p_i/P_i in the universal gravity framework and the welfare in a number of seminal models. We then show how the real output price in the universal gravity framework relates to the observed own expenditure share. Combining the two results allow ones to write the welfare in each of these models as a function of observed own expenditure share, as in Arkolakis et al. (2012a).

3.10.10.1. Real output prices and welfare. In this subsection, we provide a mapping between real output prices and the welfare of a unit of labor for the trade introduced and the economic geography model in Section 3.2.

The trade model. In the trade model, the output price p_i is $w_i^{\zeta} P_i^{1-\zeta}/A_i$. As a result we have the welfare of each worker Ω_i can be expressed as a function of the real output price in the universal gravity framework as follows:

$$\frac{w_i}{P_i} = \underbrace{\left(\frac{p_i A_i}{P_i^{1-\gamma}}\right)^{\frac{1}{\zeta}}}_{=w_i} \frac{1}{P_i} = A_i^{\frac{1}{\gamma}} \left(\frac{p_i}{P_i}\right)^{\frac{1}{\zeta}}.$$

Or equivalently, we can express the welfare in terms of the supply elasticity.

$$\frac{w_i}{P_i} = A_i^{1+\psi} \left(\frac{p_i}{P_i}\right)^{1+\psi}.$$

The economic geography model. In the economic geography model, the welfare is $\frac{w_i}{P_i} u_i$,and the price p_i is $\frac{w_i}{\overline{A}_i L_i^a}$. Therefore the welfare is

$$\Omega = \overline{A}_i \overline{u}_i L_i^{a+b} \left(\frac{p_i}{P_i}\right).$$

Welfare equalization and the labor market clearing condition implies

$$\Omega = \left(\overline{L}\right)^{a+b} \left[\sum_{i \in S} \left[\overline{A}_i \overline{u}_i \left(\frac{p_i}{P_i}\right)\right]^{-\frac{1}{a+b}}\right]^{-(a+b)}.$$

3.10.10.2. Real expenditure, real output prices and the openness to trade. In this subsection, we show we can express real expenditure and real output prices in any model within the universal gravity framework as a function of openness to trade and the gravity constants, as in Arkolakis et al. (2012a).

We begin by defining $\lambda_{ii} \equiv \frac{X_{ii}}{E_i}$ as location i's own expenditure share. From equation (3.10), we can express the real output price $\frac{p_i}{P_i}$ in a location as a function of its own expenditure share:

$$X_{ij} = \frac{p_{ij}^{-\phi}}{\sum_{k\in S} p_{kj}^{-\phi}} E_j \implies$$

$$\frac{p_i}{P_i} = \lambda_{ii}^{-\frac{1}{\phi}}. \tag{3.78}$$

Moreover, given C. 3, C. 4 and C. 5, we can write total real expenditure $W_i \equiv \frac{E_i}{P_i}$ as a function of its own expenditure share as well:

$$W_i = \frac{E_i}{P_i} \iff$$

$$W_i = \left(\frac{p_i}{P_i}\right) Q_i \iff$$

$$W_i = \left(\frac{p_i}{P_i}\right)\left(\kappa C_i \left(\frac{p_i}{P_i}\right)^{\psi}\right) \iff$$

$$W_i = \kappa C_i \left(\frac{p_i}{P_i}\right)^{1+\psi}. \tag{3.79}$$

Combining equations (3.78) and (3.79) yields:

$$W_i = \kappa C_i \left(\lambda_{ii}\right)^{-\frac{1+\psi}{\phi}}.$$

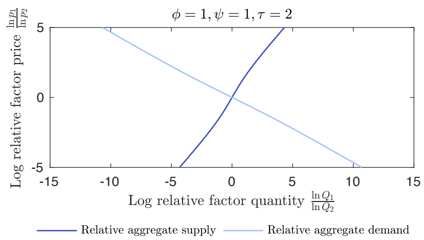
Note that a positive aggregate supply elasticity ($\psi > 0$) increases the elasticity of total real expenditure to own expenditure share, thereby amplifying the gains from trade. Note too that the derivations above imply that:

$$\frac{\partial \ln W_i}{\partial \ln \tau_{ij}} = (\psi + 1) \frac{\partial \ln \left(\frac{p_i}{P_i} \right)}{\partial \ln \tau_{ij}} + \frac{\partial \ln \kappa}{\partial \ln \tau_{ij}},$$

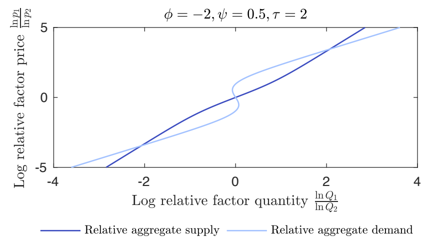
i.e. we can recover the elasticity of the total real expenditure (to-scale) to the trade friction shock from the elasticity of the real output price to the trade friction shock by simply multiplying by $\psi + 1$.

3.10.11. Additional Figures

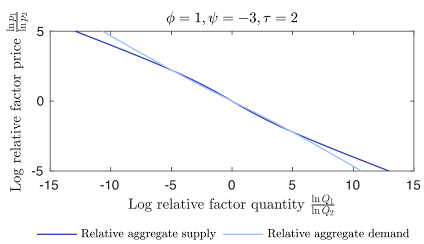
Figure 3.10.1. Examples of multiplicity and uniqueness in two locations



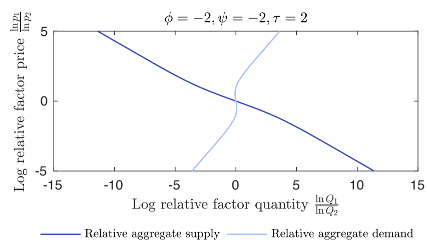
(a) Positive supply and demand elasticities



(b) Positive supply elasticity, negative demand elasticity



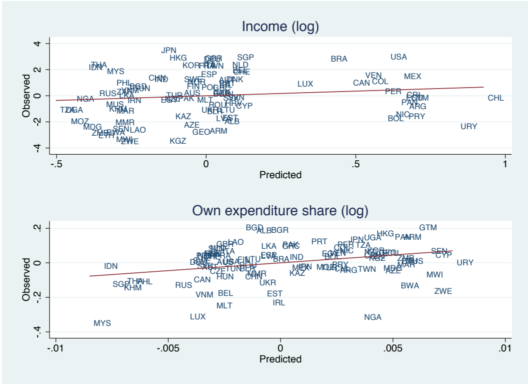
(c) Positive demand elasticity, negative supply elasticity



(d) Negative supply and demand elasticities (both ≤ -1)

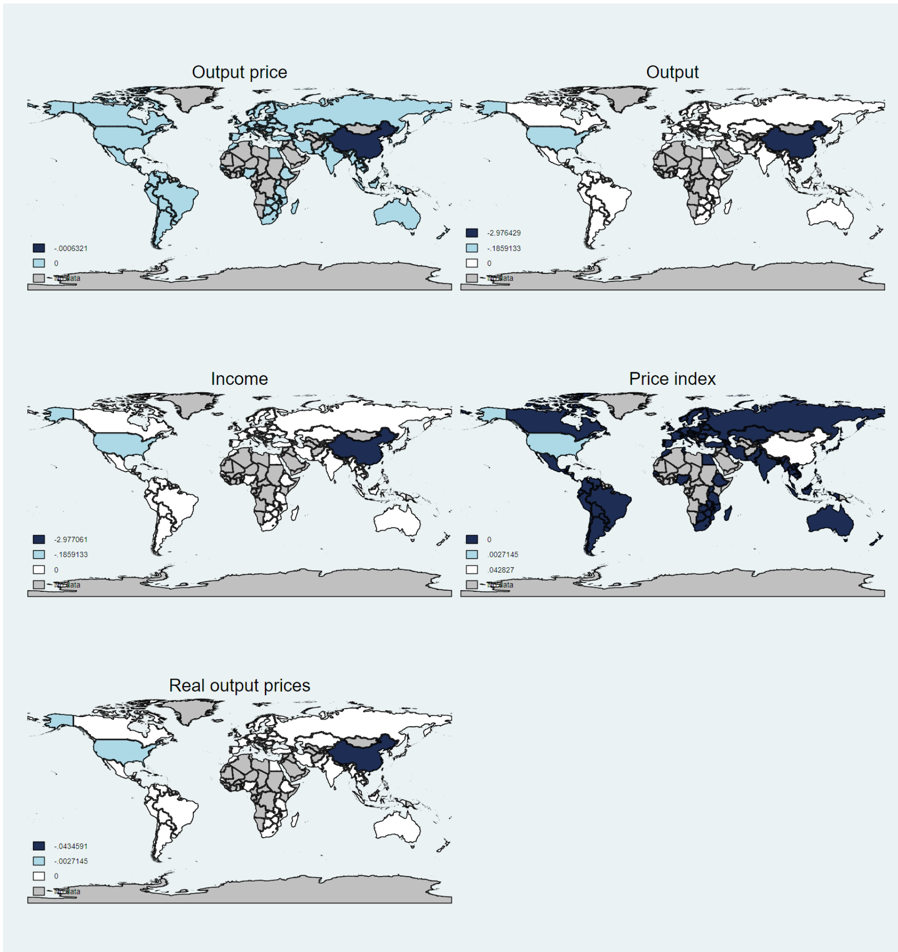
Notes: This figure shows examples of relative supply curve and relative demand curves for a two location world for different combinations of supply and demand elasticities; see Section 3.10.7 for a discussion.

Figure 3.10.2. Correlation between observed income and own expenditure shares and the equilibrium values from the gravity model



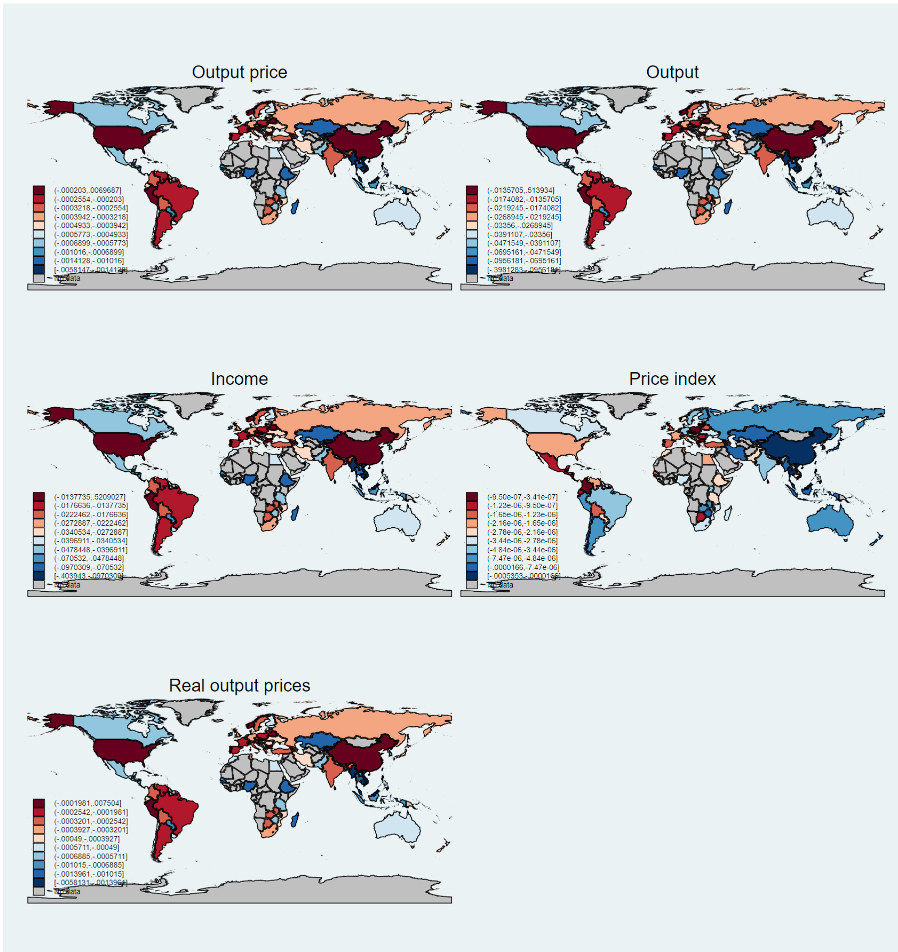
Notes: This figure shows the relationship between the observed and predicted income and own expenditure shares, respectively. The predicted incomes and own expenditure shares are the equilibrium values from the general equilibrium gravity model where bilateral frictions are those estimated from a fixed effects gravity regression and the supply shifters are estimated from a regression of log income on geographic and institutional controls. The scatter plots are plots of the residuals after controlling for the direct effect of the geographic, historical, and institutional observables.

Figure 3.10.3. The network effect of a U.S.-China trade war: Degree 0



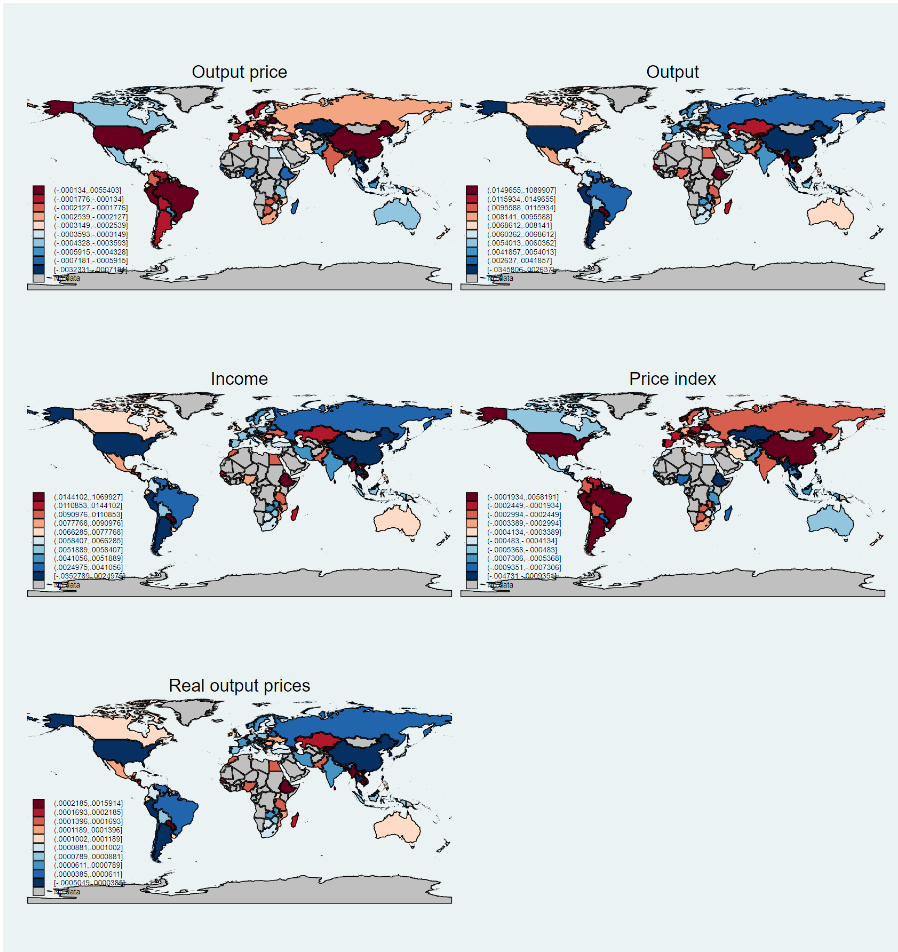
Notes: This figure depicts the "degree 0" effect of an increase in the bilateral trade frictions between the U.S. and China (a "trade war") in all countries. The "degree 0" effect is the direct impact of the trade war on the U.S. and China, holding constant the price and output in all other countries. Note that output prices, output, and the price index effects are identified only to scale, whereas the level of income and real output prices are known (see the discussion in Section 3.2).

Figure 3.10.4. The network effect of a U.S.-China trade war: Degree 1



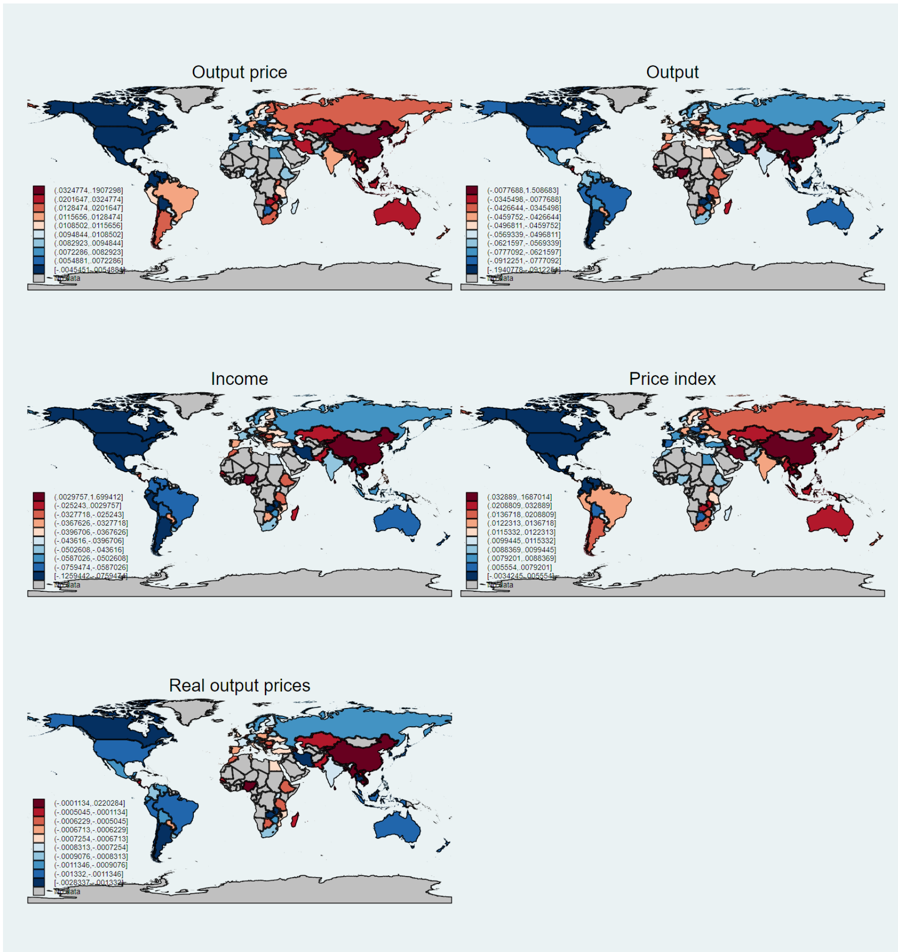
Notes: This figure depicts the "degree 1" effect of an increase in the bilateral trade frictions between the U.S. and China (a "trade war") in all countries. The "degree 1" effect is the impact of the "degree 0" shock on all countries through the trade network, holding constant the prices and output of their trading partners. Note that output prices, output, and the price index effects are identified only to scale, whereas the level of income and real output prices are known (see the discussion in Section 3.2).

Figure 3.10.5. The network effect of a U.S.-China trade war: Degree 2



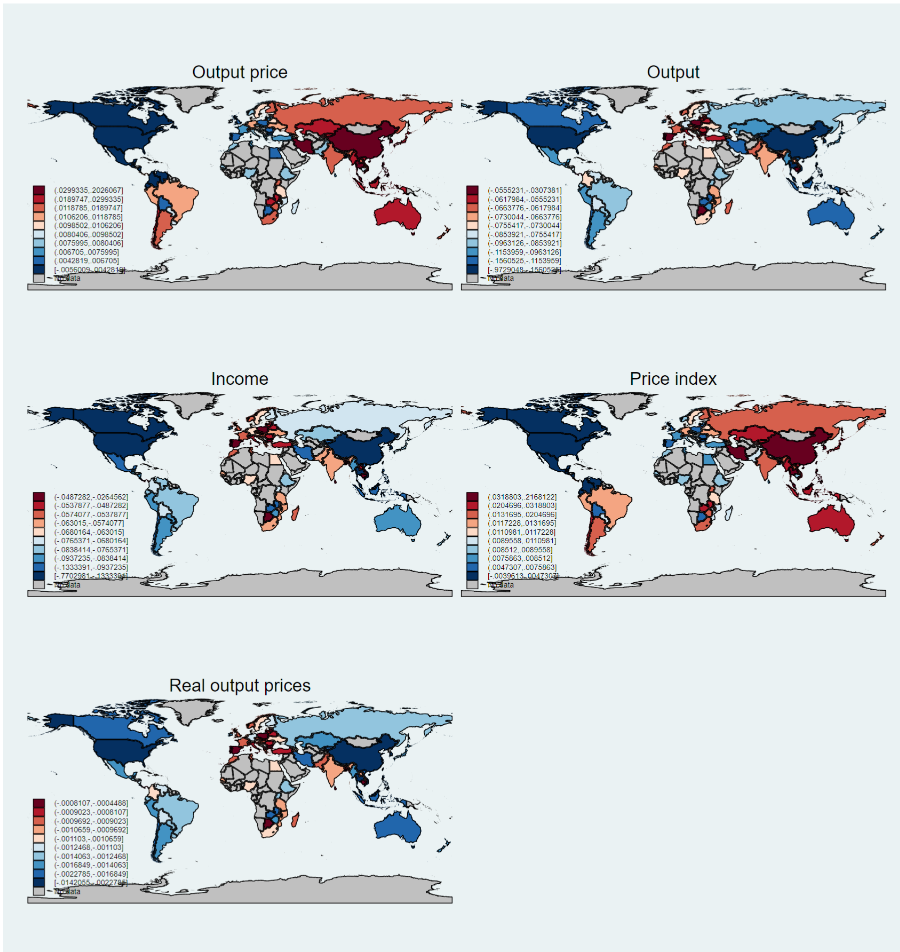
Notes: This figure depicts the "degree 2" effect of an increase in the bilateral trade frictions between the U.S. and China (a "trade war") in all countries. The "degree 2" effect is the impact of the "degree 1" shock on all countries through the trade network, holding constant the prices and output of their trading partners. Note that output prices, output, and the price index effects are identified only to scale, whereas the level of income and real output prices are known (see the discussion in Section 3.2).

Figure 3.10.6. The network effect of a U.S.-China trade war: Degrees >2



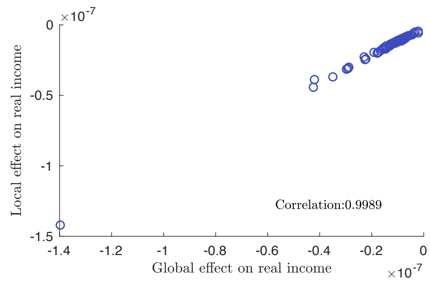
Notes: This figure depicts the cumulative effect of all degrees greater than two of an increase in the bilateral trade frictions between the U.S. and China (a "trade war") in all countries. A degree k effect is the impact of a degree $k-1$ shock on all countries through the trade network, holding constant the prices and output of their trading partners. Note that output prices, output, and the price index effects are identified only to scale, whereas the level of income and real output prices are known (see the discussion in Section 3.2).

Figure 3.10.7. The network effect of a U.S.-China trade war: Total effect



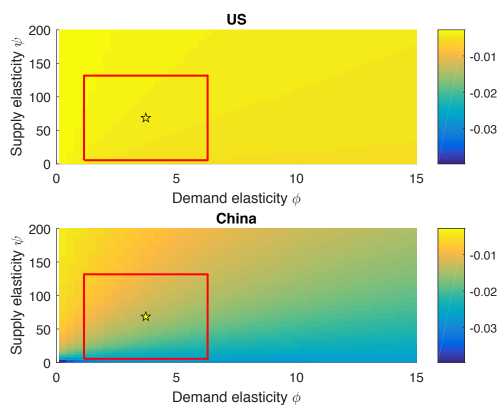
Notes: This figure depicts the total effect of an increase in the bilateral trade frictions between the U.S. and China (a "trade war") in all countries. This is the infinite sum of all degree k effects. Note that output prices, output, and the price index effects are identified only to scale, whereas the level of income and real output prices are known (see the discussion in Section 3.2).

Figure 3.10.8. Local versus global effects of a U.S.-China trade war



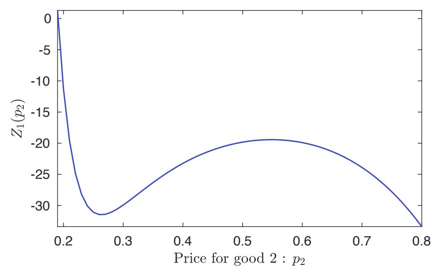
Notes: This figure depicts the correlation of the local (infinitesimal) elasticities and the global (50% increase) impacts of a trade war on the real output price in each country.

Figure 3.10.9. The effect of a U.S.-China trade war on real output prices in the U.S. and China: Robustness



Notes: This figure depicts the elasticity of real output prices to an increase bilateral trade frictions between the U.S. and China (a "trade war") for many constellations of demand and supply elasticities ϕ and ψ, respectively. The star indicates the estimated supply and demand elasticity constellation, and the red box outlines the 95% confidence interval of the two parameters.

Figure 3.10.10. Excess non-monotonic demand function for 1, $Z_1(p_2)$



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