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Multi-indexed Deligne Extensions and Multiplier Subsheaves

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ABSTRACT

Multi-indexed Deligne Extensions and Multiplier Subsheaves

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We define multi-indexed Deligne extensions and multi-indexed log-variations of Hodge structures in the category of (filtered) logarithmic $\mathcal{D}$-modules, via the idea of Bernstein–Sato polynomials and Kashiwara–Malgrange filtrations, generalizing the Deligne canonical extensions of flat vector bundles. We also obtain many comparison results with perverse sheaves via the logarithmic *de Rham* functor.

Based on multi-indexed Deligne extensions, we define multiplier subsheaves for pure Hodge modules (geometrically for higher direct images of dualizing sheaves for projective families) on algebraic varieties, which specialize to multiplier ideals when the pure Hodge module is trivial. From Kodaira–Sato vanishing for Hodge modules, we obtain a Nadel-type vanishing theorem for multiplier subsheaves, which in the geometric case generalizes both Kollár vanishing for higher direct images of dualizing sheaves and Nadel vanishing for multiplier ideals.

As an application, we use it to deduce a Fujita-type effective global generation theorem extending a result of Kawamata for higher direct images of dualizing sheaves.

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To My Father, Chang-Lu

Table of Contents

ABSTRACT 3

Acknowledgements 4

Chapter 1. Introduction 8
- 1.1. Summary of results 9

Chapter 2. Preliminaries on $V_0^D\mathcal{D}_X$-modules 16
- 2.1. $\mathcal{O}_X$-modules with Logarithmic Connections. 16
- 2.2. $\otimes$ and $\mathcal{H}om$ over $\mathcal{O}_X$ and Side-changing Operations 17
- 2.3. Filtered $V_0^D\mathcal{D}_X$-modules 18
- 2.4. Log-de Rham and Log-Spencer Complexes 19
- 2.5. Residue Maps for Left $V_0^D\mathcal{D}_X$-modules 22
- 2.6. Logarithmic Extentions and Minimal Polynomials of Residue Maps 23

Chapter 3. Multi-indexed Deligne Extensions 26
- 3.1. Local extensions on Δ^n 26
- 3.2. Sheaves of Sections of Moderate Growth 28
- 3.3. Bernstein-Sato Polynomials and Multi-indexed Deligne Extensions 35
- 3.4. Multi-indexed Deligne Extensions and Parabolic Structures of $\mathcal{V}_{\mathrm{mod}}$ 40

Chapter 4. Global Calculus of $\mathcal{V}^{(\tau_i)}$ 44
- 4.1. Holonomicity of $\mathcal{V}(*D)$ and Middle extensions 44
- 4.2. Duality 47
- 4.3. Regularity of $\mathcal{D}_X \otimes_{V_0^D\mathcal{D}_X} \mathcal{V}^{(\tau_i)}$ 58
- 4.4. Comparison Theorems 59

Chapter 5. Multi-indexed *Log*-VHS 66
- 5.1. Filtration Structures for Log-VHS and Unipotent Reductions 66
- 5.2. Canonical Filtrations for $\mathcal{V}(*D)$ and $\mathcal{V}_{\min}$ 71

5.3. Filtered Comparison Theorems 72
5.4. Base change 80

Chapter 6. Vanishing Theorems for Hodge modules 85
6.1. GAGA Principle and the Algebraicity of $F_\bullet^{(t_i)}$ 85
6.2. Hodge modules and Vanishing theorems 87
6.3. An $\mathbb{R}$-Kawamata–Viehweg Type Vanishing 90

Chapter 7. Multiplier Subsheaves, Nadel-type Vanishing And Applications 94
7.1. Multiplier subsheaves for pure Hodge modules 94
7.2. Multiplier subsheaves in Normal crossing case 97
7.3. Jumping numbers 98
7.4. A Fujita's Freeness-type Theorem 99
References 101

Appendix A. Variations of Hodge structures 103
A.1. Local Systems, Flat Vector Bundles and representations of π_1 103
A.2. Variations of Hodge structures 104

CHAPTER 1

Introduction

Morihiko Saito's theory of mixed Hodge modules is a deep generalization of classical Hodge theory. Roughly speaking, mixed Hodge modules are filtered $\mathcal{D}$-modules with properties analogous to those of mixed Hodge structures. They have many good homological properties. For instance, Grothendieck's six functor between the derived category of mixed Hodge modules can be naturally defined which are compatible with the functors for coherent $\mathcal{O}$-modules. In the pure case, Hodge modules are generically defined variations of Hodge structures (VHS) with certain boundary conditions; that is Hodge modules can be identified with generically defined VHS. This natural statement is indeed very hard to prove. This identification can be understood as the minimal extension functors $j_{!*}$ for open embeddings analogous to minimal extensions of local systems in the category of perverse sheaves. Saito's strategy towards it is as follow. By the decomposition theorem and the direct image theorem for pure Hodge modules, after taking log resolution of the singular locus of the VHS, it is enough to obtain the identification in normal crossing case. But in normal crossing case, the construction of Hodge modules from the VHS is explicit and depending on the Deligne canonical extensions. See [**Sai90**] or [**Sch14**] for details.

In this thesis, the first part is to understand Deligne canonical extensions for flat vector bundles (not only vector bundles underlying VHS). Deligne canonical extensions have naturally defined flat logarithmic connections. This motivated us to study log $\mathcal{D}$-modules (or $V_0^D\mathcal{D}_X$-modules) systematically. This is the content of Chapter 2. By using the idea of Bernstein–Sato polynomials, in the category of log $\mathcal{D}$-modules, multi-indexed Deligne extensions are defined. In trivial case, they are exactly multiplier ideals for $\mathbb{Q}$-divisors with normal crossing supports. Chapter 3 is constituted of their construction and basic properties. Chapter 4 consists of global properties of multi-indexed Deligne extensions and $\mathcal{D}$-modules induced from multi-indexed Deligne extensions.

The second part is about construction of multiplier subsheaves. The primary motivation is vanishing theorems. Kodaira vanishing is one of the most famous vanishing theorem in algebraic geometry. In the theory of Hodge modules, it has a natural generalization which is called Kodaira-Saito vanishing. In birational geometry, Kodaira vanishing has a natural birational generalization, Kawamata-Viehweg vanishing. By using multiplier ideals, $\mathbb{Q}$-Kawamata-Viehweg vanishing is equivalent to Nadel vanishing. On the other hand, in [**Wu15**] we obtained a Kawamata-Viehweg type vanishing theorem for pure Hodge modules, i.e. Kodaira–Saito vanishing also has a birational generalization. But it only works for integral divisors. In order to obtain a full generalization along this line, objects analogous to multiplier ideals are needed. All these vanishing results reflect the fact that the V-filtrations for Hodge modules are rationally indexed.

In Chapter 5, we discuss multi-indexed log-VHS, and how to construct Hodge modules from them. Chapter 6 is about vanishing theorems for Hodge modules. A $\mathbb{Q}$-Kawamata-Viehweg type vanishing will be proved. After all these, multilpier subsheaves for Hodge modules can be defined. This will be done in Chapter 7. Besides the basic properties, we also discuss some advanced properties analogous to that of multiplier ideals like jumping numbers and subadditions (at least in normal crossing case). At last, as an application, a Fujita-type effective global generation theorem for Hodge modules, parallel to Kawamata's relative Fujita's freeness conjecture in geometric case, is proved.

We begin by summarizing the main results in the thesis.

1.1. Summary of results

Let X be a complex manifold and let $D = \sum_i D_i$ be a normal crossing divisor with (possibly singular) irreducible components D_i; i.e., we assume that D is locally defined by $z_1 \cdots z_r = 0$ where $(z_1, \dots, z_n)$ are local coordinates. Suppose $(\mathcal{V}, \nabla)$ is a holomorphic vector bundle with an integrable connection on $X \setminus D$. Then it is well-known that $\mathcal{V}$ can be extended to $\tilde{\mathcal{V}}$, the Deligne canonical extension, and $\mathcal{V}(*D)$, the Deligne meromorphic extension. Moreover, $\tilde{\mathcal{V}}$ is an $\mathcal{O}_X(*D)$ lattice of $\mathcal{V}(*D)$, where $\mathcal{O}_X(*D)$ is the sheaf of meromorphic functions that are holomorphic on $X \setminus D$, and ∇ extends to a logarithmic connection $\tilde{\nabla}$ for $\tilde{\mathcal{V}}$.

If $B = \sum_i a_i D_i$ is a divisor supported on D, then $\mathcal{O}_X(B)$ is defined by, for each open set U of X

$$\mathcal{O}_X(B)(U) \coloneqq \{f \in \mathcal{O}_X(*D)(U) \text{ such that } \mathrm{ord}_{D_i|U}(f) \geq -a_i \text{ for all } i\}.$$

Namely, the natural embedding

$$\mathscr{O}_X(B) \hookrightarrow \mathscr{M}_X$$

factors through $\mathscr{O}_X(*D)$, where $\mathscr{M}_X$ is the sheaf of meromorphic functions on X. The holomorphic differential extends naturally to a logarithmic connection on $\mathscr{O}_X(B)$. If D_i is an irreducible component of D, then the residue of $\mathscr{O}_X(B)$ along D_i is exactly $-a_i$. Based on this observation, using the idea of Bernstein–Sato polynomials, we define the orders of sections of $\mathcal{V}(*D)$. Roughly speaking, for a section m of $\mathcal{V}(*D)$, $\mathrm{ord}_{D_i}(m)$ is the smallest real part of the roots of the Bernstein–Sato polynomial of m along D_i. See Definition 3.3.5 and Corollary 3.3.8. If $\mathcal{V} = \mathscr{O}_{X\setminus D}$ and $f \in \mathscr{O}_X(*D)(X)$, then $\mathrm{ord}_{D_i}(f)$ is exactly the usual order of f along D_i.

Use τ to denote a unit interval like $[a, a+1)$ or $(a, a+1]$ for some $a \in \mathbb{R}$, and (τ_i) a family of such intervals such that each i runs over the set of irreducible components of D; that is, the family (τ_i) is a locally finite family. For a fixed such (τ_i), we consider the following sheaves.

Definition 1.1.1. For each open set U of X,

$$\mathcal{V}^{(\tau_i)}(U) := \{m \in \mathcal{V}(*D)(U) \mid \mathrm{ord}_{D_i}(m) \geq a_i \text{ if } \tau_i = [a_i, a_i+1) \text{ or } > a_i \text{ if } \tau_i = [a_i, a_i+1)\}.$$

In particular, we get subsheaves $\mathcal{V}^{(\tau_i)}$ of $\mathcal{V}(*D)$ for each (τ_i). If D is smooth and irreducible, then all $\mathcal{V}^{\tau}$'s together are exactly the V-filtration (Kashiwara–Milgrange filtration) of $\mathcal{V}(*D)$, where τ corresponds to the only component D.

If we use the natural partial order for all families (τ_i) (see Notations in §3.1), then all $\mathcal{V}^{(\tau_i)}$'s give a multi-indexed filtration of $\mathcal{V}(*D)$. We also prove that $\mathcal{V}^{(\tau_i)}$ are locally free with (naturally defined) logarithmic connections, and that they are characterized by the eigenvalues of the residues of the logarithmic connections. This is summarized as follow.

Theorem 1.1.2. *With the above notation, for each family (τ_i), there exist logarithmic extensions (locally free with a logarithmic connection) $(\mathcal{V}', \nabla')$ of $\mathcal{V}$ such that*

$$\{\text{real part of eigenvalues of } \mathrm{Res}_{D_i}\nabla'\} \subseteq \tau_i.$$

Moreover, $(\mathcal{V}^{(\tau_i)}, \nabla^{(\tau_i)})$ is universal among all of them.

The above theorem extends Deligne's theorem on canonical extensions; see for instance [**HTT08**, Theorem 5.2.17] and gives a more precise description of Deligne extensions of

$\mathcal{V}$. We call such $\mathcal{V}^{(\tau_i)}$ multi-indexed Deligne extensions of $\mathcal{V}$. In particular, under such notation, we have $\tilde{\mathcal{V}} = \mathcal{V}^{(\tau_i)}$ where $\tau_i = (-1, 0]$ for each i. It is worth noticing that $\mathcal{V}^{(\tau_i)}$ can also be constructed via L^2 cohomology and all multi-indexed Deligne extensions give a parabolic structure of $\mathcal{V}(*D)$.

For a fixed family (τ_i), the logarithmic connection $\nabla^{(\tau_i)}$ induces naturally a $V_0^D\mathcal{D}_X$-module structure for $\mathcal{V}^{(\tau_i)}$, where $V_i^D\mathcal{D}_X$ is the i-th term of the V-filtration (also known as the Kashiwara-Malgrange filtration) of $\mathcal{D}_X$ along D. All these extensions constitute a full sub-category of the category of left $V_0^D\mathcal{D}_X$-modules, denoted as $\mathrm{Conn}^{(\tau_i)}(X; D)$, which is equivalent to the category of local systems on $X \setminus D$. This is the Riemann-Hilbert correspondence for log $\mathcal{D}$-modules (see Corollary 3.3.12).

For vector bundles with integrable connection, their $\mathcal{D}_X$-duals are also vector bundles with integrable connections and are equal to their $\mathcal{O}_X$-duals. How about the same story but for $\mathcal{V}^{(\tau_i)}$? Indeed, the $V_0^D\mathcal{D}_X$-dual of $\mathcal{V}^{(\tau_i)}$ is $\mathcal{V}^{*(-\tau_i)}$, where $\mathcal{V}^*$ is the $\mathcal{O}$-dual of $\mathcal{V}$. See Theorem 4.2.16. Furthermore, the $V_0^D\mathcal{D}_X$-dual of $\mathcal{V}^{(\tau_i)}$ is exactly the $\mathcal{O}_X$-dual of $\mathcal{V}^{(\tau_i)}$.

By the well-known comparison theorem of Grothendieck-Deligne, we have a quasi-isomorphism

$$\mathrm{DR}\,(\mathcal{V}(*D)) \simeq Rj_*K[n],$$

where $j : X \setminus D \hookrightarrow X$ is the open embedding and K is the local system corresponding to $\mathcal{V}$ on $X \setminus D$. Along this line, we obtain a number of comparison formulas between $\mathrm{DR}_D(\mathcal{V}^{(\tau_i)})$ and some interesting perverse sheaves derived from K. See Theorem 4.4.9 and Theorem 5.3.13.

Since $\mathcal{V}(*D)$ is a regular holonomic $\mathcal{D}_X$-module, it is also interesting to study its submodules $\mathcal{D}_X\mathcal{V}^{(\tau_i)}$; that is $\mathcal{D}_X$-submodules of $\mathcal{V}(*D)$ generated by $\mathcal{V}^{(\tau_i)}$ for different (τ_i).

Theorem 1.1.3. *Asume that $D = \sum_{i=1}^r D_i$ with D_i irreducible. Let I be a subset of $\{1, ..., r\}$. For any r-tuple (τ_i) such that $\tau_i \le [-1, 0)$ for $i \in I$, and $(-1, 0] \le \tau_i$ for $i \notin I$,*

$$\mathcal{D}_X\mathcal{V}^{(\tau_i)} = \mathcal{D}_X\mathcal{V}^{(\tau_i')},$$

where $\tau_i' = [-1, 0)$ for $i \in I$, and $\tau_i' = (-1, 0]$ for $i \notin I$.

This theorem tells that $\mathcal{D}_X\mathcal{V}^{(\tau_i)}$ changes when τ_i goes across the critical point 0.[1] For instance, if $B = \sum_i a_i D_i$ is an integral divisor such that $a_i > 0$ for $i \in I$ and $a_i \leq 0$ for $i \notin I$, then $\mathcal{D}_X\mathcal{O}_X(B) = \mathcal{O}_X(*D_I)$, where $D_I = \sum_{i\in I} D_i$.

For a subset I, denote the $\mathcal{D}_X\mathcal{V}^{(\tau_i)}$ as in the above theorem by $\mathcal{V}_I$ The first extremal case is that $\mathcal{V}_\emptyset$ corresponds to the minimal extension of the local system by Riemann-Hilbert correspondence; hence also denoted by $\mathcal{V}_{min}$ (see Proposition 4.4.11). See also [**Bjö93**, Definition 3.1.12 and Proposition 3.1.13] for an alternative perspective. The other extreme is $\mathcal{V}_{\{1,\dots,r\}} = \mathcal{V}(*D)$ (see Proposition 4.1.4). It is also worth noticing that if 1 is not an eigenvalue for any local monodromies, then all $\mathcal{V}_I$ are the same. This point becomes clearer if one looks at the corresponding perverse sheaves.

If, moreover, suppose $\mathcal{V}$ underlies a polarizable variation of Hodge structure (PVHS), $V = (\mathcal{V}, F_\bullet, \mathbb{V})$, then the multi-index Deligne extensions $\mathcal{V}^{(\tau_i)}$ induces extensions of $F_\bullet$. To be precise,

$$F_\bullet^{(\tau_i)} := \mathcal{V}^{(\tau_i)} \cap j_* F_\bullet.$$

The above sheaf is possibly quite badly behaved in general (for instance, not even coherent). Therefore, we also assume that all the local monodromies of $\mathbb{V}$ along the local irreducible component of D are quasi-unipotent. This is a standard assumption made for Hodge modules; see [**Sai90**]. In fact, by the Monodromy Theorem, if the PVHS are Gauss-Manin connections of proper families, then this assumption is automatically fulfilled. We make this assumption for all PVHS used in this paper. Under this assumption, using the local unipotent reduction and Schmid's nilpotent orbit theorem, we prove the following local freeness result for $F_\bullet^{(\tau_i)}$.

Proposition 1.1.4. *$F_\bullet^{(\tau_i)}$ are sub-bundles of $\mathcal{V}^{(\tau_i)}$ for every (τ_i).*

The proof is inspired by some idea in [**Kol86**]. See Theorem 5.1.3. Furthermore, $F_\bullet^{(\tau_i)}$ satisfy the logarithmic Griffith transversality condition. See [**Sai90**, §3.b] for a different approach by using compatibility of V-filtrations and Hodge filtrations.

Inspired by Saito's idea of filtered $\mathcal{D}_X$-modules, we obtain a filtered comparison theorem.

[1]This is a phenomenon analogous to real Morse theory (if we go through a critical value, then the homotopy of the level set changes).

Theorem 1.1.5. *For every (τ_i), the natural morphism*

$$\mathrm{DR}_D(\mathcal{V}^{(\tau_i+1)}, F_\bullet^{(\tau_i+1)}) \xrightarrow{\overset{q.i.}{\simeq}} \mathrm{DR}\left((\mathcal{D}_X, F_\bullet) \otimes_{(V_0^D\mathcal{D}_X, F_\bullet)} (\mathcal{V}^{(\tau_i)}, F_\bullet^{(\tau_i)})\right)$$

is a filted quasi-isomorphism of filtered complexes of $\mathbb{C}_X$-modules.

For some special family (τ_i), the above theorem is contained in [**Sai90**, Proposition 3.11]. Its significance is that since for some special family (τ_i),

$$(\mathcal{D}_X, F_\bullet) \otimes_{(V_0^D\mathcal{D}_X, F_\bullet)} (\mathcal{V}^{(\tau_i)}, F_\bullet^{(\tau_i)})$$

underlies a mixed Hodge module (see §5.2 or [**Sai90**]), we can understand such mixed Hodge module in terms of relatively simple log-PVHS.

The second part of this paper is devoted to the construction of multiplier subsheaves and to the study of their properties in the algebraic category. This can be seen as the application of multi-indexed Deligne extensions to algebraic geometry.

We first need to discuss a Kawamata–Viehweg type vanishing theorem, which is essential for the construction of multiplier subsheaves. Suppose, for the moment, that (X, D) is a pair consisting of a smooth complex algebraic variety X and a reduced SNC divisor $D = \sum_{i=1}^r D_i$. Let V be a PVHS on the underlying analytic space of $X \setminus D$. Set

$$S(V) := F_{\mathrm{lowest}}\mathcal{V}.$$

Namely, $S(V)$ is the initial term in the Hodge filtration of V. If $B = \sum_{i=1}^r a_i D_i$ is an $\mathbb{R}$-divisor supported on D (zero coefficients are held in B), then we write

$$S(V)^B = F_{p(V)}^{((a_1-1,a_1],\ldots,(a_r-1,a_r])};$$

i.e., we make τ_i consistently left open and right closed unit intervals for all i and write the indices in terms of $\mathbb{R}$-divisors. For instance, if V is the trivial VHS, then

$$S(V)^B = \mathcal{O}_X(-\lfloor B \rfloor)$$

. The following $\mathbb{R}$-Kawamata–Viehweg type vanishing is proved.

Theorem 1.1.6. *Assume X is projective. Let L be an integral divisor and B an $\mathbb{R}$-divisor supported on D. Suppose $L - B$ is nef and big. Then*

$$H^q(X, S(V)^B \otimes \omega_X(L)) = 0$$

for $q > 0$.

The proof is greatly inspired by that of [**Kaw02**, Theorem 4.2] which deals with only geometric cases; see Theorem 6.3.2.

Now we come to the definition of multiplier subsheaves. Suppose Y is a smooth algebraic variety over $\mathbb{C}$, M a pure Hodge module strictly supported on Y, extending a generically defined PVHS V, and B an $\mathbb{R}$-divisor on Y. Then the multiplier subsheaf associated to M and B is

$$\mathcal{J}(M, B) := \mu_*(S(V)^{\mu^*B} \otimes \omega_{X/Y})),$$

where μ is a log resolution of supp(B)+sing(M), sing(M) is the locus where V fails to be defined and $\omega_{X/Y}$ is the relative canonical sheaf. This definition is independent of choices of log resolutions. They are named multiplier subsheaves because $\mathcal{J}(M, B)$ possess many similar properties as that of multiplier ideals and when M is the trivial Hodge module, we have

$$\mathcal{J}(M, B) = \mathcal{I}(B),$$

where $\mathcal{I}(B)$ is the multiplier ideal associated to B; see [**Laz04**] for the definition of multiplier ideals . One of its most important properties is that $\mathcal{J}(M, B)$ satisfies Nadel-type vanishing.

Theorem 1.1.7. *Assume Y is projective. If L is an integral divisor such that $L - B$ is nef and big, then*

$$H^q(Y, \mathcal{J}(M, B) \otimes \omega_Y(L)) = 0$$

for $q > 0$.

Since $\mathcal{J}(M, 0) = S(M)$ (see for instance [**Wu15**, Corollary 3.13]), the above theorem specializes to a Kawamata–Viehweg-type vanishing result for pure Hodge modules which was proved independently and simultaneously in [**Suh15**] and in [**Wu15**]; see for instance [**Wu15**, Theorem 1.4]. It also specializes to the Nadel vanishing theorem for multiplier ideals when M is the trivial Hodge module and Kollár vanishing for higher direct images of dualizing sheaves.

Kawamata proposed a relative version of Fujita's freeness conjecture in [**Kaw02**], and proved a criterion to guarantee the global generation involving the higher direct image of dualizing sheaves in the normal crossing case. Parallel to the geometric case, we get a similar criterion for a global generation problem involving $S(M)$ as an application of the above Nadel-type vanishing.

Theorem 1.1.8. *Let X be a smooth projective variety of dimension n and M a pure Hodge module strictly supported on X with* $\operatorname{sing}(M) = D$ *a reduced SNC divisor. Let L be an ample divisor on X, and $x \in X$ a point. Assume that for every klt pair (X, B_0), there exists an effective $\mathbb{Q}$-divisor B on X satisfying the following conditions:*

(i) $B \equiv \lambda\mathcal{L}$ *for some* $0 < \lambda < 1$*;*

(ii) $(X, B + B_0)$ *is lc at* x*;*

(iii) $\{x\}$ *is a log canonical center of* $(X, B + B_0)$.

Then the natural morphism

$$H^0(X, S(M) \otimes \omega_X(L)) \longrightarrow S(M) \otimes \omega_X(L)|_{\{x\}}$$

is surjective.

It is known from [EL93] and [Kaw97] that the assumption for kL always holds for $k \geq n+1$ and $n \leq 4$, from [AS95] for $k \geq \binom{n+1}{2}$ and n arbitrary. Therefore, we obtain the following corollary.

Corollary 1.1.9. *Let X be a smooth projective variety of dimension n and M be a pure Hodge module strictly supported on X with* $\operatorname{sing}(M) = D$ *a reduced SNC divisor. Let $\mathcal{L}$ be an ample line bundle on X. Then the locally free sheaf $S(M) \otimes \omega_X(L)$ is globally generated when $k \geq n+1$ if $n \leq 4$, or $k \geq \binom{n+1}{2}$ in general.*

CHAPTER 2

Preliminaries on $V_0^D\mathcal{D}_X$-modules

In this Chapter, the theory of log $\mathcal{D}$-modules (or $V_0^D\mathcal{D}_X$-modules) is introduced and discussed systematically, which recovers $\mathcal{D}$-modules theory in the usual sense when the divisor D is trivial.

2.1. $\mathcal{O}_X$-modules with Logarithmic Connections.

Let X be a complex manifold of dimension n, let D be a normal crossing divisor and let $\mathcal{M}$ be a sheaf of $\mathcal{O}_X$-module. An integrable logarithmic connection ∇ on $\mathcal{M}$ along D is a $\mathbb{C}$-linear morphism

$$\nabla : \mathcal{M} \longrightarrow \Omega^1_X(\log D) \otimes \mathcal{M} \tag{2.1.1}$$

satisfying the Lebniz rule and $\nabla^2 = 0$. When $D = 0$, we call ∇ an integrable connection for short.

Definition 2.1.2 (V-filtration of $\mathcal{D}_X$ along D)**.** The V-filtration on $\mathcal{D}_X$ along D indexed by $\mathbb{Z}$ is defined by

$$V_i^D\mathcal{D}_X = \{P \in \mathcal{D}_X | P\mathcal{I}_D^j \subset \mathcal{I}^{j-i} \text{ for any } j \in \mathbb{Z}\},$$

where $\mathcal{I}_D$ is the ideal sheaf of D and

$$\mathcal{I}_D^j = \begin{cases} \mathcal{I}_D^j & \text{if } j > 0 \\ \mathcal{O}_X & \text{otherwise} \end{cases}.$$

From definition, it is obvious that $V_0^D\mathcal{D}_X$ is a sheaf of ring (it is a nice sheaf of ring, for instance $V_0^D\mathcal{D}_X$ is a coherent sheaf of ring with noetherian stalks; the proof is similar to that of $\mathcal{D}_X$). Let $(z_1, z_2, \ldots, z_n)$ be a local chart of X on a open neighborhood U and assume that D is defined by $z_1 \cdots z_r = 0$ on U. Then we have

$$V_0^D\mathcal{D}_U = \mathcal{O}_U < z_1\partial_1, ..., z_r\partial_r, \partial_{r+1}, ..., \partial_n >$$

where $\partial_i = \frac{\partial}{\partial z_i}$ and for $j > 0$

$$V_{-j}^D \mathcal{D}_U = (z_1 \cdot \ldots \cdot z_r)^j V_0^D \mathcal{D}_U, \text{ and } V_j^D \mathcal{D}_U = \sum_{k=0}^{j} V_0^D \mathcal{D}_U \cdot \partial_i^k.$$

In particular, for any $j \in \mathbb{Z}$, we have $V_j^D \mathcal{D}_X|_{X \setminus D} = \mathcal{D}_{X \setminus D}$.

We have the following interpretation of left $V_0^D \mathcal{D}_X$-modules.

Lemma 2.1.3. *Let $\mathcal{M}$ be an $\mathcal{O}_X$-module. Giving a left $V_0^D \mathcal{D}_X$-module structure on $\mathcal{M}$ extending the $\mathcal{O}_X$-module structure is equivalent to giving an integrable logarithmic connection on $\mathcal{M}$ along D.*

2.2. $\otimes$ and $\mathcal{H}om$ over $\mathcal{O}_X$ and Side-changing Operations

Since $V_0^D \mathcal{D}_X$ is non-commutative, it is also interesting to consider right $V_0^D \mathcal{D}_X$-modules. It is well-know that the *Lie-derivation* gives a right $\mathcal{D}_X$-module structure for ω_X. In particular, ω_X is a right $V_0^D \mathcal{D}_X$-module. We denote the category of left $V_0^D \mathcal{D}_X$-module by $\mathrm{Mod}(V_0^D \mathcal{D}_X)$, and the category of right $V_0^D \mathcal{D}_X$-module by $\mathrm{Mod}(V_0^D \mathcal{D}_X^{\mathrm{op}})$.

Set $\Theta_X(\log D) := \mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_X, \Omega^1(\log D))$. Namely $\Theta(\log D)$ is the dual of the sheaf of logarithmic 1-forms. I The following lemma is easy to check (its proof is essentially the same as that of [HTT08, Proposition 1.2.9]).

Lemma 2.2.1. *Let $\mathcal{M}$, $\mathcal{M}' \in \mathrm{Mod}(V_0^D \mathcal{D}_X)$ and $\mathcal{N}$, $\mathcal{N}' \in \mathrm{Mod}(V_0^D \mathcal{D}_X^{op})$. Then*
(1) $\mathcal{M} \otimes_{\mathcal{O}} \mathcal{M}' \in \mathrm{Mod}(V_0^D \mathcal{D}_X)$; $(s \otimes s')\theta = \theta s \otimes s' + s \otimes \theta s'$,
(2) $\mathcal{M} \otimes_{\mathcal{O}} \mathcal{N} \in \mathrm{Mod}(V_0^D \mathcal{D}_X^{op})$; $\theta(s \otimes t) = -\theta s \otimes t + s \otimes t\theta$,
(3) $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{M}, \mathcal{M}') \in \mathrm{Mod}(V_0^D \mathcal{D}_X)$; $(\theta\phi)(s) = \theta(\phi(s)) - \phi(\theta s)$,
(4) $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{M}, \mathcal{N}) \in \mathrm{Mod}(V_0^D \mathcal{D}_X^{op})$; $(\phi\theta)(s) = \phi(s)\theta + \phi(\theta s)$,
(5) $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{N}, \mathcal{N}') \in \mathrm{Mod}(V_0^D \mathcal{D}_X)$; $(\phi\theta)(s) = -\phi(s)\theta + \phi(s\theta)$,
where $\theta \in \Theta(\log D)$.

This lemma can be treated as generalizations of product rules and chain rules for differentiations. Because of Oda's rule (see for instance [HTT08, Remark 1.2.10]), the other two scenarios are excluded from the above lemma.

By Lemma 2.2.1, we can easily get the following corollary.

Corollary 2.2.2. *Let $\mathcal{M}$, $\mathcal{M}' \in \mathrm{Mod}(V_0^D\mathcal{D}_X)$ and $\mathcal{N} \in \mathrm{Mod}(V_0^D\mathcal{D}_X^{op})$. Then we have canonical isomorphisms*

$$(\mathcal{M} \otimes_{\mathcal{O}} \mathcal{N}) \otimes_{V_0^D\mathcal{D}_X} \mathcal{M}' \simeq \mathcal{N} \otimes_{V_0^D\mathcal{D}_X} (\mathcal{M} \otimes_{\mathcal{O}} \mathcal{M}') \simeq (\mathcal{M}' \otimes_{\mathcal{O}} \mathcal{N}) \otimes_{V_0^D\mathcal{D}_X} \mathcal{M}$$

$$m \otimes n \otimes m' \mapsto n \otimes m \otimes m' \mapsto m' \otimes n \otimes m,$$

as $\mathbb{C}$-modules.

If B is an integral divisor supported on D, then $\mathcal{O}_X(B)$ is a left $V_0^D\mathcal{D}_X$-module. By tensor product rules, we have that $\omega_X(B) = \mathcal{O}_X(B) \otimes \omega_X$ is a right $V_0^D\mathcal{D}_X$-module. In particular $\omega_X(D)$ is a right $V_0^D\mathcal{D}_X$-module. Therefore, by Lemma 2.2.1 we have functors

$$\omega_X(D) \otimes \bullet : \mathrm{Mod}(V_0^D\mathcal{D}_X) \longrightarrow \mathrm{Mod}(V_0^D\mathcal{D}_X^{\mathrm{op}})$$

and

$$\bullet \otimes \omega_X^{-1}(-D) = \mathcal{H}om_{\mathcal{O}_X}(\omega_X(D), \bullet) : \mathrm{Mod}(V_0^D\mathcal{D}_X^{\mathrm{op}}) \longrightarrow \mathrm{Mod}(V_0^D\mathcal{D}_X).$$

It is easy to check that these two functors are quasi-inverse to each other. Hence, we have proved that $\omega_X(D) \otimes \bullet$ induces an equivalence between $\mathrm{Mod}(V_0^D\mathcal{D}_X)$ and $\mathrm{Mod}(V_0^D\mathcal{D}_X^{\mathrm{op}})$. They are side-changing operations.

2.3. Filtered $V_0^D\mathcal{D}_X$-modules

The order filtration of $\mathcal{D}_X$ (see for instance [**HTT08**, §1])) induces an order filtration of $V_0^D\mathcal{D}_X$, denoted by $F_\bullet V_0^D\mathcal{D}_X$. A filtered (right or left) $V_0^D\mathcal{D}_X$-module is a pair $(\mathcal{M}, F_\bullet)$ with a $\mathcal{D}_X$-module $\mathcal{M}$ and an inceasing filtration $F_\bullet$ (bounded from below) of $\mathcal{M}$ which is compatible with the order filtration of $V_0^D\mathcal{D}_X$, i.e.

$$F_pV_0^D\mathcal{D}_X \cdot F_l\mathcal{M} (\text{or } F_l\mathcal{M} \cdot F_pV_0^D\mathcal{D}_X) \subset F_{p+l}\mathcal{M}$$

for all $p, l \in \mathbb{Z}$. The filtration is said to be coherent if

$$F_pV_0^D\mathcal{D}_X \cdot F_l\mathcal{M} (\text{or } F_l\mathcal{M} \cdot F_pV_0^D\mathcal{D}_X) = F_{p+l}\mathcal{M}$$

for $l \gg 0$ and $p \in \mathbb{Z}_{>0}$. Then obviously,

Lemma 2.3.1. *$F_\bullet$ is coherent if and only if $\mathrm{Gr}^F\mathcal{M}$ is coherent over $\mathrm{Gr}^F V_0^D\mathcal{D}_X$.*

It is easy to see that we have the canonical isomorphism

$$\mathrm{Gr}^F V_0^D\mathcal{D}_X \simeq \mathrm{Sym}(\Theta_X(\log D))$$

as $\mathscr{O}_X$-algebras.

We can also define filtered side-changing operations. For a filtered left $\mathcal{D}_X$-module $(\mathcal{M}, F_\bullet)$, define

$$F_\bullet(\omega_X(D) \otimes \mathcal{M}) = \omega_X(D) \otimes F_{\bullet-n}\mathcal{M}.$$

Clearly, the filtered side-changing operations induce an equivalence between filtered left and right $V_0^D\mathcal{D}_X$-modules.

It is worth mentioning that if $D = 0$, all the above results (side-changing operations and Lemma 2.1.3 and the above lemma etc.) are just that for $\mathcal{D}_X$-modules.

2.4. Log-de Rham and Log-Spencer Complexes

Let $M \in \mathrm{Mod}(V_0^D\mathcal{D}_X)$. By Lemma 2.1.3, from (2.1.1), we obtain a $\mathbb{C}$-linear complex,

$$\mathrm{DR}_D(M) \coloneqq [0 \longrightarrow \mathcal{M} \xrightarrow{\nabla} \Omega^1_X(\log D) \otimes \mathcal{M} \xrightarrow{\nabla^1} \ldots \xrightarrow{\nabla^{n-1}} \Omega^n(\log D) \otimes \mathcal{M} \longrightarrow 0][n],$$

which is called the logarithmic de Rham complex of $\mathcal{M}$ along D. When $D = 0$, the above complex is the usual de Rham complex of M, denoted by $\mathrm{DR}(M)$.

The left $V_0^D\mathcal{D}_X$-module structure of $V_0^D\mathcal{D}_X$ gives us $\mathrm{DR}_D(V_0^D\mathcal{D}_X)$, the log de Rham complex of $V_0^D\mathcal{D}_X$. The right $V_0^D\mathcal{D}_X$-module structure of $V_0^D\mathcal{D}_X$ makes $\mathrm{DR}_D(V_0^D\mathcal{D}_X)$ be a complex of right $V_0^D\mathcal{D}_X$-modules. Therefore, we have

$$\mathrm{DR}_D(M) \simeq \mathrm{DR}_D(V_0^D\mathcal{D}_X) \otimes_{V_0^D\mathcal{D}_X} M.$$

This means that $\mathrm{DR}_D(\bullet)$ defines a functor from the derived category of $\mathrm{Mod}(V_0^D\mathcal{D}_X)$ to the derived category of $\mathbb{C}$-sheaves.

Since $V_0^D\mathcal{D}_X$ is a $V_0^D\mathcal{D}_X$-bimodule, after side-changing (use the right $V_0^D\mathcal{D}_X$-module structure of $V_0^D\mathcal{D}_X$)

$$V_0^D\mathcal{D}_X \otimes_{\mathcal{O}} \omega_X^{-1}(-D)$$

has two compatible left $V_0^D\mathcal{D}_X$-module structures. One is induced from the left $V_0^D\mathcal{D}_X$-module structure of $V_0^D\mathcal{D}_X$, denoted by $(V_0^D\mathcal{D}_X \otimes_{\mathcal{O}} \omega_X^{-1}(-D))_{\mathrm{triv}}$. The other is induced from the right $V_0^D\mathcal{D}_X$-module structure of $V_0^D\mathcal{D}_X$, denoted by $(V_0^D\mathcal{D}_X \otimes_{\mathcal{O}} \omega_X^{-1}(-D))_{\mathrm{tens}}$. Then we have an involution (see also [Sai88, Lemme 2.4.2])

$$(V_0^D\mathcal{D}_X \otimes_{\mathcal{O}} \omega_X^{-1}(-D))_{\mathrm{triv}} \longrightarrow (V_0^D\mathcal{D}_X \otimes_{\mathcal{O}} \omega_X^{-1}(-D))_{\mathrm{tens}} \tag{2.4.1}$$

$$P \otimes \phi \mapsto P \cdot (1 \otimes \phi).$$

The involution map interchanges the two left $V_0^D\mathcal{D}_X$-module structures. Therefore, the involution induces a natural isomorphism

$$\mathrm{DR}_D((V_0^D\mathcal{D}_X \otimes_{\mathcal{O}} \omega_X^{-1}(-D))_{\mathrm{triv}})_{\mathrm{tens}} \simeq \mathrm{DR}_D((V_0^D\mathcal{D}_X \otimes_{\mathcal{O}} \omega_X^{-1}(-D))_{\mathrm{tens}})_{\mathrm{triv}}.$$

of complexes of left $V_0^D\mathcal{D}_X$-modules. In other words, we have an isomorphism

$$\mathrm{DR}_D(V_0^D\mathcal{D}_X) \otimes_{\mathcal{O}} \omega_X^{-1}(-D) \simeq \mathrm{DR}_D((V_0^D\mathcal{D}_X \otimes_{\mathcal{O}} \omega_X^{-1}(-D))_{\mathrm{tens}}) \tag{2.4.2}$$

between complexes of left $V_0^D\mathcal{D}_X$-modules.

The contraction map gives an isomorphism

$$\Omega_X^k(\log D) \otimes_{\mathcal{O}} \omega^{-1}(-D) \longrightarrow \wedge^{n-k}\Theta_X(\log D). \tag{2.4.3}$$

It induces an isomorphism

$$\Omega_X^k(\log D) \otimes_{\mathcal{O}} (V_0^D\mathcal{D}_X \otimes_{\mathcal{O}} \omega_X^{-1}(-D))_{\mathrm{tens}} \simeq V_0^D\mathcal{D}_X \otimes_{\mathcal{O}} \wedge^{n-k}\Theta_X(\log D) \tag{2.4.4}$$

of left $V_0^D\mathcal{D}_X$-modules. The $(\bullet)_{\mathrm{tens}}$ means we use the $\mathcal{O}_X$-structure induced from the tensor product for the first tensor product over $\mathcal{O}_X$ in (2.4.4). Moreover, we can see $(V_0^D\mathcal{D}_X \otimes_{\mathcal{O}} \omega_X^{-1}(-D))_{\mathrm{triv}}$ gives the left $V_0^D\mathcal{D}_X$-module structure of

$$\Omega_X^k(\log D) \otimes_{\mathcal{O}} (V_0^D\mathcal{D}_X \otimes_{\mathcal{O}} \omega_X^{-1}(-D))_{\mathrm{tens}}.$$

It is easy to check that the above isomorphism induces an isomorphism of complexes of left $V_0^D\mathcal{D}_X$-modules

$$\mathrm{SP}_D(V_0^D\mathcal{D}_X) := [0 \to V_0^D\mathcal{D}_X \otimes_{\mathcal{O}} \wedge^n\Theta_X(\log D) \to ... V_0^D\mathcal{D}_X \otimes_{\mathcal{O}} \Theta_X(\log D) \longrightarrow V_0^D\mathcal{D}_X \to 0],$$

with the first term of degree $-n$, which is called the logarithmic *Spenser* complex of $V_0^D\mathcal{D}_X$. Since $\mathrm{SP}_D(V_0^D\mathcal{D}_X)$ is induced from (2.4.4), we obtain an isomorphism

$$\mathrm{SP}_D(V_0^D\mathcal{D}_X) \simeq \mathrm{DR}_D((V_0^D\mathcal{D}_X \otimes_{\mathcal{O}} \omega_X^{-1}(-D))_{\mathrm{tens}}) \tag{2.4.5}$$

as complexes of left $V_0^D\mathcal{D}_X$-modules. Therefore, by (2.4.2) and (2.4.5) we obtain an isomorphism between the *de Rham* complex and the *Spencer* complex.

Proposition 2.4.6. *We have isomorphisms*

$$\mathrm{SP}_D(V_0^D\mathcal{D}_X) \simeq \mathrm{DR}_D(V_0^D\mathcal{D}_X) \otimes_{\mathcal{O}} \omega_X^{-1}(-D) \simeq DR_D((V_0^D\mathcal{D}_X \otimes_{\mathcal{O}} \omega_X^{-1}(-D))_{\mathrm{tens}}),$$

of complexes of left $V_0^D\mathcal{D}_X$-modules.

Suppose $\mathcal{N} \in \mathrm{Mod}(V_0^D \mathcal{D}_X^{\mathrm{op}})$. Define

$$\mathrm{SP}_D(\mathcal{N}) \coloneqq \mathcal{N} \otimes_{V_0^D \mathcal{D}_X} \mathrm{SP}_D(V_0^D \mathcal{D}_X),$$

the logarithmic Spenser complex of $\mathcal{N}$. By Proposition 2.4.6, we have isomorphisms

$$\mathrm{SP}_D(\mathcal{N}) \simeq \mathcal{N} \otimes_{V_0^D \mathcal{D}_X} \mathrm{DR}_D((V_0^D \mathcal{D}_X \otimes_{\mathcal{O}} \omega_X^{-1}(-D))_{\mathrm{tens}}) \simeq \mathrm{DR}_D(\mathcal{N} \otimes_{\mathcal{O}} \omega_X^{-1}(-D))$$

of complexes of $\mathbb{C}$-sheaves. Symmetrically, replacing $\mathcal{N}$ by $\mathcal{M} \otimes_{\mathcal{O}} \omega_X(D)$ (because of side-changing), we also have an isomorphism

$$\mathrm{DR}_D(\mathcal{M}) \simeq \mathrm{SP}_D(\mathcal{M} \otimes_{\mathcal{O}} \omega_X(D))$$

of complexes of $\mathbb{C}$-sheaves.

Now assume $\mathcal{M}^\bullet$ ($\mathcal{N}^\bullet$ resp.) is complex in $D^b(V_0^D \mathcal{D}_X)$ ($D^b(V_0^D \mathcal{D}_X^{\mathrm{op}})$ resp.), the bounded derived category of left (right resp.) $V_0^D \mathcal{D}_X$-modules. Define

$$\mathrm{DR}_D(\mathcal{M}^\bullet) \coloneqq \mathrm{DR}_D(V_0^D \mathcal{D}_X) \overset{\mathbf{L}}{\otimes}_{V_0^D \mathcal{D}_X} \mathcal{M}^\bullet,$$

and

$$\mathrm{SP}_D(\mathcal{N}^\bullet) \coloneqq \mathcal{N}^\bullet \overset{\mathbf{L}}{\otimes}_{V_0^D \mathcal{D}_X} \mathrm{SP}_D(V_0^D \mathcal{D}_X).$$

Since every term of $\mathrm{DR}_D((V_0^D \mathcal{D}_X)$ and $\mathrm{SP}_D((V_0^D \mathcal{D}_X)$ is locally free over $V_0^D \mathcal{D}_X$,

$$\mathrm{DR}_D(\mathcal{M}^\bullet) = \mathrm{DR}_D((V_0^D \mathcal{D}_X) \otimes_{V_0^D \mathcal{D}_X} \mathcal{M}^\bullet,$$

and

$$\mathrm{SP}_D(\mathcal{N}^\bullet) = \mathcal{M}^\bullet \otimes_{V_0^D \mathcal{D}_X} \mathrm{SP}_D((V_0^D \mathcal{D}_X).$$

Therefore, by Proposition 2.4.6 we obtain the following isomorphism.

Corollary 2.4.7. *There exists a functorial isomorphism*

$$\mathrm{DR}_D(\mathcal{M}^\bullet) \simeq \mathrm{SP}_D(\mathcal{M}^\bullet \otimes_{\mathcal{O}} \omega_X(D)).$$

In other words, we have a commutative diagram

$$\begin{array}{ccc} D^b(V_0^D \mathcal{D}_X) & & \\ {\scriptstyle \bullet \otimes \omega_X(D)} \downarrow & \searrow^{\mathrm{DR}_D(\bullet)} & \\ D^b(V_0^D \mathcal{D}_X^{\mathrm{op}}) & \xrightarrow[\mathrm{SP}_D(\bullet)]{} & D^b(\mathbb{C}). \end{array}$$

2.5. Residue Maps for Left $V_0^D\mathcal{D}_X$-modules

To define residue maps, we assume that D is not only normal crossing but also globally

$$D = \sum_{i=1}^{r} D_i$$

with each D_i irreducible and nonsingular (following conventions from algebraic geometry, such divisors are called reduced simple normal crossings (SNC) divisors). Since $\mathcal{O}_X$ is a left $V_0^D\mathcal{D}_X$-module, we obtain the logarithmic de Rham complex,

$$\mathrm{DR}_D(\mathcal{O}_X) = [0 \longrightarrow \mathcal{O}_X \xrightarrow{d} \Omega^1_X(\log\ D) \longrightarrow ... \longrightarrow \Omega^n(\log\ D) \longrightarrow 0][n].$$

The k-th Poincaré residue map

$$\Omega^k(\log\ D) \xrightarrow{\beta_k} \Omega^{k-1}_{D_i}(\log\ (D - D_i)|_{D_i})$$

is defined by

$$\beta_k(\phi) = \beta_k(\phi_1 + \phi_2 \wedge \frac{dz_i}{z_i}) = \phi_2|_{D_i},$$

where z_i is the local defining function of D_i and $\phi_1 + \phi_2 \wedge \frac{dz_i}{z_i}$ is the local decomposition of the section ϕ of $\Omega^k(\log\ D)$. See also [**EV92**, §2]. It is obvious that the above Poincaré residue map induces a short exact sequence of $\mathcal{O}$-modules,

$$0 \longrightarrow \Omega^k(\log\ (D - D_i)) \longrightarrow \Omega^k(\log\ D) \xrightarrow{\beta_k} \Omega^{k-1}_{D_i}(\log\ (D - D_i)|_{D_i}) \longrightarrow 0.$$

Let $(\mathcal{M}, \nabla)$ be a $\mathcal{O}_X$-module with an integrable logarithmic connection (that means $\mathcal{M}$ is a left $V_0^D\mathcal{D}_X$-module if we do not want to emphasis ∇). The Poincaré residue map induces a commutative diagram as follow and the residue maps for $\mathcal{M}$, denoted by $\mathrm{Res}_{D_i}\nabla^k$,

$$\begin{array}{ccc}
\Omega^k(\log\ (D - D_i)) \otimes \mathcal{M} & \xrightarrow{\nabla^k} & \Omega^{k+1}_X(\log\ D) \otimes \mathcal{M} \\
\downarrow & & \downarrow \beta_{k+1} \\
\Omega^k_{D_i}(\log\ (D - D_i)|_{D_i}) \otimes \mathcal{M} & \xrightarrow{\mathrm{Res}_{D_i}\nabla^k} & \Omega^k_{D_i}(\log\ (D - D_i)|_{D_i}) \otimes \mathcal{M}
\end{array} \tag{2.5.1}$$

for each i. Here $\dashrightarrow$ denotes that the morphism is induced by the commutative diagram. To be precise, the residue map $\mathrm{Res}_{D_i}\nabla^k$ is induced by $\beta_{k+1} \circ \nabla^k$ It is clear that it factors through $\Omega^k_{D_i}(\log\,(D-D_i)|_{D_i})\otimes\mathcal{M}$ and

$$\mathrm{Res}_{D_i}\nabla^k\in\mathrm{End}_{\mathcal{O}_{D_i}}(\Omega^k_{D_i}(\log\,(D-D_i)|_{D_i})\otimes\mathcal{M}).$$

In particular, when $k=0$, we get $\mathrm{Res}_{D_i}\nabla\in\mathrm{End}_{\mathcal{O}_{D_i}}(\mathcal{M}|_{D_i})$. From (2.5.1), it is easy to see that

$$\mathrm{Res}_{D_i}\nabla^k=(-1)^k\cdot id\wedge\mathrm{Res}_{D_i}\nabla.$$

If we fix coordinates $(z_1,...,z_n)$ on a neighborhood U' such that $D|'_U=(z_1\cdot...\cdot z_r=0)$, then locally we can define an integrable logarithmic connection for $\mathcal{M}|_{D_i}$ induced from the following diagram,

$$\begin{array}{ccc}\Omega^1_X(\log\,D)\otimes\mathcal{M} & \xrightarrow{\nabla^1} & \Omega^2_X(\log\,D)\otimes\mathcal{M}\\ \downarrow{\scriptstyle\beta_1} & & \downarrow{\scriptstyle\beta_2}\\ \mathcal{M}|_{D_i} & \overset{\nabla|_{D_i}}{\dashrightarrow} & \Omega^1_{D_1}(\log\,(D-D_i)|_{D_i})\otimes\mathcal{M}|_{D_i}.\end{array}\tag{2.5.2}$$

To be precise, for a section $\overline{m}\in\mathcal{M}|_{D_1}(U)$, the connection is defined by

$$\nabla|_{D_i}(\overline{m})=\beta_2(\nabla^1(\frac{dz_i}{z_i}\otimes m)),$$

where m is a lift of $\overline{m}$ in $\mathcal{M}(U)$. It is obvious that the definition of $\nabla|_{D_i}$ does not depend on both the lift m and the local defining function of D_i. It is also not hard to see $\nabla|_{D_i}$ is integrable. Therefore, we obtain an $\mathcal{O}_{D_i}$-module with an integrable logarithmic connection $(\mathcal{M}|_{D_i},\nabla|_{D_i})$ on D_i along $D^{(i)}$, where $D^{(i)}=(D-D_i)|_{D_i}$.

2.6. Logarithmic Extentions and Minimal Polynomials of Residue Maps

Now assume that $(\mathcal{E},\nabla)$ is a holomorphic vector bundle of finite rank on X with an integrable logarithmic connection along D. In this case, $\mathcal{E}|_{X\setminus D}$ is a flat holomorphic vector bundle on U. In some situation, such a pair (or just $\mathcal{E}$ if ∇ is obvious) is called a logarithmic extension of $\mathcal{E}|_{X\setminus D}$ if we want to emphasis $\mathcal{E}|_{X\setminus D}$.

Example 2.6.1. Let $B=\sum\alpha_iD_i$ be an integral divisor supported on D. We know that $(\mathcal{O}_X(B),d)$ is a logarithmic extension of $\mathcal{O}_{X\setminus D}$ with the holomorphic differential d. The residue of $\mathcal{O}_X(B)$ along D_i is $-\alpha_i\cdot\mathrm{Id}$.

Fix coordinates $(z_1,...,z_n)$ on a open neighborhood U' so that $D=(z_1\cdot...\cdot z_r=0)$. By diagram (2.5.2), $(\mathcal{E}|_{D_i},\nabla|_{D_i})$ is a holomorphic vector bundle on D_i with a logarithmic connection along $D^{(i)}$ on $U' \cap D_i$. By Lemma 2.2.1(3), $\mathcal{E}\mathrm{nd}_{\mathcal{O}_{D_i}}(\mathcal{E}|_{D_i})$ has an induced integrable logarithmic connection $\nabla^{(i)}$ along $D^{(i)}$ on $U' \cap D_i$; that is

$$(\nabla^{(i)}\phi)e = (\mathrm{id} \otimes \phi)(\nabla|_{D_i}(e)) - \nabla|_{D_i}(\phi(e))$$

for a test section e.

Lemma 2.6.2. *There exists a polynomial $b(s) \in \mathbb{C}[s]$ such that*

$$b(Res_{D_i}\nabla) = 0$$

for each i.

Proof. With the above local chart, by local calculation (see for instance the proof of [**HTT08**, Proposition 5.2.15]), we know that $\mathrm{Res}_{D_i}\nabla|_{U\cap D_i}$ is $\nabla^{(i)}$-flat, which means

$$\nabla^{(i)}(\mathrm{Res}_{D_i}\nabla|_U) = 0.$$

Hence, $\mathrm{Res}_{D_i}\nabla|_{U\cap D_i\setminus D^{(i)}}$ is locally constant with respect to $\nabla^{(i)}$. Take $b_i(s)$ to be the minimal polynomial of $(\mathrm{Res}_{D_i}\nabla)(z) \in \mathrm{End}_{\mathbb{C}}(\mathcal{E}(z))$ for some point x on $U \cap D_i \setminus D^{(i)}$, where $\mathcal{E}(z)$ is the fiber of $\mathcal{E}$ at z. Covering X by a family of such U's, we obtain

$$b_i(\mathrm{Res}_{D_i}\nabla|_{D_i\setminus D^{(i)}}) = 0.$$

After picking a test section $e \in \Gamma(D_i, \mathcal{E}|_{D_i})$, set

$$e' = b_i(\mathrm{Res}_{D_i}\nabla)e \in \Gamma(D_i, \mathcal{E}|_{D_i}).$$

Hence $e'|_{D_i\setminus D^{(i)}} = 0$. But $\mathcal{E}|_{D_i}$ is torsion-free, which tells us $e' = 0$. Therefore,

$$b_i(\mathrm{Res}_{D_i}\nabla) = 0.$$

□

It is not hard to see that $b_i(s)$ in the above proof is the minimal polynomial of $\mathrm{Res}_{D_i}\nabla$. We denote it by $b_{D_i}(s)$. Clearly, the roots of $b_{D_i}(s)$ are the eigenvalues of $(\mathrm{Res}_{D_i}\nabla)(z)$ for any $z \in X$ (which means eigenvalues of $\mathrm{Res}_{D_i}\nabla$ are locally constant). Therefore, it makes sense discuss eigenvalues of the residue map along each irreducible component of D for any logarithmic extensions globally.

If $B = \sum \alpha_i D_i$ is a divisor supported on D, then by Lemma 2.2.1(1)

$$\mathcal{E}(B) = \mathcal{E} \otimes \mathcal{O}_X(B)$$

has an integrable logarithmic connection ∇^B. It is not hard to see that the residue is given by

$$\mathrm{Res}_{D_i}\nabla^B = \mathrm{Res}_{D_i}\nabla - \alpha_i \cdot \mathrm{Id}. \tag{2.6.3}$$

See [**EV92**, Lemma 2.7] for detail. Thus, the roots of minimal polynomial of $\mathrm{Res}_{D_i}\nabla^B$ are shifted by α_i to the right.

CHAPTER 3

Multi-indexed Deligne Extensions

In this chapter, we construct a special kind of log-$\mathcal{D}$-modules from local systems defined on the complement of a normal crossing divisor, called multi-indexed Deligne extensions. We begin by local extensions first.

3.1. Local extensions on Δ^n

Notation. We use τ to denote an interval $[\alpha, \alpha+1)$ or $(\alpha, \alpha+1]$ for some $\alpha \in \mathbb{R}$, and $(\tau_i)_{i=1}^r = (\tau_1, ..., \tau_i)$ an r-tuple of such intervals for some $r \in \mathbb{Z}$. When r is clear from the context, simply write (τ_i) for short. We say $\tau < \tau'$ if interval τ is on the left side of interval τ' (we also say $\tau' \geq \tau$ if τ' is not less than τ) and $\beta < \tau$ if β is on the left side of (not in) the interval τ for some $\beta \in \mathbb{R}$ (we also say $\beta \geq \tau$ if β is not less than τ). This order gives a partial order on the set of all $(\tau_i)_{i=1}^r$.

Suppose $X = \Delta^n$ is the product of n-copies of unit disks with a complex coordinate system $(z_1, \ldots, z_n)$ and $U = \Delta^{*r} \times \Delta^{n-r}$ with the open embedding $j : U \hookrightarrow X$. Let L be a local system of rank m on U. From L we get the induced monodromy representation

$$\rho_L : \pi_1(U) \longrightarrow GL(V, \mathbb{C}),$$

where V is any fiber of L, and the induced vector bundle $\mathcal{V} = L \otimes_{\mathbb{C}} \mathcal{O}_U$, with a natural flat connection ∇. Indeed, for $s \in \mathcal{V}$, $s = \sum f_i c_i$ in a basis c_i of a local trivialization of L,

$$\nabla(s) = \sum df_i \otimes c_i \in \Omega^1_X \otimes \mathcal{V}.$$

Use γ_i to denote the monodromy operators along z_i; that is the image of loops around z_i under ρ_L.

Take the universal covering of U with the induced complex structure,

$$\begin{array}{c} \widetilde{X} := \mathbb{H}^r \times \Delta^{n-r} \\ \Big\downarrow \exp \\ U = \Delta^{*r} \times \Delta^{n-r}, \end{array}$$

where $\mathbb{H}^r$ is the product of r-copies of upper half planes with coordinates $(w_1, \ldots, w_r)$, so that for $i = 1, \ldots, r$, we have $z_i = e^{-2\pi\sqrt{-1}w_i}$.

Under this setting, we know that $\exp^{-1} L$ is trivial on $\widetilde{X}$. Choose a basis $\{c_j\}_{j=1}^m$ of $H^0(\widetilde{X}, \exp^{-1} L) \simeq V$. Then we have

$$c_j(w_1, \ldots, w_i + 1, \ldots, w_r, z_{r+1}, \ldots, z_n) = \gamma_i \cdot c_j(w_1, \ldots, w_i, \ldots, w_r, z_{r+1}, \ldots, z_n). \tag{3.1.1}$$

Lemma 3.1.2. *Assume (τ_i) is a fixed r-tuple of intervals. There exists a unique $\Gamma_j^{\tau_j} \in \mathfrak{gl}(V, \mathbb{C})$ for each $j = 1, \ldots, r$, such that*

(1) $\exp(-2\pi\sqrt{-1}\Gamma_i^{\tau_i}) = \gamma_i$,

(2) $\mathrm{Re}\{\text{all eigenvalues of } \Gamma_i^{\tau_i}\} \subset \tau_i$,

(3) $\Gamma_i^{\tau_i} (i = 1, \ldots, r)$ *mutually commute.*

Proof. Since $\gamma_\bullet$ mutually commute, there exists a decomposition

$$V = \oplus V_k$$

such that $\gamma_j(V_k) \subset V_k$ and $\gamma_j|_{V_k}$ has a single eigenvalue for every pair (j, k). By looking at each V_k, it suffices to assume that each γ_j has a single eigenvalue. Suppose the Jordan decomposition of γ_j for all j are

$$\gamma_j = \gamma_{j,s}\gamma_{j,u},$$

where $\gamma_{j,s}$ is the semi-simple part of γ_j and $\gamma_{j,u}$ is the unipotent part of γ_j.

Under such setting, $\log(\gamma_{j,u})$ exists and

$$\log(\gamma_{j,u}) = (\gamma_{j,u} - \mathrm{Id}) - \frac{(\gamma_{j,u} - \mathrm{Id})^2}{2} + \ldots.$$

The above is a finite sum, because $\gamma_{j,u}$ are unipotent. $\log(\gamma_{j,s})$ also exist, and are unique up to $2n\pi\sqrt{-1}$. Hence, there exist such $\Gamma_j^{\tau_j}$ uniquely because of condition (2). The third condition can also be satisfied because of the assumption that each γ_j has a single eigenvalue. □

For a fixed r-tuple (τ_i), we define

$$s_j = e^{\sum_{i=1}^r \Gamma_i^{\tau_i} \log\, z_i} \cdot c_j, \tag{3.1.3}$$

for $j = 1, \ldots, m$, where $\log z_i = -2\pi\sqrt{-1}w_i$. By (3.1.1) and Lemma 3.1.2(2), each s_j descends to a global holomorphic section of $\mathcal{V}$ on U. Namely, $\{s_j\}_{j=1}^m$ gives a trivialization of $\mathcal{V}$. Then we define,

$$\mathcal{V}^{(\tau_i)} := \oplus_{j=1}^m \mathcal{O}_X s_j, \tag{3.1.4}$$

which is naturally a subsheaf of $j_*\mathcal{V}$. By definition, all s_j are singular along z_i $(i = 1, \ldots, r)$ as sections of $j_*\mathcal{V}$. Moreover, the connection ∇ of $\mathcal{V}$ on U induces an integral logarithmic connection $\nabla^{(\tau_i)}$ of $\mathcal{V}^{(\tau_i)}$ on X along $D = (z_1 \cdots z_r = 0)$ by

$$\nabla^{(\tau_i)} s_j = \nabla^{(\tau_i)} e^{\sum_{i=1}^r \Gamma_i^{\tau_i} \log\, z_i} \cdot c_j = \sum_{i=1}^r \frac{dz_i}{z_i} \otimes \Gamma_i^{\tau_i} \cdot s_j \tag{3.1.5}$$

Thanks to Lemma 3.1.2(3), $\nabla^{(\tau_i)}$ is integrable. Therefore, we see that $\mathcal{V}^{(\tau_i)}$ is a logarithmic extension of $\mathcal{V}$. We also see

$$\mathrm{Res}_{D_i}(\nabla^{(\tau_i)}) = \Gamma_i^{\tau_i}$$

, where D_i is the divisor defined by $z_i = 0$. Moreover, Since $\Gamma_i^{\tau_i} + k_i \cdot \mathrm{Id} = \Gamma_i^{\tau_i + k_i}$ and $z_i^{k_i} = e^{k_i \log\, z_i}$, we obtain

$$\mathcal{V}^{(\tau_i)}(-D') = \mathcal{V}^{(\tau_i + k_i)},$$

where $D' = \sum_{i=1}^r k_i D_i$.

By construction, we know that $(\bullet)^{(\tau_i)}$ is functorial. In particular, if L_1 is a sub-local system of L, then we have a short exact sequence of $V_0^D \mathcal{D}_X$-modules,

$$0 \longrightarrow \mathcal{V}_1^{(\tau_i)} \longrightarrow \mathcal{V}^{(\tau_i)} \longrightarrow (\frac{\mathcal{V}}{\mathcal{V}_1})^{(\tau_i)} \longrightarrow 0, \tag{3.1.6}$$

where $\mathcal{V}_1 = L_1 \otimes_{\mathbb{C}} \mathcal{O}_U$.

3.2. Sheaves of Sections of Moderate Growth

The local construction of $\mathcal{V}^{(\tau_i)}$ depends on the choice of coordinate systems of Δ^n. We need a more intrinsic way to define extensions of $\mathcal{V}$ globally. In this section, I will recall the construction of Deligne's meromorphic extension of $\mathcal{V}$ following [**Bjö93**].

Suppose that X is a complex manifold of dimension n with a normal crossing divisor $D = \sum D_i$ in this section.

Definition 3.2.1 (Good Coverings)**.** An open covering $\{U_\alpha\}$ of $X \setminus D$ is said to be good if the following conditions are satisfied

(1) Every U_α is simply connected
(2) For every relatively compact subset K of X, there are only finitely many U_α having non-empty intersections with K.

Remark 3.2.2. The definition of good coverings is not the paracompactness of $X \setminus D$, because K in condition (2) is a subset of X but not of $X \setminus D$. Furthermore, good coverings for $X \setminus D$ always exist. To be precise, take a local chart $(z_1, \ldots, z_n)$ of a polydisk neighborhood Δ^n so that $D = (z_1 \cdots z_r = 0)$. Define

$$U_i^+ = \{z \in \Delta^n \mid \mathrm{Re}(z_i) < |\mathrm{Im}(z_i)|\} \text{ and } U_i^- = \{z \in \Delta^n \mid \mathrm{Re}(z_i) > -|\mathrm{Im}(z_i)|\}.$$

For every subset C of $\{1, \ldots, r\}$, define

$$U_C = \cap_{i \in C} U_i^+ \bigcap \cap_{i \notin C} U_i^-.$$

Then $\{U_C\}$ is a good covering of $\Delta^n \setminus D$ when C goes over all subset of $\{1, \ldots, r\}$. Covering X by polydisk neighborhoods, we obtain a good covering of $X \setminus D$.

Let L be a local system on $X \setminus D$ of rank m. Assume that $(\mathcal{V} = L \otimes_{\mathbb{C}} \mathcal{O}_{X \setminus D}, \nabla)$ is the induced flat vector bundle. Fix a good covering $\{U_\alpha\}$ of $X \setminus D$. Suppose locally $\{c_j^\alpha\}_{j=1}^m$ trivialize $L|_{U_\alpha}$ for each α. Suppose $s \in \Gamma(U, j_*(\mathcal{V}))$ for some open subset of X. Then

$$s|_{U \cap U_\alpha} = \sum_j f_j^\alpha c_j^\alpha, \tag{3.2.3}$$

where $f_j^\alpha \in \Gamma(U \cap U_\alpha, \mathcal{O}_X)$.

Definition 3.2.4. A section $s \in \Gamma(U, j_*(\mathcal{V}))$ is said to be of moderate growth along $U \cap D$ if for every $z_0 \in U \cap D$ there exists a polydisk neighborhood Δ^n of z_0 with coordinates $(z_1, \ldots, z_n)$ and a pair of constants $C, k \geq 0$ such that

$$|f_j^\alpha(z)| \leq A \cdot d(z, \Delta^n \cap D)^{-k}$$

for every pair of α, j and any $z \in U_\alpha \cap \Delta^n$. Here d is the standard distance function on Δ^n and f_j^α's are as in (3.2.3). Then we obtain $\mathcal{V}_{\mathrm{mod}}$, a subsheaf of $j_*(\mathcal{V})$, consisting of moderate sections along D.

It is easy to check that the moderate growth condition is independent of choices of good coverings and local trivializations of L (or see [**Bjö93**, Remark 4.1.5]). Thus, so is $\mathcal{V}_{\mathrm{mod}}$. We followed [**Bjö93**, §IV] for the construction of $\mathcal{V}_{\mathrm{mod}}$. Indeed, $\mathcal{V}_{\mathrm{mod}}$ can be constructed for any hypersurface D similarly like this; see [**Bjö93**, §IV].

Example 3.2.5. We denote by $\mathcal{O}_X(*D)$ the sheaf of meromorphic functions on X that are holomorphic on $X \setminus D$. When $L = \mathbb{C}_{X\setminus D}$, by Riemann extension theorem,

$$\mathcal{O}_{X\setminus D,\ \mathrm{mod}} = \mathcal{O}_X(*D).$$

It is well-known that $\mathcal{O}_X(*D)$ is a coherent sheaf of rings (see e.g. [**Kas03**, Appendix A.1]). It is clear from definition that $\mathcal{V}_{\mathrm{mod}}$ is an $\mathcal{O}_X(*D)$-module. By Cauchy integral formula for several complex variables, $\mathcal{V}_{\mathrm{mod}}$ is a left $\mathcal{D}_X$-module (with the D_X-module structure induced from that of $j_*(\mathcal{V})$). In this way, $\mathcal{V}_{\mathrm{mod}}$ is a left $\mathcal{D}_X(*D)$-module, where

$$\mathcal{D}_X(*D) \coloneqq \mathcal{D}_X \otimes \mathcal{O}_X(*D) \simeq \mathcal{O}_X(*D) \otimes \mathcal{D}_X.$$

Lemma 3.2.6. *Locally on a polydisk neighborhood Δ^n of X,*

$$\mathcal{V}_{mod}|_{\Delta^n} = \mathcal{V}^{(\tau_i)} \otimes_{\mathcal{O}} (\mathcal{O}_X(*D)|_{\Delta^n}),$$

for any $\mathcal{V}^{(\tau_i)}$.

Proof. Assume that $X = \Delta^n$, with a local coordinates $(z_1, \ldots, z_n)$ and $D = (z_1 \cdots z_r = 0)$. It is enough to prove that

$$\Gamma(X, \mathcal{V}_{\mathrm{mod}}) = \Gamma(X, \mathcal{V}^{(\tau_i)} \otimes \mathcal{O}_X(*D))$$

for any fixed r-tuple (τ_i). Since each $s_j^{(\tau_i)} = e^{\sum_{i=1}^{r} \Gamma_i^{\tau_i} \log\, z_i} \cdot c_j$ is of moderate growth along D, we know

$$\Gamma(X, \mathcal{V}^{(\tau_i)} \otimes \mathcal{O}_X(*D)) \subset \Gamma(X, \mathcal{V}_{\mathrm{mod}}).$$

Conversely, assume $s \in \Gamma(X, \mathcal{V}_{\mathrm{mod}})$. Take the good covering $\{U_C\}$ of X as in Remark 3.2.2. On each U_C, s can be written as

$$s|_{U_C} = \sum_j f_j^C c_j,$$

where f_j^C are holomorphic functions on U_C with moderate growth. But

$$s|_{U_C} = \sum_j f_j^C c_j = \sum_j f_j^C e^{-\sum_{i=1}^{r} \Gamma_i^{\tau_i} \log\, z_i} \cdot s_j|_{U_C} = \sum_j g_j^C s_j.$$

Since s_j's are globally defined on $X\setminus D$ and linear independent, for a fixed j all g_j^C glue to a holomorphic function g_j on $X\setminus D$ for each $j=1,...,m$. Since f_j^C and all entries of the matrix

$$e^{-\sum_{i=1}^{r}\Gamma_i^{\tau_i}\log\, z_i}$$

are of moderate growth along D, g_j are of moderate growth along D. Hence, g_j are meromorphic along D as explained in Example 3.2.5. So $s\in\Gamma(X,\mathcal{V}^{(\tau_i)}\otimes_{\mathcal{O}}\mathcal{O}_X(*D))$. Therefore

$$\mathcal{V}_{\mathrm{mod}}=\mathcal{V}^{(\tau_i)}\otimes_{\mathcal{O}}\mathcal{O}_X(*D).$$

□

We also denote $\mathcal{V}_{\mathrm{mod}}$, the Deligne meromorphic extension of $\mathcal{V}$, by $\mathcal{V}(*D)$ in order to be consistent with $\mathcal{O}_X(*D)$ (see Example 3.2.5). By the above lemma, $\mathcal{V}(*D)$ is a locally free $\mathcal{O}_X(*D)$-module, and each $\mathcal{V}^{(\tau_i)}$ gives a local trivialization.

Assume

$$f:(Y,E)\longrightarrow(X,D)$$

is a morphism of pairs, i.e. f is a morphism of complex manifolds, and $(f^*D)_{\mathrm{red}}=E$ and $\mathcal{M}$ is an $\mathcal{O}_X(*D)$-module. Since

$$f^*\mathcal{M}=\mathcal{O}_Y\otimes_{f^{-1}\mathcal{O}_X}f^{-1}(\mathcal{O}_X(*D)\otimes_{\mathcal{O}_X(*D)}\mathcal{M})=\mathcal{O}_Y(*E)\otimes_{f^{-1}\mathcal{O}_X(*D)}f^{-1}\mathcal{M},$$

$f^*(\mathcal{M})$ is an $\mathcal{O}_Y(*E)$-module. In this definition, it is not necessary to assume that D is SNC. Similarly, if moreover, $\mathcal{M}$ is a left $\mathcal{D}_X(*D)$-module, then $f^*\mathcal{M}$ is also a left $\mathcal{D}_Y(*E)$-module.

If E is also an SNC divisor, and $\mathcal{N}$ is a left $V_0^D\mathcal{D}_X$-module, then similarly $f^*\mathcal{N}$ is a left $V_0^E\mathcal{D}_Y$-module with the left $V_0^E\mathcal{D}_Y$-action induced by the morphism

$$\Theta_Y(\log\, E)\longrightarrow f^*\Theta_X(\log\, D).$$

More generally than Lemma 3.2.6, the following lemma tells us that every logarithmic extension is a $\mathcal{O}_X(*D)$-lattice of $\mathcal{V}_{\mathrm{mod}}$.

Lemma 3.2.7. *If $\overline{\mathcal{V}}$ is a logarithmic extension of $\mathcal{V}$ (assuming existence), then*

$$\mathcal{V}_{mod}\simeq\overline{\mathcal{V}}\otimes_{\mathcal{O}}\mathcal{O}_X(*D)$$

as left $\mathcal{D}_X(*D)$-modules.

Proof. Since $\overline{\mathcal{V}}|_{X\setminus D} = \mathcal{V}$ and $\overline{\mathcal{V}} \otimes \mathcal{O}_X(*D)$ is torsion-free, the natural map

$$\overline{\mathcal{V}} \otimes \mathcal{O}_X(*D) \longrightarrow j_*(\mathcal{V})$$

is injective. First, we prove that all sections of $\overline{\mathcal{V}} \otimes \mathcal{O}_X(*D)$ are of moderate growth.

First we assume $X = \Delta$, the unit disk, and $D = \{0\}$ first. In this case, by some classical theory about complex ODE's (see [**HTT08**, Theorem 5.1.4] for instance), sections of $\overline{\mathcal{V}} \otimes_{\mathcal{O}} \mathcal{O}_X(*D)$ are of moderate growth.

In general, we assume $X = \Delta^n$. If s is a section of $\overline{\mathcal{V}} \otimes_{\mathcal{O}} \mathcal{O}_X(*D)$, then since $j_*(\mathcal{V}) = \mathcal{V}^{(\tau_i)} \otimes_{\mathcal{O}_X} j_*\mathcal{O}_U$, we can write $s = \sum_j f_j s_j$ for some sections f_j of $j_*\mathcal{O}_U$ and sections s_j as in (3.1.3). For any inclusion $\Delta \hookrightarrow X$ satisfying $\Delta \cap D = \{0\}$, as in 1-dimensional case we know $f_j|_\Delta$ are meromorphic at 0. Hence f_j are meromorphic along D. Hence s is of moderate growth.

Now we know that particularly sections of $\overline{\mathcal{V}}$ are of moderate growth. By Lemma 3.2.6, this means

$$\overline{\mathcal{V}} \hookrightarrow \mathcal{V}^{(\tau_i)} \otimes \mathcal{O}_X(*D).$$

Hence, for some $k \gg 0$, we have a short exact sequence both as $\mathcal{O}_X$-modules and $V_0^D\mathcal{D}_X$-modules,

$$0 \longrightarrow \overline{\mathcal{V}} \longrightarrow \mathcal{V}^{(\tau_i)}(kD) \longrightarrow \frac{\mathcal{V}^{(\tau_i)}(kD)}{\overline{\mathcal{V}}} \longrightarrow 0.$$

Since $\mathcal{O}_X(*D)$ is flat over $\mathcal{O}_X$, after tensoring with $\mathcal{O}_X(*D)$, we obtain another short exact sequence both as $\mathcal{O}_X(*D)$-modules and $V_0^D\mathcal{D}_X$-modules (the $V_0^D\mathcal{D}_X$-module structures are induced by product rules (see Lemma 2.2.1)(1)),

$$0 \longrightarrow \overline{\mathcal{V}} \otimes_{\mathcal{O}} \mathcal{O}_X(*D) \longrightarrow \mathcal{V}_{\mathrm{mod}} \longrightarrow \frac{\mathcal{V}^{(\tau_i)}(kD)}{\overline{\mathcal{V}}} \otimes_{\mathcal{O}} \mathcal{O}_X(*D) \longrightarrow 0.$$

It is clear that $\dfrac{\mathcal{V}^{(\tau_i)}(kD)}{\overline{\mathcal{V}}}$ is supported on the divisor D. By analytic nullstellensatz,

$$\frac{\mathcal{V}^{(\tau_i)}(kD)}{\overline{\mathcal{V}}} \otimes_{\mathcal{O}} \mathcal{O}_X(*D) = 0.$$

Therefore, we get

$$\mathcal{V}_{\mathrm{mod}} \simeq \overline{\mathcal{V}} \otimes_{\mathcal{O}} \mathcal{O}_X(*D),$$

as $\mathcal{O}_X(*D)$-modules and left $V_0^D\mathcal{D}_X$-modules and hence also as left $\mathcal{D}_X(*D)$-modules. ☐

Suppose $\mathcal{M}$ and $\mathcal{N}$ are two left $\mathcal{D}_X(*D)$-modules. By the product rule similar to Lemma 2.2.1(1), $\mathcal{M} \otimes_{\mathcal{O}_X(*D)} \mathcal{N}$ has an induced left $\mathcal{D}_X(*D)$-module structure. Similarly, by the chain rule similar to Lemma 2.2.1(3), $\mathcal{H}om_{\mathcal{O}_X(*D)}(\mathcal{M}, \mathcal{N})$ has an induced left $\mathcal{D}_X(*D)$-module structure.

Lemma 3.2.8. *Assume that $\mathcal{V}$ and $\mathcal{V}'$ are two vector bundles with integrable connections on $X \setminus D$. With the induced integrable connections, as $\mathcal{D}_X(*D)$-modules*
(1) $\mathcal{V}_{\text{mod}} \otimes_{\mathcal{O}_X(*D)} \mathcal{V}'_{\text{mod}} \simeq (\mathcal{V} \otimes_{\mathcal{O}_{X\setminus D}} \mathcal{V}')_{\text{mod}}$
(2) $\mathcal{E}nd_{\mathcal{O}_X(*D)}(\mathcal{V}_{\text{mod}}) \simeq \mathcal{E}nd_{\mathcal{O}_{X\setminus D}}(\mathcal{V})_{\text{mod}}$
(3) $\mathcal{H}om_{\mathcal{O}_X(*D)}(\mathcal{V}_{\text{mod}}, \mathcal{V}'_{\text{mod}}) \simeq \mathcal{H}om_{\mathcal{O}_{X\setminus D}}(\mathcal{V}, \mathcal{V}')_{\text{mod}}$.

Proof. We prove the first isomorphism. The other two can be proved similarly.

Since sections of $\mathcal{V}_{\text{mod}} \otimes_{\mathcal{O}_X(*D)} \mathcal{V}'_{\text{mod}}$ are of moderate growth,

$$\mathcal{V}_{\text{mod}} \otimes_{\mathcal{O}_X(*D)} \mathcal{V}'_{\text{mod}} \hookrightarrow (\mathcal{V} \otimes_{\mathcal{O}_{X\setminus D}} \mathcal{V}')_{\text{mod}}.$$

Clearly the cokernal of the above inclusion is a coherent $\mathcal{O}_X(*D)$-module supported on D. Therefore, the cokernal is 0 by the analytic nullstellensatz. So

$$\mathcal{V}_{\text{mod}} \otimes_{\mathcal{O}_X(*D)} \mathcal{V}'_{\text{mod}} \simeq (\mathcal{V} \otimes_{\mathcal{O}_{X\setminus D}} \mathcal{V}')_{\text{mod}}.$$

□

Denote the category of $\mathcal{D}_X$-modules by $\text{Mod}(\mathcal{D}_X)$. Assume $\mathcal{M} \in \text{Mod}(\mathcal{D}_X)$. Set

$$\mathcal{M}^{\nabla} := \ker \nabla = ker(\mathcal{M} \xrightarrow{\nabla} \Omega^1_X \otimes \mathcal{M}).$$

Lemma 3.2.9.

$$\mathcal{V}^{\nabla}_{\text{mod}} = j_* L.$$

Proof. By definition, we have

$$\begin{array}{ccc} \mathcal{V}_{\text{mod}} & \xrightarrow{\nabla} & \Omega^1_X \otimes \mathcal{V}_{\text{mod}} \\ \downarrow & & \downarrow \\ j_*\mathcal{V} & \xrightarrow{\nabla} & \Omega^1_X \otimes j_*\mathcal{V} \end{array}$$

By the left exactness of j_*,

$$(j_*\mathcal{V})^{\nabla} = j_* L.$$

Since sections of j_*L are of moderate growth by definition, clearly we have

$$\mathcal{V}_{\mathrm{mod}}^{\nabla} = j_*L.$$

□

Denote the category of $\mathcal{O}_X(*D)$-coherent (locally finite presented) modules with $\mathcal{D}_X$-module structures extending the $\mathcal{O}_X$-module structures by

$$\mathrm{Conn}(X; D).$$

Morphisms are just morphisms of the corresponding $\mathcal{D}_X$-modules. Hence $\mathrm{Conn}(X; D)$ is naturally a full subcategory of $\mathrm{Mod}(\mathcal{D}_X)$. Notice that in this definition, it is not necessary to assume that D is SNC. Moreover, It is easy to see that $\mathrm{Conn}(X; D)$ is closed under bifunctors $\bullet \otimes_{\mathcal{O}_X(*D)} \bullet$ and $\mathcal{H}om_{\mathcal{O}_X(*D)}(\bullet, \bullet)$.

Definition 3.2.10. Suppose $\mathcal{M} \in \mathrm{Conn}(X; D)$. By the analytic Nullstellensatz, the nature morphism

$$\mathcal{M} \hookrightarrow j_*(\mathcal{M}|_{X\setminus D})$$

is injective. It is called regular along D if

$$\begin{array}{ccc} \mathcal{M} & \hookrightarrow & j_*(\mathcal{M}|_{X\setminus D}) \\ & \searrow \quad \nearrow & \\ & (\mathcal{M}|_{X\setminus D})_{\mathrm{mod}} & \end{array}$$

i.e. the above natural morphism factor throught $(\mathcal{M}|_{X\setminus D})_{\mathrm{mod}}$ ($\mathcal{M}|_{X\setminus D}$ is an integrable connection on $X \setminus D$). All such $\mathcal{M}$ form a full subcategory of $\mathrm{Conn}(X; D)$. Denote this subcategory by

$$\mathrm{Conn}^{\mathrm{reg}}(X; D).$$

The above definition of *regularity* is coincide with the regularity defined in [**Bjö93**, §V.3] (see [**Bjö93**, Proposition 5.3.9]). See also Theorem 4.1.7.

By analytic Nullstellensatz, the following lemma is easy to prove.

Lemma 3.2.11. *If* $\mathcal{M} \in \mathrm{Conn}^{\mathrm{reg}}(X; D)$*, then*

$$\mathcal{M} = (\mathcal{M}|_{X\setminus D})_{\mathrm{mod}}.$$

By the above lemma and Lemma 3.2.8, we see that $\mathrm{Conn}^{\mathrm{reg}}(X; D)$ is also closed under bifunctors $\bullet \otimes_{\mathcal{O}_X(*D)} \bullet$ and $\mathcal{H}om_{\mathcal{O}_X(*D)}(\bullet, \bullet)$.

Theorem 3.2.12 (Deligne [Del70]). *The functor $ker\nabla|_{X\setminus D}$ induces an equivalence of categories*

$$\mathrm{Conn}^{\mathrm{reg}}(X;D) \xrightarrow{\simeq} \mathrm{Loc}(X\setminus D),$$

where $\mathrm{Loc}(X\setminus D)$ *is the category of* $\mathbb{C}$*-local systems on* $X\setminus D$*.*

Proof. Suppose $\mathcal{M}$ and $\mathcal{M}'\in \mathrm{Conn}^{\mathrm{reg}}(X;D)$. Then, by definition

$$\mathrm{Hom}_{\mathrm{Conn}^{\mathrm{reg}}(X;D)}(\mathcal{M},\mathcal{M}') = \Gamma(X, \mathcal{H}om_{\mathcal{O}_X(*D)}(\mathcal{M},\mathcal{M}')^{\nabla}))$$

Use L and L' to denote the underlying local system of $\mathcal{M}|_U$ and $\mathcal{M}'|_U$ respectively. By Lemma 3.2.8 (3), Lemma 3.2.9 and Lemma 3.2.11,

$$\Gamma(X, \mathcal{H}om_{\mathcal{O}_X(*D)}(\mathcal{M},\mathcal{M}')^{\nabla}) \simeq \mathrm{Hom}_{\mathbb{C}_{X\setminus D}}(L,L').$$

Therefore, $ker\nabla|_{X\setminus D}$ induces

$$\mathrm{Hom}_{\mathrm{Conn}^{\mathrm{reg}}(X;D)}(\mathcal{M},\mathcal{M}') \simeq \mathrm{Hom}_{\mathbb{C}_{X\setminus D}}(L,L'),$$

which means the functor is fully faithful.

For essential surjectivity, suppose $L\in \mathrm{Loc}(X\setminus D)$. Then by the definition of regularity

$$\mathcal{V}_{\mathrm{mod}} \in \mathrm{Conn}^{\mathrm{reg}}(X;D),$$

where $\mathcal{V} = L\otimes \mathcal{O}_U$. ☐

The above theorem is Deligne's Riemann-Hilbert correspondence for Deligne's meromorphic extensions. Riemann-Hilbert correspondence for regular holonomic $\mathcal{D}$-modules is based on this theorem.

Clearly, $ker\nabla$ induces

$$\mathrm{Conn}(X\setminus D)\simeq \mathrm{Loc}(X\setminus D).$$

We thus also have the following equivalence.

Corollary 3.2.13. *The functor* $\bullet|_{X\setminus D}$ *induces an equivalence of categories*

$$\mathrm{Conn}^{\mathrm{reg}}(X;D) \xrightarrow{\simeq} \mathrm{Conn}(X\setminus D).$$

3.3. Bernstein-Sato Polynomials and Multi-indexed Deligne Extensions

We have constructed the canonical extension, $\mathcal{V}_{\mathrm{mod}} = \mathcal{V}(*D)$ of $\mathcal{V}$. From now on, we forget the moderate-growth structure $\mathcal{V}_{\mathrm{mod}}$ (because the moderate-growth condition is not needed anymore), but only memorize the meromorphic structure $\mathcal{V}(*D)$. In this section, we will canonically and globally construct $\mathcal{V}^{(\tau_i)}$ via Bernstein-Sato polynomials.

Suppose that m is a local section of $\mathcal{V}(*D)$ on a open neighborhood of $z \in D_i$ for some fixed i. We focus on the $V_0^D(\mathcal{D}_X)_z$(stalk of $V_0^D(\mathcal{D}_X)$ at z)-submodule of $\mathcal{V}(*D)_z$ generated by the germ of m at z. By Lemma 3.2.6, we have the inclusion

$$(V_0^D\mathcal{D}_X)_z \cdot m_z \subset \mathcal{V}^{(\tau_i)}(kD)_z,$$

for some $k \gg 0$ and a fixed r-tuple (τ_i). Since $\mathcal{O}_{X,z}$ is a noetherian ring, $(V_0^D\mathcal{D}_X)_z \cdot m_z$ is finite generated over $\mathcal{O}_{X,z}$. Suppose locally D is defined by $z_1 \cdots z_r = 0$ around the point z. Then we have the following result about the minimal polynomial of $z_i\partial_i$-action.

Proposition 3.3.1. *For each D_i, there exists a polynomial $b_{D_i}(s) \in \mathbb{C}[s]$ such that*

$$b_{D_i}(z_i\partial_i) \cdot \frac{(V_0^D\mathcal{D}_X)_z \cdot m_z}{z_i \cdot (V_0^D\mathcal{D}_X)_z \cdot m_z} = 0.$$

Proof. For simplicity, set

$$\mathcal{W}_0 = (V_0^D\mathcal{D}_X)_z \cdot m_z.$$

Assume $\mathcal{W}_0 \subset \mathcal{V}^{(\tau_i)}(kD)_z$ for some $k \in \mathbb{Z}$. Then we have a short exact sequence

$$0 \longrightarrow \frac{\mathcal{W}_{j+1}}{z_i\mathcal{W}_j} \longrightarrow \frac{\mathcal{W}_j}{z_i\mathcal{W}_j} \longrightarrow \frac{\mathcal{W}_j}{\mathcal{W}_{j+1}} \longrightarrow 0, \tag{3.3.2}$$

where $\mathcal{W}_{j+1} = \mathcal{W}_j \cap \mathcal{V}^{(\tau_i)}(kD - (j+1)D_i)_z$ for $j \in \mathbb{Z}_{\geq 0}$. We also have short exact sequences

$$0 \longrightarrow z_i \cdot \frac{\mathcal{W}_j}{\mathcal{W}_{j+1}} \longrightarrow \frac{\mathcal{W}_{j+1}}{z_i\mathcal{W}_{j+1}} \longrightarrow \frac{\mathcal{W}_{j+1}}{z_i\mathcal{W}_j} \longrightarrow 0. \tag{3.3.3}$$

Since

$$\frac{\mathcal{W}_j}{\mathcal{W}_{j+1}} \hookrightarrow \frac{\mathcal{V}^{(\tau_i)}(kD - jD_i)_z}{\mathcal{V}^{(\tau_i)}(kD - (j+1)D_i)_z},$$

the $z_i\partial_i$-action on $\frac{\mathcal{W}_j}{\mathcal{W}_{j+1}}$ is the restriction of the residue $\mathrm{Res}_{D_i}\nabla$ of $\mathcal{V}^{(\tau_i)}(kD - jD_i)$ on $\frac{\mathcal{W}_j}{\mathcal{W}_{j+1}}$. Hence, by Lemma 2.6.2, there exists some $b_j(s) \in \mathbb{C}[s]$ such that $b_j(z_i\partial_i)$ annihilates $\frac{\mathcal{W}_j}{\mathcal{W}_{j+1}}$ for each j . Similarly, there exists some $b'_j(s) \in \mathbb{C}[s]$ such that $b'_j(z_i\partial_i)$ annihilates $z_i \cdot \frac{\mathcal{W}_j}{\mathcal{W}_{j+1}}$ for each j.

On the other hand, by Artin-Rees lemma

$$\frac{\mathcal{W}_{j+1}}{z_i\mathcal{W}_j} = 0 \tag{3.3.4}$$

for $j \gg 0$. By (3.3.2), (3.3.3) and (3.3.4), the assertion follows inductively. □

Definition 3.3.5. Suppose $0 \neq m_z \in \mathcal{V}(*D)_z$ for some point $z \in D_i$. All the polynomials annihilating $\frac{V_0^D(\mathcal{D}_X)_z \cdot m_z}{z_i V_0^D(\mathcal{D}_X)_z \cdot m_z}$ is an ideal of $\mathbb{C}[s]$. The generator of this ideal (since $\mathbb{C}[s]$ is a PID) is called the Bernstein-Sato polynomial of m along the germ of D_i, denoted by $b_{D_i,m_z}(s)$. The order of m_z along the germ of D_i is defined by

$$\operatorname{ord}_{D_i}(m_z) = \min\{\text{real parts of roots of } b_{D_i,m_z}(s)\}.$$

Conventionally, set

$$\operatorname{ord}_{D_i}(0) = +\infty,$$

and the root of 0 polynomial to be also $+\infty$.

When $\mathcal{V} = \mathcal{O}_{X \setminus D}$ and f is a meromorphic function which is holomorphic on $X \setminus D$, it is clear that $\operatorname{ord}_{D_i}(f)$ is exactly the order of vanishing of f along the germ of D_i.

Corollary 3.3.6. *For some fixed r-tuple of intervals (τ_i) and fixed point $z \in X$,*

$$\mathcal{V}_z^{(\tau_i)} = \{m \in \mathcal{V}(*D)_z \mid \operatorname{ord}_{D_i}(m) \geq \tau_i \text{ for all } D_i \text{ passing } z\}.$$

Proof. By (2.6.3) and the proof of Proposition 3.3.1,

$$\mathcal{V}_z^{(\tau_i)} \subseteq \{m \in \mathcal{V}(*D)_z \mid \operatorname{ord}_{D_i}(m) \geq \tau_i \ \forall i\}.$$

On the other hand, assume that for $m \in \mathcal{V}(*D)_z$ $\operatorname{ord}_{D_i}(m) \geq \tau_i$ for all i. Let k be the smallest integer so that $m \in \mathcal{V}^{(\tau_i)}(kD)_z$. The proof is over if $k \leq 0$. Assume on the contrary $k > 0$. Then there exist an i so that,

$$\mathcal{W}_0 \nsubseteq \mathcal{V}^{(\tau_i)}(kD - D_i)_z.$$

This particularly implies $\frac{\mathcal{W}_0}{\mathcal{W}_1} \neq 0$. Since $\frac{\mathcal{W}_0}{\mathcal{W}_1}$ is a quotient of $\frac{\mathcal{W}_0}{z_i \mathcal{W}_0}$, $b_{D_i,m}(z_i \partial_i)$ annihilates $\frac{\mathcal{W}_0}{\mathcal{W}_1}$. On the other hand,

$$\frac{\mathcal{W}_0}{\mathcal{W}_1} \hookrightarrow \frac{\mathcal{V}^{(\tau_i)}(kD)_z}{\mathcal{V}^{(\tau_i)}(kD - D_i)_z}.$$

Hence, $\frac{\mathcal{W}_0}{\mathcal{W}_1}$ is killed by $b_i(z_i \partial_i)$, where $b_i(s)$ is the minimal polynomial of $\mathcal{V}^{(\tau_i)}(kD)$ along D_i (see Lemma 2.6.2). However, by (2.6.3) and the local construction of $\mathcal{V}^{(\tau_i)}$, $b_i(s)$ and $b_{D_i,m}(s)$ have no common roots. Therefore, by Bézout's lemma, 1 annihilates $\frac{\mathcal{W}_0}{\mathcal{W}_1}$. So $\frac{\mathcal{W}_0}{\mathcal{W}_1} = 0$, which is a contradiction. □

Remark 3.3.7. If $\mathcal{M}$ is a finite generated $(V_0^D \mathcal{D}_X)_z$-submodule of $\mathcal{V}(*D)_z$, by the same method, their exists a polynomial $b(s) \in \mathbb{C}[s]$ such that $b(z_i \partial_i)$ annihilates $\frac{M}{z_i M}$.

Corollary 3.3.8. *For any section m of $\mathcal{V}(*D)$, $\mathrm{ord}_{D_i}(m_z)$ does not depend on $z \in D_i$.*

Proof. It is enough to proof that $\mathrm{ord}_{D_i}(m_z)$ is locally constant for $z \in D_i$. Hence it is sufficient to assume that X is a small polydisk with coordinate $(z_1, ..., z_n)$ and D is defined by $z_1 \cdot ... \cdot z_r = 0$. Assume $m_z \in \mathcal{V}_z^{\tau_i}$ for some r-tuple (τ_i).

If $\mathcal{V}$ is of rank 1, then $m_z = f \cdot s$ where f is a holomorphic function round z and s is the generator of $\mathcal{V}^{(\tau_i)}$ as in (3.1.4). Let k be the order of f along D_i. Then we know $(V_0^D\mathcal{D}_X)_z \cdot m_z \in \mathcal{V}^{(\tau_i)}(-kD_i)_z$, but $(V_0^D\mathcal{D}_X)_z \cdot m_z \notin \mathcal{V}^{(\tau_i)}(-(k+1)D_i)_z$. Hence,

$$\mathrm{ord}_{D_i}(m_z) = \alpha + k$$

where α is the eigenvalue of $\mathrm{Res}_{D_i}\nabla^{(\tau_i)}$. By Weierstrass Preparation Theorem, k is locally constant around $z \in D_i$, and by Lemma 2.6.2, α is also locally constant. Hence, in this case $\mathrm{ord}_{D_i}(m_z)$ is locally constant.

In general, suppose ρ_L is the π_1-representation on $X \setminus D$ corresponding to the flat bundle $\mathcal{V}$. Take an irreducible sub-representation ρ_1 of ρ_L. Then we have a short exact sequence,

$$0 \longrightarrow \mathcal{V}_1^{(\tau_i)} \longrightarrow \mathcal{V}^{(\tau_i)} \longrightarrow \mathcal{V}_2^{(\tau_i)} \longrightarrow 0,$$

where $\mathcal{V}_1^{(\tau_i)}$ is the extension constructed from ρ_1 and $\mathcal{V}_2^{(\tau_i)}$ is the quotient constructed from the quotient representation. By the induction assumption, $\mathrm{ord}_{D_i}(\overline{m}_z)$ is locally constant, where $\overline{m}_z$ is the image of m_z in $\mathcal{V}_{2,z}^{(\tau_i)}$. The above short exact sequence induces another short exact sequence

$$0 \longrightarrow \mathcal{M} \longrightarrow \mathcal{W} \longrightarrow \overline{\mathcal{W}} \longrightarrow 0,$$

where $\mathcal{W} = (V_0^D\mathcal{D}_X)_z \cdot m_z$, $\overline{\mathcal{W}} = (V_0^D\mathcal{D}_X)_z \cdot \overline{m}_z$, and $\mathcal{M} = \mathcal{W} \cap \mathcal{V}_{1,z}^{(\tau_i)}$. By Remark 3.3.7, the minimal polynomial of $\mathcal{M}$ exist. Moreover, since the rank of $\mathcal{V}_1^{(\tau_i)}$ is 1, the smallest root of the minimal polynomial of $\mathcal{M}$ is locally constant. Since the Bernstein-Sato polynomial of m_z is the product of that of $\overline{m}_z$ and the minimal polynomial of $\mathcal{M}$, we conclude that $\mathrm{ord}_{D_i}(m_z)$ is locally constant. □

Therefore, we can define $\mathrm{ord}_{D_i}(m)$ to be this common value. Then for a fixed r-tuple (τ_i), we can globally define $\mathcal{V}^{(\tau_i)}$ as follow:

Definition 3.3.9. For each open subset U of X,

$$\mathcal{V}^{(\tau_i)}(U) := \{m \in \mathcal{V}(*D)(U) \mid \mathrm{ord}_{D_i}(m) \geq \tau_i \text{ for all } D_i \text{ intersects } U\}.$$

If P is a section of $V_0^D\mathcal{D}_X$, then it is easy to see

$$\mathrm{ord}_{D_i}(P\cdot m)\geq \mathrm{ord}_{D_i}(m).$$

Hence, $\mathcal{V}^{(\tau_i)}$ is a $V_0^D\mathcal{D}_X$-module. Moreover, by Corollary 3.3.6, (3.1.4) gives a local trivialization of $\mathcal{V}^{(\tau_i)}$ as locally free $\mathcal{O}_X$-modules. Hence,

Proposition 3.3.10. *$\mathcal{V}^{(\tau_i)}$ is a logarithmic extension of $\mathcal{V}$ globally.*

Assume (τ_i) is a fixed family of unit intervals for divisor D globally. From local description, we know $\mathcal{V}^{(\tau_i)}$ has the following characterization (by Lemma 3.1.2)

$$\{\text{real parts of eigenvalues of } \mathrm{Res}_{D_i}\nabla^{(\tau_i)}\}\subseteq \tau_i$$

for each i. Therefore, when $\tau_i=(-1,0]$ for each i, $\mathcal{V}^{(\tau_i)}$ is the Deligne canonical extension. Motivated by this observation, for arbitrary family (τ_i) we call $\mathcal{V}^{(\tau_i)}$ the multi-indexed Deligne extension of $\mathcal{V}$ with index (τ_i). If $\tau_i=\tau$ for every i, write $\mathcal{V}^{\tau}$ instead.

For a fixed family (τ_i), now we denote category of all logarithmic extensions with the characterization

$$\{\text{real parts of eigenvalues of } \mathrm{Res}_{D_i}\nabla\}\subseteq \tau_i$$

by $\mathrm{Conn}^{(\tau_i)}(X;D)$. Morphisms are $V_0^D\mathcal{D}_X$-linear sheaf morphisms. Hence $\mathrm{Conn}^{(\tau_i)}(X;D)$ is naturally a full subcategory of $\mathrm{Mod}(V_0^D\mathcal{D}_X)$.

For a morphism $\mathcal{V}\longrightarrow\mathcal{W}$ in the category of $\mathrm{Conn}(X\setminus D)$, by the Corollary 3.2.13, it corresponds to a morphism

$$\psi:\mathcal{V}(*D)\longrightarrow\mathcal{W}(*D).$$

The restriction of ψ on $\mathcal{V}^{(\tau_i)}$ gives us a $V_0^D\mathcal{D}_X$-linear morphism

$$\psi|_{\mathcal{V}^{(\tau_i)}}:\mathcal{V}^{(\tau_i)}\longrightarrow\mathcal{W}(*D).$$

Suppose m is a section of $\mathcal{V}^{(\tau_i)}$. Then for each D_i, $\frac{V_0^D\mathcal{D}_X\cdot\psi(m)}{V_0^D\mathcal{D}_X(-D_i)\cdot\psi(m)}$ is a quotient of $\frac{V_0^D\mathcal{D}_X\cdot m}{V_0^D\mathcal{D}_X(-D_i)\cdot m}$. Hence the Bernstein-Sato polynomial of m annihilates $\frac{V_0^D\mathcal{D}_X\cdot\psi(m)}{V_0^D\mathcal{D}_X(-D_i)\cdot\psi(m)}$. Therefore, $\psi(m)\in\mathcal{W}^{(\tau_i)}$. Namely, $\psi|_{\mathcal{V}^{(\tau_i)}}$ factors through $\mathcal{W}^{(\tau_i)}$. Conversely, for a morphism in $\mathrm{Conn}^{(\tau_i)}(X;D)$, tensoring by $\mathcal{O}_X(*D)$ gives a morphism in $\mathrm{Conn}^{\mathrm{reg}}(X;D)$. Hence,

$$\mathrm{Hom}_{\mathrm{Conn}(X;D)}(\mathcal{V}(*D),\mathcal{W}(*D))\simeq\mathrm{Hom}_{\mathrm{Conn}^{(\tau_i)}(X;D)}(\mathcal{V}^{(\tau_i)},\mathcal{W}^{(\tau_i)}).$$

However, by Corollary 3.2.13, we have an isomorphism

$$\mathrm{Hom}_{\mathrm{Conn}(X;D)}(\mathcal{V}(*D),\mathcal{W}(*D)) \simeq \mathrm{Hom}_{\mathrm{Conn}(X\setminus D)}(\mathcal{V},\mathcal{W}).$$

Since it is induced by $\bullet|_{X\setminus D}$, the morphism

$$\mathrm{Hom}_{\mathrm{Conn}(X;D)}(\mathcal{V}(*D),\mathcal{W}(*D)) \longrightarrow \mathrm{Hom}_{\mathrm{Conn}(X\setminus D)}(\mathcal{V},\mathcal{W})$$

also factors through $\mathrm{Hom}_{\mathrm{Conn}^{(\tau_i)}(X;D)}(\mathcal{V}^{(\tau_i)},\mathcal{W}^{(\tau_i)})$. In short, we have proved the following lemma.

Lemma 3.3.11.

$$\mathrm{Hom}_{\mathrm{Conn}^{(\tau_i)}(X;D)}(\mathcal{V}^{(\tau_i)},\mathcal{W}^{(\tau_i)}) \simeq \mathrm{Hom}_{\mathrm{Conn}(X\setminus D)}(\mathcal{V},\mathcal{W}).$$

By the above lemma, we obtain a refine version of Riemann-Hilbert correspondence for log $\mathcal{D}$-modules.

Theorem 3.3.12. *For a fixed family (τ_i), the functor $\bullet|_{X\setminus D}$ induces an equivalence of categories*

$$\mathrm{Conn}^{(\tau_i)}(X;D) \simeq \mathrm{Conn}(X\setminus D).$$

As a byproduct, we also know for any (τ_i) the inclusion

$$\mathcal{V}^{(\tau_i)} \hookrightarrow \mathcal{V}(*D)$$

is natural.

There may exist logarithmic extension of $\mathcal{V}$ other than multi-indexed Deligne extensions. For instance, assume

$$f:(Y,E)\longrightarrow(X,D)$$

is a morphism of smooth log pairs, then for some τ

$$f^*(\mathcal{V}^\tau)$$

is a logarithmic extension of $f^*\mathcal{V}$, but can hardly be a multi-indexed Deligne extension because the eigenvalues of its residues have been significantly changed after pull-back.

3.4. Multi-indexed Deligne Extensions and Parabolic Structures of $\mathcal{V}_{\mathrm{mod}}$

With the same notation as in the previous section, immediately, we obtain

Proposition 3.4.1. $\mathcal{V}^{(\tau_i)} \subseteq \mathcal{V}^{(\tau_i')}$ *if $\tau_i \geq \tau_i'$ for all i.*

The above proposition means with the natural partial order for all families of unit intervals, $\mathcal{V}(*D)$ is decreasingly filtered by the multi-indexed Deligne extensions. Since the set of eigenvalues of the residue map along any D_i is finite (see Lemma 2.6.2), such filtration is discrete in the following sense,

$$\mathcal{V}^{(\tau_i)} = \mathcal{V}^{(\tau_i + (-1)^{a(\tau_i)}\epsilon_i)},$$

for $0 \leq \epsilon_i \ll 1$, where $a(\tau_i) = 1$ when τ_i is left closed and right open unit interval, and $a(\tau_i) = 0$ otherwise. Moreover, by (2.6.3), we have

$$\mathcal{V}^{(\dots,\tau_j+1,\dots)} = \mathcal{V}^{(\tau_i)} \otimes \mathcal{O}_X(-D_j). \tag{3.4.2}$$

If $D = \sum_{i=1}^r D_i$ is finite, for any $(t_i) = (t_1, \dots, t_r) \in \mathbb{R}^r$ or $\mathbb{R}$-divisor $B = \sum_{i=1}^{r} t_i D_i$, then we also write

$$\mathcal{V}^{(t_i)} = \mathcal{V}^B := \mathcal{V}^{((t_i-1,t_i])},$$

and

$${}^{(t_i)}\mathcal{V} = {}^{B}\mathcal{V} := \mathcal{V}^{([t_i,t_i+1))}.$$

If $t_i = t$ for all i, then denote $\mathcal{V}^{(t,\dots,t)}$ and ${}^{(t,\dots,t)}\mathcal{V}$ by $\mathcal{V}^t$ and ${}^t\mathcal{V}$ respectively. For instance, ${}^0\mathcal{V}$ and $\mathcal{V}^0$ are the lower and upper canonical extensions respectively in [**Kol86**]. In [**Sai90**], ${}^0\mathcal{V}$ and $\mathcal{V}^0$ are denoted by $\mathcal{V}^{\geq 0}$ and $\mathcal{V}^{>-1}$ respectively.

When the local monodromies are unipotent, the order of any section along each D_i can only be integers. Hence,

$$\mathcal{V}^{(t_i)} = \mathcal{V}^{(\lfloor t_i \rfloor)},$$

and

$$\mathcal{V}^{(k_i)} = {}^{(k_i)}\mathcal{V},$$

when $(k_i) \in \mathbb{Z}^r$.

Definition 3.4.3. For any nonzero section m of $\mathcal{V}(*D)$, the total order of m is an $\mathbb{R}$-divisor defined by

$$\mathrm{ord}(m) = \sum_i \mathrm{ord}_{D_i}(m) D_i,$$

Proposition 3.4.4. *For a section m, we have*

$$\mathrm{ord}(m) = \max\{\sum t_i D_i \mid m \in {}^{(t_i)}\mathcal{V}\},$$

and

$$\mathrm{ord}(m) = \sup\{\sum (t_i - 1)D_i \mid m \in \mathcal{V}^{(t_i)}\}.$$

Proof. We prove the first equation; the second is similar and left to interested readers. Without loss of generality, we assume that D is finite.

$$B = \sup\{\sum_{i=1}^{r} t_i D_i \mid m \in {}^{(t_i)}\mathcal{V}\} = \sum_{i=1}^{r} t_i Di.$$

If $t_i > \mathrm{ord}_{D_i}(m)$ for each i, then m is not a section of ${}^{(\tau_i)}\mathcal{V}$. Hence B is a well-defined divisor. By definition of ${}^{(t_i)}\mathcal{V}$, $B = \mathrm{ord}(m)$. Hence the sup is a max. □

Example 3.4.5. If $\mathcal{V} = \mathcal{O}_U$, then $\mathcal{V}^B = \mathcal{O}_X(-\lfloor B \rfloor)$, and ${}^B\mathcal{V} = \mathcal{O}_X(-\lceil B \rceil)$, where B is an $\mathbb{R}$-divisor supported on D.

For a good covering $\{U_\alpha\}$ of $X \setminus D$. Suppose $\{c_j^\alpha\}_{j=1}^m$ trivialize $L|_{U_\alpha}$ for each α (since U_α is simply connected). Suppose $s \in \Gamma(U, j_*(\mathcal{V}))$ for some open subset U of X. Then

$$s|_{U \cap U_\alpha} = \sum_j f_j^\alpha c_j^\alpha, \tag{3.4.6}$$

where $f_j^\alpha \in \Gamma(U \cap U_\alpha, \mathcal{O}_X)$.

Definition 3.4.7. For any $\mathbb{R}$-divisor B supported on D, a section $s \in \Gamma(U, j_*(\mathcal{V}))$ is said to be locally L^2 along D with weight B if for every $z_0 \in U \cap D$ there exists a polydisk neighborhood Δ^n of z_0 with $B|_{\Delta^n} = \sum t_i D_i|_{\Delta^n}$ and $D_i|_{\Delta^n}$ defined by holomorphic function z_i, such that

$$\int_{\Delta^n \cap U_\alpha} \frac{|f_j^\alpha|^2}{\Pi |z_i|^{2t_i}} du < \infty$$

for every pair of α, j. Here du is the standard volume form on Δ^n and f_j^α's are as in (3.4.6).

Similar to sections of moderate growth, the condition for a section to be locally L^2 is also independent of choices of good covers and local trivializaitions of L. Denote by $L^2(L, D; B)$ the subsheaf of $j_*\mathcal{V}$ consisting of locally L^2 sections with weight B.

When $L = \mathbb{C}_{X\setminus D}$ and B is effective,

$$L^2(L, D; B) = \mathcal{I}(\psi_B) = \mathcal{O}_X(-\lfloor B \rfloor),$$

where $\psi_B = \sum t_i \log|g_i|$ is the plurisubharmonic function associated to B locally, and $\mathcal{I}(\psi_B)$ is the analytic multiplier ideal associated to ψ_B. See e.g. [**Dem**].

Proposition 3.4.8. *For any $\mathbb{R}$-divisor B supported on D, we have an identification*

$$\mathcal{V}^B = L^2(L, D; B).$$

In particular, when $B = 0$ we have $\mathcal{V}^0 = L^2(L, D)$.[1]

Proof. Assume that $X = \Delta^n$, the coordinate of X is $z = (z_1, \ldots, z_n)$, $D = (z_1 \cdots z_r = 0)$ and $B = \sum_{i=1}^r t_i D_i$. It is enough to prove that

$$\Gamma(X, \mathcal{V}_B) = \Gamma(X, L^2(L, D; B)).$$

Since each $s_j^{((t_i-1,t_i])} = e^{\sum_{i=1}^r \Gamma_i^{(t_i-1,t_i]} \log\, z_i} \cdot c_j$ (as in (3.1.3)) is L^2 with weight B,

$$\Gamma(X, \mathcal{V}^B) \subseteq \Gamma(X, L^2(L, D; B)).$$

Conversely, assume $s \in \Gamma(X, \mathcal{V}_{\mathrm{mod}})$. Take the good covering $\{U_C\}$ of X as in Remark 3.2.2. On each U_C, s can be written as

$$s|_{U_C} = \sum_j f_j^C c_j,$$

where f_j^C's are holomorphic functions on U_C. But

$$s|_{U_C} = \sum_j f_j^C c_j = \sum_j f_j^C e^{-\sum_{i=1}^r \Gamma_i^{\tau_i} \log\, z_i} \cdot s_j|_{U_C} = \sum_j g_j^C s_j.$$

Since s_j's are globally defined on $X \setminus D$ and linear independent, for a fixed j all g_j^C glue to a holomorphic function g_j on $X \setminus D$ for each $j = 1, \ldots, m$. Since

$$\int_{X \cap U_C} \frac{|f_j^C|^2}{\Pi |z_i|^{2t_i}} du < \infty,$$

by Cauchy-Schwarz inequality and the eigenvalue assumption on $\Gamma^{(t_i-1,t_i]}$ (see Lemma 3.1.2) we obtain

$$\int_X |g_j|^2 du < \infty$$

Hence, g_j extends to a holomorphic function on X for each j. So $s \in \Gamma(X, \mathcal{V}^B)$. □

The above proposition means that all $\mathcal{V}^B$ form a parabolic structure of $\mathcal{V}(*D)$.

[1] This explains why the upper canonical extension is usually called the Deligne canonical extension of $\mathcal{V}$.

CHAPTER 4

Global Calculus of $\mathcal{V}^{(\tau_i)}$

In this Chapter, we discuss some global properties of $\mathcal{V}^{(\tau_i)}$ and $\mathcal{D}$-modules induced from $\mathcal{V}^{(\tau_i)}$.

4.1. Holonomicity of $\mathcal{V}(*D)$ and Middle extensions

Let (X, D) be a complex manifold with a normal crossing divisor D. To make notation simpler, we assume D is finite and every irreducible component is smooth; that is D is a simple normal crossing (SNC) divisor. Suppose that L is a local system on $X \setminus D$, and $\mathcal{V} = L \otimes \mathcal{O}_{X\setminus D}$ as before. From the previous construction, we obtain a bunch of Deligne extensions of $\mathcal{V}$, $\mathcal{V}(*D)$ and $\mathcal{V}^{(\tau_i)}$. We already know that $\mathcal{V}^{(\tau_i)}$ is a $V_0^D\mathcal{D}_X$-module which is also locally free over $\mathcal{O}_X$ for every index (τ_i). In this section, we will find out generators of $\mathcal{V}(*D)$ as a $\mathcal{D}_X$-module in terms of $\mathcal{V}^{(\tau_i)}$, and prove the holonomicity of $\mathcal{V}(*D)$.

Lemma 4.1.1. *For some r-tuple of intervals (τ_i),*
(1) if $\tau_k \leq [-1, 0)$ for some k, then for $z \in D_k$ the morphism

$$\partial_k : \left(\frac{\mathcal{V}^{(\tau_i)}}{\mathcal{V}^{(\tau_i)}(\lambda_k D_k)}\right)_z \longrightarrow \left(\frac{\mathcal{V}^{(\tau_i)}(D_k)}{\mathcal{V}^{(\tau_i)}((\lambda_k + 1)D_k)}\right)_z$$

is isomorphic, where λ_k is the only integer in τ_k;
(2) if $[0, 1) < \tau_j$ for some j, then for $z \in D_j$ the morphism

$$\partial_j : \mathcal{V}_z^{(\tau_i)} \longrightarrow \mathcal{V}^{(\tau_i)}(D_j)_z$$

is isomorphic.

Proof. By functorality, we can use inductions on the rank of $\mathcal{V}$ to prove both of the two statement. Therefore, by picking an irreducible sub-representation of ρ_L, it is sufficient to assume that $\mathcal{V}$ is of rank 1. Assume s is the local generator of $\mathcal{V}^{(\tau_i)}$ defined as in (3.1.3). Then we have for each i, there exists $\lambda_i \in \mathbb{C}$

$$\partial_i s = \lambda_i \frac{s}{z_i}.$$

By the assumption on the index (τ_i), we know $\lambda_k < 0$ and $\lambda_j > 0$. Thereafter, for some holomorphic function f on a neighborhood of z (assume $z = 0$),

$$\partial_i(fs) = \partial_i \sum_{j=0}^{\infty} f_j z_i^j s = \sum_{j=0}^{\infty} (j + \lambda_i) f_j z_i^j \frac{s}{z_i}.$$

From above, if $\lambda_i \notin \mathbb{Z}_{\leq 0}$,

$$\partial_i : \mathcal{V}_z^{(\tau_i)} \longrightarrow \mathcal{V}^{(\tau_i)}(D_i)_z \tag{4.1.2}$$

is isomorphic. This proves the second statement. If $\lambda_k \in \mathbb{Z}_{<0}$, only $z_k^{-\lambda_k - 1} s$ is not in the image of ∂_k. Therefore, in this case,

$$\partial_k : (\frac{\mathcal{V}^{(\tau_i)}}{\mathcal{V}^{(\tau_i)}(\lambda_k D_k)})_z \longrightarrow (\frac{\mathcal{V}^{(\tau_i)}(D_k)}{\mathcal{V}^{(\tau_i)}((\lambda_k + 1) D_k)})_z \tag{4.1.3}$$

is an isomorphism of $\mathbb{C}$-vector spaces. Consequently, combining (4.1.2) and (4.1.3), the first statement is also proved. □

Proposition 4.1.4. *For any r-tuple of intervals (τ_i) so that $\tau_i \leq [-1, 0$ for all i, we have*

$$\mathcal{D}_X \mathcal{V}^{(\tau_i)} = \mathcal{V}(*D).$$

Proof. From Lemma 4.1.1(1) and (3.4.2), for any $k \geq 0$ we have locally

$$\mathcal{V}^{\tau_i}((k+1)D) = \sum_{k=1}^{n} \partial_k \mathcal{V}^{(\tau_i)}(kD) + \mathcal{V}^{(\tau_i)}(kD).$$

On the other hand

$$\mathcal{V}(*D) = \lim_{k \to \infty} \mathcal{V}^{(\tau_i)}(kD).$$

Therefore, $\mathcal{V}^{(\tau_i)}$ generates $\mathcal{V}(*D)$ over $\mathcal{D}_X$. □

By the second part of Lemma 4.1.1, we also get the following proposition.

Proposition 4.1.5. *For any r-tuple of intervals (τ_i) satisfying $[0, 1) < \tau_i$ for all i, we have*

$$\mathcal{D}_X \mathcal{V}^{(\tau_i)} = \mathcal{D}_X \mathcal{V}^{(-1,0]}.$$

Proof. By the assumption on (τ_i),

$$\mathcal{D}_X \mathcal{V}^{(k,k+1]} \subseteq \mathcal{D}_X \mathcal{V}^{(\tau_i)} \subseteq \mathcal{D}_X \mathcal{V}^{(-1,0]}$$

for $k \gg 0$. But by the second part of Lemma 4.1.1, we know $\mathcal{D}_X\mathcal{V}^{(k,k+1]} = \mathcal{D}_X\mathcal{V}^{(-1,0]}$. Therefore, we have $\mathcal{D}_X\mathcal{V}^{(\tau_i)} = \mathcal{D}_X\mathcal{V}^{(-1,0]}$. □

Combining the arguments of the proof of Proposition 4.1.4 and Proposition 4.1.5, we obtain the following proposition about $\mathcal{D}_X$-submodules generated by Multi-indexed extensions.

Proposition 4.1.6. *Asume $I \subseteq \{1, \ldots, r\}$. For any r-tuple (τ_i) such that $\tau_i \leq [-1, 0)$ for $i \in I$, and $[0, 1) < \tau_i$ for $i \notin I$, we have*

$$\mathcal{D}_X\mathcal{V}^{(\tau_i)} = \mathcal{D}_X\mathcal{V}^{(\tau_i')},$$

where $\tau_i' = [-1, 0)$ for $i \in I$, and $\tau_i' = (-1, 0]$ for $i \notin I$.

Inspired by the above proposition, for the same index (τ_i) set

$$\mathcal{V}_I \coloneqq \mathcal{D}_X\mathcal{V}^{(\tau_i)}.$$

We call the first extremal case $\mathcal{V}_\emptyset$ the minimal extension of $\mathcal{V}$, thus it is also denoted by $\mathcal{V}_{\min}$. The other extreme is $\mathcal{V}_{\{1,\ldots,r\}} = \mathcal{V}(*D)$. Thus $\mathcal{V}(*D)$ is also called the maximal extension of $\mathcal{V}$. All $\mathcal{V}_I$ other than these two extremal cases are called middle extensions associated to proper subset I of $\{1, \ldots, r\}$. This phenomenon is also analogous to real Morse theory (if we go through a critical value, then the homotopy of the level set changes).

Theorem 4.1.7. *$\mathcal{V}(*D)$ is holonomic. Thus, so is $\mathcal{V}_I$ for every $I \subset \{1, \ldots, r\}$.*

Proof. Since holonomicity is local, we can assume $X = \Delta^n$ with coordinate system $(z_1, \ldots, z_n)$ and $D = (z_1 \cdots z_r = 0)$. First, we prove the statement when the rank of $\mathcal{V}$ is 1. By Lemma 3.2.6, we can assume that $\mathcal{V}(*D)$ is a free $\mathcal{O}_X(*D)$-module with generator s satisfying $\partial_i s = \lambda_i \frac{s}{z_i}$ for $i = 1, \ldots, r$, and $\lambda_i \in \mathbb{C}$ such that $\mathrm{Re}(\lambda_i) \in [-1, 0)$, and $\partial_i s = 0$ for $i = r+1, \ldots, n$. By Proposition 4.1.4, $\mathcal{V}(*D)$ is generated by s as a $\mathcal{D}_X$-module. Hence,

$$\mathcal{V}(*D) \simeq \frac{\mathcal{D}_X}{\sum_{i=1}^r \mathcal{D}_X(z_i\partial_i - \lambda_i) + \sum_{i=r+1}^n \mathcal{D}_X\partial_i}.$$

Therefore, the characteristic variety of $\mathcal{V}(*D)$ is

$$\mathrm{ch}(\mathcal{V}(*D)) = \Lambda = \{(x, \xi) \mid z_1\xi_1 = \cdots = z_r\xi_r = \xi_{r+1} = \cdots = \xi_n = 0\},$$

where ξ_i is the symbol of ∂_i. Since Λ is Lagrangian, $\mathcal{V}(*D)$ is holonomic in this case.

In general, by functoriality after picking an irreducible sub representation of ρ_L on $\Delta^{*r} \times \Delta^{n-r}$, we get a short exact sequence of integrable connections on $\Delta^{*r} \times \Delta^{n-r}$,

$$0 \longrightarrow \mathcal{V}_1 \longrightarrow \mathcal{V} \longrightarrow \frac{\mathcal{V}}{\mathcal{V}_1} \longrightarrow 0,$$

where $\mathcal{V}_1$ corresponds to the irreducible sub representation. Hence, because of the equivalence in Theorem 3.2.12, we also get a short exact sequence of Deligne meromorphic extensions,

$$0 \longrightarrow \mathcal{V}_1(*D) \longrightarrow \mathcal{V}(*D) \longrightarrow \frac{\mathcal{V}(*D)}{\mathcal{V}_1(*D)} \longrightarrow 0.$$

Since its rank is 1, the statement for $\mathcal{V}_1$ is proved. Consequently, by additivity of characteristic varieties and inductions, the statement is proved. □

4.2. Duality

In this section, we are dealing with dualities of Deligne extensions. For simplicity, we assume D is finite.

We set $\mathcal{V}^* = \mathcal{H}om_{\mathcal{O}_{X\setminus D}}(\mathcal{V}, \mathcal{O}_{X\setminus D})$. By Lemma 2.2.1(3), $\mathcal{V}^*$ has an induced integrable connection.

Lemma 4.2.1. *For any fixed r-tuple (τ_i), we have a natural isomorphism*

$$\mathcal{H}om_{\mathcal{O}_X}(\mathcal{V}^{(\tau_i)}, \mathcal{O}_X) \simeq \mathcal{V}^{*(-\tau_i)}$$

as $V_0^D\mathcal{D}_X$-modules.

Proof. Since $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{V}^{(\tau_i)}, \mathcal{O}_X)|_{X\setminus D} \simeq \mathcal{V}^*$, $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{V}^{(\tau_i)}, \mathcal{O}_X)$ is a logarithmic extension of $\mathcal{V}^*$. By the residue characterization of $\mathcal{V}^{(\tau_i)}$, it is sufficient to check the statement locally. Assume that $\{s_j^*\}$ is the local basis of $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{V}^{(\tau_i)}, \mathcal{O}_X)$ dual to the basis of $\mathcal{V}^{(\tau_i)}$ defined in (3.1.3). Since $s_j^*(s_k) = \delta_{jk}$, we know

$$s_j^* = c_j^* \cdot e^{\sum_{i=1}^r -\Gamma_i^{\tau_i} \log\, z_i},$$

where c_j^*'s are multi-valued flat basis of $\mathcal{V}^*$ dual to c_j's. Hence,

$$\nabla s_j^* = \sum_{i=1}^r \frac{dz_i}{z_i} \otimes (-\Gamma_i^{\tau_i})^T \cdot s_j^*,$$

where $\bullet^T$ is the transpose operator. Hence, we have $\mathrm{Res}_{D_i}\nabla = (-\Gamma_i^{\tau_i})^T$. Since real parts of eigenvalues of $(-\Gamma_i^{\tau_i})^T$ are in $-\tau_i$, the proof is finished thanks to the eigenvalue-characterization of multi-indexed Deligne extensions. $\square$

The above duality is the duality of multi-indexed Deligne extensions as $\mathcal{O}_X$-modules. By Lemma 3.2.8(3), $\mathcal{H}om_{\mathcal{O}_X(*D)}(\mathcal{V}(*D), \mathcal{O}_X(*D)) \simeq \mathcal{V}^*(*D)$ as $\mathcal{D}_X(*D)$-modules. This gives us the $\mathcal{O}_X(*D)$-dual of $\mathcal{V}(*D)$.

By Theorem 4.1.7, we already know that $\mathcal{V}(*D)$ is holonomic. Hence, it is interesting to ask what the $\mathcal{D}_X$-dual of $\mathcal{V}(*D)$ is and what the $V_0^D\mathcal{D}_X$-dual of $\mathcal{V}^{(\tau_i)}$'s are.

Koszul complexes. Before we proceed, let us recall Koszul complexes for future use. Suppose A is an abelian group, and

$$\phi_i : A \longrightarrow A$$

are pairwise commuting endomorphisms of A for $i = 1, \dots, l$. Then the Koszul complex associated to $\phi_1, \dots, \phi_l$, denoted by $K(\phi_1, \dots, \phi_l; A)$, is inductively defined as follow:
(1) $K(\phi_1; A)$ is defined to be the complex

$$0 \longrightarrow A \xrightarrow{\phi_1} A \longrightarrow 0,$$

with the first A of degree -1;
(2) $K(\phi_1, \dots, \phi_{i+1}; A)$ is the total complex of the double complex

$$\begin{array}{c} K(\phi_1, \dots, \phi_i; A) \\ {\scriptstyle \phi_{i+1}} \big\uparrow \\ K(\phi_1, \dots, \phi_i; A). \end{array}$$

Clearly from the above definition, $K(\phi_1, \dots, \phi_l; A)$ is independent of the order of ϕ_i's and $K(\bullet; \bullet)$ is functorial. If furthermore, A is a $\mathbb{C}$-vector space, then the Koszul complex becomes

$$K(\phi_1, \dots, \phi_l; A) = [A \otimes_C \wedge^l \mathbb{C}^l \longrightarrow \dots \longrightarrow A \otimes_{\mathbb{C}} \mathbb{C}^l \longrightarrow A],$$
$$a \otimes e_1 \wedge e_2 \wedge \dots \wedge e_k \mapsto \sum_{i=1}^{k} (-1)^k \phi_i(a) \otimes e_1 \wedge \dots \wedge \widehat{e_i} \wedge \dots \wedge e_k.$$

The following lemma is easy but useful. So the proof is omitted.

Lemma 4.2.2. *Suppose* $\{\phi_i\}_{i=1}^l$ *and* $\{\psi_i\}_{i=1}^l$ *are two families of pairwise commuting endomorphisms of abelian groups* A *and* B *respectively. If for each* i *we assume*

$$K(\phi_i) \overset{q.i.}{\simeq} K(\psi_i)$$

where $\overset{q.i.}{\simeq}$ *denotes the quasi-isomorphism of complexes, then we have*

$$K(\phi_1, \ldots, \phi_l; A) \overset{q.i.}{\simeq} K(\psi_1, \ldots, \psi_l; B).$$

Furthermore, if one of the ϕ_i*'s is an isomorphism, then we have*

$$K(\phi_1, \ldots, \phi_l; A) \overset{q.i.}{\simeq} 0.$$

After the general theory about Koszul complexes, we start to prove duality theorems. First, we will construct Koszul resolutions of $\mathcal{V}^{(\tau_i)}$ locally.

Temporarily, we assume $X = \Delta^n$ with coordinate system $(z_1, \ldots, z_n)$, and $D = (z_1 \cdots z_r = 0)$. For any fixed r-tuple (τ_i), define $V_0^D\mathcal{D}_X$-morphisms

$$\phi_i : V_0^D\mathcal{D}_X \otimes_{\mathcal{O}} \mathcal{V}^{(\tau_i)} \longrightarrow V_0^D\mathcal{D}_X \otimes_{\mathcal{O}} \mathcal{V}^{(\tau_i)}$$

by

$$\phi_i(P \otimes s_j) = Px_i\partial_i \otimes s_j - P \otimes \Gamma_i^{\tau_i} \cdot s_j$$

for $i = 1, \ldots, r$, where s_j and $\Gamma_i^{\tau_i}$ are as that in (3.1.3), and

$$\phi_i(P \otimes s_j) = P\partial_i \otimes s_j$$

for $i = r+1, \ldots, n$. Since $\mathcal{V}^{(\tau_i)}$ is a $V_0^D\mathcal{D}_X$-module, we have a natural morphism of $V_0^D\mathcal{D}_X$-modules

$$V_0^D\mathcal{D}_X \otimes_{\mathcal{O}} \mathcal{V}^{(\tau_i)} \longrightarrow \mathcal{V}^{(\tau_i)}$$

defined by

$$P \otimes s \longmapsto P \cdot s.$$

By the construction of ϕ_i's, the above morphism induces a morphism of complexes of $V_0^D\mathcal{D}_X$-modules,

$$K(\phi_1, \ldots, \phi_n; V_0^D\mathcal{D}_X \otimes_{\mathcal{O}} \mathcal{V}^{(\tau_i)}) \longrightarrow \mathcal{V}^{(\tau_i)}.$$

Globally, this is nothing but the canonical morphism

$$\mathrm{SP}(V_0^D\mathcal{D}_X \otimes_{\mathcal{O}} \mathcal{V}^{(\tau_i)}) \longrightarrow \mathcal{V}^{(\tau_i)}.$$

Theorem 4.2.3. *The morphism*

$$\mathrm{SP}_D(V_0^D\mathcal{D}_X \otimes_{\mathcal{O}} \mathcal{V}^{(\tau_i)}) \xrightarrow[\simeq]{q.i.} \mathcal{V}^{(\tau_i)}$$

is a quasi-isomorphism as complexes of left $V_0^D\mathcal{D}_X$-modules. Thus, $\mathrm{SP}(V_0^D\mathcal{D}_X \otimes_{\mathcal{O}} \mathcal{V}^{(\tau_i)})$ *is a locally free resolution of $\mathcal{V}^{(\tau_i)}$. In particular, we have*

$$\mathrm{SP}_D(V_0^D\mathcal{D}_X) \xrightarrow[\simeq]{q.i.} \mathcal{O}_X.$$

Proof. Since we have already established the canonical morphism globally, it is enough to prove the statement locally. Hence, assume $X = \Delta^n$ and $D = (z_1 \cdots z_r = 0)$.

First, we assume $\mathcal{V}$ is of rank 1. By identifying $\mathcal{V}^{(\tau_i)}$ with $\mathcal{O}_X$ locally, we have to prove that

$$K(\phi_1, \ldots, \phi_n; V_0^D\mathcal{D}_X) \xrightarrow{q.i.} \mathcal{O}_X,$$

where $\phi_i : V_0^D\mathcal{D}_X \longrightarrow V_0^D\mathcal{D}_X$ is defined by

$$\phi_i(P) = P(x_i\partial_i - \lambda_i)$$

with $\mathrm{Re}(\lambda_i) \in \tau_i$ for $i = 1, \ldots, r$, and

$$\phi_i(P) = P\partial_i$$

for $i = r+1, \ldots, n$.

We also need the order filtration of $V_0^D\mathcal{D}_X$, $F_\bullet V_0^D\mathcal{D}_X$. It induces a filtration of $K(\phi_1, ..., \phi_n; V_0^D\mathcal{D}_X)$,

$$F_\bullet K(\phi_1, \ldots, \phi_n; V_0^D\mathcal{D}_X).$$

It is not hard to see

$$\mathrm{Gr}_F^\bullet K(\phi_1, \ldots, \phi_n; V_0^D\mathcal{D}_X) \simeq K(\xi_1, \ldots, \xi_n; \mathrm{Gr}_F^\bullet V_0^D\mathcal{D}_X),$$

where ξ_i's are symbols of $x_i\partial_i$ or ∂_i in $\mathrm{Gr}_F^\bullet V_0^D\mathcal{D}_X$. On the other hand, since $\{\xi_1, \ldots, \xi_n\}$ is a regular sequence in $\mathrm{Gr}_F^\bullet V_0^D\mathcal{D}_X$, we know

$$K(\xi_1, \ldots, \xi_n; \mathrm{Gr}_F^\bullet V_0^D\mathcal{D}_X) \xrightarrow{q.i.} \mathcal{O}_X. \tag{4.2.4}$$

This implies, for $p \geq 1$ ($K(\xi_1, \ldots, \xi_n; \mathrm{Gr}_F^\bullet V_0^D\mathcal{D}_X)$ is a complex of graded modules)

$$K(\xi_1, \ldots, \xi_n; \mathrm{Gr}_F^\bullet V_0^D\mathcal{D}_X)_p \overset{q.i.}{\simeq} 0. \tag{4.2.5}$$

By (4.2.4) and (4.2.5), for any $0 \le p < q$, we have quasi-isomorphisms

$$F_pK(\phi_1,\dots,\phi_n;V_0^D\mathcal{D}_X) \xrightarrow{q.i.} F_qK(\phi_1,\dots,\phi_n;V_0^D\mathcal{D}_X).$$

Since

$$K(\phi_1,\dots,\phi_n;V_0^D\mathcal{D}_X) = \lim_{p\to+\infty} F_pK(\phi_1,...,\phi_n;V_0^D\mathcal{D}_X)$$

and

$$F_0K(\phi_1,...,\phi_n;V_0^D\mathcal{D}_X) = \mathcal{O}_X,$$

by exactness of direct limit functors, we obtain a quasi-isomorphism

$$K(\phi_1,\dots,\phi_n;V_0^D\mathcal{D}_X) \xrightarrow{q.i.} \mathcal{O}_X. \tag{4.2.6}$$

In general, by picking an irreducible sub-representation of ρ_L, we obtain a short exact sequence of $V_0^D\mathcal{D}_X$-modules,

$$0 \longrightarrow \mathcal{V}_1^{(\tau_i)} \longrightarrow \mathcal{V}^{(\tau_i)} \longrightarrow \frac{\mathcal{V}^{(\tau_i)}}{\mathcal{V}_1^{(\tau_i)}} \longrightarrow 0.$$

Since $V_0^D\mathcal{D}_X$ is flat over $\mathcal{O}_X$ (locally free indeed), tensoring $V_0^D\mathcal{D}_X$, we also obtain another short exact sequence of $V_0^D\mathcal{D}_X$-modules,

$$0 \longrightarrow V_0^D\mathcal{D}_X \otimes_{\mathcal{O}} \mathcal{V}_1^{(\tau_i)} \longrightarrow V_0^D\mathcal{D}_X \otimes_{\mathcal{O}} \mathcal{V}^{(\tau_i)} \longrightarrow V_0^D\mathcal{D}_X \otimes_{\mathcal{O}} \frac{\mathcal{V}^{(\tau_i)}}{\mathcal{V}_1^{(\tau_i)}} \longrightarrow 0.$$

From it, we obtain a short exact sequence of complexes,

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathrm{SP}_D(V_0^D\mathcal{D}_X \otimes_{\mathcal{O}} \mathcal{V}_1^{(\tau_i)}) & \longrightarrow & \mathcal{V}_1^{(\tau_i)} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathrm{SP}_D(V_0^D\mathcal{D}_X \otimes_{\mathcal{O}} \mathcal{V}^{(\tau_i)}) & \longrightarrow & \mathcal{V}^{(\tau_i)} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathrm{SP}_D(V_0^D\mathcal{D}_X \otimes_{\mathcal{O}} \frac{\mathcal{V}^{(\tau_i)}}{\mathcal{V}_1^{(\tau_i)}}) & \longrightarrow & \frac{\mathcal{V}^{(\tau_i)}}{\mathcal{V}_1^{(\tau_i)}} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & .
\end{array}$$

Since $\mathcal{V}_1$ is of rank 1, we have proved that the first row is exact. The third row is also exact by inductive assumptions. Therefore, by the Snake Lemma, so is the second row. □

By Proposition 2.4.6, in particular we have the following quasi-isomorphism.

Corollary 4.2.7. *The natural morphism*

$$\mathrm{DR}_D(V_0^D\mathcal{D}_X) \xrightarrow[\simeq]{q.i.} \omega_X(D)$$

is a quasi-isomorphism.

By Theorem 4.2.3, we have

$$\mathcal{D}_X \overset{\mathbf{L}}{\otimes}_{V_0^D\mathcal{D}_X} \mathcal{V}^{(\tau_i)} \overset{q.i.}{\simeq} \mathcal{D}_X \otimes_{V_0^D\mathcal{D}_X} \mathrm{SP}_D(V_0^D\mathcal{D}_X \otimes_{\mathcal{O}} \mathcal{V}^{(\tau_i)}) \simeq \mathrm{SP}_D(\mathcal{D}_X \otimes_{\mathcal{O}} \mathcal{V}^{(\tau_i)}).$$

More than this we have the following resolution.

Proposition 4.2.8. *The morphism*

$$\mathrm{SP}_D(\mathcal{D}_X \otimes_{\mathcal{O}} \mathcal{V}^{(\tau_i)}) \xrightarrow[\simeq]{q.i.} \mathcal{D}_X \otimes_{V_0^D\mathcal{D}_X} \mathcal{V}^{(\tau_i)}$$

is a quasi-isomorphism as complexes of left $\mathcal{D}_X$-modules; that is $\mathrm{SP}_D(\mathcal{D}_X \otimes_{\mathcal{O}} \mathcal{V}^{(\tau_i)})$ is a locally free resolution of $\mathcal{D}_X \otimes_{V_0^D\mathcal{D}_X} \mathcal{V}^{(\tau_i)}$ as left $\mathcal{D}_X$-modules.

Proof. By a similar inductive argument as in the proof of Theorem 4.2.3, it is sufficient to assume that $\mathcal{V}^{(\tau_i)}$ is of rank 1. In this case, by identifying $\mathcal{V}^{(\tau_i)}$ with $\mathcal{O}_X$ locally, we have

$$\mathrm{SP}_D(\mathcal{D}_X \otimes_{\mathcal{O}} \mathcal{V}^{(\tau_i)}) \simeq K(\phi_1, ..., \phi_n; \mathcal{D}_X)$$

where $\phi_i : \mathcal{D}_X \longrightarrow \mathcal{D}_X$ is defined by

$$\phi_i(P) = P(x_i\partial_i - \lambda_i)$$

with $\mathrm{Re}(\lambda_i) \in \tau_i$ for $i = 1, \ldots, r$, and

$$\phi_i(P) = P\partial_i$$

for $i = r + 1, \ldots, n$. Then the symbols of $x_1\partial_1, \ldots, x_r\partial_r, \partial_{r+1}, \ldots, \partial_n$ form a regular sequence in $\mathrm{Gr}^\bullet_F\mathcal{D}_X$. Therefore, by arguments similar to the proof of (4.2.6), we obtain

$$K(\phi_1, \ldots, \phi_n; \mathcal{D}_X) \xrightarrow{q.i.} \frac{\mathcal{D}_X}{\sum_{i=1}^r \mathcal{D}_X(z_i\partial_i - \lambda_i) + \sum_{i=r+1}^n \mathcal{D}_X\partial_i} \simeq \mathcal{D}_X \otimes_{V_0^D\mathcal{D}_X} \mathcal{V}^{(\tau_i)}.$$

□

We also have another resolution in terms of *de Rham* complexes.

Corollary 4.2.9. *For any* (τ_i)*, we have*

$$\mathrm{DR}_D(\mathcal{V}^{(\tau_i)}\otimes_{\mathcal{O}}\mathcal{D}_X)\xrightarrow[\simeq]{q.i.}\omega_X\otimes_{\mathcal{O}}(\mathcal{D}_X\otimes_{V_0^D\mathcal{D}_X}\mathcal{V}^{(\tau_i-1)})$$

as complexes of right $\mathcal{D}_X$*-modules.*

Proof. By Corollary 4.2.7, we know

$$\mathrm{DR}_D(V_0^D\mathcal{D}_X\otimes_{\mathcal{O}}\mathcal{V}^{(\tau_i)})\xrightarrow[\simeq]{q.i.}\omega_X(D)\otimes_{\mathcal{O}}\mathcal{V}^{(\tau_i)}.$$

Thus we have

$$\mathrm{DR}_D(V_0^D\mathcal{D}_X\otimes_{\mathcal{O}}\mathcal{V}^{(\tau_i)})\otimes_{V_0^D\mathcal{D}_X}\mathcal{D}_X\xrightarrow{\simeq}(\omega_X(D)\otimes_{\mathcal{O}}\mathcal{V}^{(\tau_i)})\overset{\mathbf{L}}{\otimes}_{V_0^D\mathcal{D}_X}\mathcal{D}_X. \tag{4.2.10}$$

By Corollary 2.2.2, we see

$$(\omega_X(D)\otimes_{\mathcal{O}}\mathcal{V}^{(\tau_i)})\overset{\mathbf{L}}{\otimes}_{V_0^D\mathcal{D}_X}\mathcal{D}_X\simeq(\omega_X\otimes_{\mathcal{O}}\mathcal{D}_X)\overset{\mathbf{L}}{\otimes}_{V_0^D\mathcal{D}_X}\mathcal{V}^{(\tau_i-1)}. \tag{4.2.11}$$

Meanwhile, by the involution (2.4.1) (taking $D=0$), we also know

$$(\omega_X\otimes_{\mathcal{O}}\mathcal{D}_X)\overset{\mathbf{L}}{\otimes}_{V_0^D\mathcal{D}_X}\mathcal{V}^{(\tau_i-1)}\simeq\omega_X\otimes_{\mathcal{O}}(\mathcal{D}_X\overset{\mathbf{L}}{\otimes}_{V_0^D\mathcal{D}_X}\mathcal{V}^{(\tau_i-1)}). \tag{4.2.12}$$

On the other hand, similar to the involution (2.4.1), the following canonical morphism

$$V_0^D\mathcal{D}_X\otimes_{\mathcal{O}}\mathcal{V}^{(\tau_i)}\longrightarrow\mathcal{V}^{(\tau_i)}\otimes_{\mathcal{O}}V_0^D\mathcal{D}_X$$

$$(P\otimes v)\cdot Q\mapsto P\cdot(v\otimes Q)$$

gives an isomorphism between $V_0^D\mathcal{D}_X\otimes_{\mathcal{O}}\mathcal{V}^{(\tau_i)}$ and $\mathcal{V}^{(\tau_i)}\otimes_{\mathcal{O}}V_0^D\mathcal{D}_X$ as bi-$V_0^D\mathcal{D}_X$-modules. Thus, we obtain

$$\mathrm{DR}_D(V_0^D\mathcal{D}_X\otimes_{\mathcal{O}}\mathcal{V}^{(\tau_i)})\simeq\mathrm{DR}_D(\mathcal{V}^{(\tau_i)}\otimes_{\mathcal{O}}V_0^D\mathcal{D}_X) \tag{4.2.13}$$

as complexes of right $V_0^D\mathcal{D}_X$-modules. Combining (4.2.11) (4.2.12) and (4.2.13), we obtain

$$\mathrm{DR}_D(\mathcal{V}^{(\tau_i)}\otimes_{\mathcal{O}}\mathcal{D}_X)\xrightarrow{\simeq}\omega_X\otimes_{\mathcal{O}}(\mathcal{D}_X\overset{\mathbf{L}}{\otimes}_{V_0^D\mathcal{D}_X}\mathcal{V}^{(\tau_i-1)}). \tag{4.2.14}$$

By Proposition 4.2.8, we can get rid of $\mathbf{L}$ in (4.2.14). Therefore,

$$\mathrm{DR}_D(\mathcal{V}^{(\tau_i)}\otimes_{\mathcal{O}}\mathcal{D}_X)\xrightarrow[\simeq]{q.i.}\omega_X\otimes_{\mathcal{O}}(\mathcal{D}_X\otimes_{V_0^D\mathcal{D}_X}\mathcal{V}^{(\tau_i-1)})$$

as complexes of right $\mathcal{D}_X$-modules. ∎

Theorem 4.2.3 enable us to calculate

$$R\mathcal{H}om_{V_0^D\mathcal{D}_X}(\mathcal{V}^{(\tau_i)}, V_0^D\mathcal{D}_X)\otimes_{\mathcal{O}}\omega_X^{-1}(-D). \tag{4.2.15}$$

Since $R\mathcal{H}om_{V_0^D\mathcal{D}_X}(\mathcal{V}^{(\tau_i)}, V_0^D\mathcal{D}_X)$ is naturally a complex of right $V_0^D\mathcal{D}_X$-modules, after tensoring $\omega_X^{-1}(-D)$ (the side-changing operation), it becomes a complex of left $V_0^D\mathcal{D}_X$-modules.

Theorem 4.2.16. *For any r-tuple (τ_i), we have*

$$R\mathcal{H}om_{V_0^D\mathcal{D}_X}(\mathcal{V}^{(\tau_i)}, V_0^D\mathcal{D}_X)\otimes_{\mathcal{O}}\omega_X^{-1}(-D)[n]\xrightarrow{\overset{q.i.}{\simeq}}\mathcal{V}^{*(-\tau_i)}.$$

Proof. By Theorem 4.2.3,

$$R\mathcal{H}om_{V_0^D\mathcal{D}_X}(\mathcal{V}^{(\tau_i)}, V_0^D\mathcal{D}_X)\simeq\mathcal{H}om_{V_0^D\mathcal{D}_X}(\mathrm{SP}_D(V_0^D\mathcal{D}_X\otimes_{\mathcal{O}}\mathcal{V}^{(\tau_i)}), V_0^D\mathcal{D}_X).$$

We have the following isomorphisms of right $V_0^D\mathcal{D}_X$-modules,

$$\mathcal{H}om_{V_0^D\mathcal{D}_X}(V_0^D\mathcal{D}_X\otimes_{\mathcal{O}}\mathcal{V}^{(\tau_i)}\otimes_{\mathcal{O}}\wedge^k\Theta_X(\log D), V_0^D\mathcal{D}_X)\simeq\mathcal{H}om_{\mathcal{O}}(\mathcal{V}^{(\tau_i)}\otimes_{\mathcal{O}}\wedge^k\Theta_X(\log D), V_0^D\mathcal{D}_X)$$
$$\simeq\mathcal{H}om_{\mathcal{O}}(\mathcal{V}^{(\tau_i)}, \Omega_X^k(\log D)\otimes_{\mathcal{O}}V_0^D\mathcal{D}_X)\simeq\mathcal{H}om_{\mathcal{O}}(\mathcal{V}^{(\tau_i)}, \mathcal{O}_X)\otimes_{\mathcal{O}}\Omega_X^k(\log D)\otimes_{\mathcal{O}}V_0^D\mathcal{D}.$$

The first two"$\simeq$" follow from the tensor-hom adjunction. It is because $V_0^D\mathcal{D}_X$ is locally free over $\mathcal{O}_X$ that the last "$\simeq$" is true. By Lemma 4.2.1, we know

$$\mathcal{H}om_{\mathcal{O}}(\mathcal{V}^{(\tau_i)}, \mathcal{O}_X)=\mathcal{V}^{*(-\tau_i)}.$$

Therefore,

$$\mathcal{H}om_{V_0^D\mathcal{D}_X}(\mathrm{SP}_D(V_0^D\mathcal{D}_X\otimes_{\mathcal{O}}\mathcal{V}^{(\tau_i)}), V_0^D\mathcal{D}_X)[n]\simeq\mathrm{DR}_D(\mathcal{V}^{*(-\tau_i)}\otimes_{\mathcal{O}}V_0^D\mathcal{D}_X). \tag{4.2.17}$$

Moreover, we also have two canonical isomorphisms

$$\mathcal{V}^{*(-\tau_i)}\otimes_{\mathcal{O}}V_0^D\mathcal{D}_X\simeq V_0^D\mathcal{D}_X\otimes_{\mathcal{O}}\mathcal{V}^{*(-\tau_i)}$$
$$P\cdot(v\otimes 1)\mapsto P\otimes v$$

as left $V_0^D\mathcal{D}_X$-modules and

$$\mathcal{V}^{*(-\tau_i)}\otimes_{\mathcal{O}}V_0^D\mathcal{D}_X\simeq V_0^D\mathcal{D}_X\otimes_{\mathcal{O}}\mathcal{V}^{*(-\tau_i)}$$
$$v\otimes P\mapsto(v\otimes 1)\cdot P$$

as right $V_0^D\mathcal{D}_X$-modules. By the above two isomorphisms, we obtain

$$\mathrm{DR}_D(\mathcal{V}^{*(-\tau_i)} \otimes_{\mathcal{O}} V_0^D\mathcal{D}_X) \simeq \mathrm{DR}_D(V_0^D\mathcal{D}_X \otimes_{\mathcal{O}} \mathcal{V}^{*(-\tau_i)}) \tag{4.2.18}$$

as complexes of right $V_0^D\mathcal{D}_X$-modules. Since we use the trivial left $V_0^D\mathcal{D}_X$-module structure of $V_0^D\mathcal{D}_X \otimes_{\mathcal{O}} \mathcal{V}^{*(-\tau_i)}$ for $\mathrm{DR}_D(V_0^D\mathcal{D}_X \otimes_{\mathcal{O}} \mathcal{V}^{*(-\tau_i)})$, clearly we have

$$\mathrm{DR}_D(V_0^D\mathcal{D}_X \otimes_{\mathcal{O}} \mathcal{V}^{*(-\tau_i)}) \simeq \mathrm{DR}_D(V_0^D\mathcal{D}_X) \otimes_{\mathcal{O}} \mathcal{V}^{*(-\tau_i)} \tag{4.2.19}$$

as complexes of right $V_0^D\mathcal{D}_X$-modules. By Corollary 4.2.7, we also know

$$\mathrm{DR}_D(V_0^D\mathcal{D}_X) \otimes_{\mathcal{O}} \mathcal{V}^{*(-\tau_i)} \xrightarrow[\simeq]{q.i.} \omega_X(D) \otimes_{\mathcal{O}} \mathcal{V}^{*(-\tau_i)} \tag{4.2.20}$$

Combining (4.2.17), (4.2.18), (4.2.19) and (4.2.20) together, we get the desired quasi-isomorphism.

□

So we can define the $V_0^D\mathcal{D}_X$-dual of $\mathcal{V}^{(\tau_i)}$ by

$$\mathcal{E}xt^n_{V_0^D\mathcal{D}_X}(\mathcal{V}^{(\tau_i)}, V_0^D\mathcal{D}_X) \otimes_{\mathcal{O}} \omega_X^{-1}(-D) = \mathcal{V}^{*(-\tau_i)}.$$

This theorem tells us that the $\mathcal{O}_X$-dual and $V_0^D\mathcal{D}_X$-dual of $\mathcal{V}^{(\tau_i)}$ are canonically isomorphism.

For a left $\mathcal{D}_X$-module $\mathcal{M}$, recall that the $\mathcal{D}_X$-dual of $\mathcal{M}$ is defined by

$$\mathbb{D}(\mathcal{M}) := R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{D}_X) \otimes_{\mathcal{O}} \omega^{-1}[n].$$

By Theorem 4.2.3, we have

$$\begin{aligned} R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{D}_X \otimes_{V_0^D\mathcal{D}_X} \mathcal{V}^{(\tau_i)}, \mathcal{D}_X) &\simeq \mathcal{H}om_{V_0^D\mathcal{D}_X}(SP_D(V_0^D\mathcal{D}_X \otimes_{\mathcal{O}} \mathcal{V}^{(\tau_i)}), \mathcal{D}_X) \\ &\simeq \mathcal{H}om_{V_0^D\mathcal{D}_X}(SP_D(V_0^D\mathcal{D}_X \otimes_{\mathcal{O}} \mathcal{V}^{(\tau_i)}), V_0^D\mathcal{D}_X) \otimes_{V_0^D\mathcal{D}_X} \mathcal{D}_X. \end{aligned}$$

By Theorem 4.2.16, then we have

$$\mathbb{D}(\mathcal{D}_X \otimes_{V_0^D\mathcal{D}_X} \mathcal{V}^{(\tau_i)}) \xrightarrow[\simeq]{q.i.} (\mathcal{V}^{*(-\tau_i)} \otimes_{\mathcal{O}} \omega_X(D)) \overset{\mathbf{L}}{\otimes}_{V_0^D\mathcal{D}_X} \mathcal{D}_X \otimes_{\mathcal{O}} \omega_X^{-1}[n].$$

By Corollary 2.2.2, we also have

$$(\mathcal{V}^{*(-\tau_i)} \otimes_{\mathcal{O}} \omega_X(D)) \overset{\mathbf{L}}{\otimes}_{V_0^D\mathcal{D}_X} \mathcal{D}_X \otimes_{\mathcal{O}} \omega_X^{-1} \simeq (\omega_X \otimes_{\mathcal{O}} \mathcal{D}_X \otimes_{\mathcal{O}} \omega_X^{-1}) \overset{\mathbf{L}}{\otimes}_{V_0^D\mathcal{D}_X} \mathcal{V}^{*(-\tau_i)}(D).$$

However, by (2.4.1) taking $D = 0$, we have

$$(\omega_X \otimes_{\mathcal{O}} \mathcal{D}_X \otimes_{\mathcal{O}} \omega_X^{-1}) \simeq \mathcal{D}_X$$

as $\mathcal{D}_X$-bi-modules. Thanks to Proposition 4.2.8, we have proved the following duality.

Theorem 4.2.21. *For any r-tuple (τ_i),*

$$\mathbb{D}(\mathcal{D}_X \otimes_{V_0^D \mathcal{D}_X} \mathcal{V}^{(\tau_i)}) \xrightarrow[\simeq]{q.i.} \mathcal{D}_X \otimes_{V_0^D \mathcal{D}_X} \mathcal{V}^{*(-\tau_i - 1)}.$$

We have a natural morphism of $\mathcal{D}_X$-modules

$$\mathcal{D}_X \otimes_{V_0^D \mathcal{D}_X} \mathcal{V}^{(\tau_i)} \longrightarrow \mathcal{V}(*D) \tag{4.2.22}$$

defined by

$$P \otimes s \longmapsto P \cdot s.$$

Its image is just $\mathcal{D}_X \mathcal{V}^{(\tau_i)}$.

Lemma 4.2.23. *For any r-tuple (τ_i) satisfying*

$$\tau_i \leq [-1, 0)$$

for all i, the natural $\mathcal{D}_X$-morphism

$$\mathcal{D}_X \otimes_{V_0^D \mathcal{D}_X} \mathcal{V}^{(\tau_i)} \longrightarrow \mathcal{V}(*D)$$

is isomorphic.

Proof. Suppose that we have a short exact sequence of flat vector bundles on $X \setminus D$,

$$0 \longrightarrow \mathcal{V}_1 \longrightarrow \mathcal{V} \longrightarrow \mathcal{V}_2 \longrightarrow 0.$$

Then by Theorem 3.3.12, we have a short exact sequence of $V_0^D \mathcal{D}_X$-modules

$$0 \longrightarrow \mathcal{V}_1^{(\tau_i)} \longrightarrow \mathcal{V}^{(\tau_i)} \longrightarrow \mathcal{V}_2^{(\tau_i)} \longrightarrow 0.$$

Tensoring $\mathcal{D}_X$ over $V_0^D \mathcal{D}_X$, by Proposition 4.2.8 we have a short exact sequence

$$0 \longrightarrow \mathcal{D}_X \otimes_{V_0^D \mathcal{D}_X} \mathcal{V}_1^{(\tau_i)} \longrightarrow \mathcal{D}_X \otimes_{V_0^D \mathcal{D}_X} \mathcal{V}^{(\tau_i)} \longrightarrow \mathcal{D}_X \otimes_{V_0^D \mathcal{D}_X} \mathcal{V}_2^{(\tau_i)} \longrightarrow 0.$$

However, by Theorem 3.2.12, we also have another short exact sequence of $\mathcal{D}_X$-modules

$$0 \longrightarrow \mathcal{V}_1(*D) \longrightarrow \mathcal{V}(*D) \longrightarrow \mathcal{V}_2(*D) \longrightarrow 0.$$

Hence, by induction and Five-Lemma, it is sufficient to assume that $\mathcal{V}$ is of rank 1. In this case, assume s is the generator of $\mathcal{V}^{(\tau_i)}$ as in (3.1.3) satisfying

$$z_i\partial_i \cdot s = \lambda_i s$$

with $\mathrm{Re}(\lambda_i) \in \tau_i$ for $i = 1, ..., r$, and

$$\partial_i \cdot s = 0$$

for $i = r + 1, ..., n$. Thus, we see

$$\mathcal{V}^{(\tau_i)} \simeq \frac{V_0^D\mathcal{D}_X}{\sum_{i=1}^r V_0^D\mathcal{D}_X(z_i\partial_i - \lambda_i) + \sum_{i=r+1}^n V_0^D\mathcal{D}_X\partial_i}.$$

Since tensor product is right exact, we know

$$\mathcal{D}_X \otimes_{V_0^D\mathcal{D}_X} \mathcal{V}^{(\tau_i)} \simeq \frac{\mathcal{D}_X}{\sum_{i=1}^r \mathcal{D}_X(z_i\partial_i - \lambda_i) + \sum_{i=r+1}^n \mathcal{D}_X\partial_i}.$$

But by Proposition 4.1.4, we also know

$$\mathcal{V}(*D) = \mathcal{D}_X\mathcal{V}^{(\tau_i)} = \mathcal{D}_X s \simeq \frac{\mathcal{D}_X}{\sum_{i=1}^r \mathcal{D}_X(z_i\partial_i - \lambda_i) + \sum_{i=r+1}^n \mathcal{D}_X\partial_i}.$$

Since in rank 1 case, the natural morphisms are

$$\frac{\mathcal{D}_X}{\sum_{i=1}^r \mathcal{D}_X(z_i\partial_i - \lambda_i) + \sum_{i=r+1}^n \mathcal{D}_X\partial_i} \xrightarrow{\simeq} \mathcal{D}_X \otimes_{V_0^D\mathcal{D}_X} \mathcal{V}^{(\tau_i)} \longrightarrow \mathcal{D}_X\mathcal{V}^{(\tau_i)}$$

$$1 \mapsto 1 \otimes s \mapsto s,$$

we have isomorphism

$$\mathcal{D}_X \otimes_{V_0^D\mathcal{D}_X} \mathcal{V}^{(\tau_i)} \longrightarrow \mathcal{D}_X\mathcal{V}^{(\tau_i)} = \mathcal{V}(*D).$$

□

Remark 4.2.24. If $\mathcal{V} = \mathcal{O}_{X\setminus D}$, then we know

$$\mathcal{V}^{[0,1)} = \mathcal{O}_X \simeq \frac{V_0^D\mathcal{D}_X}{\sum_{i=1}^r V_0^D\mathcal{D}_X(z_i\partial_i) + \sum_{i=r+1}^n V_0^D\mathcal{D}_X\partial_i}.$$

Tensoring $\mathcal{D}_X$, we get

$$\mathcal{D}_X \otimes_{V_0^D\mathcal{D}_X} \mathcal{O}_X \simeq \frac{\mathcal{D}_X}{\sum_{i=1}^r \mathcal{D}_X(z_i\partial_i) + \sum_{i=r+1}^n \mathcal{D}_X\partial_i}.$$

However, we know as $\mathcal{D}_X$-modules

$$\mathcal{O}_X = \mathcal{D}_X \cdot \mathcal{O}_X \simeq \frac{\mathcal{D}_X}{\sum_{i=1}^n \mathcal{D}_X \partial_i}.$$

This is why we have to put the requirement on (τ_i) in Lemma 4.2.23. If $0 \in \tau_i$ for some i, the natural morphism (4.2.22) is neither injective nor surjective in general. This in particular tells us that $\mathcal{D}_X$ is not flat over $V_0^D \mathcal{D}_X$.

By Theorem 4.2.21 and Lemma 4.2.23, we obtain the following duality for $\mathcal{V}(*D)$.

Theorem 4.2.25. *For any r-tuple of intervals (τ_i) satisfying $\tau_i \leq [-1, 0)$ for all i,*

$$\mathbb{D}(\mathcal{V}(*D)) \xrightarrow[]{\overset{q.i.}{\simeq}} \mathcal{D}_X \otimes_{V_0^D \mathcal{D}_X} \mathcal{V}^{*(-\tau_i - 1)}.$$

By Corollary 4.2.9 and Lemma 4.2.23, we obtain the *de Rham* resolution of the right $\mathcal{D}$-module $\omega_X \otimes_{\mathcal{O}} \mathcal{O}_X(*D)$ as follow.

Proposition 4.2.26. *For any r-tuple of intervals (τ_i) satisfying $\tau_i \leq [0, 1)$ for all i,*

$$\mathrm{DR}_D(\mathcal{V}^{(\tau_i)} \otimes_{\mathcal{O}} \mathcal{D}_X) \xrightarrow[]{\overset{q.i.}{\simeq}} \omega_X \otimes_{\mathcal{O}} \mathcal{V}(*D)$$

as complexes of right $\mathcal{D}_X$-modules. In particular, we have

$$\mathrm{DR}_D(\mathcal{D}_X) \xrightarrow[]{\overset{q.i.}{\simeq}} \omega_X(*D) := \omega_X \otimes_{\mathcal{O}} \mathcal{O}_X(*D).$$

Remark 4.2.27. In [**MP16**], the authors proved the quasi-isomorphism

$$\mathrm{DR}_D(\mathcal{D}_X) \xrightarrow[]{\overset{q.i.}{\simeq}} \omega_X(*D)$$

by using the Eagon-Northcott complex associated to the logarithmic tangent bundle. See [**MP16**, Proposition 3.1] for details.

4.3. Regularity of $\mathcal{D}_X \otimes_{V_0^D \mathcal{D}_X} \mathcal{V}^{(\tau_i)}$

In this section, regularity of $\mathcal{D}_X \otimes_{V_0^D \mathcal{D}_X} \mathcal{V}^{(\tau_i)}$ will be proved.

Definition 4.3.1. Suppose $\mathcal{M}$ is a holonomic $\mathcal{D}_X$-module. Let Λ be its characteristic variety in T^*X, and $\mathcal{I}_\lambda$ the ideal sheaf of $\mathrm{Gr}_F \mathcal{D}_X$ of functions vanishing on Λ. $\mathcal{M}$ is regular if there exists locally a good filtration $F_\bullet$ on $\mathcal{M}$ such that $\mathcal{I}_\Lambda \cdot \mathrm{Gr}_F \mathcal{M} = 0$.

Proposition 4.3.2. *For any index (τ_i), $\mathcal{D}_X \otimes_{V_0^D \mathcal{D}_X} \mathcal{V}^{(\tau_i)}$ is regular holonomic.*

Proof. After picking an irreducible sub-representation of ρ_L on $X \setminus D$ at least locally, we obtain a short exact sequence of $V_0^D\mathcal{D}_X$-modules,

$$0 \longrightarrow \mathcal{V}_1^{(\tau_i)} \longrightarrow \mathcal{V}^{(\tau_i)} \longrightarrow (\frac{\mathcal{V}}{\mathcal{V}_1})^{(\tau_i)} \longrightarrow 0.$$

Tensoring $\mathcal{D}_X$ over $V_0^D\mathcal{D}_X$ from the left, by Proposition 4.2.8, we get a short exact sequence of $\mathcal{D}_X$-modules

$$0 \longrightarrow \mathcal{D}_X \otimes_{V_0^D\mathcal{D}_X} \mathcal{V}_1^{(\tau_i)} \longrightarrow \mathcal{D}_X \otimes_{V_0^D\mathcal{D}_X} \mathcal{V}^{(\tau_i)} \longrightarrow \mathcal{D}_X \otimes_{V_0^D\mathcal{D}_X} (\frac{\mathcal{V}}{\mathcal{V}_1})^{(\tau_i)} \longrightarrow 0.$$

Since regularity and holonomicity are all stable by extensions, by induction on rank of $\mathcal{V}$, it is enough to assume $\mathcal{V}$ is or rank 1. In this case, locally

$$\mathcal{D}_X \otimes_{V_0^D\mathcal{D}_X} \mathcal{V}^{*(\tau_i)} \simeq \frac{\mathcal{D}_X}{\sum_{i=1}^r \mathcal{D}_X(z_i\partial_i - \lambda_i) + \sum_{i=r+1}^n \mathcal{D}_X\partial_i},$$

with $\lambda_i \in \tau_i$, which is obviously regular holonomic. □

By Lemma 4.2.23, immediately we obtain

Corollary 4.3.3. *$\mathcal{V}(*D)$ is regular.*

This corollary implies that the regularity defined in §3.2 is compatible with the regularity of holonomic $\mathcal{D}_X$-modules.

4.4. Comparison Theorems

By the classical Riemann-Hilbert correspondence, $L \overset{q.i.}{\simeq} \mathrm{DR}(\mathcal{V})$. Since $X\setminus D$ is Stein, by a well-know theorem of Cartan and Oka, $\Omega^k_{X\setminus D}\otimes\mathcal{V}$ is acyclic for j_*, where $j : X\setminus D \hookrightarrow X$ is the open embedding. Therefore,

$$j_*\mathrm{DR}(\mathcal{V}) \overset{q.i.}{\simeq} Rj_*L[n]. \tag{4.4.1}$$

Also, by adjunction between j_* and j^{-1}, we have the natural morphism

$$\mathrm{DR}(\mathcal{V}(*D)) \hookrightarrow j_*\mathrm{DR}(\mathcal{V}). \tag{4.4.2}$$

Then we have the following well-know comparison theorem due to Grothendieck and Deligne. The idea of the following proof is due to Malgrange.

Theorem 4.4.3 (Grothendieck-Deligne)**.** *The morphism* (4.4.2) *induces a quasi-isomorphism*

$$\mathrm{DR}(\mathcal{V}(*D)) \overset{q.i.}{\simeq} j_*\mathrm{DR}(\mathcal{V}).$$

Proof. Since the statement is local, we can assume X is the germ at 0 of $\mathbb{C}^n$ and D is defined by $z_1 \cdots z_r = 0$. Thereafter, it is enough to prove that the natural morphism

$$\mathrm{DR}(\mathcal{V}(*D))_0 \longrightarrow (j_*\mathrm{DR}(\mathcal{V}))_0$$

is a quasi-isomorphism. By picking an irreducible sub-representation of ρ_L as in the proof of Theorem 4.1.7, we get a diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{DR}(\mathcal{V}_1(*D)) & \longrightarrow & \mathrm{DR}(\mathcal{V}(*D)) & \longrightarrow & \mathrm{DR}(\frac{\mathcal{V}(*D)}{\mathcal{V}_1(*D)}) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & j_*\mathrm{DR}(\mathcal{V}_1) & \longrightarrow & j_*\mathrm{DR}(\mathcal{V}) & \longrightarrow & j_*\mathrm{DR}(\mathcal{V}/\mathcal{V}_1) & \longrightarrow & 0, \end{array}$$

where $\mathcal{V}_1$ corresponds to the irreducible sub representation (hence of rank 1). It is clear that the two rows in the above diagram are exact. Hence, by 5-lemma, the statement can be proved inductively on the rank of L.

After these simplifications, it is sufficient to assume that $\mathcal{V}(*D)_0$ is free of rank 1 over $\mathcal{O}_X(*D)_0$ (assume the generator is s). By Lemma 3.2.6 and (3.1.5), we can assume

$$\partial_i s = \lambda_i \frac{s}{z_i}$$

for $i = 1, \ldots, r$, where $\mathrm{Re}(\lambda_i) \in [0, 1)$, and

$$\partial_i s = 0$$

for $i = r+1, \ldots, n$. Therefore, $\mathrm{DR}(\mathcal{V}(*D))_0$ is just the Koszul complex

$$K(\delta_1, \ldots, \delta_r, \partial_{r+1}, \ldots, \partial_n; \mathcal{O}_X(*D)_0),$$

where $\delta_i = \partial_i + \frac{\lambda_i}{z_i}$.

On the other hand, it is clear that s also generates $(j_*\mathcal{V})_0$ over $(j_*\mathcal{O}_{X\setminus D})_0$ freely. Since

$$(j_*\mathrm{DR}(\mathcal{V}))_0 \simeq \mathrm{DR}(j_*\mathcal{V})_0,$$

$(j_*\mathrm{DR}(\mathcal{V}))_0$ is the Koszul complex

$$K(\delta_1, \ldots, \delta_r, \partial_{r+1}, \ldots, \partial_n; (j_*\mathcal{O}_{X\setminus D})_0).$$

Therefore, the statement is reduced to compare the complex

$$0 \longrightarrow \mathcal{O}_X(*D)_0 \xrightarrow{\delta_i} \mathcal{O}_X(*D)_0 \longrightarrow 0,$$

and the complex

$$0 \longrightarrow (j_*\mathcal{O}_{X\backslash D})_0 \xrightarrow{\delta_i} (j_*\mathcal{O}_{X\backslash D})_0 \longrightarrow 0$$

thanks to Lemma 4.2.2. For a function f (meromorphic or of essential singularities along D), write its Laurent expansion as

$$f(z_1, \ldots, z_n) = \sum_{k\in\mathbb{Z}} f_k(z_1, \ldots, z_{i-1}, z_{i+1}, \ldots, z_n) z_i^k.$$

Then

$$\delta_i(f) = \delta_i(\sum_{k\in\mathbb{Z}} f_k z_i^k) = \sum_{k\in\mathbb{Z}} (k + 1 + \lambda_i) f_k z_i^k. \tag{4.4.4}$$

If $\lambda_i \neq 0$ for some i, then by (4.4.4), δ_i for both $\mathcal{O}_X(*D)_0$ and $(j_*\mathcal{O}_{X\backslash D})_0$ are isomorphisms (as $\mathbb{C}$-vector spaces). Hence, by Lemma 4.2.2,

$$\mathrm{DR}(\mathcal{V}(*D))_0 \overset{q.i.}{\simeq} (j_*\mathrm{DR}(\mathcal{V}))_0 \overset{q.i.}{\simeq} 0,$$

in this case. Otherwise, $\lambda_i = 0$ for all $i = 1, ..., r$. In this case, we are left to prove

$$K(\partial_1, ..., \partial_n; \mathcal{O}_X(*D)_0) \overset{q.i.}{\simeq} K(\partial_1, ..., \partial_n; (j_*\mathcal{O}_{X\backslash D})_0).$$

This can be proved by comparing cohomologies of the complex

$$0 \longrightarrow \mathcal{O}_X(*D)_0 \xrightarrow{\partial_i} \mathcal{O}_X(*D)_0 \longrightarrow 0,$$

and the complex

$$0 \longrightarrow (j_*\mathcal{O}_{X\backslash D})_0 \xrightarrow{\partial_i} (j_*\mathcal{O}_{X\backslash D})_0 \longrightarrow 0$$

for all i, via similar use of Laurent expansions. Details are left for interested readers. □

Immediately, from the above theorem and quasi-isomorphism (4.4.1), we get

Corollary 4.4.5.

$$\mathrm{DR}(\mathcal{V}(*D)) \overset{q.i.}{\simeq} Rj_*L[n].$$

Before we prove other comparison theorems, we need the following lemma. See also [**EV92**, Lemma 2.10].

Lemma 4.4.6. *Assume that* $(\mathcal{E},\nabla)$ *is a logarithmic extension of* $\mathcal{V}$*, such that* 0 *is not a eigenvalue of* $\mathrm{Res}_{D_1}\nabla$*. Then the natural inclusion*

$$\mathrm{DR}_D(\mathcal{E}(-D_1)) \hookrightarrow \mathrm{DR}_D(\mathcal{E})$$

is a quasi-isomorphim.

Proof. First, define $\mathrm{DR}^l_D(\mathcal{E})$ to be the complex

$$\mathcal{E}(-D_1) \longrightarrow \Omega^1(log\ D)\otimes\mathcal{E}(-D_1)) \longrightarrow ... \longrightarrow \Omega^{l-1}(log\ D)\otimes\mathcal{E}(-D_1) \longrightarrow$$

$$\longrightarrow \Omega^l(log\ (D-D_1))\otimes\mathcal{E} \longrightarrow \Omega^{l+1}(log\ D)\otimes\mathcal{E} \longrightarrow ... \longrightarrow \Omega^n(log\ D)\otimes\mathcal{E}.$$

Then we have $\mathrm{DR}^0_D(\mathcal{E}) = \mathrm{DR}_D(\mathcal{E})$ and $\mathrm{DR}^n_D(\mathcal{E}) = \mathrm{DR}_D(\mathcal{E}(-D_1))$, and inclusions

$$\mathrm{DR}^{l+1}_D(\mathcal{E}) \hookrightarrow \mathrm{DR}^l_D(\mathcal{E}).$$

By (2.5.1), the quotient of the above inclusion is the complex

$$\Omega^l_{D_i}(\log\ (D-D_i)|_{D_i})\otimes\mathcal{E} \xrightarrow{\mathrm{Res}_{D_1}\nabla^l} \Omega^l_{D_i}(\log\ (D-D_i)|_{D_i})\otimes\mathcal{E}.$$

Since 0 is not a eigenvalue of $\mathrm{Res}_{D_1}\nabla$, $\mathrm{Res}_{D_1}\nabla^l$ is isomorphic for each l. Hence, all $\mathrm{DR}^l_D(\mathcal{E})$ are quasi-isomorphic. Therefore,

$$\mathrm{DR}_D(\mathcal{E}(-D_1)) \hookrightarrow \mathrm{DR}_D(\mathcal{E})$$

is quasi-isomorphic. □

Inspired by the above Grothendieck-Deligne comparison theorem, it is also interested to compare $\mathrm{DR}_D(\mathcal{V}^{(\tau_i)})$ and $Rj_*L[n]$ (or some other sheaves associated to L).

Proposition 4.4.7. *For any index* r*-tuple* (τ_i) *such that* $\tau_i \leq [0,1)$ *for all* i*, the natural inclusion*

$$\mathrm{DR}_D(\mathcal{V}^{(\tau_i)}) \xhookrightarrow{q.i.} \mathrm{DR}(\mathcal{V}(*D))$$

is a quasi-isomorphism. Thus, the natural morphism

$$\mathrm{DR}_D(\mathcal{V}^{[0,1)}) \xrightarrow{q.i.} Rj_*L[n]$$

is quasi-isomorphic.

Proof. Since

$$\mathcal{V}(*D) = \lim_{k\to+\infty}\mathcal{V}^{(\tau_i)}(kD),$$

we have an induced direct system of De Rham complexes, and

$$\mathrm{DR}(\mathcal{V}(*D)) = \lim_{k\to+\infty} \mathrm{DR}_D(\mathcal{V}^{(\tau_i)}(kD)).$$

Since filtered direct limit functors are exact, it is enough to proved that for such (τ_i) and all $k \in \mathbb{Z}_{\geq 0}$, the natural inclusion

$$\mathrm{DR}_D(\mathcal{V}^{(\tau_i)}(kD)) \hookrightarrow \mathrm{DR}_D(\mathcal{V}^{(\tau_i)}((k+1)D)) \tag{4.4.8}$$

is a quasi-isomorphism. Because of Lemma 4.4.6 and the assumption that $\tau_i \leq [0,1)$ for all i, the inclusion (4.4.8) is quasi-isomorphic. Therefore, the proof is complete. □

Assume $I \subseteq \{1, \ldots, n\}$. Define

$$D_I = \sum_{i\notin I} D_i.$$

Then we have two open embeddings,

$$X \setminus D \xhookrightarrow{j_1} X \setminus D_I \xhookrightarrow{j_2} X.$$

Thereafter,

Theorem 4.4.9. *For any r-tuple (τ_i) such that $\tau_i \leq [0,1)$ for $i \in I$, and $[0,1) < \tau_i$ for $i \notin I$,*

$$\mathrm{DR}_D(\mathcal{V}^{(\tau_i)}) \overset{q.i.}{\underset{\longleftarrow}{\simeq}} j_{2!}Rj_{1*}L[n].$$

In particular,

$$\mathrm{DR}_D(\mathcal{V}^{[0,1)}(-D_I)) \overset{q.i.}{\underset{\longleftarrow}{\simeq}} j_{2!}Rj_{1*}L[n].$$

Proof. First, we need a natural morphism between $\mathrm{DR}_D(\mathcal{V}^{(\tau_i)})$ and $j_{2!}Rj_{1*}L$. By Proposition 4.4.7 and the assumption on (τ_i), we see

$$\mathrm{DR}_{D|_{X\setminus D_I}}(\mathcal{V}^{(\tau_i)}|_{X\setminus D_I}) \overset{q.i.}{\simeq} Rj_{1*}L.$$

Hence, by the adjunction between $j_2^{-1} = j_2^!$ and $j_{2!}$, we have a natural morphism

$$j_{2!}Rj_{1*}L[n] \longrightarrow \mathrm{DR}_D(\mathcal{V}^{(\tau_i)}).$$

Clearly, it is enough to prove that for any $z \in D_I$ the natural morphism

$$(j_{2!}Rj_{1*}L[n])_z \longrightarrow \mathrm{DR}_D(\mathcal{V}^{(\tau_i)})_z$$

is quasi-isomorphic. But for any $z \in D_I$, we have $(j_{2!}Rj_{1*}L[n])_z \simeq 0$. Hence it is sufficient to prove that for any $z \in D_I$, $\mathrm{DR}_D(\mathcal{V}^{(\tau_i)})_z$is acyclic. Suppose the coordinate system around z is $(z_1, ..., z_n)$ and $D = (z_1 \cdot ... \cdot z_r = 0)$. Then we know

$$\mathrm{DR}_D(\mathcal{V}^{(\tau_i)})_z = K(z_1\partial_1, ..., z_r\partial_r, \partial_{r+1}, ...\partial_n; \mathcal{V}_z^{(\tau_i)}).$$

But by Lemma 4.1.1(2), the assumption on (τ_i), for $i \notin I$, the morphism

$$\mathcal{V}_z^{(\tau_i)} \xrightarrow{z_i\partial_i} \mathcal{V}_z^{(\tau_i)}$$

is an isomorphism. Hence, by the second part of Lemma 4.2.2, the Koszul complex

$$K(z_1\partial_1, ..., z_r\partial_r, \partial_{r+1}, ...\partial_n; \mathcal{V}_0^{(\tau_i)})$$

is acyclic. □

Since $\mathbf{D}^V \circ \mathrm{DR} \simeq \mathrm{DR} \circ \mathbb{D}$, and $j_! \simeq \mathbf{D}^V \circ Rj_* \circ \mathbf{D}^V$ ($\mathbf{D^V}$ is the Verdier duality functor; see [**KS90**] for these two isomorphisms of functors), applying Proposition 4.4.7 for $\mathcal{V}^*(*D)$, we obtain

$$j_!L[n] \overset{q.i.}{\simeq} \mathbf{D}^V \circ Rj_* \circ \mathbf{D}^V L \xrightarrow{\overset{q.i.}{\simeq}} \mathrm{DR}(\mathbb{D}(\mathcal{V}^*(*D)).$$

Applying Theorem 4.2.25 for $\mathcal{V}^*$, we get

$$\mathbb{D}(\mathcal{V}^*(*D)) \overset{q.i.}{\simeq} \mathcal{D}_X \otimes_{V_0^D\mathcal{D}_X} \mathcal{V}^{(-1,0]}.$$

Hence, we have

$$\mathrm{DR}(\mathcal{D}_X \otimes_{V_0^D\mathcal{D}_X} \mathcal{V}^{(-1,0]}) \overset{q.i.}{\simeq} \mathrm{DR}(\mathbb{D}(\mathcal{V}^*(*D))).$$

Also, applying Theorem 4.4.9 for $I = \{1, ..., r\}$, we get $j_!L[n] \xrightarrow{\overset{q.i.}{\simeq}} \mathrm{DR}_D(\mathcal{V}^{(0,1]})$. Moreover, it is clear that the natural morphism of $V_0^D\mathcal{D}_X$-modules,

$$\mathcal{V}^{(0,1]} \longrightarrow \mathcal{D}_X \otimes_{V_0^D\mathcal{D}_X} \mathcal{V}^{(-1,0]}$$

induces a morphism

$$\mathrm{DR}_D(\mathcal{V}^{(0,1]}) \longrightarrow \mathrm{DR}(\mathcal{D}_X \otimes_{V_0^D\mathcal{D}_X} \mathcal{V}^{(-1,0]}).$$

Combining all these quasi-isomorphisms together, we obtain the following comparisons.

Proposition 4.4.10. *The natural morphisms*

$$j_!L[n] \xrightarrow{\overset{q.i.}{\simeq}} DR_D(\mathcal{V}^{(0,1]}) \xrightarrow{\overset{q.i.}{\simeq}} DR(\mathcal{D}_X \otimes_{V_0^D\mathcal{D}_X} \mathcal{V}^{(-1,0]})$$

are quasi-isomorphisms.

The natural morphism

$$\mathcal{D}_X \otimes_{V_0^D\mathcal{D}_X} \mathcal{V}^{(-1,0]} \longrightarrow \mathcal{D}_X \otimes_{V_0^D\mathcal{D}_X} \mathcal{V}^{[-1,0)} \simeq \mathcal{V}(*D)$$

induces a morphism

$$\mathrm{DR}(\mathcal{D}_X \otimes_{V_0^D\mathcal{D}_X} \mathcal{V}^{(-1,0]}) \longrightarrow \mathrm{DR}(\mathcal{V}(*D)).$$

After using the quasi-isomorphisms in Proposition 4.4.7 and Proposition 4.4.10, it is easy to see that the above morphism is just the canonical morphism

$$j_!L[n] \longrightarrow Rj_*L[n]$$

in the $D^b(\mathbb{C}_X)$, where $D^b(\mathbb{C}_X)$ is the derived category of bounded $\mathbb{C}_X$-complexes. By Riemann-Hilbert correspondence (see for instance [**Kas03**, Theorem 5.7]), the De Rham functor induces an equivalence of abelian categories between the category of regular holonomic D_X-modules and the category of perverse sheaves (which is an abelian category). Hence, by Proposition 4.4.7 and Proposition 4.4.10, $j_!L[n]$ and $Rj_*L[n]$ are perverse. Hence, we get a morphism

$$j_!L[n] \longrightarrow Rj_*L[n]$$

in the category of perverse sheaves. The minimal extension of $L[n]$ is defined as the image of the above morphism,

$$j_{!*}(L[n]) := im^p(j_!L[n] \longrightarrow Rj_*L[n])$$

in the category of perverse sheaves (see [**HTT08**, Definition 8.2.2] or [**BBD82**]). However, by construction, we also know

$$\mathcal{V}_{\min} = im(\mathcal{D}_X \otimes_{V_0^D\mathcal{D}_X} \mathcal{V}^{(-1,0]} \longrightarrow \mathcal{V}(*D)).$$

In summary, we have proved the following proposition.

Proposition 4.4.11.

$$j_{!*}(L[n]) \overset{q.i.}{\simeq} DR(\mathcal{V}_{min}).$$

This proposition also explains why $\mathcal{D}_X\mathcal{V}^{(-1,0]}$ is called the minimal extension of $\mathcal{V}$.

CHAPTER 5

Multi-indexed *Log*-VHS

We already know how to make nice extensions across normal crossing boundaries from flat vector bundles. If the flat vector bundle underlies a VHS, it would be interesting to get nice extensions for the filtrations. The induced extensions turn out to work. All these extensions are crucial for the study of pure Hodge modules.

5.1. Filtration Structures for Log-VHS and Unipotent Reductions

As before, let (X, D) be a pair of a complex manifold and $D = \sum_{i=1}^{r} D_i$ a reduced SNC divisor. Suppose $\mathcal{V}$ underlies a PVHS (polarized variation of Hodge structure) $V = (\mathcal{V}, F_\bullet, \mathbb{V})$ on $X \setminus D$. From now on, we assume all the local monodromies of $\mathbb{V}$ are quasi-unipotent.

Naively, we make the following definition.

Definition 5.1.1. For any index (τ_i), we define

$$F_\bullet^{(\tau_i)} := \mathcal{V}^{(\tau_i)} \cap j_* F_\bullet,$$

where the intersection happens inside $j_*\mathcal{V}$.

From the above definition, $F_\bullet^{(\tau_i)}$ have an induced filtration from that of $\mathcal{V}^{(\tau_i)}$. Moreover, they also inherit the discreteness property,

$$F_\bullet^{(t_i)} = F_\bullet^{(t_i+\epsilon)}$$

and

$${}^{(t_i)}F_\bullet = {}^{(t_i-\epsilon)}F_\bullet$$

for $(t_i) \in \mathbb{R}^n$ and $0 < \epsilon \ll 1$. But it is not clear whether $F_\bullet^{(\tau_i)}$ is coherent or not. However, we will see that $F_\bullet^{(\tau_i)}$ are locally free with the quasi-unipotent assumption on local monodromies of $\mathbb{V}$. To prove this, we need the following easy but useful lemma.

Lemma 5.1.2. *If $\mathcal{F}$ and $\mathcal{G}$ are coherent subsheaves of a holomorphic vector bundle $\mathcal{E}$ on a complex manifold X satisfying $\mathcal{F}|_U = \mathcal{G}|_U$ for some open dense U, and $\mathcal{F}$ is a subbundle of $\mathcal{E}$, then*
(1) $\mathcal{G} \subseteq \mathcal{F}$;
(2) $\mathcal{F} = \mathcal{E} \cap j_*\mathcal{F}|_U$, where $j : U \hookrightarrow X$.

Proof. Since $\mathcal{G}$ is torsion-free (as a subsheaf of a holomophic vector bundle), the natural map

$$\mathcal{G} \longhookrightarrow j_*\mathcal{G}|_U = j_*\mathcal{F}|_U$$

is injective. Hence, (1) follows from (2). For (2), consider the following commutative diagram,

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{F} & \longrightarrow & \mathcal{E} & \longrightarrow & \mathcal{E}/\mathcal{F} & \longrightarrow & 0 \\ & & \alpha\big\downarrow & & \big\downarrow & & \beta\big\downarrow & & \\ 0 & \longrightarrow & \mathcal{E} \cap j_*\mathcal{F}|_U & \longrightarrow & \mathcal{E} & \longrightarrow & \mathcal{E}/(\mathcal{E} \cap j_*\mathcal{F}|_U) & \longrightarrow & 0 \end{array}$$

Since $\mathcal{E}/\mathcal{F}$ is locally free, $\ker\beta = 0$. Therefore the natural morphism α is an isomorphism by the Snake Lemma.

□

Theorem 5.1.3. *For any (τ_i), $F_\bullet^{(\tau_i)}$ are subbundles of $\mathcal{V}^{(\tau_i)}$.*

Proof. This is a local question. Hence we can assume $X = \Delta^n$ and $U = \Delta^{*r} \times \Delta^{n-r}$, where Δ and Δ^* are disk and punctured disk respectively. Assume $(z_1, \ldots, z_n)$ are coordinates of Δ^n so that $D = (z_1 \cdots z_r = 0)$. Denote the monodromies of V around z_i counterclockwise by γ_i and denote the multivalued flat sections that generate $\mathcal{V}$ by $c_1, \ldots, c_m$, as in (3.1.1). Suppose the Jordan decomposition of γ_i is

$$\gamma_i = \gamma_{i,s}\gamma_{i,u},$$

and m_i is any fixed positive integer such that

$$\gamma_{i,s}^{m_i} = 1, \tag{5.1.4}$$

for $i = 1, \ldots, r$. Then as in (3.1.4) $\mathcal{V}^{(\tau_i)}$ is generated freely by

$$\{e^{\sum\limits_{i=1}^{r} \Gamma_i^{\tau_i} \log(z_i)} v_j\}$$

for any (τ_i) and $i = 1, ..., m$. By Lemma 3.1.2, we know

$$e^{-2\pi\sqrt{-1}\Gamma_i^{\tau_i}} = \gamma_i,$$

and the eigenvalues of $\Gamma_i^{\tau_i}$ are in τ_i. Write the Jordan decomposition of $\Gamma_i^{\tau_i}$ as

$$\Gamma_i^{\tau_i} = S_i^{\tau_i} + N_i.$$

So

$$N_i = \frac{-1}{2\pi\sqrt{-1}}\log(\gamma_{i,u}).$$

Suppose $\pi : \Delta^n \longrightarrow \Delta^n$ is the branched covering satisfying,

$$\pi^* z_i = \begin{cases} w_i^{m_i} & \text{if} \quad i = 1, \dots, r \\ w_i & \quad\text{otherwise}, \end{cases} \tag{5.1.5}$$

where $(w_1, \dots, w_n)$ are coordinates upstairs. Then the monodromies of π^*V around each w_i are unipotent. Thereafter, by Lemma 3.1.2 $S_i^{\tau_i}$ for π^*V is $0 \in \mathfrak{gl}(V, \mathbb{C})$. Therefore $(\pi^*\mathcal{V})^0$ is generated freely by

$$e^{\sum_{i=1}^{r} m_i N_i \log(w_i)} \pi^* v_j$$

for $j = 1, ..., m$. Furthermore, $(\pi^*\mathcal{V})^{(t_i)}$ is generated freely by all

$$e^{\sum_{i=1}^{r} m_i N_i \log(w_i)} \prod_{i=1}^{r} w_i^{\lfloor t^i \rfloor} \pi^* v_j.$$

The Galois group G of π is isomorphic to $\prod_{i=1}^{r} \mathbb{Z}/(m_i)$. Let $g_1, ..., g_r$ be generator of G such that the Galois action on Δ_w^n is

$$w_k^{g_j} = \zeta_{m_j}^{\delta_{kj}} w_k$$

where ζ_{m_j} is the m_j-th root of unit for all j and k. Clearly, G acts on $\pi^* v_j$ and

$$(\pi^* v_j)^{g_i} = \gamma_i(\pi^* v_j).$$

Hence G acts on $(\pi^*\mathcal{V})^{(t_i)}$. A simple computation would show that

$$\pi_*((\pi^*\mathcal{V})^{(t_i)})^G \simeq {}^{(\lfloor t_i \rfloor / m_i)}\mathcal{V}. \tag{5.1.6}$$

Furthermore, since the monodromies are unipotent, by Schimd's Nilpotent Orbit Theorem([Sch73]), $F_{\bullet w} := F_\bullet \pi^* \mathcal{V}$ extend to subbundles $F_{\bullet w}^0$ of $(\pi^*\mathcal{V})^0$ upstairs.

Since $F^0_{\bullet w}\prod_{i=1}^r w_i^{\lfloor t_i\rfloor}$ are subbundles of $(\pi^*\mathcal{V})^{(t_i)}$, we know

$$F^{(t_i)}_{\bullet w} \simeq F^0_{\bullet w}\prod_{i=1}^r w_i^{\lfloor t_i\rfloor}.$$

Claim. *$F^0_{\bullet w}$ also have G-actions.*

Hence $F^{(t_i)}_{\bullet w}$ have G-actions as well.

Proof of claim. Suppose $e \in F^{(t_i)}_{\bullet w}$. If $e^g \notin F^{(t_i)}_{\bullet w}$ for some $g \in G$, then

$$0 \neq \overline{e^g} \in \frac{(\pi^*\mathcal{V})^{(t_i)}}{F^{(t_i)}_{\bullet w}}.$$

On the other hand, G acts on $F_{\bullet w}$. Hence $\overline{e^g}$ is supported on $X\backslash U$. Therefore $\overline{e^g}$ generates a torsion $\mathcal{O}_X$ submodule of $\dfrac{(\pi^*\mathcal{V})^{(t_i)}}{F^{(t_i)}_{\bullet w}}$. This is a contradiction, because $F^{(t_i)}_{\bullet w}$ are subbundles $(\pi^*\mathcal{V})^{(t_i)}$. □

Since π is finite and flat, by the Grauert's base change Theorem, $\pi_*F^{(t_i)}_{\bullet w}$ are subbundles of $\pi_*(\pi^*\mathcal{V})^{(t_i)}$. The $\mathcal{O}_{\Delta^n_z}$-linear morphism

$$\phi : \pi_*F^{(t_i)}_{\bullet w} \longrightarrow (\pi_*F^{(t_i)}_{\bullet w})^G$$

defined by

$$\phi(s) = \bar{s} = \frac{1}{\prod_{i=1}^r m_i}\sum_{i=1}^r\sum_{k_i=0}^{m_i-1} s^{(\prod_{k=1}^r g_k^{k_i})}$$

splits the inclusion

$$(\pi_*F^{(t_i)}_{\bullet w})^G \hookrightarrow \pi_*F^{(t_i)}_{\bullet w}.$$

Thus as a direct summands of $\pi_*F^{(t_i)}_{\bullet w}$, $(\pi_*F^{(t_i)}_{\bullet w})^G$ are subbundles of ${}^{(\lfloor t_j\rfloor/m_j)}\mathcal{V}$. Moreover, it is obvious that

$$(\pi_*F^{(t_i)}_{\bullet w})^G|_U = F_\bullet.$$

Hence, by Lemma 5.1.2 (2), we have

$$ {}^{(\lfloor t_j\rfloor/m_j)}F_\bullet = (\pi_*F^{(t_i)}_{\bullet w})^G. \tag{5.1.7}$$

Since the multi-indexed filtration of $\mathcal{V}^{(\tau_i)}$ is discrete ($\mathbb{Q}^r$ indexed indeed, because the monodromies are quasi-unipotent), for any (τ_i) there exists a $(\dfrac{n_i}{m_i}) \in \mathbb{Q}^r$ such that

(5.1.8) $$\mathcal{V}^{(\tau_i)} = {}^{(n_i/m_i)}\mathcal{V}.$$

Hence, we also get $F_\bullet^{(\tau_i)}$ as subbundles of $\mathcal{V}^{(\tau_i)}$.

□

Example 5.1.9. $\mathcal{O}_{\Delta^n}$ underlies the trivial PVHS on Δ^n. We know

$$(\pi_*\mathcal{O}_{\Delta^n})_0 = \bigoplus_{i=1}^{r}\bigoplus_{k_i=0}^{m_i-1}\prod_{l=1}^{r} w_l^{k_i}\mathcal{O}_{\Delta^n,0},$$

and $(\pi_*\mathcal{O}_{\Delta^n})^G = \mathcal{O}_{\Delta^n}$, where π and other notations are as that in the proof of Theorem 5.1.3. More generally, if $B = \sum_{i=1}^r t_i D_i$ is an $\mathbb{R}$-divisor where D_i is the divisor defined by $w_i = 0$, then

$$\pi_*(\mathcal{O}_U^B)^G = \pi_*(\mathcal{O}_{\Delta^n}(-\lfloor B\rfloor))^G = \mathcal{O}_{\Delta^n}(-\sum_{i=1}^r \lceil\frac{\lfloor t_i\rfloor}{m_i}\rceil D_i'),$$

where D_i' is the divisor defined by $z_i = 0$.

Since $F_\bullet^{(t_i)}$ are subbundles of $\mathcal{V}^{(t_i)}$, by (3.4.2) and Lemma 5.1.2 (1), we have

(5.1.10) $$F_\bullet^{(t_1,\dots,t_j+1,\dots,t_r)} = F_\bullet^{(t_i)} \otimes \mathcal{O}_X(-D_j).$$

More generally we can depict $F_\bullet^{(t_i)}$ in a consistent way for all $(t_i) \in \mathbb{R}^r$ as follow. The following lemma is inspired by [**Kaw02**, Lemma 3.5]. The author used the parabolic structures with respect to Hodge metrics of higher direct images of relative dualizing sheaves.

Lemma 5.1.11. *For any $z \in X$, there exist a polydisk neighborhood U of z and a free basis $\{s_1, \dots, s_k\}$ of $F_\bullet^0|_U$ such that*

$$(\prod_i z_i^{\lfloor t_i - \mathrm{ord}_{D_i}(s_1)\rfloor})s_1, \dots, (\prod_i z_i^{\lfloor t_i - \mathrm{ord}_{D_i}(s_m)\rfloor})s_k$$

generates $F_\bullet^{(t_i)}|_U$ freely for any index (t_i), where z_i is the local defining equation of D_i.

Proof. When $x \notin D$, the statement is trivial. It is enough to assume $x \in D$. Take U to be a polydisk neighborhood of x, and construct a Galois covering π of U as in the proof of Theorem 5.1.3. Notations are as that in the proof of Theorem 5.1.3. Suppose w is a point lying over z via the Galois covering π, and m_w is the maximal ideal sheaf of w in $\mathcal{O}_{\Delta^n}$.

By the definition of G-action on $F_{\bullet w}^{(t_i)}$, G acts on $m_w F_{\bullet w}^0$. Hence G acts on $F_{\bullet w}^0 \otimes \kappa(w)$ where $\kappa(w)$ is the residue field of w. Since G is abelian, $F_{\bullet w}^0 \otimes \kappa(w)$ (as a $\mathbb{C}$-vector space) is decomposed into simultaneous eigenspaces with respect to the G-action. Pick s_w to be a simultaneous generalized eigenvector in $F_{\bullet w}^0 \otimes \kappa(w)$ such that

$$s_w^{g_i} = \zeta_{m_i}^{a_i} s_w$$

for some integers $0 \leq a_i < m_i$ and $i = 1, \ldots, r$. This can be achieved because $g_i^{m_i} = 1$. Lift s_w to a section s of $F_{\bullet w}^0$ (shrink U if necessary). Then the mean section

$$\bar{s} = \frac{1}{\prod_{i=1}^{r} m_i} \sum_{i=1}^{r} \sum_{k_i=1}^{m_i-1} \frac{s^{(\prod_{k=1}^{r} g_k^{k_i})}}{\prod_{k=1}^{r} \zeta_{m_k}^{a_k k_i}}$$

satisfying $\bar{s}(w) = s_w$ and $\bar{s}^{g_i} = \zeta_{m_i}^{a_i} \bar{s}$. So $s' := (\prod_i w_i^{-a_i})\bar{s}$ is G-invariant and

$$s' \in F_{\bullet w}^{(a_i)} \subseteq F_{\bullet w}^{(-m_i+1)}.$$

On the other hand, by (5.1.7) and (5.1.8),

$$F_{\bullet}^0 = (\pi_* F_{\bullet w}^{(-m_i+1)})^G.$$

Hence s' descends to a section of $F_{\bullet}^0$ downstairs.

If one lets s_w run over a basis of $F_{\bullet w}^0 \otimes \kappa(w)$, the corresponding sections s' form a free basis of

$$(\pi_* F_{\bullet w}^{(-m_i+1)})^G = F_{\bullet}^0.$$

Similarly, since $-\frac{a_i}{m_i} + \lfloor t_i + \frac{a_i}{m_i} \rfloor \in (t_i - 1, t_i]$, the sections $(\prod_{i=1}^r z_i^{\lfloor t + \frac{a_i}{m_i} \rfloor})s'$ form a free basis of $F_{\bullet}^{(t_i)}$, when s_w runs over a basis of $F_{\bullet w}^0 \otimes \kappa(w)$.

Finally, we have to calculate the order of s'. By (5.1.7) again, we know $s' \in {}^{(-a_i/m_i)}F_{\bullet}$, but by the construction of s', we can easily see $s' \notin {}^{(t_i)}F_{\bullet}$ whenever $t_i > -a_i/m_i$ for some i. It follows that $s' \notin {}^{(t_i)}\mathcal{V}$ whenever $t_i > -a_i/m_i$ for some i. Therefore, we also get $\mathrm{ord}_{D_i}(s') = -a_i/m_i$, thanks to Proposition 3.4.4. □

5.2. Canonical Filtrations for $\mathcal{V}(*D)$ and $\mathcal{V}_{\mathbf{min}}$

Let (X, D) be a pair of a complex manifold and a reduced SNC divisor $D = \sum_{i=1}^r D_i$ as before. Suppose $\mathcal{V}$ underlies a PVHS (polarized variation of Hodge structures) $V =$

$(\mathcal{V}, F_\bullet, \mathbb{V})$ on $X \setminus D$ with quasi-unipotent local monodromies. In §5.1, for any index (τ_i), we have established filtered subbundles $F_\bullet^{(\tau_i)}$ of $\mathcal{V}^{(\tau_i)}$ respectively.

By the definition of VHS, the connection of $\mathcal{V}$ satisfies the Griffiths' transversality condition; that is.

$$\nabla(F_p) \subset \Omega^1_{X\setminus D} \otimes F_{p+1}.$$

By definition, $F_\bullet^{(\tau_i)}$ and $\nabla^{(\tau_i)}$ inherit this condition. Namely, $\nabla^{(\tau_i)}$ satisfying the logarithmic Griffiths' transversality condition,

$$\nabla^{(\tau_i)}(F_p^{(\tau_i)}) \subset \Omega^1_X(\log D) \otimes F_{p+1}^{(\tau_i)}.$$

This means $F_\bullet^{(\tau_i)}$ give a coherent filtration for $\mathcal{V}^{(\tau_i)}$.

For $\mathcal{V}(*D)$ and index of r-tuple (τ_i) such that $\tau_i \leq [-1, 0)$, we set

$$F_p^{(\tau_i)}\mathcal{V}(*D) = \sum_i F_{p-i}\mathcal{D}_X \cdot F_i^{(\tau_i)}. \tag{5.2.1}$$

By Proposition 4.1.4, $F_\bullet^{(\tau_i)}\mathcal{V}(*D)$ give a coherent filtration of $\mathcal{V}(*D)$ for each such (τ_i). Denote them by $(\mathcal{V}(*D), F_\bullet^{(\tau_i)})$ for all such index (τ_i). Among them, the canonical one is

$$F_p\mathcal{V}(*D) = \sum_i F_{p-i}\mathcal{D}_X \cdot {}^{-1}F_i.$$

This filtration for $\mathcal{V}(*D)$ is canonical in the sense that it defines the Hodge filtration for the underlying mixed Hodge module of $\mathcal{V}(*D)$ (see [Sai90]). For $\mathcal{V}_{\min}$, define

$$F_p\mathcal{V}_{\min} = \sum_i F_{p-i}\mathcal{D}_X \cdot F_i^0. \tag{5.2.2}$$

$F_\bullet\mathcal{V}_{\min}$ also give a coherent filtration of $\mathcal{V}_{\min}$. The construction of these two Hodge filtrations is due to Saito. With such filtrations, $(\mathcal{V}(*D), F_\bullet)$ underlies a mixed hodge module, and $(\mathcal{V}_{\min}, F_\bullet)$ underlies a pure Hodge module. See [Sai90, Theorem 3.20 and §3.c]. We will go back to this again in Chapter 6.

5.3. Filtered Comparison Theorems

Filtered Modules and Graded modules. First, let us recall some definitions about filtered modules and graded modules. Suppose $G_\bullet$ is a $\mathbb{Z}$-graded ring (bounded from below and possibly not commutative), and S is a bi-$G_\bullet$-module and S' is left $G_\bullet$-module. If we forget the grading structures, $S \otimes_G S'$ is a left G-module, where G is $G_\bullet$ without grading. Define the grading by

$$(S \otimes_G S')_p = \{\sum_i s_{p-i} \otimes s'_i \mid s_{p-i} \in S_{p-i} \text{ and } s'_i \in S'_i\}.$$

Then it is clear that

$$S \otimes_{G_\bullet} S' = \bigoplus_p (S \otimes_G S')_p,$$

which is a left $G_\bullet$-module. If S is only a right $G_\bullet$-module, then $S \otimes_{G_\bullet} S'$ is a graded abelian group.

Suppose $(A, F_\bullet)$ is a filtered ring (always assume that the filtration is bounded from below and exaustive, but A is not necessarily commutative). A filtered A-module $(M, F_\bullet)$ is A-modules M with a filtration compatible with that of A; i.e.

$$F_pA \cdot F_lM \subseteq F_{p+l}M.$$

For simplicity, we call such modules $(A, F_\bullet)$-modules. The Rees ring of $(A, F_\bullet)$ is a graded ring

$$R_F(A) = \bigoplus_p F_pA \cdot z^p \subset A[z, \frac{1}{z}].$$

The symbol variable z is introduced to help memorize the order. From definition, $R_F(A)$ is a $\mathbb{Z}[z]$-algebra. We can recover (A, F) from $R_F(A)$ by

$$A \simeq R_F(A) \otimes_{\mathbb{Z}[z]} \mathbb{Z}[z]/(z-1)\mathbb{Z}[z],$$

and

$$F_pA = \mathrm{im}(R_F(A)_p \to R_F(A) \otimes_{\mathbb{Z}[z]} \mathbb{Z}[z]/(z-1)\mathbb{Z}[z]).$$

Moreover, the associated graded ring of $(A, F_\bullet)$ is

$$\mathrm{Gr}^F_\bullet A \simeq R_F(A) \otimes_{\mathbb{Z}[z]} \mathbb{Z}[z]/z\mathbb{Z}[z].$$

Similarly, the Rees module of $(M, F_\bullet)$ is

$$R_F(M) = \bigoplus_p F_pM \cdot z^p$$

which is naturally a graded $R_F(A)$-module. For instance, $(V_0^D\mathcal{D}_X, F_\bullet)$ with the order filtration (induced from that of $\mathcal{D}_X$) is a filtered sheaf of ring.

If $N_\bullet$ is graded $R_F(A)$-bi-module (left or right) which is in priori a $\mathbb{Z}[z]$-module, then $N_\bullet$ is *strict* if it is z-torsion free. Denote the category of filtered $(A, F_\bullet)$-modules (with obvious morphisms) by $\mathrm{FMod}(A, F_\bullet)$ and the category of *strict* graded $R_F(A)$-modules by $\mathrm{GMod}^s(R_F(A))$ (as a full subcategory of $\mathrm{GMod}(R_F(A))$, the category of graded $R_F(A)$-modules). Then, we have the following easy but useful lemma.

Lemma 5.3.1. *The functor $R_F(\bullet)$ gives an equivalence of categories*

$$\mathrm{FMod}(A, F_\bullet) \simeq \mathrm{GMod}^s(R_F(A)).$$

If $(M, F_\bullet)$ a filtered $(A, F_\bullet)$-module and $(N, F_\bullet)$ is a left $(A, F_\bullet)$-module, then the filtered tensor product $(M, F_\bullet) \otimes_{(A,F_\bullet)} (N, F_\bullet)$ is given by the left-filtered the strict graded $R_F(A)$-module $R_F(M) \otimes_{R_F(A)} R_F(N)$ as in Lemma 5.3.1. In particular, if $(M, F_\bullet)$ is only a right $(A, F_\bullet)$-module, $(M, F_\bullet) \otimes_{(A,F_\bullet)} (N, F_\bullet)$ is a filtered abelian group.

The following lemma is obvious.

Lemma 5.3.2.

$$\mathrm{Gr}^F_\bullet((M, F_\bullet) \otimes_{(A,F_\bullet)} (N, F_\bullet)) \simeq R_F(M) \otimes_{R_F(A)} R_F(N) \otimes_{\mathbb{Z}[z]} \frac{\mathbb{Z}[z]}{z\mathbb{Z}[z]} \simeq \mathrm{Gr}^F_\bullet M \otimes_{\mathrm{Gr}^F_\bullet A} \mathrm{Gr}^F_\bullet N.$$

Remark 5.3.3. The filtered tensor product can be generalized to filtered complexes. To be precise, assume that $(M^\bullet, F_\bullet)$ is a complex of right $(A, F_\bullet)$-modules and $(N^\bullet, F_\bullet)$ is a complex of left $(A, F_\bullet)$-modules. Then $R_F(M^\bullet)$ is a complex of right $R_F(A)$-modules and $R_F(N^\bullet)$ is a complex of left $R_F(A)$-modules. Thereafter, $R_F(M^\bullet) \otimes_{R_F(A)} R_F(N^\bullet)$ is a complex of strict $\mathbb{Z}[z]$-modules. Hence, $(M^\bullet, F_\bullet) \otimes_{(A,F_\bullet)} (N^\bullet, F_\bullet)$ is defined to be the filtered complex of abelian groups induced by $R_F(M^\bullet) \otimes_{R_F(A)} R_F(N^\bullet)$.

Filtered resolution of $(\mathcal{D}_X, F_\bullet) \otimes_{(V_0^D\mathcal{D}_X, F_\bullet)} (\mathcal{V}^{(\tau_i)}, F_\bullet^{(\tau_i)})$. Now, we are going to apply the above abstract nonsense to filtered log $\mathcal{D}$-modules.

Let (X, D) be a pair of a complex manifold and a reduced SNC divisor $D = \sum_{i=1}^r D_i$. Suppose $\mathcal{V}$ underlies a PVHS (polarizable variation of Hodge structures) $V = (\mathcal{V}, F_\bullet, \mathbb{V})$ on $X \setminus D$ with quasi-unipotent local monodromies.

The Canonical filtrations for $\Theta_X(\log D)$ and $\Omega^1_X(\log D)$. $(\mathcal{O}_X, F_\bullet)$ is a filtered sheaf of rings with trivial filtration; i.e.

$$F_p\mathcal{O}_X = \begin{cases} \mathcal{O}_X & \text{if } p \geq 0 \\ 0 & \text{otherwise} \end{cases}.$$

Since $\Theta_X(\log D) \subset F_1 V_0^D \mathcal{D}_X$, define a filtration of $\Theta_X(\log D)$ by

$$F_p\Theta_X(\log D) = \begin{cases} \Theta_X(\log D) & \text{if } p \geq 1 \\ 0 & \text{otherwise} \end{cases}.$$

Dually, define a filtration of $\Omega^1_X(\log D)$ by

$$F_p\Omega^1_X(\log D) = \begin{cases} \Omega^1_X(\log D) & \text{if } p \geq -1 \\ 0 & \text{otherwise} \end{cases}.$$

With these filtrations, we have a filtered isomorphism

$$(\mathcal{O}_X, F_\bullet) \simeq (\Theta_X(\log D), F_\bullet) \otimes_{(\mathcal{O}_X, F_\bullet)} (\Omega^1_X(\log D), F_\bullet).$$

Then naturally we have $(\wedge^k\Theta_X(\log D), F_\bullet) = \wedge^n(\Theta_X(\log D), F_\bullet)$; i.e.

$$F_p \wedge^k \Theta_X(\log D) = \begin{cases} \wedge^k\Theta_X(\log D) & \text{if } p \geq k \\ 0 & \text{otherwise} \end{cases},$$

and dually $(\Omega^k_X(\log D), F_\bullet) = \wedge^k(\Omega^1_X(\log D), F_\bullet)$; i.e.

$$F_p\Omega^k_X(\log D) = \begin{cases} \Omega^k_X(\log D) & \text{if } p \geq -k \\ 0 & \text{otherwise} \end{cases}.$$

In particular, $(\omega_X(D), F_\bullet) = (\Omega^n_X(\log D), F_\bullet)$ is a right $(V_0^D\mathcal{D}_X, F_\bullet)$-module.

From §5.1, we have known that $(\mathcal{V}^{(\tau_i)}, F_\bullet^{(\tau_i)})$ is a left $(V_0^D\mathcal{D}_X, F_\bullet)$-module for each index (τ_i). So the filtered tensor product $(\mathcal{D}_X, F_\bullet) \otimes_{(\mathcal{O}_X, F_\bullet)} (\mathcal{V}^{(\tau_i)}, F_\bullet^{(\tau_i)})$ is a $(\mathcal{D}_X, F_\bullet)$-$(V_0^D\mathcal{D}_X, F_\bullet)$-bimodule. Under this pattern, the complex $\mathrm{SP}_D(\mathcal{D}_X \otimes_{\mathcal{O}} \mathcal{V}^{(\tau_i)})$ is filtered by

$$\begin{aligned} F_p(\mathrm{SP}_D(\mathcal{D}_X \otimes_{\mathcal{O}} \mathcal{V}^{(\tau_i)})) = & F_p(\mathcal{A} \otimes_{(\mathcal{O}_X, F_\bullet)} (\wedge^n\Theta_X(\log D), F_\bullet)) \longrightarrow \cdots \\ & \longrightarrow F_p(\mathcal{A} \otimes_{(\mathcal{O}_X, F_\bullet)} (\Theta_X(\log D), F_\bullet)) \longrightarrow F_p\mathcal{A} \end{aligned}$$

with the first term of degree $-n$, where $\mathcal{A} = (\mathcal{D}_X, F_\bullet) \otimes_{(\mathcal{O}_X, F_\bullet)} (\mathcal{V}^{(\tau_i)}, F_\bullet^{(\tau_i)})$. With this filtration, we have that

$$\mathrm{SP}_D((\mathcal{D}_X, F_\bullet) \otimes_{(\mathcal{O}_X, F_\bullet)} (\mathcal{V}^{(\tau_i)}, F_\bullet^{(\tau_i)})) := (\mathrm{SP}_D(\mathcal{D}_X \otimes_{\mathcal{O}} V_0^D\mathcal{D}_X), F_\bullet)$$

is a complex of left $(\mathcal{D}_X, F_\bullet)$-modules. Then we have the following filtered quasi-isomorphism.

Lemma 5.3.4. *The natural morphism*

$$\mathrm{SP}_D((\mathcal{D}_X, F_\bullet) \otimes_{(\mathcal{O}_X, F_\bullet)} (\mathcal{V}^{(\tau_i)}, F_\bullet^{(\tau_i)})) \xrightarrow[\simeq]{q.i.} (\mathcal{D}_X, F_\bullet) \otimes_{(V_0^D\mathcal{D}_X, F_\bullet)} (\mathcal{V}^{(\tau_i)}, F_\bullet^{(\tau_i)})$$

is a filtered quasi-isomorphism of complexes of $(\mathcal{D}_X, F_\bullet)$*-modules.*

Proof. It is sufficient to prove that

$$\mathrm{Gr}_\bullet^F(\mathrm{SP}_D((\mathcal{D}_X, F_\bullet) \otimes_{(\mathcal{O}_X, F_\bullet)} (\mathcal{V}^{(\tau_i)}, F_\bullet^{(\tau_i)}))) \xrightarrow[\simeq]{q.i.} \mathrm{Gr}_\bullet^F((\mathcal{D}_X, F_\bullet) \otimes_{(V_0^D\mathcal{D}_X, F_\bullet)} (\mathcal{V}^{(\tau_i)}, F_\bullet^{(\tau_i)})).$$

However, by definition, we have

$$\mathrm{Gr}_\bullet^F((\mathcal{D}_X, F_\bullet) \otimes_{(V_0^D\mathcal{D}_X, F_\bullet)} (\mathcal{V}^{(\tau_i)}, F_\bullet^{(\tau_i)})) \simeq \mathrm{Gr}_\bullet^F\mathcal{D}_X \otimes_{\mathrm{Gr}_\bullet^F V_0^D\mathcal{D}_X} \mathrm{Gr}_\bullet^F\mathcal{V}^{(\tau_i)},$$

and

$$\mathrm{Gr}_\bullet^F((\mathcal{D}_X, F_\bullet) \otimes_{(\mathcal{O}_X, F_\bullet)} (\mathcal{V}^{(\tau_i)}, F_\bullet^{(\tau_i)})) \simeq \mathrm{Gr}_\bullet^F\mathcal{D}_X \otimes_{\mathrm{Gr}_\bullet^F \mathcal{O}_X} \mathrm{Gr}_\bullet^F\mathcal{V}^{(\tau_i)}.$$

Hence, locally we obtain

$$\mathrm{Gr}_\bullet^F(\mathrm{SP}_D((\mathcal{D}_X, F_\bullet) \otimes_{(\mathcal{O}_X, F_\bullet)} (\mathcal{V}^{(\tau_i)}, F_\bullet^{(\tau_i)}))) \simeq K(\phi_1, ..., \phi_n; \mathrm{Gr}_\bullet^F\mathcal{D}_X \otimes_{\mathrm{Gr}_\bullet^F \mathcal{O}_X} \mathrm{Gr}_\bullet^F\mathcal{V}^{(\tau_i)}),$$

where $\phi_i = -\xi_i \otimes 1 + 1 \otimes \xi_i$ for ξ_i's symbols of $z_i\partial_i$ or ∂_i in $\mathrm{Gr}_1^F V_0^D\mathcal{D}_X$. Since $\{\xi_1, ..., \xi_n\}$ is a regular sequence for $\mathrm{Gr}_F^\bullet V_0^D\mathcal{D}_X$ (hence also for $\mathrm{Gr}_F^\bullet\mathcal{D}_X$), it not hard to see that $\{\phi_1, ..., \phi_n\}$ is also a regular sequence for $\mathrm{Gr}_\bullet^F\mathcal{D}_X \otimes_{\mathrm{Gr}_\bullet^F \mathcal{O}_X} \mathrm{Gr}_\bullet^F\mathcal{V}^{(\tau_i)}$. Therefore, we have

$$\mathrm{Gr}_\bullet^F(\mathrm{SP}_D((\mathcal{D}_X, F_\bullet) \otimes_{(\mathcal{O}_X, F_\bullet)} (\mathcal{V}^{(\tau_i)}, F_\bullet^{(\tau_i)}))) \xrightarrow[\simeq]{q.i.} \mathrm{Gr}_\bullet^F((\mathcal{D}_X, F_\bullet) \otimes_{(V_0^D\mathcal{D}_X, F_\bullet)} (\mathcal{V}^{(\tau_i)}, F_\bullet^{(\tau_i)})).$$

□

If the filtration is forgotten, the above lemma is exactly Proposition 4.2.8.

Remark 5.3.5. By the above lemma, we would get a filtered duality formula for

$$(\mathcal{D}_X, F_\bullet) \otimes_{(V_0^D\mathcal{D}_X, F_\bullet)} (\mathcal{V}^{(\tau_i)}, F_\bullet^{(\tau_i)});$$

that is, a filtered version of Theorem 4.2.16. In particular, it implies that $(\mathcal{D}_X, F_\bullet) \otimes_{(V_0^D\mathcal{D}_X, F_\bullet)} (\mathcal{V}^{(\tau_i)}, F_\bullet^{(\tau_i)})$ are Cohen-Macaulay. We also know that some of them are Hodge modules. This means in normal crossing case Cohen-Macaulayness of Hodge modules can be checked directly by simple calculations.

By arguments similar to the proof of Lemma 5.3.4, we obtain another filtered quasi-isomorphism but in terms of *de Rham* complexes.

Lemma 5.3.6. *The natural morphisms*

$$DR((\mathcal{D}_X, F_\bullet) \otimes_{(\mathcal{O}_X, F_\bullet)} (\mathcal{V}^{(\tau_i)}, F_\bullet^{(\tau_i)})) \xrightarrow{\overset{q.i.}{\simeq}} (\omega_X, F_\bullet) \otimes_{(\mathcal{O}, F_\bullet)} (\mathcal{V}^{(\tau_i)}, F_\bullet^{(\tau_i)})$$

is a filtered quasi-isomorphism of complexes of $(\mathcal{O}_X, F_\bullet)$*-modules (*$(V_0^D\mathcal{D}_X, F_\bullet)$*-modules indeed).*

Consequently, after tensoring the above filtered quasi-isomorphism in Lemma 5.3.6 by $(\wedge^{n-k}\Theta_X(\log D), F_\bullet)$ (since $(\wedge^{n-k}\Theta_X(\log D), F_\bullet)$ is flat over $(\mathcal{O}_X, F_\bullet)$), we get

$$\begin{array}{c} (\mathcal{V}^{(\tau_i+1)}, F_\bullet^{(\tau_i+1)}) \otimes_{(\mathcal{O}, F_\bullet)} (\Omega_X^k(\log D), F_\bullet) \\ \simeq\Big\uparrow \\ (\omega_X, F_\bullet) \otimes_{(\mathcal{O}, F_\bullet)} (\mathcal{V}^{(\tau_i)}, F_\bullet^{(\tau_i)}) \otimes_{(\mathcal{O}, F_\bullet)} (\wedge^{n-k}\Theta_X(\log D), F_\bullet) \\ \overset{q.i.}{\simeq}\Big\uparrow \\ DR((\mathcal{D}_X, F_\bullet) \otimes_{(\mathcal{O}_X, F_\bullet)} (\mathcal{V}^{(\tau_i)}, F_\bullet^{(\tau_i)})) \otimes_{(\mathcal{O}_X, F_\bullet)} (\wedge^{n-k}\Theta_X(\log D), F_\bullet), \end{array} \tag{5.3.7}$$

as filtered complexes of $(\mathcal{O}_X, F_\bullet)$-modules.

The following theorem is the filtered version of Theorem 4.4.7.

Theorem 5.3.8. *For any* (τ_i)*, the natural morphism*

$$\mathrm{DR}_D(\mathcal{V}^{(\tau_i+1)}, F_\bullet^{(\tau_i+1)}) \xrightarrow{\overset{q.i.}{\simeq}} \mathrm{DR}\,((\mathcal{D}_X, F_\bullet) \otimes_{(V_0^D\mathcal{D}_X, F_\bullet)} (\mathcal{V}^{(\tau_i)}, F_\bullet^{(\tau_i)}))$$

is a filtered quasi-isomorphism of filtered complexes of $\mathbb{C}_X$*-modules. In particular, if the index* (τ_i) *satisfies that for all* i $\tau_i \leq [-1, 0)$*, then we have*

$$\mathrm{DR}_D(\mathcal{V}^{(\tau_i+1)}, F_\bullet^{(\tau_i+1)}) \xrightarrow{\overset{q.i.}{\simeq}} \mathrm{DR}\,(\mathcal{V}(*D), F_\bullet^{(\tau_i)}).$$

Proof. First, by (5.3.7), we know that

$$\mathrm{DR}\,(\mathrm{SP}_D((\mathcal{D}_X, F_\bullet) \otimes_{(\mathcal{O}_X, F_\bullet)} (\mathcal{V}^{(\tau_i)}, F_\bullet^{(\tau_i)}))) \xrightarrow{\overset{q.i.}{\simeq}} \mathrm{DR}_D(\mathcal{V}^{(\tau_i+1)}, F_\bullet^{(\tau_i+1)}), \tag{5.3.9}$$

as filtered complex of $\mathbb{C}_X$-modules.

For simplicity, set

$$(\mathcal{B}^\bullet, F_\bullet) := \mathrm{SP}_D((\mathcal{D}_X, F_\bullet) \otimes_{(\mathcal{O}_X, F_\bullet)} (\mathcal{V}^{(\tau_i)}, F_\bullet^{(\tau_i)}) \longrightarrow (\mathcal{D}_X, F_\bullet) \otimes_{(V_0^D\mathcal{D}_X, F_\bullet)} (\mathcal{V}^{(\tau_i)}, F_\bullet^{(\tau_i)}).$$

Hence, by Lemma 5.3.4, $\mathcal{B}^\bullet$ is an exact complex of $(\mathcal{D}_X, F_\bullet)$-modules (i.e. every subcomplex in the filtration is also exact). So $\mathrm{Gr}_\bullet^F\mathcal{B}^\bullet$ is an exact complex of $\mathrm{Gr}_\bullet^F\mathcal{D}_X$-modules.

By definition of associated graded modules, we easily see

$$\mathrm{Gr}^F_\bullet \mathrm{DR}\,(\mathcal{B}^\bullet, F_\bullet) \simeq \mathrm{Gr}^F_\bullet \mathrm{DR}\,(\mathcal{D}_X, F_\bullet) \otimes_{\mathrm{Gr}^F_\bullet \mathcal{D}_X} \mathrm{Gr}^F_\bullet \mathcal{B}^\bullet.$$

On the other hand, we also know

$$\begin{aligned}\mathrm{Gr}^F_\bullet \mathrm{DR}\,(\mathcal{D}_X, F_\bullet) = [\mathrm{Gr}^F_\bullet \mathcal{D}_X \longrightarrow \mathrm{Gr}^F_\bullet((\mathcal{D}_X, F_\bullet) \otimes_{(\mathcal{O}_X, F_\bullet)} (\Omega^1_X, F_\bullet)] \longrightarrow \cdots \\ \longrightarrow \mathrm{Gr}^F_\bullet((\mathcal{D}_X, F_\bullet) \otimes_{(\mathcal{O}_X, F_\bullet)} (\Omega^n_X, F_\bullet)][n].\end{aligned}$$

From it, we see each term of $\mathrm{Gr}^F_\bullet \mathrm{DR}\,(\mathcal{D}_X, F_\bullet)$ is a locally free $\mathrm{Gr}^F_\bullet \mathcal{D}_X$-module. Therefore, $\mathrm{Gr}^F_\bullet \mathrm{DR}\,(\mathcal{D}_X, F_\bullet) \otimes_{\mathrm{Gr}^F_\bullet \mathcal{D}_X} \bullet$ is exact. So $\mathrm{DR}\,(\mathcal{D}_X, F_\bullet) \otimes_{(\mathcal{D}_X, F_\bullet)} \bullet$ is filtered exact. Equivalently

$$\begin{array}{c}\mathrm{DR}\,((\mathcal{D}_X, F_\bullet) \otimes_{(V^D_0 \mathcal{D}_X, F_\bullet)} (\mathcal{V}^{(\tau_i)}, F^{(\tau_i)}_\bullet)) \\ \stackrel{q.i.}{\simeq}\Big\uparrow \\ \mathrm{DR}\,(\mathrm{SP}_D((\mathcal{D}_X, F_\bullet) \otimes_{(\mathcal{O}_X, F_\bullet)} (\mathcal{V}^{(\tau_i)}, F^{(\tau_i)}_\bullet))).\end{array} \tag{5.3.10}$$

Combining (5.3.9) and (5.3.10), the proof is finished. □

Trivial Filtrations of $\mathcal{V}^{(\tau_i)}$. Since $\mathcal{V}^{(\tau_i)}$ is a locally free $\mathcal{O}_X$-module for each (τ_i), $\mathcal{V}^{(\tau_i)}$ has a trivial (coherent) filtration as a $V^D_0 \mathcal{D}_X$-module; that is

$$G_p \mathcal{V}^{(\tau_i)} = \begin{cases} \mathcal{V}^{(\tau_i)} & \text{if } p \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

For indices (τ_i) satisfying $\tau_i \leq [-1, 0)$ for all i, we set

$$(\mathcal{V}(*D), G^{(\tau_i)}_\bullet) := (\mathcal{D}_X, F_\bullet) \otimes_{(V^D_0 \mathcal{D}_X, F_\bullet)} (\mathcal{V}^{(\tau_i)}, G_\bullet).$$

It is not hard to see that Lemma 5.3.4, Lemma 5.3.6 and (5.3.7) still hold if $F^{(\tau_i)}_\bullet$ is replaced by $G_\bullet$. Therefore, the above filtered comparison theorem still holds by the same proof if we use $G_\bullet$ instead.

Proposition 5.3.11. *For any (τ_i), the natural morphism*

$$\mathrm{DR}_D(\mathcal{V}^{(\tau_i+1)}, G_\bullet) \xrightarrow{\stackrel{q.i.}{\simeq}} \mathrm{DR}\,((\mathcal{D}_X, F_\bullet) \otimes_{(V^D_0 \mathcal{D}_X, F_\bullet)} (\mathcal{V}^{(\tau_i)}, G_\bullet))$$

is a filtered quasi-isomorphisms of filtered complexes of $\mathbb{C}_X$-modules. In particular, if the index (τ_i) satisfies that for all i $\tau_i \le [-1,0)$, then we have

$$\mathrm{DR}_D(\mathcal{V}^{(\tau_i+1)}, G_\bullet) \xrightarrow[]{\overset{q.i.}{\simeq}} \mathrm{DR}\,(\mathcal{V}(*D), G_\bullet^{(\tau_i)})).$$

Remark 5.3.12. Since only $G_\bullet$ are used, above theorem works for any flat bundle $\mathcal{V}$ on U. Therefore, if the filtration structures are forgotten, we have

$$\mathrm{DR}_D(\mathcal{V}^{(\tau_i+1)}) \xrightarrow[]{\overset{q.i.}{\simeq}} \mathrm{DR}\,(\mathcal{D}_X \otimes_{V_0^D\mathcal{D}_X} \mathcal{V}^{(\tau_i)})$$

which specializes to Proposition 4.4.7 and Proposition 4.4.10.

For $I \subseteq \{1,\dots,n\}$, we set

$$D_I = \sum_{i\notin I} D_i.$$

With this notation, we have two open embeddings,

$$X \setminus D \xhookrightarrow{j_1} X \setminus D_I \xhookrightarrow{j_2} X.$$

Theorem 4.4.9 tells us how to present $j_{2!}Rj_{1*}L$ in forms of *Log De Rham* complexes of $\mathcal{V}^{(\tau_i)}$. How about $Rj_{1*}j_{2!}L$?

Well, by Verdier duality of constructible sheaves,

$$Rj_{1*}j_{2!}L[n] \simeq \mathbf{D}^V(j_{1!}Rj_{2*}L^*[n]),$$

where $\mathbf{D}^V$ is the Verdier duality functor and L^* is the dual local system of L. For any r-tuple (τ_i) such that $\tau_i \le [0,1)$ for $i \in I$, and $[0,1) < \tau_i$ for $i \notin I$, by Theorem 4.4.9 and the above Remark, we obtain

$$Rj_{2*}j_{1!}L[n] \overset{q.i.}{\simeq} \mathrm{DR}\,(\mathbb{D}(\mathcal{D}_X \otimes_{V_0^D\mathcal{D}_X} \mathcal{V}^{*(\tau_i-1)})).$$

By Theorem 4.2.21, we also know

$$\mathbb{D}(\mathcal{D}_X \otimes_{V_0^D\mathcal{D}_X} \mathcal{V}^{*(\tau_i-1)}) \overset{q.i.}{\simeq} \mathcal{D}_X \otimes_{V_0^D\mathcal{D}_X} \mathcal{V}^{(-\tau_i)}.$$

Combining the above two observations together, we have proved the following comparison result.

Proposition 5.3.13. *For any r-tuple (τ_i) such that $\tau_i \leq [0,1)$ for $i \notin I$, and $[0,1) < \tau_i$ for $i \in I$,*

$$Rj_{2*}j_{1!}L[n] \overset{q.i.}{\simeq} \mathrm{DR}_D(\mathcal{V}^{(-\tau_i+1)}) \xrightarrow{\overset{q.i.}{\simeq}} \mathrm{DR}\,(\mathcal{D}_X \otimes_{V_0^D\mathcal{D}_X} \mathcal{V}^{(-\tau_i)}).$$

In particular,

$$Rj_{2*}j_{1!}L[n] \overset{q.i.}{\simeq} \mathrm{DR}_D(\mathcal{V}^{(0,1]})(D_I).$$

If D_I and D_{I^C} are considered together (I^C is the complement of I in {1,. . . ,n}), we have commutative diagram

$$\begin{array}{ccc} X \setminus D & \xhookrightarrow{j_1} & X \setminus D_I \\ \Big\downarrow{\scriptstyle k_1} & & \Big\downarrow{\scriptstyle j_2} \\ X \setminus D_{I^C} & \xhookrightarrow{k_2} & X. \end{array}$$

After combining Theorem 4.4.9 and Proposition 5.3.13, we obtain a constructive proof of the following easy corollary (see also [**KS90**, Proposition 3.1.9(ii)])

Corollary 5.3.14. *If L is a local system on $X \setminus D$, then*

$$Rk_{2*}k_{1!}L[n] \overset{q.i.}{\simeq} j_{2!}Rj_{1*}L[n].$$

In short, we have known all interesting perverse sheaves (with stratification (X, D)) coming from $L[n]$ in forms of Log *de Rham* complexes of $\mathcal{V}^{(\tau_i)}$ and their images under the *Rieman-Hilbert* correspondence.

5.4. Base change

In this section, following the idea of Kawamata, we will obtain a base change formula for log-VHS which will be used in the proof of the vanishing theorem in §6.3.

Assume $f : (Y, E) \longrightarrow (X, D)$ is a morphism of smooth log pairs; i.e. f is morphism of pairs such that D and $E = f^{-1}D$ are both reduced SNC divisors. Assume that $E = \sum_{i'=1}^{r'} E_{i'}$ and $D = \sum_{i=1}^{r} D_i$. Suppose $V = (\mathcal{V}, F_\bullet, \mathbb{V})$ is PVHS defined on $X \setminus D$. The pull-back f^*V is also a PVHS on $Y \setminus E$. To simplify notation, we write

$$F_{\bullet Y} := F_\bullet f^*V.$$

Other such pullbacks will also be denoted similarly.

Example 5.4.1 (Unipotent Case)**.** If all the local monodromies of V are assumed to be unipotent, then so are those of f^*V by local calculation. In this case the filtration descends to be $\mathbb{Z}^r$-indexed. Namely,

$$F_\bullet^B = F_\bullet^0 \otimes \mathcal{O}_X(-\lfloor B \rfloor).$$

By calculating the residues locally and Lemma 5.1.2 (2), we have

$$F_{\bullet Y}^0 = f^* F_\bullet^0,$$

and

$$F_{\bullet Y}^{B'} = f^* F_\bullet^B,$$

where $B' = f^* \lfloor B \rfloor$.

Inspired by Kawamata's idea, we obtain the following base-change lemma. See [**Kaw02**, Lemma 4.1].

Lemma 5.4.2. *Suppose* $\{s_1, \ldots, s_k\}$ *is the local basis of* $F_\bullet^{(t_i)}$ *on* U *in* Lemma 5.1.11 *for some fixed* $(t_i) \in \mathbb{R}^r$*. Then*
(1) $\mathrm{ord}(f^* s_j) = f^* \mathrm{ord}(s_j)$*;*
(2) the basis $\{f^* s_1, \ldots, f^* s_k\}$ *of* $f^*(F_\bullet^{(t_i)})$ *on* U' *(*$f(U') \subset U$*) satisfy that*

$$(\prod_{i'}^{r'} y_{i'}^{\lfloor d_{i'} - \mathrm{ord}_{i'}(f^* s_1) \rfloor}) f^* s_1, \ldots, (\prod_{i'}^{r'} y_{i'}^{\lfloor d_{i'} - \mathrm{ord}_{i'}(f^* s_k) \rfloor}) f^* s_k$$

generates $F_{\bullet Y}^{(d_{i'})}$ *freely for any* $(d_{i'}) = (d_{i'})_{i'=1}^{r'} \in \mathbb{R}^{r'}$*. Here the* $y_{i'}$ *are defining functions of* $E_{i'}$ *on* U'*.*

Proof. First, clearly $f^*(\mathcal{V}^{(\tau_i)})$ is also a logarithmic extension of $f^*\mathcal{V}$. Hence, by Lemma 3.2.7

$$f^*(\mathcal{V}^{(\tau_i)}) \otimes \mathcal{O}_Y(*E) \simeq (f^*\mathcal{V})(*E).$$

Hence orders of $f^* s_j$ make sense.

We can assume $X = \Delta^n$ and $Y = \Delta^{n'}$. Let $\pi : X' \longrightarrow X$ be the branched covering (branched along D_i's) making the monodromies unipotent upstairs as in (5.1.5). By construction, s_j is a G-invariant section of $F_{\bullet X'}^{\pi^* \mathrm{ord}(s_i)} = F_{\bullet X'}^0(-\pi^* \mathrm{ord}(s_i))$ (see the proof of Lemma 5.1.11).

Claim. *There exists another branched covering*

$$\pi' : Y' \longrightarrow Y$$

branched along $E_{i'}$'s such that the following diagram

$$\begin{array}{ccc} Y' & \xrightarrow{f'} & X' \\ {\scriptstyle \pi'}\downarrow & & \downarrow{\scriptstyle \pi} \\ Y & \xrightarrow[f]{} & X \end{array}$$

commutes and f' also is a morphism of smooth log pairs $(Y', (\pi'^*E)_{\text{red}})$ and $(X', (\pi^*D)_{\text{red}})$.

Proof of Claim. Suppose the coordinates of X, X', Y, and Y' are $(x_1, \dots, x_n)$ $(z_1, \dots, z_n)$, $(y_1, \dots, y_{n'})$ and $(w_1, \dots, w_{n'})$ respectively, and suppose $D = (x_1 \cdot ... \cdot x_r = 0)$ and $E = (y_1 \cdot ... \cdot y_{r'} = 0)$.

First, we assume π is the m-th cyclic covering along D_1, i.e.

$$\pi^* x_i = \begin{cases} z_i^m & \text{if} \quad i = 1 \\ z_i & \quad \text{otherwise,} \end{cases}$$

Since f is a morphism of smooth log pairs, we can assume that

$$f^* D_1 = g \cdot y_1^{\beta_1} \cdots y_k^{\beta_k}$$

for some $k \leq r'$, and a nowhere vanishing holomorphic function g on Y. Take π' to be the branched covering corresponding to $\sqrt[m]{g}$, $\sqrt[m]{y_1}, ...,$ $\sqrt[m]{y_k}$. Then f' can be defined to be $\sqrt[m]{f}$ accordingly. By construction, $\pi'^* E$ is also normal crossing.

In general, since π is a composition of a series of cyclic coverings, π' and f' can be constructed inductively. Furthermore, π' is also a composition of a series of cyclic coverings and unbranched coverings. □

Assume the Galois group of π' is G'. Since the monodromies are unipotent,

$$f'^*(F_{\bullet X'}^{\pi^* \text{ord}(s_j)}) = F_{\bullet Y'}^{f'^* \pi^* \text{ord}(s_j)} = F_{\bullet Y'}^0(-f'^* \pi^* \text{ord}(s_j)).$$

Hence,

$$\text{ord}(f'^* s_j) = f'^* \pi^* \text{ord}(s_j).$$

On the other hand, since Y is the quotient space of Y' with respect to G'-action, $f'^* s_j$ descends to $f^* s_j$. Moreover, since π' is a composition of a series of cyclic coverings and unbranched coverings (but unbranched coverings just bring extra copies of Y since Y is simply connected), G' acts on $F^{B}_{\bullet Y'}$ for any $\mathbb{R}-$divesor B supported on $(\pi'^{*}E)_{\text{red}}$. Hence by (5.1.7)

$$(\pi_* F_{\bullet Y'}^{f'^*\pi^*\mathrm{ord}(s_j)})^{G'} = {}^{f^*\mathrm{ord}(s_j)}F_{\bullet Y}.$$

Therefore, we have

$$\mathrm{ord}(f^* s_j) = f^*\mathrm{ord}(s_j).$$

For the second statement, we write

$$\mathrm{ord}(f'^* s_j) = \sum_{i'} \alpha_{i'j} E'_{i'},$$

where $E'_{i'} = \pi'^{*}(E_{i'})_{red}$. Since $\mathrm{ord}(f'^* s_j) = f'^*\pi^*\mathrm{ord}(s_j)$, $\{(\prod_{i'}^{r'} w_{i'}^{-\alpha_{i'j}}) f'^* s_j\}_{j=1}^{k}$ form a basis of $F^0_{\bullet Y'}$, where $w_{i'}$ is the defining function of $E'_{i'}$. Since $f'^* s_j$'s are G'-invariant, the second statement follows. □

Corollary 5.4.3.

$$F_{\bullet Y}^{(d_{i'})} = \sum_{(t_i)\in\mathbb{R}^r} f^*(F_{\bullet}^{(t_i)}) \otimes \mathcal{O}_Y(-\lfloor \sum_{i'=1}^{r'} d_{i'} E_{i'} + \sum_i (1-t_i) f^* D_i \rfloor),$$

where $(d_{i'}) \in \mathbb{R}^{r'}$. Here, the sum of the right hand side of the equation happens inside the Deligne meromorphic extension of $f^*\mathcal{V}$.

Proof. In sense of Lemma 5.1.2 (2), it is enough to prove the above equation locally. Suppose $\{e_1, \ldots, e_k\}$ are the local basis of $F^0_{\bullet}$ as in Lemma 5.1.11. Set

$$f^* D_i = \sum_{i'=1}^{r'} a_{ii'} E_{i'}.$$

By Lemma 5.4.2, the left hand side is generated by

$$\{(\prod_{i'} y_{i'}{}^{\lfloor d_{i'} - \mathrm{ord}_{i'}(f^* e_j)\rfloor}) f^* e_j\}_{j=1}^{k},$$

where $y_{i'}$ are the local defining function of $E_{i'}$. Meanwhile, the right hand side is generated by

$$\{(\prod_{i'} y_{i'}{}^{\sum_i \lfloor t_i - \mathrm{ord}_i(e_j)\rfloor a_{ii'} + \lfloor d_{i'} + \sum_i (1-t_i) a_{ii'}\rfloor}) f^* e_j\}.$$

By Lemma 5.4.2 (1), we know

$$\mathrm{ord}_{i'}(\pi^* e_j) = \sum_i a_{ii'} \mathrm{ord}_i(e_j),$$

hence

$$\sum_i \lfloor t_i - \mathrm{ord}_i(e_j) \rfloor a_{ii'} + \lfloor d_{i'} + \sum_i (1 - t_i) a_{ii'} \rfloor - \lfloor d_{i'} - \mathrm{ord}_{i'}(f^* e_j) \rfloor \geq 0,$$

for any $(t_i) \in \mathbb{R}^r$. So we have one inclusion

$$F_{\bullet Y}^{(d_i)} \supseteq \sum_t F_{\bullet}^{(t_i)} \otimes \mathcal{O}_Y(\lfloor \sum_{i'} d_{i'} E_{i'} + \sum_i (1 - t_i) f^* D_i \rfloor). \tag{5.4.4}$$

On the other hand, for a fixed j, if $(t_i^j) = (\mathrm{ord}_i(e_j) - \epsilon + 1)$ for $0 < \epsilon \ll 1$, then

$$\sum_i \lfloor t_i^j - \mathrm{ord}_i(e_j) \rfloor a_{ii'} + \lfloor d_{i'} + \sum_i (1 - t_i^j) a_{ii'} \rfloor = \lfloor d_{i'} - \mathrm{ord}_{i'}(f^* e_j) \rfloor.$$

Therefore, the right hand side of (5.4.4) includes the left hand side, because it contains the basis of the left hand side. □

CHAPTER 6

Vanishing Theorems for Hodge modules

From this chapter, we go back to algebraic category. Throughout this section, we fix a PVHS $V=(\mathcal{V},F_\bullet,\mathbb{V})$ on U^{an} (the corresponding analytic space of U), where $U=X\setminus D$ for an algebraic smooth log pair $(X,D=\sum_{i=1}^{r}D_i)$. For complete varieties, it is well known that coherent analytic sheaves and coherent algebraic sheaves make no difference by GAGA principle. We will see that even for non-complete varieties, PVHS satisfy the GAGA principle automatically because of the extendability.

6.1. GAGA Principle and the Algebraicity of $F_\bullet^{(t_i)}$

By taking resolutions of singularities, we have an embedding of smooth log pairs,

$$i:(X,D)\hookrightarrow(\overline{X},\overline{D})$$

with $\overline{X}$ complete such that $\overline{X}\setminus\overline{D}=U$ and $\overline{D}|_X=D$. By Theorem 5.1.3, we have multi-indexed (holomorphic) extensions of $F_\bullet$ on both X^{an} and $\overline{X}^{\mathrm{an}}$ indexed by $\mathbb{R}$-divisors supported on D and $\overline{D}$ respectively. Denote them by $F_\bullet^B$ on X^{an} for any $\mathbb{R}$-divisor B supported on D and $F_\bullet^{\overline{B}}$ on $\overline{X}^{\mathrm{an}}$ for any $\mathbb{R}$-divisor $\overline{B}$ supported on $\overline{D}$. Namely, they are distinguished by their index divisors. By GAGA principle, $F_\bullet^{\overline{B}}$ are algebraic on $\overline{X}$. Clearly, for any $\mathbb{R}$-divisor B supported on D, take $\overline{B}_1$ to be the closure of B in $\overline{X}$. Then

$$F_\bullet^B=i^*F_\bullet^{\overline{B}_1}.$$

Hence, $F_\bullet^B$ are also algebraic on X. In particular, $\mathcal{V}$ and $F_\bullet$ are algebraic on U. This tells us that the quasi-unipotency of all monodromies along the boundary divisor D forces $\mathcal{V}$ and $F_\bullet$ becoming algebraic.

Example 6.1.1. Let $f:Y\longrightarrow X$ be a surjective morphism between smooth projective varieties. Assume there exists a SNC divisor D such that f is smooth over $U=X\setminus D$. Set $d=\dim Y-\dim X$. Then $\mathcal{V}=R^{d+i}f_*(\mathbb{Q}_Y)|_U\otimes\mathcal{O}_U$ underlies a PVHS, denoted by V_f^i

(the Gauss-Manin connection). It is proved in [Kol86] that

$$R^i f_*\omega_{Y/X} = F^0_{-d}V^i_f.$$

Adding indices. If C is a reduced simple normal crossings divisor such that $C+D$ is also SNC, then clearly $F_\bullet$ have extensions indexed by $\mathbb{R}$-divisors supported on $\mathrm{supp}(C+D)$ by extending $F_\bullet|_{U\setminus C}$. Since V has trivial local monodromies along C_i, irreducible components of C, the filtrations along C_i are just C_i-adic filtration, i.e.

$$F_\bullet^{C'+D'} = F_\bullet^{D'} \otimes \mathcal{O}(-\lfloor C' \rfloor)$$

and

$${}^{C'+D'}F_\bullet = {}^{D'}F_\bullet \otimes \mathcal{O}(-\lceil C' \rceil),$$

where C' and D' are $\mathbb{R}$-divisors supported on C and D respectively.

It is well known that $R^i f_*\omega_{Y/X}$ is birational invariant. More generally, $F_\bullet^B$ are also birational invariant. To be precise, we have

Proposition 6.1.2. *Suppose B is an $\mathbb{R}$-divisor supported on D. If $\mu : Y \longrightarrow X$ is a log resolution of D with $E = (\mu^*D)_{red}$, then*

$$\mu_*(F_{\bullet Y}^{B'} \otimes \omega_{Y/X}) = F_\bullet^B,$$

where $B' = \mu^*B$, and $F_{\bullet Y}^{B'}$ is the extenstion of $\pi^*_{|Y\setminus E}F_\bullet$ on Y with index B'.

Proof. By the projection formula and (5.1.10), it suffices to assume $\lfloor B \rfloor = 0$. Write

$$E = D' + E',$$

where D' is the birational transform of D, and $E' = \sum_{i'} E'_{i'}$ is exceptional. For simplicity, we can assume that E' has only one component. There is no essential difference in general. If $B = \sum_i t_i D_i$, then

$$B' = \sum_i t_i D'_i + \sum_i a_i t_i E'.$$

Suppose $\{s_1, ..., s_k\}$ is the local basis of $F_\bullet^B$ in Lemma 5.4.2 and

$$\mathrm{ord}_i(s_j) = \alpha_{ij} \in (t_i - 1, t_i].$$

Hence, by Lemma 5.4.2 (1),

$$\mathrm{ord}_{E'}(\mu^* s_j) = \sum_i a_i \alpha_{ij}.$$

Therefore, by Lemma 5.4.2 (2) we know that

$$y^{\lfloor\sum_i a_i(t_i-\alpha_{i1})\rfloor}\mu^*s_1, .., y^{\lfloor\sum_i a_i(t_i-\alpha_{ik})\rfloor}\mu^*s_k$$

generate $F^{B'}_{\bullet Y}$, where y is the defining function of E'. Since $t_i - \alpha_{ij} \geq 0$,

$$F^{B'}_{\bullet Y} \subseteq \mu^*(F^B_\bullet).$$

On the other hand, by local calculation (see for instance [**Laz04**, Lemma 9.2.19]), we see

$$\omega_{Y/X} \geq (\sum_i a_i - 1)E'.$$

Moreover since $0 \leq t_i - \alpha_{ij} < 1$, we get an inequality

$$(\sum_i a_i - 1)E' \geq \lfloor\sum_i a_i(t_i - \alpha_{ij})\rfloor E'.$$

Consequently, we have

$$\mu^* F^B_\bullet \subseteq F^{B'}_{\bullet Y} \otimes \omega_{Y/X} \subseteq \mu^*(F^B_\bullet) \otimes \omega_{Y/X}.$$

After pushing forward the above inclusions, the proof is finished by the projection formula.

□

6.2. Hodge modules and Vanishing theorems

In previous chapters, we have already mentioned Hodge modules. Now we move to a recall of the main notions and results (especially vanishing theorems) from the theory of mixed Hodge modules that are used in this thesis.

Let X be a smooth algebraic variety over $\mathbb{C}$. A pure Hodge modules[1] M is a triple $M = (\mathcal{M}, F_\bullet, K_\mathbb{Q})$ consisting of

(1) A coherent filtered regular holonomic $\mathcal{D}_X$-module $(\mathcal{M}, F_\bullet)$;
(2) A $\mathbb{Q}$-perverse sheaf K on X whose complexification corresponds to $\mathcal{M}$ via the Riemann-Hilbert correspondence, so that there is an isomorphism

$$\mathrm{DR}(M) \xrightarrow{\simeq} K \otimes \mathbb{C}.$$

In addition, they are subject to a list of conditions, which are defined by induction on the dimension of the support of M. If X is a point, a pure Hodge module is simply a

[1]Since we are in algebraic category, pure Hodge modules are assumed to be polarizable as in Saito's original definition; see [**Sai90**].

polarizable Hodge structure. In general it is required, roughly speaking, that the nearby and vanishing cycles associated to M with respect to any locally defined holomorphic function are again Hodge modules, now on a variety of smaller dimension; See [**Sch14**] for a readable discussion or originally [**Sai88**].

Furthermore, M. Saito introduced in [**Sai90**] mixed Hodge modules on X. In addition to data as in (1) and (2) above, in this case a third main component is:

(3) A finite increasing weight filtration $W_\bullet M$ of M by objects of the same kind, such that the graded quotients are pure Hodge modules.

Again, a mixed Hodge module on a point is a graded-polarizable mixed Hodge structure, while in general these components are subject to several conditions defined by induction on the dimension of the support of M, involving the graded quotients of the nearby and vanishing cycles of M; see [**Sai90**] for detail.

Hodge modules (pure or mixed) form an abelian category. In pure case, the abelian category are semi-simple, with simple objects consisting of pure Hodge modules with strict support. A pure Hodge module supported precisely along a subvariety Z is said to have strict support if it has no nontrivial subobjects or quotient objects whose support is Z. Roughly speaking, pure Hodge modules with strict support Z are just PVHS generically defined on the smooth locus of Z. To be precise, Saito proved the following structure theorem for pure Hodge modules.

Theorem 6.2.1 (Simple objects, [**Sai90**, Theorem 3.21])**.** *Let X be a smooth complex variety, and Z an irreducible closed subvariety of X. Then*

(1) every polarizable VHS defined on a nonempty open set of Z extends uniquely to a pure Hodge module with strict support Z;

(2) Conversely, every pure Hodge module with strict support Z is obtained in this way.

Immediate from the above theorem, we know all pure Hodge modules on X are algebraic because we can always make X complete and smooth by adding boundary divisors.

Example 6.2.2 (Examples of Hodge modules)**.**

(1) All PVHS on X are pure Hodge modules strictly supported on X, in particular the trivial one $\mathbb{Q}_X^H =: (\mathcal{O}_X, F_\bullet, \mathbb{Q}_X)$ (immediate from the above theorem);

(2) when V is a PVHS on $X \setminus D$ for a SNC divisor D, $M = (\mathcal{V}_{\min}, F_\bullet, j_{!*}K_{\mathbb{Q}})$ is a pure Hodge modules strictly supported on X; $F_\bullet$ is defined as in §5.2;

(3) $(\mathcal{V}(*D), F_\bullet, Rj_*K_\mathbb{Q})$ is a mixed Hodge module; again see §5.2.

Usually, it is almost impossible to check a filtered $\mathcal{D}_X$ underlies a Hodge module directly by the inductive definition, because the nearby cycle and vanishing cycle might be more complicated even in trivial case.

Now assume $M = (\mathcal{M}, F_\bullet, K_\mathbb{Q})$ is a Hodge module (mixed or pure. Set

$$S(M) = F_{p(M)}\mathcal{M}$$

where $p(M)$ is defined to be

$$p(M) = min\{q|F_q\mathcal{M} \neq 0\}.$$

Theorem 6.2.3 (Saito Vanishing)**.** *Let X be a projective smooth variety over $\mathbb{C}$ and L an ample divisor. Then*

$$H^i(X, S(M) \otimes \omega_X(L)) = 0$$

for all $i > 0$.

In fact, the above theorem just a small part of the complete Saito vanishing, but it is enough for the use of this thesis. See [**Sai90**, §2.g] for the complete statement.

Example 6.2.4. Let $f : Y \longrightarrow X$ be a surjective morphism between smooth projective varieties of relative dimension d. Since f is generically smooth, $R^{d+i}f_*(\mathbb{Q}_Y)$ generically underlies a PVHS (the Gauss-Manin connection) on X. By Theorem 6.2.1, it uniquely determines a pure Hodge module M^i strictly supported on X. Saito proved in [**Sai91**]

$$S(M^i) = R^i f_*\omega_{Y/X}.$$

By the above example, geometrically Saito vanishing specializes to Kollár vanishing for higher direct images of dualizing sheaves. More generally, we proved a Kawamata-Viehweg type generalization of Saito vanishing in [**Wu15**] (independently by Suh in [**Suh15**]).

Theorem 6.2.5. *Let X be a projective smooth variety over $\mathbb{C}$ and L an nef and big divisor. If M is a pure Hodge modules strictly supported on X, then*

$$H^i(X, S(M) \otimes \omega_X(L)) = 0$$

for all $i > 0$.

Indeed, smoothness of X are not necessary in both of the above vanishing theorems, because pure Hodge modules can be naturally defined on singular varieties. We will see the above vanishing can be further generalized by using multiplier subsheaves.

6.3. An $\mathbb{R}$-Kawamata–Viehweg Type Vanishing

In this section, we go back to normal crossing case and prove an $\mathbb{R}$-Kawamata–Viehweg Type Vanishing.

Let $V = (\mathcal{V}, F_\bullet, \mathbb{V})$ be a PVHS on $U = X \setminus D$. By Theorem 6.2.1, V extends to a pure Hodge module $M = (\mathcal{V}_{\min} \otimes \omega_X, F_\bullet, j_{!*}\mathbb{V})$ strict supported on X.

As in (5.2.2), we know

$$S(M) = F^0_{p(V)}V, \tag{6.3.1}$$

where $p(V)$ is defined to be

$$p(V) = min\{q|F_qV \neq 0\}.$$

In order to simplify notations, we set

$$S(V)^B = F^B_{p(V)}V$$

and

$$S(V)^{(t_i)} = F^{(t_i)}_{p(V)}V$$

for an $\mathbb{R}$-divisor B supported on D and index $(t_i) \in \mathbb{R}^r$. Furthermore, if $\mu : X' \longrightarrow X$ is a log-resolution of D, for B' an $\mathbb{R}$-divisor supported on $\mathrm{supp}(\mu^*D)$, then use $S(V)^{B'}$ to denote the corresponding extension on X' (since μ is identical outside D).

Theorem 6.3.2. *Assume X is projective. Let L be an integral divisor and B an $\mathbb{R}$-divisor supported on D. Suppose $L - B$ is nef and big. Then*

$$H^q(X, S(V)^B \otimes \omega_X(L)) = 0$$

for $q > 0$.

Proof. First, we assume that B is a $\mathbb{Q}$-divisor supported on D and $L - B$ is ample. Take the Kawamata covering

$$\pi : Y \longrightarrow X$$

so that π^*B is integral. By the construction of Kawamata covering (see [**Kaw81**, Theorem 17]), there exists an SNC divisor $\overline{B}$ such that $\mathrm{supp}(\overline{B}) \supseteq \mathrm{supp}(D)$ and π is étale over $U = X \setminus \overline{B}$. By adding indices, we can assume $D = \overline{B}$. Let G be the Galois group of π.

Therefore, by Corollary 5.4.3 and (5.1.10),

$$S(V')^0 = \sum_{(t_i)\in[0,1)^r} \pi^*(S(V)^{(t_i)}) \otimes \mathcal{O}_Y(-\lfloor\sum_{i=1}^{r}(1-t_i)\pi^*D_i\rfloor), \tag{6.3.3}$$

where V' is the pull-back PVHS of V via $\pi_{|U}$. By (6.3.1),

$$S(V')^0 = S(M'),$$

where M' is the pure Hodge module extended from V'. By Theorem 6.2.3,

$$H^q(Y, S(V')^0 \otimes \omega_Y(\pi^*L - \pi^*B)) = 0$$

for $q > 0$. Again by the construction of the Kawamata covering,

$$\pi^*\omega_X(D) = \omega_Y(E),$$

where $E = (\pi^*D)_{\mathrm{red}}$. Therefore,

$$H^q(Y, \sum_{(t_i)\in[0,1)^r} \pi^*(S(V)^{(t_i)} \otimes \omega_X(L)) \otimes \mathcal{O}_Y(-E - \lfloor\sum_{i=1}^{r}(1-t_i)\pi^*D_i\rfloor + \pi^*(D-B))) = 0$$

for $q > 0$ by (6.3.3).

On the other hand, from the construction of Kawamata covering, it is not hard to see that if A is a integral divisor supported on E, then $\mathcal{O}_Y(A)$ has a G-action and the G-invariant part of $\pi_*\mathcal{O}_Y(A)$ is

$$\pi_*\mathcal{O}_Y(A)^G = \mathcal{O}_X(A'),$$

where A' is a divisor on X such that

$$\pi^*A' \leq A.$$

So $S(V)^{(t_i)} \otimes \omega_X(L) \otimes \pi_*(\mathcal{O}_Y(-E - \lfloor\sum_{i=1}^{r}(1-t_i)\pi^*D_i\rfloor) + \pi^*(D-B))$ has a G-action, and its G-invariant part is

$$S(V)^{(t_i)} \otimes \omega_X(L - \lfloor B + \sum_{i=1}^{r}(1-t_i)D_i\rfloor).$$

Therefore, $\pi_*(S(V')^0 \otimes \omega_Y(\pi^*L - \pi^*B))$ also has a G-action and its G-invariant part is

$$\sum_{(t_i)\in[0,1)^r} S(V)^{(t_i)} \otimes \omega_X(L - \lfloor B + \sum_{i=1}^{r}(1-t_i)D_i\rfloor).$$

By (5.1.10) and the discreteness of the filtration, it is not hard to see that

$$\sum_{(t_i)\in[0,1)^r} S(V)^{(t_i)} \otimes \omega_X(L - \lfloor B + \sum_{i=1}^{r}(1-t_i)D_i\rfloor) = S(V)^B \otimes \omega_X(L).$$

Since π is finite, we know

$$H^q(X, \pi_*(S(V')^0 \otimes \omega_Y(\pi^*L - \pi^*B))) = 0$$

for $q > 0$. Since $\pi_*(S(V')^0 \otimes \omega_Y(\pi^*L - \pi^*B))^G$ is a direct summand of $\pi_*(S(V')^0 \otimes \omega_Y(\pi^*L - \pi^*B))$,

$$H^q(X, S(V)^B \otimes \omega_X(L)) = 0$$

for $q > 0$. In consequence, we have proved the theorem when $L - B$ is an ample $\mathbb{Q}$-divisor.

Furthermore, if $L - B$ is an ample $\mathbb{R}$-divisor, by the openness of the amplitude of $\mathbb{R}$-divisor, we can find a $\mathbb{Q}$-divisor B' by perturbing the coefficients of B such that $L - B'$ is still ample and

$$S(V)^B = S(V)^{B'}.$$

Again, we obtain

$$H^q(X, S(V)^B \otimes \omega_X(L)) = 0$$

for $q > 0$.

In general, if $L - B$ is a nef and big $\mathbb{R}$-divisor, then

$$L - B \equiv_{\text{num}} A + N$$

where A is an ample and N is an effective $\mathbb{R}$-divisor. Take a log resolution of $D + N$,

$$\mu : X' \longrightarrow X.$$

We can assume

$$\mu^*(L - B) \equiv_{\text{num}} A' + N'$$

where A' is an ample and N' is an effective $\mathbb{R}$-divisor on X'. Since $\mu^*(L - B)$ is nef,

$$\mu^*L - \mu^*B - \frac{1}{k}N$$

is ample for any sufficiently large k. After picking k large enough, by the discreteness of the multi-indexed filtration,

$$S(V)^{\mu^*B} = S(V)^{\mu^*B+\frac{1}{k}N}.$$

But since $\mu^*L - \mu^*B - \frac{1}{k}N$ is ample, we already know

$$H^q(X', S(V)^{\mu^*B+\frac{1}{k}N} \otimes \omega_{X'}(\mu^*L)) = 0$$

for $q > 0$. Choose a integral divisor H sufficiently ample, so that

$$R^q\mu_*(S(V)^{\mu^*B+\frac{1}{k}N} \otimes \omega_{X'}(\mu^*(L+H))$$

is globally generated and the Leray spectral sequence degenerates. Then

$$H^0(X, R^q\mu_*(S(V)^{\mu^*B+\frac{1}{k}N} \otimes \omega_{X'}(\mu^*(L+H))) = H^q(X', S(V)^{\mu^*B+\frac{1}{k}N} \otimes \omega_{X'}(\mu^*(L+H)))$$

But since $\mu^*(L+H) - \mu^*B - \frac{1}{k}N$ is also ample, we see

$$H^q(X', S(V)^{\mu^*B+\frac{1}{k}N} \otimes \omega_{X'}(\mu^*(L+H))) = 0,$$

for $q > 0$. Therefore,

$$R^q\mu_*(S(V)^{\mu^*B+\frac{1}{k}N} \otimes \omega_{X'}) = 0$$

for $q > 0$. By Proposition 6.1.2, we have

$$\mu_*(S(V)^{\mu^*B+\frac{1}{k}N} \otimes \omega_{X'}) = \mu_*(S(V)^{\mu^*B} \otimes \omega_{X'}) = S(V)^B \otimes \omega_X.$$

Consequently, by the degeneracy of the Leray spectral sequence again, we get

$$H^q(X, S(V)^B \otimes \omega_X(L)) = 0$$

for $q > 0$. □

If $M = \mathbb{Q}_X^H$ the trivial pure Hodge module, then the above theorem is exactly the $\mathbb{R}$-Kawamata-Viehweg Vanishing Theorem (see for instance [**Laz04**, Theorem 9.1.18]), because we know $S(\mathbb{Q}_X^H)^B = \mathcal{O}_X(-\lfloor B \rfloor)$.

CHAPTER 7

Multiplier Subsheaves, Nadel-type Vanishing And Applications

7.1. Multiplier subsheaves for pure Hodge modules

At this point, we established all we need to define multiplier submodules for Pure Hodge modules.

Let X be a smooth algebraic variety over $\mathbb{C}$ and M pure Hodge module strictly supported on X. Use V to denote the PVHS generically defined on X corresponding to M. By abuse of notations, V will also be used to denote the PVHS generically defined on X' whenever X' is birational to X.

Definition 7.1.1 (Multiplier subsheaves)**.** If B is an $\mathbb{R}$-divisor on X, then the multiplier submodules of M, $\mathcal{J}(M, B)$ associated to M and B is defined to be

$$\mathcal{J}(M, B) = \mu_*(S(V)^{\mu^* B} \otimes \omega_{Y/X}),$$

where $\mu : Y \longrightarrow X$ is a fixed log resolution of $\text{supp}(B) \cup \text{sing}(M)$.

By Proposition 6.1.2, it is obvious that $\mathcal{J}(M, B)$ is independent of choices of log resolutions. By birational invariance of $S(M)$ Furthermore, if B is effective, $\mathcal{J}(M, B)$ is canonically a coherent subsheaf of $\mathcal{J}(M, 0) = \mu_*(S(V)^0 \otimes \omega_{Y/X})$ by definition. By (6.3.1) and birational invariance of $S(M)$ (See e.g. [**Wu15**, Corollary 3.13]),

$$S(M) = \mathcal{J}(M, 0).$$

Hence, if B is effective, $\mathcal{J}(M, B)$ is a coherent subsheaves of $S(M)$.

Example 7.1.2. (1) If $M = \mathbb{Q}_X^H$ is the constant pure Hodge module, then $\mathcal{J}(\mathbb{Q}_X^H, B)$ is the usual multiplier ideal associated to B. This is the reason why they are named multiplier subsheaves.
(2) Suppose $f : Y \longrightarrow X$ is a surjective morphism of projective varieties with Y smooth as in Example 6.2.4. Thereafter, $J(M^i, B)$ gives a coherent submodule of $R^i f_* \omega_{Y/X}$ when

B is an effective $\mathbb{R}$-divisor.

More generally, we can also define Multiplier submodules for ideal sheaves.

Definition 7.1.3 (Multiplier submodules associated to an ideal sheaf)**.** Let I be a non-zero ideal sheaf and $b \in \mathbb{R}_+$. Take a log resolution $\mu : Y \longrightarrow X$ of $Z(I) \cup \mathrm{sing}(M)$ with $I \cdot \mathcal{O}_Y = \mathcal{O}_Y(-F)$, where $Z(\mathcal{I})$ is the scheme defined by I. Then the multiplier submodule associated to M and $\mathcal{I}^b$ is

$$\mathcal{J}(M, I^b) = \mu_*(S(V)^{bF} \otimes \omega_{Y/X}).$$

Again, $\mathcal{J}(M, I^b)$ does not depend on μ either and is canonically a subsheaf of $S(M)$. When $M = \mathbb{Q}^H_X$ and X is smooth,

$$\mathcal{J}(\mathbb{Q}^H_X, I^b) = \mathcal{I}(I^b)$$

the usual multiplier ideal associated to I^b.

The following proposition is about some easy properties of mutliplier subsheaves.

Proposition 7.1.4. *With notations as above, we have*

(i) If B and B' are two $\mathbb{R}$-divisors on X with B' effective, then

$$\mathcal{J}(M, B) = \mathcal{J}(M, B + \epsilon B')$$

for all $0 < \epsilon \ll 1$.

(ii) If B is an integral divisor, then

$$J(M, B) = S(M)(-B).$$

(iii) If $B_2 \leq B_1$ are $\mathbb{R}$-divisors on X, then

$$\mathcal{J}(M, B_1) \subseteq \mathcal{J}(M, B_2).$$

Proof. The statement (i) is clear from the discreteness of $F^{(t_i)}_\bullet$. (ii) follows from the definition of $S(M)$ and projection formula. (iii) follows from the filtration structure of the log-VHS. □

Theorem 7.1.5. *Let B be an $\mathbb{R}$-divisor on X, and let $\mu : Y \longrightarrow X$ be a log resolution of* $\mathrm{supp}(B) \cup \mathrm{sing}(M)$*. Then*

$$R^q\mu_*(S(V)^{\mu^*B} \otimes \omega_Y) = 0$$

for $q > 0$.

Proof. First assume that X and Y are projective. Pick a sufficiently ample integral divisor A such that $A-B$ is also ample. Hence, $\mu^*(A-B)$ is nef and big. By Theorem 6.3.2,

$$H^q(Y, S(V)^{\mu^*B} \otimes \omega_Y(\mu^*A)) = 0$$

for $q > 0$. Using the same trick as in the third part of the proof of Theorem 6.3.2 (or [**Laz04**, Lemma 4.3.10]), we obtain

$$R^q\mu_*(S(V)^{\mu^*B} \otimes \omega_Y) = 0$$

for $q > 0$. We have proved the statement under the projective hypothesis.

For the general situation, we are going to reduce it to the case just treated. Since this statement is local on X, we can assume X is affine. Then we can constructed the following fiber square,

$$\begin{array}{ccc} Y & \hookrightarrow & \overline{Y} \\ \downarrow{\scriptstyle \mu} & & \downarrow{\scriptstyle \overline{\mu}} \\ X & \hookrightarrow & \overline{X}, \end{array}$$

with $\overline{X}$ and $\overline{Y}$ projective. Since μ is a log resolution of $\mathrm{supp}(B) \cup \mathrm{sing}(M)$ (we can assume μ is a series of blow-ups along smooth center supported over singularities of $\mathrm{supp}(B) \cup \mathrm{sing}(M)$), we can also assume that $\overline{\mu}$ is a log resolution of $\overline{\mathrm{supp}(B) \cup \mathrm{sing}(M)} \cup (\overline{X} \setminus X)$. Since the projective case has been proved,

$$R^q\overline{\mu}_*(S(V)^{\overline{\mu}^*\overline{B}} \otimes \omega_{\overline{Y}}) = 0$$

for $q > 0$. Since

$$R^q\overline{\mu}_*(S(V)^{\overline{\mu}^*\overline{B}} \otimes \omega_{\overline{Y}})|_X = R^q\mu_*(S(V)^{\mu^*B} \otimes \omega_Y),$$

we know

$$R^q\mu_*(S(V)^{\mu^*B} \otimes \omega_Y) = 0.$$

□

Remark 7.1.6. A similar local vanishing for ideal sheaves is also true by imitating the proof of Theorem 7.1.5.

By Theorem 6.3.2 and Theorem 7.1.5, and the degeneracy of the Leray spectral sequence, we obtain the following Nadel type vanishing theorem:

Theorem 7.1.7. *Suppose X is projective. Let B be an $\mathbb{R}$-divisor on X, and let L be an integral divisor so that $L - B$ is nef and big. Then*

$$H^q(X, \mathcal{J}(M, B) \otimes \omega_X(L)) = 0$$

for $q > 0$.

When X is smooth and $M = \mathbb{Q}_X^H$, the above theorem is exacly the Nadel vanishing theorem for multiplier ideals. Because the proof of Theorem 6.3.2 only uses Saito vanishing, taking $B = 0$, it gives another proof of Theorem 6.2.5.

7.2. Multiplier subsheaves in Normal crossing case

When X is smooth of dimension n and $\mathrm{sing}(M) = D$ is a normal crossing divisor, by Theorem 5.1.3 $S(M) = S(V)^0$ are locally free. Under such setting, we have the following subaddition.

Proposition 7.2.1. *Let X be a smooth algebraic variety and M be a pure Hodge module strictly supported on X with $\mathrm{sing}(M) = D$ a reduced simple normal crossings divisor. Suppose B is an $\mathbb{R}$-divisor supported on D and F is another $\mathbb{R}$-divisor. Then*

$$\mathcal{J}(M, B + F) \subseteq \mathcal{J}(M, B) \otimes_{\mathcal{O}} \mathcal{I}(F)$$

where $\mathcal{I}(F)$ is the multiplier ideal associated to F. Furthermore, if all the local monodromies of M are unipotent and B is integral, then the equality holds, i.e.

$$\mathcal{J}(M, B + F) = \mathcal{J}(M, B) \otimes_{\mathcal{O}} \mathcal{I}(F).$$

Proof. Take a log resolution of $D + \mathrm{supp}(F)$, $\mu : X' \longrightarrow X$. Suppose $\{s_1, \ldots, s_k\}$ is the local basis $S(V)^B$ as in Lemma 5.1.11. Since $\mathrm{ord}(s_j) \leq B$, by Lemma 5.4.2(1) we know

$$\mu^*(F + B) - K_{X'/X} - \mathrm{ord}(\mu^* s_j) \geq \mu^* F - K_{X'/X},$$

where $K_{X'/X}$ is the relative canonical divisor of μ. Therefore, by Lemma 5.4.2(2), we know

$$S(V)^{\mu^*(B+F)} \otimes_{\mathcal{O}} \omega_{X'/X} \subseteq \mu^*(S(V)^B) \otimes_{\mathcal{O}} \omega_{X'/X}(-\lfloor \mu^* F \rfloor).$$

So the first assertion follows by projection formula.

For the second statement, since the monodromies are unipotent and B is integral

$$S(V)^{\mu^*(B+F)} \otimes_{\mathcal{O}} \omega_{X'/X} = \mu^*(S(V)^B) \omega_{X'/X}(-\lfloor \mu^* F \rfloor).$$

Hence the second statement is proved by projection formula again. □

By the above containment, we obtain that points of high multiplicity ensure non-triviality of multiplier submodules.

Corollary 7.2.2. *With notations as above, if F is effective and $mult_x(F) \geq n$ at some point $x \in X$, then*

$$\mathcal{J}(M, F) \subsetneqq S(M).$$

Proof. Under the assumption, by [Laz04, Proposition 9.3.2], $\mathcal{I}(F)$ is non-trivial at x. Therefore, by Proposition 7.2.1, $\mathcal{J}(M, F)$ is non-trivial. □

7.3. Jumping numbers

If we fix an effective $\mathbb{Q}$-divisor, by definition $\mathcal{J}(M, rB)$ discretely decrease as r grows generalizing that of multiplier ideals. Namely, jumping numbers can be defined for multiplier subsheaves too. To be precise,

Lemma 7.3.1. *Let X be an algebraic variety (not necessarily smooth), and M a pure Hodge module strictly supported on X, and B be an effective $\mathbb{Q}$-divisor on X with x on the support of B. Then there is an increasing discrete sequence*

$$0 = \xi_0 < \xi_1 < \xi_2 < \dots$$

of rational numbers depending on M, B and x characterized by the properties that

$$\mathcal{J}(M, rB)_x = \mathcal{J}(M, \xi_i B)_x$$

for $r \in [\xi_i, \xi_{i+1})$, while

$$\mathcal{J}(M, \xi_{i+1} B)_x \subsetneqq \mathcal{J}(M, \xi_i B)_x$$

for all i.

Proof. Fix a log resolution of sing$(M)+$ supp(B), $\mu : X' \longrightarrow X$. By the discreteness of $F_\bullet^{(t_i)}$, $S^{\mu^*(rB)}(V)$ remains the same if c increases slightly, where V is the generically defined PVHS corresponding to M. Therefore, the corresponding multiplier submodules are constant on the interval of the indicated shape. Since we are assuming that all the local monodromies of V are all quasi-unipotent, $F_\bullet^{(t_i)}$ is indexed by $\mathbb{Q}^r$ indeed. Hence the ξ_i are rational numbers. □

Definition 7.3.2. The rational numbers ξ_i are the jumping numbers associated to the pair (M, D) at x. ξ_1 is the log-canonical threshold of (M, D) at x, denoted by $\mathrm{lct}(M, D; x)$.

7.4. A Fujita's Freeness-type Theorem

In [**Kaw02**], Kawamata has proposed a relative version of Fujita's freeness conjecture and proved it under a stronger numerical condition. In this section, we extend it abstractly to Hodge modules under similar conditions with the help of multiplier subsheaves and Theorem 7.1.7.

Theorem 7.4.1. *Let X be a smooth projective variety of dimension n and M be a pure Hodge module strictly supported on X with $sing(M) = D$ a reduced simple normal crossings divisor. Let L be an ample divisor on X, and $x \in X$ a point. Assume that for any klt pair (X, B_0), there exists an effective $\mathbb{Q}$-divisor B on X satisfying the following conditions:*

(i) $B \equiv \lambda L$ for some $0 < \lambda < 1$;
(ii) $(X, B + B_0)$ is lc at x;
(iii) $\{x\}$ is a log canonical center of $(X, B + B_0)$.

Then the nature morphism

$$H^0(X, S(M) \otimes \omega_X(L)) \longrightarrow S(M) \otimes \omega_X(L)|_{\{x\}}$$

is surjective.

Proof. Let $\{s_1, ..., s_k\}$ be the basis of $S(V)^0$ on a neighborhood $x \in U$ as in Lemma 5.1.11. It is sufficient to prove that the image of the morphism

$$H^0(X, S(M) \otimes \omega_X(L)) \longrightarrow S(M) \otimes \omega_X(L)|_{\{x\}}$$

contains $s_j \otimes t|_{\{x\}}$ for any j, where t is the generator of $\omega_X \otimes L$ on U.

Recall that $\mathrm{ord}(s_j) = \sum_i \mathrm{ord}_i D_i|_U$ is a $\mathbb{Q}$-divisor on U. Then by definition, $(X, -\mathrm{ord}(s_j))$ is klt. By the assumption of the theorem, there exists an effective $\mathbb{Q}$-divisor B such that

(i) $B \equiv \lambda L$ for some $0 < \lambda < 1$;
(ii) $(X, B - \mathrm{ord}(s_j))$ is lc at x;
(iii) $\{x\}$ is a log canonical center of $(X, B - \mathrm{ord}(s_j))$.

Take a general element B' of $|mL|$ for $m \gg 0$ passing through x. After replacing B by

$$(1 - \epsilon_1)B + \epsilon_2 B'$$

for some suitable $0 < \epsilon_i \ll \frac{1}{m}$, we can assume that $\{x\}$ is the only log canonical center of $(X, B - \mathrm{ord}(s_j))$ passing through x. Let $\mu : X' \longrightarrow X$ be a log resolution of $D + \mathrm{supp}(B)$. Then

$$\mu^*(B - \mathrm{ord}(s_j)) = K_{X'/X} + E + F$$

where $K_{X'/X}$ the relative canonical divisor for μ and E is a reduced divisor such that $\mu(E) = \{x\}$ and F is effective and a boundary divisor on $\mu^{-1}U$ (shrink U if necessary). Then we have a short exact sequence

$$0 \longrightarrow S(V)^{\mu^*\mathrm{ord}(s_j)+E+F} \longrightarrow S(V)^{\mu^*\mathrm{ord}(s_j)+F} \longrightarrow S(V)^{\mu^*\mathrm{ord}(s_j)+F}|_E \longrightarrow 0.$$

But we also know

$$S(V)^{\mu^*\mathrm{ord}(s_j)+E+F} = S(V)^{\mu^*B - K_{X'/X}} \simeq S(V)^{\mu^*B} \otimes \omega_{X'/X}.$$

By Proposition 7.1.5, after pushing-forward the above short exact sequence and twisting by $\omega_X(L)$, we obtain another short exact sequence on X

$$0 \longrightarrow \mathcal{J}(M, D) \otimes \omega_X(L) \longrightarrow Q \longrightarrow Q' \longrightarrow 0,$$

where $Q = \mu_*(S(V)^{\mu^*\mathrm{ord}(s_j)+F}) \otimes \omega_X \otimes L$ and $Q' = \mu_*(S(V)^{\mu^*\mathrm{ord}(s_j)+F}|_E) \otimes \omega_X \otimes L$. Moreover, since

$$\mu^*\mathrm{ord}(s_j) - K_{X'/X} \leq \mu^*\mathrm{ord}(s_j) + F,$$

we have

$$Q \subseteq \mu_*(S(V)^{\mu^*\mathrm{ord}(s_j) - K_{X'/X}}) \otimes \omega_X \otimes L \simeq \mathcal{J}(M, \mathrm{ord}(s_j)) \otimes L.$$

By Proposition 7.2.1, we also have

$$\mathcal{J}(M, B) \subseteq \mathcal{J}(M, \mathrm{ord}(s_j)) \otimes \mathcal{J}(B - \mathrm{ord}(s_j)).$$

Hence we have nature morphism

$$Q' \longrightarrow \mathcal{J}(M, \mathrm{ord}(s_j)) \otimes \omega_X(L)|_Z,$$

where Z is the scheme defined by $\mathcal{J}(B - \mathrm{ord}(s_j))$. By our assumption, $\{x\}$ is an isolated support of Z. Since F is a boundary divisor on $\mu^{-1}U$, by Lemma 5.4.2

$$\mu^* s_j \in S(V)^{\mu^*\mathrm{ord}(s_j)+F}|_{\mu^{-1}U}.$$

Hence by projection formula $s_j \otimes t \in Q(U)$. But by Lemma 5.1.11, $s_j \otimes t$ is a member of the local basis $J(M, \mathrm{ord}(s_j))$ induced by the local basis $\{s_1, ..., s_j\}$. Therefore, $s_j \otimes t$ is not a local section of $\mathcal{J}(M,B)\otimes\omega_X(L)$. Since x is an isolated support of Z and $\{x\}$ is the support of Q', $s_j\otimes t|_{\{x\}}$ lifts to a global section of Q'. By Theorem 7.1.7, we have

$$H^1(X,\mathcal{J}(M,D)\otimes\omega_X(L))=0,$$

and hence $s_j\otimes t|_{\{x\}}$ lifts to a global section of Q . On the other hand, since E is exceptional and B is effective,

$$H^0(X,Q)=H^0(X',S(V)^{\mu^*B}\otimes\omega_{X'}(\pi^*L+E))\subseteq H^0(X,S(M)\otimes\omega_X(L)),$$

which makes the proof accomplished. □

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APPENDIX A

Variations of Hodge structures

For the reader's convienience, in this paragraph we give a brief overview of some basic facts about variations of Hodge structures.

A.1. Local Systems, Flat Vector Bundles and representations of π_1

Let X be a complex manifold and let $\mathbb{V}$ be a locally constant sheaf of $\mathbb{Q}$-vector spaces ($\mathbb{Q}$-local system). For local systems, usually we consider $\mathbb{Q}$-coefficients, but $\mathbb{Q}$ can be replaced by $\mathbb{R}$, $\mathbb{C}$ or even any other rings. Then $\mathcal{V} = \mathbb{V} \otimes \mathcal{O}_X$ is a holomorphic vector bundle on X. Since $\mathbb{V}$ is locally constant, for local sections c and f of $\mathbb{V}$ and $\mathcal{O}_X$ respectively the assignment

$$\nabla : \mathcal{V} \longrightarrow \Omega^1_X \otimes_{\mathcal{O}} \mathcal{V}$$

$$c \otimes f \mapsto df \otimes c$$

defines a $\mathbb{C}$-linear map for which the Leibniz rule holds

$$\nabla(fs) = f\nabla(s) + df \otimes s,$$

for s a local section of $\mathcal{V}$. This is an example of a holomorphic connection:

Definition A.1.1. Let $\mathcal{V}$ be a holomorphic vector bundle on X. A holomorphic connection on $\mathcal{V}$ is a $\mathbb{C}$-linear map

$$\nabla : \mathcal{V} \longrightarrow \Omega^1_X \otimes_{\mathcal{O}} \mathcal{V}$$

such that the Leibniz rule as above holds. A local section s of $\mathcal{V}$ with $\nabla(s) = 0$ is called flat or horizontal. We use the notation $\mathcal{V}^{\nabla}$ for $\ker\nabla$.

The composition of cup product of holomorphic forms and ∇ gives

$$\nabla : \Omega^p \otimes_{\mathcal{O}} \mathcal{V} \longrightarrow \Omega^{p+1} \otimes_{\mathcal{O}} \mathcal{V}.$$

The curvature of the connection is the $\mathcal{O}_X$-linear map ∇^2. The connection ∇ is flat or integrable if the curvature is 0.

Clearly if $(\mathcal{V}, \nabla)$ comes from a local system, then ∇ is integrable. Conversely, we have the following theorem.

Theorem A.1.2. *Let $(\mathcal{V}, \nabla)$ be a holomorphic vector bundle on X with an integrable connection. Then*

$$\mathbb{V} = \mathcal{V}^{\nabla}$$

is a local system on X and $(\mathcal{V}, \nabla) \simeq (\mathbb{V} \otimes \mathcal{O}_X, \nabla)$.

The theorem is classic. For a proof see for instance [**Voi02**, §9.2]. It tells us that the category of complex local system on X is equivalent to the category of flat holomorphic vector bundles. This is the prototype of the Riemann-Hilbert correspondence.

Assume X is connected. Starting from an arbitrary local system L, for a fixed base point x, we have the monodromy representation of π_1

$$\rho_L : \pi_1(X, x) \longrightarrow Aut(L_x).$$

For a loop γ, $\rho_L(\gamma)$ is the composition of the following isomorphism

$$L_0 \simeq \Gamma([0, 1], r^{-1}L) \simeq L_1.$$

Conversely, if we have a π_1-representation $\rho : \pi_1(X) \longrightarrow Aut(L)$, the sheaf of locally constant sections of the principal bundle $\tilde{X} \times_\rho L$ gives a local system on X. Here $\tilde{X}$ is the universal cover of X. Therefore, we have proved the following equivalence.

Theorem A.1.3. *If X is connected, the category of local system on X and the category of π_1-representations on X are equivalent.*

A.2. Variations of Hodge structures

Definition A.2.1. Let X be a complex manifold. A $\mathbb{Q}$-variation of Hodge structure V of weight k on X consists the following data:

(1) a $\mathbb{Q}$-local system of finite type on X;
(2) a finite increasing filtration $\{F_\bullet\}$ of holomorphic vector bundle of $\mathcal{V} = \mathbb{V} \otimes \mathcal{O}_X$ by holomorphic subbundles (the Hodge filtration).[1]

Those date satisfy conditions

(1) for the $x \in X$, the fibres defines a $\mathbb{Q}$-Hodge structure of weight k;

[1]Usually, the Hodge filtration is decreasing, but to be consistent with Hodge modules we use increasing filtration.

(2) the natural connection ∇ of $\mathcal{V}$ satisfies the Griffiths' transversality condition:

$$\nabla(F_p) \subset F_{p+1} \otimes \Omega^1_X.$$

Definition A.2.2. A polarization of a variation of Hodge structure V of weight k on X is a morphism of variations

$$Q : \mathbb{V} \otimes \mathbb{V} \longrightarrow \mathbb{Q}(-k)$$

which induces on each fibre a polarization of the corresponding Hodge structure of weight k, where $\mathbb{Q}(-k) = (2\pi i)^{-k}\mathbb{Q}$, the $-k$-th Tate twist of constant variation $\mathbb{Q}$.