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# Essays in Optimal Contracting

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Zhiguo He

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# ABSTRACT

Essays in Optimal Contracting

Zhiguo He

The first chapter of this dissertation studies a continuous-time agency model where the agent controls the drift of the geometric Brownian motion firm size. The changing firm size generates partial incentives, analogous to awarding the agent equity shares according to her continuation payoff. When the agent is as patient as investors, performance-based stock grants implement the optimal contract. My model generates a leverage effect on the equity returns, and implies that the agency problem is more severe for smaller firms. That the empirical evidence shows that grants compensation are largely based on CEO's historical performance—rather than current performance—lends support to my model model.

The second chapter studies the optimal contracting problem in the commonly used cash-flow setup, and offers a general framework to quantitatively assess the impact of agency problem. When cash-flow follows a square-root mean-reverting process, I derive the optimal contract in closed form, and provide a calibration exercise for the agency cost based on the empirical estimates of pay-performance sensitivity and cashflow mean-reverting intensity. In the geometric Brownian cash-flow setup, I embed the agency problem into Leland (1994), and find that small firms will take less leverage in choosing their optimal capital structure.

The third chapter generalizes the Leland and Pyle (1977) model to the case of multiple correlated assets. There, I study the signaling and hedging behavior of an intermediary with multi-dimensional private information who trades multiple assets in financial markets. Based on information asymmetry, the model demonstrates the intrinsic interdependence of risk management and asset selling for intermediaries, and obtains several testable empirical implications. For instance, an intermediary with a more diversified underlying portfolio will face greater liquidity (a smaller price impact) when communicating true asset qualities to the market. Several applications are discussed, including bank loan sales and selling mechanisms.

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My journey of Finance began in 1995 when my father chose the School of Economics and Management at Tsinghua University in Beijing for me after I performed well in the annual Chinese national college entrance exam. Twelve years later when I am compiling three papers into this dissertation at the Bendheim Center for Finance at Princeton University, I know that my father made the most wise decision.

Before I started my five-year Ph.D. life at Kellogg, I received the first rigorous training in Economics in Boston University from 2001 to 2003. The knowledge instilled by Jerome De-Temple (Finance Department), Pierre Perron (Economics Department), and Robert Rothensal (Economics Department, who past away in February 2002) enables me to quickly grasp key intuitions behind economic problems.

Joining the Ph.D. program at Kellogg in 2003 was the most important decision I made in my career. My first research project grew out of an in-class discussion in my first year. The class was Corporate Finance taught by Prof. Mike Fishman, who later became the Chair of my dissertation committee. When I brought my idea to him after the class, he smiled, and said, "Why not give it a shot?" The "shot" turned out to be quite successful. The paper eventually became the third chapter in my dissertation.

After my first year, I became interested in the growing continuous-time contracting literature. Coincidentally (well, there might have been a causal relationship), Mike was among the pioneers who formulated the theoretical framework of this literature. In the beginning I simply felt great to do something on a new play-ground. I soon realized that it was a real luxury to have a

top-player in the business at my side and to receive constant guidance from him. As a result, two papers on this topic later became the first and second chapters in my dissertation.

In my early experience without coauthors, the pleasure of writing papers was always accompanied with all kinds of painful sufferings. Later when I worked with Prof. Arvind Krishnamurthy, I was so excited to find out that I was provided with a reliever from almost all the usual anxiety and depression. I was able to quickly get out of downturns because once we sat down in his office and talked for some long enough time, solutions always popped out somewhere. Besides our joint work, Arvind—as well as Mike—gave me the most detailed and useful guidance about this profession, and their words in the past five years have greatly shaped my view on the Finance field.

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Even though I lived with my wife in Hyde Park which is twenty miles away from Evanston, my fellow classmates and I shared many exciting extra-curricular activities. Flavio de Andrade and I improved our squash techniques a lot over the years. The road trip to David Dicks' wedding in Missouri in the winter of 2006 was my No.1 journey with my fellow classmates. I witnessed Joey Engelberg's way onto a proud father of two children. I found out that Jared Williams was not a bad ping-pong player (remember I'm Chinese). I became good friends with Paul Gao, Itzik Kleshchelski, and Vadim de Pietro, not just because we studied for the Finance field exam together until the very night before.

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CHAPTER 1

# Optimal Executive Compensation when Firm Size Follows Geometric Brownian Motions

## 1.1. Introduction

This paper analyzes optimal executive compensation by studying a continuous-time moral hazard problem. The existing continuous-time agency models typically employ the less appealing arithmetic Brownian motion (ABM) framework which essentially entails a constant firm size. However, the relevance of firm size in the context of agency problems is widely documented.[1] Our model represents a significant departure from the previous literature in that we allow firm size to be time-varying and follow a geometric Brownian motion (GBM). We address the following questions: 1) Does time-varying firm size affect incentive provisions in the optimal contract? 2) Is the optimal contract under this environment different from the one under the ABM setting? and 3) How can the resulting optimal contract be implemented?

A large literature studies dynamic contracting under moral hazard. Formally introduced in Spear and Srivastava (1987), the agent's *continuation payoff* has been acknowledged as a powerful tool to serve as the state variable in dynamic programming. However, this literature is reluctant to bring in another state variable to capture the time-varying technology, largely for the sake of tractability. For instance, typical continuous-time moral hazard models assume an ABM output process (e.g., Holmstrom and Milgrom (1987)). A divergence exists between this specification and the one employed in the standard finance literature (see, among others,

[1]For instance, Gertler and Gilchrist (1994) find that small firms are more constrained when the monetary policy is tightening.

Goldstein et al. (2001)). By adopting the GBM framework, this paper makes the first attempt to bridge the gap between the continuous-time agency model and the conventional continuous-time finance literature.

In our model, investors hire an agent for business operation. The firm size process follows a GBM, and the agent controls *firm size* growth through unobservable effort. In contrast, the existing literature (DeMarzo and Fishman (2006), DF; DeMarzo and Sannikov (2006), DS; Biais, Mariotti, Plantin and Rochet (2006), BMPR) focuses on the setting with constant firm size, where the agent controls the drift of instantaneous ABM *cash flows*.[2] Later we refer to these models as ABM, as opposed to our GBM model. Relative to the existing literature, this paper highlights how changing firm size affects the agency problem.

In addition, in the ABM models the cash flows are unbounded from below. Consequently substantial losses can arise during any time interval and, therefore, the agent is always constrained. However, the GBM model has positive cash flows, and we show that in the optimal contract there are absorbing states in which the constraint disappears and the first-best outcome is achieved (see Section 1.3.3.2). These first-best absorbing states are attained when the agent has a long history of successes, or equivalently, when the firm has experienced rapid growth. Both the role of firm size, and the possibility that the agency issue may be resolved along the optimal path, are realistic features that are present in discrete-time models, but not in the existing continuous-time literature. Our modeling thus advances the continuous-time optimal contracting literature in important ways.

[2]DF study a discrete-time model; DS study a continuous-time model; and BMPR solve the discrete-time model first, then take the result to the continuous-time limit. In their main models, all three papers study the problem where the agent can secretly divert cash from the current output for her own consumption. Under the ABM setup, the cash-diverting problem is isomorphic to the standard moral hazard problem with binary effort. In our GBM model, since cash flows are predetermined, there is no such equivalence. However, our model is equivalent to the agency problem where the agent can "steal" the firm's assets (secretly sell part of the firm's plants and pocket the sale proceeds).

The key trade-offs in this type of setting (DF, DS, BMPR, and this paper) are as follows. Implementing high effort requires sufficient incentives, which mandate that poor results be met with penalties. As the agent's limited liability precludes negative wages, these penalties will accumulate until inefficient termination is triggered. This implies that incentive provision is potentially costly, and hence the optimal contract provides just enough incentives to induce the agent to exert effort.

Different from ABM models, the time-varying firm size in our GBM setting generates a portion of incentives through the agent's continuation payoff. Intuitively, this mechanism works as if investors grant the agent a number of equity shares according to her current continuation payoff, and this hypothetical inside stake provides some incentives for the agent when the firm size is changing (see discussion in Section 1.3.3.3). However, along the optimal path, these incentives are not sufficient to motivate the agent. Therefore, additional incentives are provided in the optimal contract (e.g., through performance-based stock grants).

Other than the trade-off between incentive provision and inefficient termination, there is a wedge between two contracting parties: the agent is, at most, as patient as investors. Therefore exchanging relative consumption timings between these two parties improves efficiency, and the optimal contract pays cash (wage) to the agent as early as possible. However, paying cash earlier to the agent, or setting a lower payment boundary in the employment contract, is potentially costly. The reason is that, by reducing the agent's continuation payoff, this might make future inefficient liquidation more likely. As a result, the optimal contract calls for investors to set the optimal cash payment boundary such that the marginal benefit equals the marginal cost. Consistent with DS and BMPR, for the case of a strictly impatient agent where the marginal benefit of paying cash earlier is positive, the payment threshold is a reflecting barrier, and a positive marginal cost of paying cash earlier is maintained.

The novel result in this paper pertains to the case of an equally patient agent under the continuous-time framework. When the agent is equally patient, most discrete-time long-term agency models derive an optimal contract with a first-best absorbing state, as agency issue will be completely resolved when the agent's stake within the contractual relationship becomes sufficiently high (see, DF, BMPR, and Albuquerque and Hopenhayn (2005), etc.). However, in the *continuous-time* ABM setting (DS and BMPR), unbounded cash flows imply that future inefficient liquidation is always possible, and the first-best state obtained in the *discrete-time* model (DF and BMPR) disappears. In fact, because an earlier cash payment has zero marginal benefit due to the irrelevance of relative consumption timings, while the marginal cost brought on by future termination is always positive, in the ABM model DS and BMPR find that investors should delay the agent's wage indefinitely to minimize the probability of inefficient liquidation. Consequently, when the agent is as patient as investors, the optimal contract fails to exist in their ABM models (see Section 1.3.3.2).

In contrast, we derive an optimal contract for the equally patient agent case in our GBM model. When the agent's continuation payoff is sufficiently high, she is granted certain equity shares and works forever in the firm; and in this situation the positive cash flows in the GBM model preclude future inefficient liquidation. Therefore, our GBM setting recovers the interesting absorbing first-best state, but with a mechanism that is distinct from the discrete-time setup studied in DF or BMPR. Furthermore, in this equally patient agent case, we derive a new optimal contract even when it is suboptimal to implement working all the time. Under the latter contract, shirking becomes another absorbing state. This extends the results in DS who only study the case of an impatient agent.

Our optimal contract can be implemented through a performance-based compensation scheme: *Incentive Points Plan*. Under this plan, the points trace the agent's scaled (by firm size) continuation payoff, and the agent can redeem those points above a prespecified threshold. Interestingly, in the case of equally patient agents, this plan corresponds to *performance-based stock grants*: once the agent has accumulated enough points, she can convert them to a prescribed number of equity shares. This implementation resembles "performance shares" that are currently used in most long-term incentive plans (see, among others, Frydman and Saks (2005)).[3]

We discuss several interesting implications of our results. Larger firms that experience a better performance history suffer less severe agency problems. And, equity returns exhibit rising volatility when the firm's performance is poor. This "leverage" effect caused by agency problem is more compelling than the one obtained in BMPR, because, in their ABM framework, a constant volatility in *levels* could lead to a leverage effect for *returns*, even without the agency problem. Using simulation, we contend that research on CEO pay-performance sensitivity should consider long-term incentives when analyzing executives' remuneration contracts. Empirical evidence that shows that, for stock and option grants CEOs are primarily compensated based on their historical achievements rather than their current performance, lends support to this paper.

The related literature on long-term agency models includes Sannikov (2004), who considers an ABM environment with an equally patient risk-averse agent, and allows for a continuum of effort levels from the agent. There, the optimal contract features an upper-absorbing retirement state without working, while, here, we find an upper-absorbing state where the agent works voluntarily forever.[4] Williams (2006) develops a general theory about the principal-agent model which accommodates both hidden-actions and hidden-states. Tchistyi (2005) extends DF by

[3]For executives' long-term compensation components, a recent survey ("*2005 CEO Compensation Survey and Trend*" conducted by Mercer Human Resource Consulting) documents a trend toward performance shares. From 2003 to 2005, the use of performance shares increases from 18% to 21%, while that of stock options drops from 72% to 52% during the period 2002-2005.

[4]This difference stems from the agent's risk aversion and accompanying income effect, which imply that providing incentives becomes extremely costly when the agent's continuation payoff is sufficiently high. Holmstrom and Milgrom (1987) also analyze a risk-averse agent, where the effort cost is in terms of *monetary* units rather than the agent's *utility* units. This specification (under CARA utility) eliminates the income effect.

allowing for correlated cash flows, and Sannikov (2006) studies a mixture of moral hazard and adverse selection problem.

The theory of optimal dynamic lending contracts (Hart and Moore (1994), Thomas and Worrall (1994), and Albuquerque and Hopenhayn (2005), etc.) is also related. This strand of literature focuses on the dynamic borrowing constraints caused by the possibility of strategic default from the borrower, and there is no inter-period agency problem as modeled in DF or BMPR. For instance, Albuquerque and Hopenhayn (2005) relate the borrowing constraint to the endogenous equity value (the borrower's continuation payoff).[5]

We present the model in Section 2, and characterize the optimal contract in Section 3. Section 4 considers the model's extensions. Section 5 discusses implementations and implications, including an empirical study about long-term grant-performance sensitivity. Finally, Section 6 concludes. Proofs are in the Appendix.

## 1.2. The Model

Our basic framework is a continuous-time principal-agent model, where risk-neutral investors of an infinitely lived firm hire a risk-neutral agent to operate the business. The firm produces cash flow $\delta_t$ per unit of time, which evolves according to a GBM

$$d\delta_t = a_t \delta_t dt + \sigma \delta_t dZ_t;$$

where $Z = \{Z_t, \mathcal{F}_t; 0 \le t < \infty\}$ is a standard Brownian motion on a complete probability space $(\Omega, \mathcal{F}, \mathcal{P})$, and $a_t \in \{0, \mu\}$ is the agent's *binary* effort choice. Here, $a_t = \mu > 0$ stands for "working," while $a_t = 0$ stands for "shirking." Investors discount future cash flows at the market

[5]Based on Holmstrom and Milgrom (1987), there is another active area on the continuous-time contracting problem where CARA utility and ABM processes are usually assumed (e.g., Ou-Yang (2005) with a constant volatility). Their framework differs fundamentally from that of this paper: instead of allowing for interim consumption and endogenous termination, the authors assume a lump-sum payment at the end of an exogenous employment horizon $[0, T]$. Furthermore, as acknowledged in Ou-Yang (2005), under that framework the adoption of a log-normal cash flow process could render the optimal contracting problem intractable.

interest rate $r > \mu > 0$. Note that if agent works all the time, then from the view of investors, the firm's first-best value at time $t$ is $\mathbb{E}_t\left[\int_t^\infty e^{-r(s-t)}\delta_s ds\right] = \frac{1}{r-\mu}\delta_t$ which follows a GBM process as well.

We interpret the cash flow rate $\delta_t$—which is proportional to the firm's first-best value—as the current firm size. Firm size process $\{\delta\}$ is observable and contractible, while the agent's effort choice $a_t$ is not. The agent derives a positive nonpecuniary private benefit $\phi\delta_t dt$ from shirking where $\phi$ is a positive constant. This benefit is proportional to the current firm size, because administering a larger firm requires more effort.[6]

The agent has no initial wealth, and negative wage is ruled out by limited liability. We assume that the agent's reservation value is zero, which ensures the scale invariance property of the model.[7,8] The agent has a discount rate $\gamma \geq r$, that is, the agent is (weakly) less patient than investors. Note that the ABM model in DS or BMPR requires $\gamma > r$ strictly.

Agent's employment starts at $t = 0$, and is terminated when the firm is liquidated. At the time of liquidation, investors recover a value $L\delta_t$ from the firm's assets, and fire the agent. We assume that $L < \frac{1}{r-\mu}$; that is, liquidation is inefficient. Later, we endogenize $L$ by allowing the firm to replace the incumbent agent with a new identical agent (see Section 1.4.1).

Assume that investors can commit to an employment contract which specifies an endogenous stochastic liquidation time $\tau$, and a right-continuous-left-limit nondecreasing *cumulative* wage process $\{U\} = \{U_t : 0 \leq t \leq \tau\}$. We denote such a contract by $\Pi \equiv \{\{U\}, \tau\}$, where both

[6]The shirking benefit (available only when the agent is hired in the firm) can be interpreted as the negation of the agent's effort cost. Note that this assumption is also consistent with the notion that the agent's private benefit is increasing with the firm size.

[7]Similar to Thomas and Worrall (1994), we can assume that the agent is able to appropriate a fraction of firm so that her reservation value is $\underline{k}\delta_t$, where $\underline{k}$ is a nonnegative constant that is sufficiently small to ensure that "stealing-absconding" is inferior to "behaving" in the optimal contract. This specification can also be interpreted as that the agent with better performance records faces a more favorable outside option. The entire analysis can be conducted by replacing 0 with $\underline{k}$.

[8]This assumption is consistent with the notion of competitive labor markets. Besides, evidence suggests that failed managers are not as competent as other candidates, even if the previous corporation failure is viewed to be beyond the manager's control. Cannella, Fraser and Lee (1995) find that these "innocent bystander" managers are 63% less likely to find banking posts compared to those at non-failed banks.

elements are $\delta$-measurable, and $\tau$ could take the value $\infty$. We impose the usual square-integrable condition on $\Pi$ as follows:

$$\mathbb{E}\left[\left(\int_0^{\tau} e^{-\gamma s} dU_s\right)^2\right] < \infty. \tag{1.1}$$

A contract $\Pi$ is incentive-compatible if it motivates the agent to work until liquidation; in other words, if $\{a_t^* = \mu : 0 \leq t < \tau\}$ solves the following agent's problem:

$$\max_{a=\{a_t \in \{0,\mu\}: 0 \leq t < \tau\}} \quad \mathbb{E}^a \left[\int_0^{\tau} e^{-\gamma t} \left(dU_t + \phi\left(1 - \frac{a_t}{\mu}\right)\delta_t dt\right)\right],$$

where $\mathbb{E}^a[\cdot]$ is the expectation operator under the probability measure over $\{\delta\}$ that is induced by any effort process $a = \{a_t \in \{0, \mu\} : 0 \leq t < \tau\}$. We assume that it is optimal to implement working all the time, and verify its optimality in Section 1.4.2. Therefore in this paper, unless otherwise stated, the expectation operator is under the measure induced by $\{a_t = \mu : 0 \leq t < \tau\}$.

Throughout, we assume that the firm possesses full bargaining power. Denote the set of incentive compatible contracts as $\mathbb{IC}$, and the firm's problem is

$$\max_{\Pi \in \mathbb{IC}} \quad \mathbb{E}\left[\int_0^{\tau} e^{-rt}\delta_t dt + e^{-r\tau} L\delta_{\tau} - \int_0^{\tau} e^{-rt} dU_t\right].$$

There is no agent's participation constraint in this problem, as the agent enjoys a positive rent once she is hired. Denote the solution for this problem as $\Pi^* = \{\{U^*\}, \tau^*\}$.

## 1.3. Model Solution and Optimal Contracting

### 1.3.1. Continuation Payoff and Incentive Compatibility

This section gives a key proposition for any incentive-compatible contract $\Pi \in \mathbb{IC}$. Fix the effort process $a = \{a_t = \mu : 0 \leq t < \tau\}$. For any contract $\Pi$, define the agent's continuation payoff at time $t$, as

$$W_t(\Pi) \equiv \mathbb{E}_t\left[\int_t^\tau e^{-\gamma(s-t)} dU_s\right].$$

In words, $W_t$ is the agent's continuation value obtained under $\Pi$ when she plans to work from $t$ onwards.

Define $\lambda \equiv \frac{\phi}{\mu}$, which relates to the minimum incentive required to motivate the agent.[9] The following proposition expresses the evolution of $W_t$ in terms of observable performance $d\delta_t$, and provides a necessary and sufficient condition for any contract $\Pi$ to be incentive-compatible.

**Proposition 1.** *For any contract $\Pi = \{\{U\}, \tau\}$, there exists a progressively measurable process $\left\{\sigma_t^W : 0 \leq t < \tau\right\}$, such that, under working (i.e., $a_t = \mu$ always), the agent's continuation value $W_t$ evolves according to*

$$dW_t = \gamma W_t dt - dU_t + \frac{\sigma_t^W}{\sigma}\left(d\delta_t - \mu\delta_t dt\right). \tag{1.2}$$

*The contract $\Pi \in \mathbb{IC}$, i.e., is incentive-compatible, if and only if $\sigma_t^W \geq \lambda\sigma$ for $t \in [0, \tau)$.*

Proposition 1 states that, the agent's instantaneous compensation—the wage ($dU_t$) plus the change of continuation payoff ($dW_t$)—has a predetermined drift part $\gamma W_t dt$ which corresponds to the Promise-Keeping condition in the discrete-time formulation, and a diffusion part

$$\delta_t \sigma_t^W dZ_t = \frac{\sigma_t^W}{\sigma}\left(d\delta_t - \mu\delta_t dt\right),$$

which links to her effort choice and provides working incentive. To motivate the agent, the instantaneous volatility of continuation payoff, $\delta_t\sigma_t^W$, must be higher than $\lambda\sigma\delta_t$. To see this, if the agent chooses to shirk, she gains a private benefit $\phi\delta_t dt$, but loses $\frac{\mu\delta_t\sigma_t^W}{\sigma} dt$ in compensation ($d\delta_t$ becomes driftless under shirking). Thus, she will work if and only if $\frac{\mu\delta_t\sigma_t^W}{\sigma} \geq \phi\delta_t$, or

[9] It is clear that the larger the personal benefit $\phi$, the more difficult it is to motivate the agent. But $\mu$ matters too; a higher drift makes it easier to detect shirking, hence less incentive is needed.

$\sigma_t^W \geq \lambda\sigma$. Therefore, by putting her at risk, the volatility of $dW_t + dU_t$ pushes the agent to exert high effort. In other words, for enough punishing and rewarding, the volatility of the agent's continuation payoff must exceed a certain threshold $\lambda\sigma\delta_t$.

As we will see shortly in the optimal contract we have $\sigma_t^W = \lambda\sigma$; that is, the incentive compatibility constraint always binds. Geometrically, on the $(\delta, W)$ plot in Figure 1.1, the (local) movement of the state-variable pair $(\delta, W)$ (which is determined by the diffusion term) must be as steep as $\lambda$.[10]

### 1.3.2. Optimality Equation and Its Solution

**1.3.2.1. Optimality Equation and Boundary Conditions.** There are two state variables in this model: firm size $\delta_t$, and the agent's continuation payoff $W_t$. The investors' value function $b(\delta, W) \in \mathbb{C}^2$ (i.e., twicely differentiable in both arguments) is the firm's highest expected future profit given these two state variables. When the agent works all the time, the firm size $\delta_t$ evolves as

$$d\delta_t = \mu\delta_t dt + \sigma\delta_t dZ_t.$$

And the agent's continuation payoff $W_t$ follows,

$$dW_t = \gamma W_t dt - dU_t + \delta_t \sigma_t^W dZ_t. \tag{1.3}$$

As we will verify in Section 1.3.4, the concavity of the investors' value function implies that the optimal contract provides just enough incentives; i.e., $\sigma_t^W = \lambda\sigma$. Also, similar to DS, the optimal cash (wage) payment policy depends on $\frac{\partial b}{\partial W}$. If $\frac{\partial b}{\partial W} > -1$, then promising one dollar of continuation payoff to the agent costs the firm less than paying one dollar cash. As a result, in this case the firm should hold the cash and promise to pay later.

[10] In the optimal contract $dW_t$ has a diffusion term $\lambda\sigma\delta_t dZ_t$. Therefore $\frac{dW_t}{d\delta_t} \simeq \frac{\lambda\sigma\delta_t dZ_t}{\sigma\delta_t dZ_t} = \lambda$.

The tractability of our GBM model hinges on the scale invariance property, which implies that the optimal policy is homogeneous in firm size $\delta_t$.[11] As a result, the investors' value function $b(\delta, W)$ must be of the form of $\delta c(k)$, where the agent's scaled continuation payoff $k \equiv W/\delta$ is the only relevant state variable, and $c(\cdot) \in \mathbb{C}^2$ is a univariate smooth function. We call $c(\cdot)$ the investors' scaled value function.

In the Appendix, after writing down the Hamilton-Jacobi-Bellman equation, we find that $c(\cdot)$ must solve the following $2^{nd}$-order ordinary differential equation (ODE) when there is no cash payment ($dU = 0$):

$$(r-\mu)\, c(k) = 1 + (\gamma-\mu)\, kc'(k) + \frac{1}{2}(\lambda - k)^2 \sigma^2 c''(k)\,. \tag{1.4}$$

This equation plays a key role in analyzing the optimal contract; we call it *Optimality Equation.*

The optimality of cash payment yields two boundary conditions at the upper end. Scale invariance implies that the optimal cash payment barrier is linear in $\delta$, i.e., $\overline{W}_t \equiv \overline{k}\delta_t$, where $\overline{k}$ is a positive constant to be solved in the optimal contract. Once $W_t$ sits above $\overline{W}_t$, investors will pay the agent $W_t - \overline{W}_t$ in cash to bring $W_t$ back to $\overline{W}_t$ (see Figure 1.1). Because paying agent cash to reduce her continuation payoff $W$ is a barrier control with linear cost, we have *Smooth-Pasting* condition $\frac{\partial b}{\partial W}\left(\delta_t, \overline{k}\delta_t\right) = -1$, and *Super-Contact* condition $\frac{\partial^2 b}{\partial W^2}\left(\delta_t, \overline{k}\delta_t\right) = 0$ (see A. Dixit (1993)). In terms of $c(\cdot)$, the conditions are

$$c'\left(\overline{k}\right) = -1; \tag{1.5}$$

$$c''\left(\overline{k}\right) = 0. \tag{1.6}$$

[11] Also, recall that both the shirking benefit $\phi\delta_t dt$ and liquidation value $L\delta_t$ are linear in the firm size, and that the agent's outside option is worth zero.

Applying these two conditions to (1.4), we find that, at $\overline{k}$, $c(\cdot)$ attaches the function $\frac{1}{r-\mu} - \frac{\gamma-\mu}{r-\mu}k$ with slope $-1$. We extend $c(\cdot)$ linearly (with slope $-1$) for $k > \overline{k}$ based on the optimal wage policy (see Figure 1.2).

Termination delivers another boundary condition at the lower end. Let $\tau$ be the first hitting time at which $W_t = 0$. Once this occurs, the agent is fired, and investors liquidate the firm for a surrender value $L\delta_\tau$. Hence,

$$c(0) = L, \tag{1.7}$$

and $c(\cdot)$ solves (1.4) with boundary conditions (1.5), (1.6), and (1.7).

In light of the Feynman-Kac formula, $c(k)$ can be written in its probabilistic representation (see Lemma 2 in the Appendix)[12]

$$c(k) = \mathbb{E}^{k_0=k}\left[\int_0^\tau e^{-(r-\mu)t}dt + e^{-(r-\mu)\tau}L - \int_0^\tau e^{-(r-\mu)t}du_t\right],$$

where the process $\{k\}$ evolves according to

$$dk_t = (\gamma-\mu)\,k_t dt + (\lambda - k_t)\,\sigma dZ_t - du_t; \tag{1.8}$$

and $u_t$ is a nondecreasing process that reflects $k_t$ at $\overline{k}$.[13] Intuitively, the scaled value function $c(k)$ equals expected scaled cash flows $1dt$, plus the scaled liquidation value $L$, minus scaled wages, all discounted by the effective discount rate $r-\mu$.

We define the first-best scaled value function $c^{fb}(k) \equiv \frac{1}{r-\mu} - k$ for later references.

[12]Note that this form does not require the Super-Contact condition (1.6), an important fact when we derive the comparative static results in Lemma 3.

[13]An interesting caveat exists regarding the evolution of process $k$. Similar to the difference between the risk-neutral and physical measures in asset pricing literature, the evolution (1.8) for $k$ is under an auxiliary measure induced by (1.4), which annihilates certain drift of $k$. Under the physical measure, without cash payment, $k_t = W_t/\delta_t$ evolves according to $dk_t = (\gamma-\mu)\,k_t dt + (k_t-\lambda)\,\sigma^2 dt + (\lambda-k_t)\,\sigma dZ_t$. This differs from (1.8) by $(k_t-\lambda)\,\sigma^2 dt$ due to the scaling of $\delta_t$ (a quadratic covariation between $k_t$ and $\delta_t$). Nevertheless, since we focus on the diffusion part which provides incentives, the drift is of less importance.

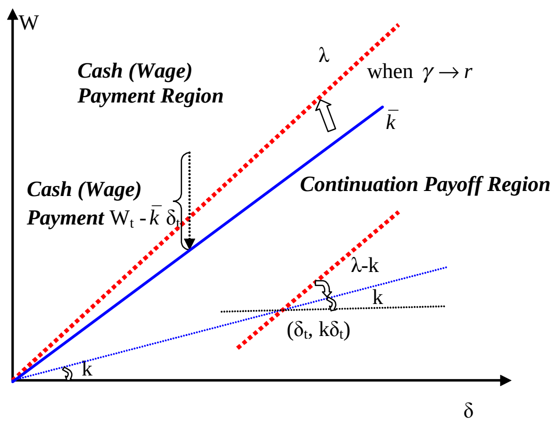


Figure 1.1. **The optimal cash (wage) payment and incentive provision policy.** There exists $\overline{k} \leq \lambda$ so that it is optimal to pay the agent cash $W_t - \overline{k}\delta_t$ once $W_t > \overline{k}\delta_t$. The incentive compatibility constraint requires a slope $\lambda$ for the local movement of $(\delta_t, W_t) = (\delta_t, k\delta_t)$, while the agent's hypothetical inside stake only contributes a slope $k < \lambda$ due to the diffusion of $\delta$. Subsequently, the optimal contract provides additional incentives to fulfill the slope discrepancy $\lambda - k$.

**1.3.2.2. Comparison to ABM Setting in DS and BMPR.** As a comparison, under the ABM setting analyzed in DS and BMPR, the agent controls the instantaneous *cash flow* $dY_t$, which can be written as (when the agent is working)

$$dY_t = \mu dt + \sigma dZ_t.$$

In contrast, in the GBM model, the agent controls the change of *firm size (cash flow rate)* $d\delta_t$, rather than the predetermined *cash flow* $\delta_t dt$. This distinction necessarily leads to different implementation mechanisms in Section 1.5.1. Also, in the GBM model the cash flow $\delta_t dt$ is positive, but in the ABM setting $dY_t$ is unbounded from below. As we will see later, this divergence affects the existence of the first-best state in optimal contracting.

Once the agent (with a reservation utility $R$) shirks to enjoy the private benefit $\phi dt$, the drift of $dY_t$ drops to 0. As before, define $\lambda = \phi/\mu$. Similar arguments as in Section 1.3.1 imply that the state variable, which is the agent's continuation payoff $W$ (as opposed to $k = W/\delta$ in our GBM model), evolves according to

$$dW_t = \gamma W_t dt + \lambda\sigma dZ_t - dU_t. \tag{1.9}$$

Denote $\overline{W}$ as the payment boundary in the optimal contract. When $W \in \left[R, \overline{W}\right]$, cash payment $dU = 0$, and the unidimensional value function $b(W)$ satisfies

$$rb(W) = \mu + \gamma W b'(W) + \frac{1}{2}\lambda^2\sigma^2 b''(W), \tag{1.10}$$

with similar boundary conditions $b'\left(\overline{W}\right) = -1$, $b''\left(\overline{W}\right) = 0$, and $b(R) = L$. The optimal contract pays out cash $dU_t > 0$ only when $W_t$ exceeds the reflecting barrier $\overline{W}$. When $W_\tau = R$, the firm is liquidated. Comparing (1.4) to (1.10), we immediately discern a difference: because in the GBM setup the drift captures the firm's growth, the parameter $\mu$ enters (1.4) by reducing both parties' discount rates.

The key difference, however, lies in the $2^{nd}$-order term in these two equations: in (1.4), the coefficient of the $2^{nd}$-order term is $(\lambda - k)\sigma$, while, in (1.10), it is $\lambda\sigma$. Because the $2^{nd}$-order term corresponds to the diffusion part of respective state variables, and the diffusion in turn captures incentives, two important implications ensue. First, note that according to (1) the required incentives are $\lambda\sigma$, while only $(\lambda - k)\sigma$ portion of incentives would lead to future inefficient liquidation—it is the diffusion that causes $k_t$ to hit the liquidation boundary 0. This fact suggests that, in the GBM model, the scaled continuation payoff $k$ itself generates some "costless" incentives along the optimal path. In Section 1.3.3.3, we will see that this finding stems from the time-varying firm size in our GBM setting.

Second, this state-dependent diffusion $(\lambda - k)\sigma$ in (1.4) leads to one significant result that contrasts drastically with the ABM model. Unlike (1.10), (1.4) involves a singular point (when

$k_t = \lambda$, the diffusion of $k$ dies), which corresponds to the absorbing state where a sufficiently high inside stake drives the agent to work voluntarily (see Section 1.3.3.2). In fact, Section 1.3.3.2 shows that when the agent is equally patient, this absorbing state, as a part of the optimal contract, achieves the first-best result.

### 1.3.3. The Optimal Contract

Consistent with BMPR, we find that the optimal contract differs for the two cases $\gamma > r$ and $\gamma = r$. As we discussed in the Introduction, postponing the agent's consumption alleviates the agency problem, and thereby improves efficiency. However, if the agent is impatient, then postponing consumption will entail a cost, as the first-best result has the agent consume as early as possible. In contrast, for an equally patient agent, the payment delay is absolutely free. Therefore, whether the cost is present or not determines the structure of optimal contract.

**1.3.3.1. When $\gamma > r$ (Impatient Agent).** If the agent is impatient, earlier wage payments tend to be optimal, and the optimal payment boundary $\overline{k}$ is always below $\lambda$ as stated in the next proposition.

**Proposition 2.** *When $\gamma > r$, we have $\overline{k} < \lambda$. There exists a unique solution $c(\cdot)$ to (1.4) with boundary conditions (1.5), (1.6) and (1.7), and the solution is strictly concave on $\left[0, \overline{k}\right]$.*

Our resulting optimal contract can be described as follows. At $t = 0$, the firm hires an agent by offering her a continuation payoff $W_0 = k^* \delta_0$, and promises the evolution of her continuation payoff $W_t$ to be

$$dW_t = \gamma W_t dt + \lambda \left( d\delta_t - \mu \delta_t dt \right). \tag{1.11}$$

When $W_t$ achieves $\overline{k}\delta_t$, investors start paying the agent cash to maintain her continuation payoff $W_t$ at $\overline{k}\delta_t$. When $W$ hits zero at time $\tau$, investors fire the agent and liquidate the firm.

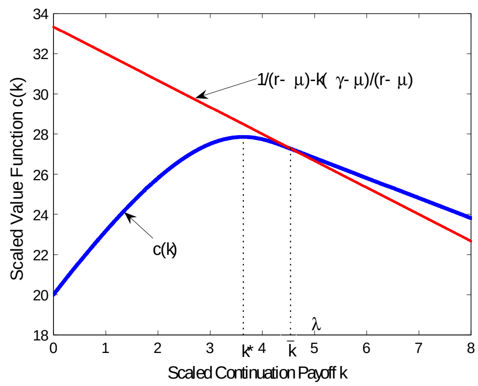


Figure 1.2. **The scaled value function $c(\cdot)$ for the case $\gamma > r$ (an impatient agent).** Parameters are $r = 4\%, \gamma = 5\%, \mu = 1\%, \sigma^2 = 10\%, \lambda = 5, L = 20$. $\overline{k} < \lambda$ is a reflecting barrier. $c(\cdot)$ attaches $\frac{1}{r-\mu} - \frac{\gamma-\mu}{r-\mu}k$ with a slope $-1$, and is extended for $k > \overline{k}$ with a slope $-1$.

This optimal contract is quite similar to that of DS and BMPR, except that the cash payment threshold $\overline{k}\delta_t$ is state-dependent, with an upper bound $\lambda\delta_t$. In addition, the result of $\overline{k} < \lambda$ implies a non-dying diffusion of $k_t$, which suggests that, along the optimal path, it is always possible to have $k_t$ drop to zero if the agent's future performance is poor. This result is due to the gap between two party's patience levels. To see this, first note that the agent's impatience implies a strictly positive marginal benefit of paying cash to the agent earlier, or setting a lower payment threshold $\overline{k}$. However, in the Appendix (Lemma 15 part 3), we show that, the *marginal cost* of setting a lower payment boundary (brought on by future inefficient liquidation) is zero at the absorbing state $\lambda$, and positive for $\overline{k} < \lambda$. To equate the marginal cost with the marginal benefit, the firm should pay the agent cash before $k_t$ reaches $\lambda$. This trade-off never exists for an equally patient agent, as we will discuss in the next section.

**1.3.3.2. When $\gamma = r$ (Equally Patient Agent).** When the issue of relative consumption timing is absent, postponing cash payments has zero cost. As a result, $\overline{k} = \lambda$ is the optimal payment boundary, which is higher than the one obtained when $\gamma > r$. In fact, $\lambda$ is the first-best absorbing state, and there will be no further chance of liquidation once $k_t$ attains $\lambda$.

**Proposition 3.** *When $\gamma = r$, without loss of generality, we have $\overline{k} = \lambda$. There exists a unique solution $c(\cdot)$ to (1.4) with boundary conditions (1.5), (1.6), and (1.7), and the solution is strictly concave on $\left[0, \overline{k}\right]$.*

Investors start the employment at $W_0 = k^*\delta_0$, and let the agent's continuation payoff evolve according to (1.11). If $W_t$ falls to zero, then investors liquidate the firm and fire the agent. However, once good fortune drives $W_t$ to attain $\lambda\delta_t$, the agent receives cash payment $dU_s = \lambda(r - \mu)\delta_s ds$ for $s \geq t$, and, as an absorbing state, her continuation payoff $W_s$ stays at $\lambda\delta_s > 0$ forever (so $k_s = \lambda$ from then on). Note that it is equivalent to granting $\lambda(r - \mu)$ shares to the agent, and these shares provide required incentives to motivate the agent.

We observe a key difference between our result and the one obtained in DS and BMPR who consider the impatient agent case only. Under their ABM setting, for however high the agent's continuation payoff, in any time interval $W_t$ can reach the agent's fixed outside option $R$ due to unbounded Brownian increments (check (1.9)), and the marginal cost of setting a lower cash payment barrier $\overline{W}$ is always positive. In other words, in their ABM model the agent is always constrained, and there is always a gain from relaxing the constraint even further.[14] However, since the benefit of paying cash earlier is absent when $\gamma = r$, investors should postpone the agent's wage indefinitely, which renders the nonexistence of the optimal contract.[15] Under our

[14]In fact, the GBM model with positive cash flows also helps us disentangle the agency problem from the agent's limited-cash-reserve constraint. Note that, in the ABM model, the costly termination is caused not only by the agent's moral hazard problem, but also by the fact that she only has a finitely "deep pocket." Specifically, even if the agent (given a fixed cash reserve) runs the firm as a proprietorship, unbounded cash flows—hence substantial losses—imply that future inefficient liquidation is always possible, and the probability of future liquidation is strictly decreasing in the level of the firm's cash reserve. Clearly the latter differs from the inefficient punishment in the standard moral hazard literature.

[15]To see this under the ABM setting we have (1.9). Recall $dU_t \geq 0$; therefore for a loss $dZ_t < 0$, $W$ has to drop, and the size of the drop is independent of the level of $W$. This implies that within any time interval there is always a positive probability for $W$ to reach the termination boundary $R$. The higher the continuation payoff $W$, the lower the liquidation probability, and the higher the efficiency. It implies that the marginal cost of paying cash early is strictly positive. Given this fact, when $\gamma = r$ so that there is zero benefit to pay the agent cash early, the optimal contract should accumulate $W$ as high as possible to approach (but never reach) the first-best outcome. In words, any contract given a payment boundary $\overline{W}$ could be improved by setting $\overline{W} + 1 > \overline{W}$, and the wage payments are further delayed. Therefore in the limit $dU_t = 0$ for $0 \leq t < \infty$, thus violating the Promise-Keeping

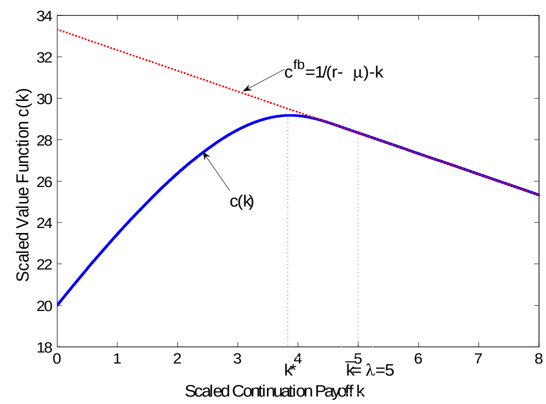


Figure 1.3. **The scaled value function $c(\cdot)$ for the case $\gamma = r$ (an equally patient agent)**. Parameters are $r = 4\%, \mu = 1\%, \sigma^2 = 10\%, \lambda = 5, L = 20$. The scaled value function $c(\cdot)$ attaches $c^{fb}(k) = \frac{1}{r-\mu} - k$ smoothly, and $\overline{k} = \lambda$ is an absorbing barrier.

GBM setup, because the firm's cash flows stay positive, we obtain an optimal contract with a first-best absorbing state $k = \lambda$ where the marginal cost of paying cash early is zero. In this state, the agent with enough equity shares works voluntarily, and future liquidation never occurs.

Note that most discrete-time agency models, including those in DF and BMPR, feature a first-best absorbing state in the optimal contract—as agency issue will be completely resolved once the agent's continuation payoff becomes sufficiently high. The driving forces, however, are different. For instance, in the binomial model in BMPR, given the time step size, the per-period loss is bounded. Therefore, there exists an upper first-best absorbing state, where the firm accumulates a large fund whose interest is sufficient to cover all potential future losses. When the time step size goes to zero as the cash flow process converges to an ABM, this absorbing state explodes. In contrast, we derive a bounded absorbing state ($\lambda$) in the GBM model.

condition ($W_t = 0$ always; investors' promise about future wages is actually void). Note that Sannikov (2006) imposes a fixed finite life-span for the firm, therefore this issue is absent.

**1.3.3.3. Discussion: Continuation Payoff and Inside Stake.** This section provides economic intuition for the optimal contract. We first discuss the optimal incentive provision policy, and, for simplicity, we focus on the equally patient agent case. The same argument applies to the $\gamma > r$ case.

It is interesting to note that, due to the time-varying firm size in the GBM framework, the agent's continuation payoff can generate a portion of incentives. Consider the following thought experiment. Suppose that, at time $t$, investors decide to reward the agent with equity shares according to her continuation payoff. Note that the agent values $\alpha$ fraction of the firm as $\frac{\alpha\delta_t}{r-\mu}$ (given that she is working all the time). Therefore, to fulfill $W_t$ the agent is qualified to own $\alpha = (r-\mu)\, k_t$ shares of this firm (recall $W_t = k_t\delta_t$), and these shares generate an instantaneous volatility of $\frac{\alpha}{r-\mu}\sigma\delta_t = k_t\sigma\delta_t$.[16] By Proposition 1, when $\alpha \geq \alpha^* \equiv \lambda\,(r-\mu)$, or $k_t \geq \lambda$, these incentives are sufficient to motivate the agent. Because the agent's continuation payoff obtained from these shares remains positive, there is no future liquidation and the first-best outcome is achieved.

Now, in the optimal contract, before reaching the absorbing state, the agent's scaled continuation payoff $k_t$ is always lower than $\lambda$. This implies that only $\alpha_t = (r-\mu)\, k_t < \alpha^*$ shares can be awarded, if investors decided to do so. Also, as suggested by the optimal wage policy, investors should *wait to reward the agent later* (as $dU = 0$ before $k_t$ reaches $\lambda$). Are the incentives described above still present in this scenario? The answer is Yes. Imagine that these "hypothetical" shares (which are held by investors at time $t$) are promised to be delivered at $t+dt$, so that the agent cannot receive any portion of current dividends $\delta_t dt$ yet. Though hypothetical, these shares still generates incentives: since current dividends are in the lower order of $dt$, when firm size $\delta_t$ diffuses, the value of these hypothetical shares exhibits the same volatility $k_t\sigma\delta_t$ as actual

[16] Note that when $\gamma > r$, these $\alpha = (\gamma-\mu)\, k_t$ shares generate the same volatility level $k_t\sigma\delta_t$, and the similar argument about *hypothetical* shares (see below) can be applied to this case. Of course, the payment boundary $\overline{k}$ will be lower, as indicated by the optimal contract.

shares.[17] Loosely speaking, these hypothetical shares represent the agent's *inside stake* in the firm, but in a *forward-looking* sense. Finally, as required by Proposition 1, the optimal contract $\Pi^*$ imposes additional incentives $(\lambda - k_t)\,\sigma\delta_t$ to motivate the agent, and the above argument can be applied to any time $s$ before $k_s$ attains $\lambda$.

We illustrate the idea in the previous paragraph graphically on the $(\delta, W)$ plot in Figure 1.1. Fix $k$; given $W_t = k\delta_t$ the local movement of $(\delta_t, k\delta_t)$ is along a ray with slope $k$, due to the diffusion of $\delta_t$. In fact, this slope $k$ just captures those incentives generated by the agent's hypothetical inside stake when the firm size $\delta_t$ diffuses. The faster the firm grows, the higher the agent's continuation payoff, and the larger the incentives generated by these hypothetical shares.

In Section 1.3.1, we also observe that, on the $(\delta, W)$ plot, the incentive compatibility constraint requires a slope $\lambda > k$ for the local movement of $(\delta_t, W_t)$, and the agent will shirk if these hypothetical shares are the only incentive scheme available. Therefore, to implement working, investors have to provide additional incentives $(\lambda - k)\,\sigma\delta_t$ to fill out the slope gap $\lambda - k$, and these incentives constitute the diffusion term of $dk_t$ in (1.8). Intuitively, these additional incentives are provided by promising a larger future stock grants if her subsequent performance is superb, or liquidating the firm otherwise.

These observations lead to implications for the optimal wage policy. First, because the agent with $\alpha^*$ inside shares works voluntarily, when $\gamma = r$ (no relative consumption timing issue), it is the first-best absorbing state. Consequently, for the case of an equally patient agent, granting $\alpha^*$ shares to the agent once $W_t$ reaches $\lambda\delta_t$ must be part of the optimal contract. Second, when the agent is less patient than investors $(\gamma > r)$, the first-best outcome not only avoids inefficient liquidation, but also pays the agent as early as possible. Hence, it is never optimal for investors to wait until $W_t = \lambda\delta_t$ to award the agent with $\alpha^*$ shares. Accordingly, there exists a $\overline{k} < \lambda$ so

[17] In contrast, in the ABM model the firm size is fixed. Hence these hypothetical shares—without current dividends—cannot generate incentives as in our GBM setup.

that the firm starts paying wage once $W_t$ reaches $\overline{k}\delta_t$. Both statements are exactly the optimal contracts derived in previous sections.

### 1.3.4. Justification for the Optimal Contract

Take any incentive-compatible contract $\Pi = \{\{U\}, \tau\}$, and, for any $t \leq \tau$, define its auxiliary gain process $\{G\}$ as

$$G_t(\Pi) = \int_0^t e^{-rs}(\delta_s ds - dU_s) + e^{-rt} b(\delta_t, W_t), \tag{1.12}$$

where the agent's continuation payoff $W_t$ evolves according to (2.5). Under the optimal contract $\Pi^*$, the associated optimal continuation payoff $W_t^*$ has a volatility $\lambda\sigma\delta_t$, and $\{U^*\}$ reflects $W_t^*$ at $\overline{W}_t^* = \overline{k}\delta_t$.

Recall that $k_t = W_t/\delta_t$ and $b(\delta_t, W_t) = \delta_t c(k_t)$. Ito's lemma implies that, for $t < \tau$,

$$e^{rt} dG_t(\Pi) = \delta_t \left\{ \begin{array}{c} \left[-(r-\mu)c(k_t) + 1 + (\gamma-\mu)k_t c'(k_t) + \frac{1}{2}\left(\sigma_t^W - k_t\sigma\right)^2 c''(k_t)\right] dt + \\ \left[-1 - c'(k_t)\right] dU_t/\delta_t + \sigma\left[c(k_t) - k_t c'(k_t) + c'(k_t)\frac{\sigma_t^W}{\sigma}\right] dZ_t \end{array} \right\}.$$

Now, let us verify that, under any $\Pi \in \mathbb{IC}$, $e^{rt} dG_t(\Pi)$ has a nonpositive drift, and zero drift for the optimal contract. The first piece is our (1.4), which, under the optimal contract, is always zero. Because we have $c''(k_t) < 0$ for $k_t < \overline{k}$, and, since $\sigma_t^W \geq \lambda\sigma$ holds for any $\Pi \in \mathbb{IC}$, this term is nonpositive. The second piece captures the optimality of the cash payment policy. It is nonpositive since $c'(k_t) \geq -1$, but equals zero under the optimal contract. Therefore, we have the following theorem.

**Theorem 4.** *Take the scaled value function $c(\cdot)$ and its corresponding payment threshold $\overline{k}$, and define $k^* \equiv \underset{k\in[0,\overline{k}]}{\arg\max} c(k)$. Under the optimal contract $\Pi^* = \{\{U^*\}, \tau^*\}$, we have*

$$dW_t^* = \gamma W_t^* dt - dU_t^* + \lambda(d\delta_t - \mu\delta_t dt),$$

*where* $dU_t^*$ *reflects* $W_t^*$ *back to* $\overline{k}\delta_t$*, and* $\tau^* = \inf\{t \geq 0 : W_t^* = 0\}$*. Given* $\delta_0$*, the firm initiates the employment by picking* $W_0^* = k^*\delta_0$*, and investors obtain an expected value* $c(k^*)\,\delta_0$*.*

## 1.4. Extensions

### 1.4.1. Optimal Contracting with Costly Replacements

As in DF and DS, in this section we endogenize the liquidation value (factor) $L$. Assume that the incumbent agent can be replaced with a new but identical agent. Replacement is costly (e.g., the entrenchment effect), and we formulate the cost as $l\delta_t$ (where $l$ is a positive constant) in order to capture the underlying size effect. The form of a pure variable cost retains the scale invariance of this model.

The same analysis as in Section 1.3.2.1 goes through; the only difference is in the lower-end boundary condition

$$c(k^*) - c(0) = l \text{ where } k^* = \arg\max_{k\in[0,\overline{k}]} c(k)\,, \tag{1.13}$$

which embeds the optimal replacement policy. We solve (1.4) with conditions (1.5), (1.6), and (1.13), and obtain the endogenous liquidation value $L$ as $c(0)$.

The optimal contract with replacement is analogous to the previous liquidation case. When the agent is impatient, the incumbent agent receives some wage whenever her continuation payoff $W_t$ exceeds $\overline{k}\delta_t$, and the firm replaces her once $W_t$ falls to zero. In the equally patient agent case, poorly performing agents are fired, until a lucky one achieves $k_t = \lambda$ and henceforth continues to work for the firm forever.

Comparative Static Analyses. Here, we carry out comparative static analyses for the replacement case. (To find the corresponding results for the liquidation case, simply replace $\infty$ with the liquidation time $\tau^*$; and the results for $L$ are listed as well.) We also examine the comparative

| | $\partial c(k)/$ | $\partial \overline{k}/$ | $\partial k^*/$ |
|---|---|---|---|
| $\partial\gamma$ | $d_2(k), \pm$ | $\left\{\frac{-\overline{k}-(r-\mu)d_2(\overline{k})}{\gamma-r} < 0\right\}$ | $\left\{\frac{d_2'(k^*)}{-c''(k^*)} < 0\right\}$ |
| $\partial\mu$ | $\{d_1(k) - d_2(k) > 0\}$ | $\left\{\frac{\overline{k}+c(\overline{k})-(r-\mu)\left[d_1(\overline{k})-d_2(\overline{k})\right]}{\gamma-r} > 0\right\}$ | $\left\{\frac{d_1'(k^*)-d_2'(k^*)}{-c''(k^*)} > 0\right\}$ |
| $\partial\sigma$ | $d_3(k) < 0$ | $\frac{-(r-\mu)d_3(\overline{k})}{\gamma-r} > 0$ | $\frac{d_3'(k^*)}{-c''(k^*)}, \pm$ |
| $\partial l$ | $-d_{rp}(k) < 0$ | $\frac{(r-\mu)d_{rp}(\overline{k})}{\gamma-r} > 0$ | $\left\{\frac{-d_{rp}'(k^*)}{-c''(k^*)} > 0\right\}$ |
| $\partial L$ | $d_L(k) > 0$ | $\frac{-(r-\mu)d_L(\overline{k})}{\gamma-r} < 0$ | $\left\{\frac{d_L'(k^*)}{-c''(k^*)} < 0\right\}$ |

Table 1.1. Comparative Static Results

statics for optimal policies, namely, the payment boundary $\overline{k}$, and the replacement point $k^*$. Two key conditions that we exploit here are $c(\overline{k}) = \frac{1}{r-\mu} - \frac{\gamma-\mu}{r-\mu}\overline{k}$, and $c'(k^*) = 0$.

As in DS, Lemma 14 in the Appendix expresses the marginal impact of any parameter on $c(k)$ in terms of the conditional expectation of a certain integral. Under the auxiliary measure induced by (1.4), let $\{N_t\}$ be the counting process for replacements,[18] and define

$$\begin{aligned}
d_1(k) &= \mathbb{E}^{k_0=k}\left[\int_0^\infty e^{-(r-\mu)t} c(k_t)\,dt\right] > 0, \quad d_2(k) = \mathbb{E}^{k_0=k}\left[\int_0^\infty e^{-(r-\mu)t} k_t c'(k_t)\,dt\right], \\
d_3(k) &= \mathbb{E}^{k_0=k}\left[\int_0^\infty e^{-(r-\mu)t} \sigma(\lambda-k_t)^2 c''(k_t)\right] < 0, \\
d_{rp}(k) &= \mathbb{E}^{k_0=k}\left[\int_0^\infty e^{-(r-\mu)t} dN_t\right] > 0, \text{ and } d_L(k) = \mathbb{E}^{k_0=k}\left[e^{-(r-\mu)\tau^*}\right] > 0.
\end{aligned}$$

The following table summarizes our results. Note that, since both $\lambda$ and $\sigma$ measure the degree of the agency problem in this model, their comparative static results share the same sign.[19]

Most signs follow easily from $d(k)$'s. The less obvious signs, especially those involving the derivative information of $d(k)$'s, are placed in $\{\cdot\}$ (see proofs in the Appendix). Two of the terms

[18] Under the auxiliary measure induced by (1.4), $k_t$ evolves as $dk_t = (\gamma-\mu)k_t + (\lambda-k_t)\sigma dZ_t - du_t + k^* dN_t$, where $du_t$ reflects $k_t$ at $\overline{k}$, and $dN_t \equiv \mathbf{1}_{\{k_t=0\}}$ is the counting process for replacements.

[19] Here, we do not show comparative static results with respect to $r$, because most of them are ambiguous. For instance, there are two offsetting effects on the payment boundary $\overline{k}$. When $r$ increases, the benefit from exchanging the relative consumption timings is smaller; investors should pay cash later as reflected by a larger $\overline{k}$. However, the cost from future terminations is also reduced due to a larger discounting effect, which makes investors less worried about inefficient turnovers, and thus lowers $\overline{k}$. As a result, the overall effect is ambiguous.

could have either sign. Finally, when $\gamma = r$, the results still hold except for those regarding $\overline{k}$; recall that $\overline{k} = \lambda$ always in this case.

Most of the results are intuitive. For instance, we have $\partial c(k)/\partial\sigma < 0$—as if investors were risk-averse—because costly termination is more likely with a riskier project. This result also implies that $\partial\overline{k}/\partial\sigma > 0$, since investors should accumulate more continuation payoff along the optimal path in order to increase the buffer capacity. The same intuition applies to $\partial\overline{k}/\partial l > 0$.

A number of interesting empirical predictions ensue. For instance, the more severe the agency problem (higher $\sigma$ or $\lambda$), the later the agent will receive incentive payouts (larger $\overline{k}$). Also, for more profitable firms—with a higher $\mu$—the new agent is offered more favorable terms (larger $k^*$), but will receive incentive payouts later (larger $\overline{k}$). A larger $\overline{k}$ follows from the fact that investors avert to costly liquidations of highly profitable projects, and therefore they set a higher payment boundary $\overline{k}$ to reduce the chance of future terminations.

### 1.4.2. When Is It Optimal to Allow Shirking?

**1.4.2.1. General Analysis.** When the agent is shirking, she enjoys a private benefit $\phi\delta_t dt$, and the firm size follows as $d\delta_t = \sigma\delta_t dZ_t$. Since no cash payment is needed, the agent's continuation payoff $W_t$ evolves according to $dW_t = (\gamma W_t - \phi\delta_t)\,dt + \sigma_t^W \delta_t dZ_t$, where $\sigma_t^W \leq \lambda\sigma$. In other words, the working incentive must be lower than the level required by Proposition 1. For the auxiliary gain process $\{G\}$ in Section 1.3.4 to remain a supermartingale given this policy, we need that

$$-rc(k) + 1 + (\gamma k - \phi)\,c'(k) + \frac{1}{2}\left(\sigma_t^W - k\sigma\right)^2 c''(k) \leq 0 \text{ for } \forall k \in \left[0, \overline{k}\right].$$

Equivalently, we can rewrite the above condition as

$$-rc(k) + 1 + (\gamma k - \phi)\,c'(k) \leq 0 \text{ for } \forall k \in \left[0, \overline{k}\right], \tag{1.14}$$

because investors can set $\sigma_t^W = k_t\sigma \leq \lambda\sigma$ in order to remove the negative $2^{nd}$-order term. This interesting fact implies that, under our GBM setup, although incentives are superfluous when shirking is allowed, the optimal incentive provision is $k_t\sigma\delta_t$, rather than zero as in the ABM framework. In fact, they are merely incentives generated by the agent's hypothetical inside stake as discussed in Section 1.3.3.3.

Similar to DS, based on (1.14) we find the following sufficient condition for the optimality of working all the time:

$$\gamma c\left(\frac{\phi}{\gamma}\right) + (r-\gamma)\, c\,(k^*) \geq 1. \tag{1.15}$$

DS also find that when the agent is impatient, if the shirking benefit $\phi$ is sufficiently high, then the contract with an absorbing shirking state is optimal. It transpires that, for the equally patient agent case, this class of contracts is indeed optimal among all contracts that involve shirking along the history.[20]

**1.4.2.2. The Optimal Contract with Shirking when $\gamma = r$.** When $\gamma = r$, one can check that $c(k) - \left(k - \frac{\phi}{r}\right) c'(k)$ is quasi-concave and achieves its minimum at $\frac{\phi}{r}$. Therefore, (1.14) implies that the necessary and sufficient condition for the optimality of working all the time is simply

$$c\left(\frac{\phi}{r}\right) \geq \frac{1}{r}.$$

Note that, by "shirking all the time," the agent has a value $\frac{\phi}{r}\delta_t$, while investors obtain $\frac{1}{r}\delta_t$. Hence, it is just the optimality condition of working at the state $k = \frac{\phi}{r}$. Interestingly, this necessary condition is also sufficient for the optimality of implementing high effort at all states.

We can go one step further. Suppose that $rc\left(\frac{\phi}{r}\right) < 1$—that is, the point $\left(\frac{\phi}{r}, \frac{1}{r}\right)$ sits above $c(\cdot)$ in Figure 1.4—hence working all the time must be suboptimal. We show below that, in

[20]Note that in the $\gamma > r$ case, if the scaled shirking benefit $\phi$ is only slightly higher than the level such that (1.15) binds, then a more sophisticated contract will be optimal (see DS for detail).

the new optimal contract with shirking, $\left(\frac{\phi}{r}, \frac{1}{r}\right)$ is another absorbing state where the agent is shirking forever, and the agent works whenever her continuation payoff $W_t \neq \frac{\phi}{r}\delta_t$. Therefore, there are two absorbing states in this optimal contract: the upper working state where $W_t = \lambda\delta_t$ (the first-best result), and the middle shirking state where $W_t = \frac{\phi}{r}\delta_t$ (not the first-best result).

In the Appendix, based on this two-absorbing-state policy, we provide details about constructing $c^S(\cdot)$, that is, the scaled value function with shirking. Moreover, we show that $c_-^{S\prime}\left(\frac{\phi}{r}\right) > c'\left(\frac{\phi}{r}\right) > c_+^{S\prime}\left(\frac{\phi}{r}\right)$, where $c_+^{S\prime}\left(\frac{\phi}{r}\right)$ (or $c_-^{S\prime}\left(\frac{\phi}{r}\right)$) denotes the right- (or left-) derivative of $c^S(\cdot)$ at $\frac{\phi}{r}$ (see Figure 1.4). Since $c^S(\cdot)$ exhibits a downward kink (relative to $c(\cdot)$) at $\frac{\phi}{r}$, the function remains strictly concave, and the similar verification argument as in Section 1.3.4 applies. We have the following proposition.

**Proposition 5.** *Suppose $\gamma = r$. When $rc\left(\frac{\phi}{r}\right) < 1$, it is suboptimal to induce working all the time. Given $c^S(\cdot)$, denote $k^{*S} = \underset{k\in[0,\lambda]}{\arg\max} c^S(k)$. Along the optimal path, investors initiate the employment contract at $W_0^* = k^{*S}\delta_0$, and their expected value is $c^S\left(k^{*S}\right)\delta_0$. If $k^{*S} \neq \frac{\phi}{r}$, investors ask the agent to work by promising her $dW_t^* = rW_t^* dt + \lambda\sigma\left(d\delta_t - \mu\delta_t dt\right)$, until $W_t$ hits $\lambda\delta_t$ where she works forever (with cash payments $\lambda(r-\mu)\delta_s ds$ for $s \geq t$), or reaches $\frac{\phi}{r}\delta_t$ where she begins shirking forever (without any wage). If $k^{*S} = \frac{\phi}{r}$, then shirking all the time is optimal.*

Figure 1.4 shows one example of scaled value function with shirking $c^S(\cdot)$, and the original scaled value function $c(\cdot)$. In this example, we assume that, at termination, investors can either liquidate their assets at a surrender value $L\delta_t$, or replace the agent at a cost $l\delta_t$. Clearly, the optimal termination policy depends on the relative magnitude of $l$ and $L$. Interestingly, because shirking reduces the possibility of future terminations, the specific optimal termination policy might depend on whether or not the employment contract allows for shirking. Figure 1.4 demonstrates that, due to a relatively large replacement cost $l$, the optimal contract with

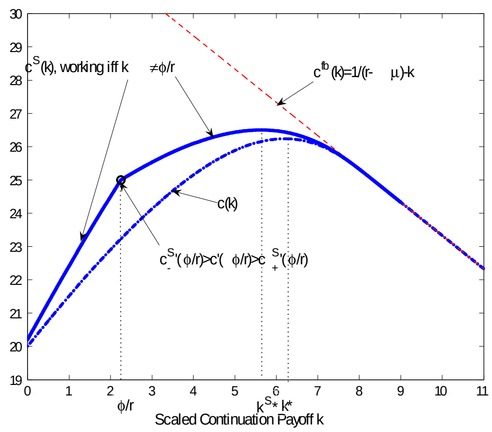


Figure 1.4. **The scaled value function $c^S(\cdot)$ with shirking**. Parameters are $r = 4\%, \mu = 1\%, \sigma^2 = 10\%, \phi = 0.09, \lambda = 9, L = 20, l = 6.3$. Since $c\left(\frac{\phi}{r}\right) = 23.23 < \frac{1}{r} = 25$, working all the time is suboptimal. The new scaled value function $c^S$ (the solid curve on top of $c$) solves (1.4) on both sides of $\left(\frac{\phi}{r}, \frac{1}{r}\right)$, and $c_-^{S\prime}\left(\frac{\phi}{r}\right) > c'\left(\frac{\phi}{r}\right) > c_+^{S\prime}\left(\frac{\phi}{r}\right)$. The agent shirks forever if and only if $k_t = \frac{\phi}{r}$. In this example, $c^S(0) = 20.21 > L$, while $c(k^*) - c(0) = 6.24 < l$. This implies that, in the absence of shirking, the optimal termination policy is liquidation, while when shirking is allowed on the optimal path, replacement becomes optimal.

working all the time stipulates liquidation as the optimal termination policy; however, if shirking is allowed, replacement becomes optimal.

## 1.5. Implementation and Applications

### 1.5.1. Implementation

To implement the optimal contract in Theorem 1, we design an *Incentive Points Plan* where the points trace the agent's scaled continuation payoff $k_t$. Specifically, the agent starts with $k^*$ points when she is hired by the firm. From then on, this plan rewards the agent with incentive points according to[21]

$$dk_t = \left[(\gamma - \mu) k_t + \sigma^2 (k_t - \lambda)\right] dt + (\lambda - k_t) \left(\frac{d\delta_t}{\delta_t} - \mu dt\right). \tag{1.16}$$

Once $k_t$ hits zero, she is fired. As featured by the payment boundary in the optimal contract, there is a redemption threshold $\overline{k}$ in this plan. When her points balance $k_t$ exceeds $\overline{k}$, the agent can redeem $k_t - \overline{k}$ points for $\left(k_t - \overline{k}\right) \delta_t$ amount of cash from the firm.

Performance-Based Stock Grants. Now, focus on the case of an equally patient agent. As suggested by Section 1.3.3.3, when $\gamma = r$, the optimal contract can be easily implemented by the performance-based stock grants, where the firm initially puts $\alpha^* = (r - \mu) \lambda$ incentive shares in the treasury. Under the incentive points plan, once the agent accumulates sufficient points ($k_t = \lambda$), she can redeem these points to obtain $\alpha^*$ incentive shares, and receive her portion of dividends $\alpha^* \delta_t dt$ onwards. On the other hand, the agent is fired if she depletes all her points before receiving the stock grants.

Note that the performance-based stock grants are merely a variant of stock options, with a zero strike price, and a nonstandard exercise boundary.[22] Also, we require those equity shares to be restricted shares, a feature consistent with what we observe in practice. In fact, our implementation resembles "performance shares" or "rights" in the long-term incentive plans of today's corporations.[23]

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[21]Check $dk_t = d(W_t^*/\delta_t)$. Note that this is under the physical measure (see discussion in footnote 13). Starting from $k_0$, before $k$ is regenerated (at 0) or regulated (at $\overline{k}$), this linear SDE admits the solution $k_t = e^{\kappa t - \sigma Z_t} \left[\lambda \sigma \int_0^t e^{-\kappa s + \sigma Z_s} dZ_s + k_0\right]$ where $\kappa \equiv \gamma - \mu + \frac{\sigma^2}{2}$. This result is useful in simulating our model.

[22]Under this framework we cannot implement the optimal contract using common stock options, because there is no one-to-one relationship between $k$ and the firm value. Besides, using American stock options leads to another potentially interesting issue: once granted some shares of stock options, the agent will solve a doubly stochastic-control problem: one is how to control the drift, and the other is the optimal exercising policy.

[23]For example, Citect—one of the top 5 technology companies in Australia—Long Term Incentive plan for 2005 states that the executive will receive a certain number of rights (to acquire an equivalent number of equity shares) on the commencement of employment. If prespecified performance targets are achieved during the employment, these rights become gradually vested and the exercise price is nil; if not, some rights lapse. Furthermore, the executive can only dispose of vested shares after three years from the date of granting rights. This is very similar to the optimal contract derived in this paper.

**1.5.1.1. Implications.** There are several implications that follow from the evolution of the scaled continuation payoff $k$ in (1.16). The incentive points, which track the agent's scaled continuation payoff, have a positive drift which is increasing in the level of $k_t$. This indicates the *positive feedback effect* of the agent's performance on her future cash payouts. This effect also shows up in the probability of the agent's future layoff: the higher the incentive points' balance, the larger the drift, and also the lower the volatility of $k_t$. As discussed in Section 1.3.3.3, less "additional" incentives are needed when the agent's continuation payoff is higher.

Second, in this model the agent's scaled continuation payoff $k_t$ measures the extent of the firm's agency problem. Since $k_t$ comoves positively with firm size growth, cross-sectionally we expect that agency issues will tend to be more severe in small firms. Note that the aforementioned positive feedback effect in this model could potentially amplify this divergence. If, in addition, the firm's value affects the firm's investment policy (not modeled here), then this amplification mechanism can be strengthened even further. Future work on this cross-sectional divergence is worth pursuing.

**1.5.1.2. Can We Have Similar Implementations as in DS or BMPR?.** We do not propose implementations that are similar to those in DS or BMPR. In their papers, a fund balance, which evolves according to the firm's cash flows, keeps track of the agent's continuation payoff. For instance, in DS, the combination of *long-term debt, equity, and credit line* implements the optimal contract, and the credit line balance traces the agent's continuation payoff.

In our GBM model, however, the agent's continuation payoff cannot be linked to actual cash flows. This difference is rooted in the fundamental control equation. In their model, the agent controls the *cash flow* $dY_t$, hence their optimal contract can use $dY_t$ to trace the agent's continuation payoff. In our model, however, the agent controls $d\delta_t$, which is the change of *firm size* rather than the *predetermined cash flow* $\delta_t dt$. This implies that our optimal contract has to rely upon $d\delta_t$ to keep track of the agent's continuation payoff, and thus cannot be implemented by standard cash flow contracts (e.g., credit lines) where only $\delta_t dt$ matters.

### 1.5.2. Applications

**1.5.2.1. Financial Distress and the Leverage Effect.** In our model, during financial distress, the firm's equity return becomes more volatile: the well-known leverage effect. To study the equity return, we first exclude the agent's nontradable stake. To accommodate corporate debt within our setting, we assume that the firm maintains a short-term debt $\rho\delta_t$ outstanding where $0 < \rho < c(k)$ for all $k \in \left[0, \overline{k}\right]$. That is to say, the firm simply rolls over and adjusts this amount of *riskless* short-term debt according to the firm size. Therefore, the equity value is $(c(k_t) - \rho)\,\delta_t$. Note that, without agency problems, the presence of such short-term riskless debt does not affect the constant volatility of equity return (the equity value is $\left(\frac{1}{r-\mu} - \rho\right)\delta_t$).

Now, under the agency problem, one can verify that the instantaneous equity return is

$$\frac{d\left[\left(c\left(k_t\right) - \rho\right)\delta_t\right]}{\left(c\left(k_t\right) - \rho\right)\delta_t} = \left[r - \frac{1 - \rho r}{c\left(k_t\right) - \rho} - \frac{\rho\mu}{c\left(k_t\right) - \rho}\right] dt + \left[1 + \frac{c'\left(k_t\right)\left(\lambda - k_t\right)}{c\left(k_t\right) - \rho}\right] \sigma dZ_t.$$

The drift term comprises three parts: 1) discount rate $r$; 2) dividend payout rate $\frac{1-\rho r}{c(k_t)-\rho}$; and 3) stock repurchase rate for new debt $\frac{\rho\mathbb{E}_t[d\delta_t]}{(c(k_t)-\rho)\delta_t} = \frac{\rho\mu}{c(k_t)-\rho}$. The diffusion term exhibits a stochastic volatility, and the volatility rises when $k \to 0$, as the firm is on the verge of liquidation. BMPR also derive the leverage-effect result under the ABM framework. However, without the agency problem, the GBM setting would result in constant volatility for the *return*, as opposed to the *level* in the ABM framework. Therefore, our predicted leverage effect is more compelling than the one obtained in BMPR.

**1.5.2.2. Executive's Pay-Performance Sensitivity.** Jensen and Murphy (1990) show that a CEO only obtains \$3.25 per \$1,000 increase in the shareholders' wealth; in the authors' terminology, this constitutes the sensitivity of CEO's "*expected wealth*" on his/her performance.

Interestingly, the concept of expected wealth in Jensen and Murphy (1990) captures an idea similar to the continuation payoff in this paper.[24] However, by only considering instant compensation (cash or new grants) and the capital gains from existing inside shares and options, most of the current empirical literature on CEO compensation might understate the long-term incentives generated by the continuation payoff in executives' remuneration contracts.[25]

What Might Be Missing in the Ongoing Empirical Work? To illustrate the importance of continuation payoff in measuring executive pay-performance responsiveness, we first simulate our model for the $\gamma = r$ case. The optimal contract is implemented by the performance-based stock grants. We choose $r = 4\%, \mu = 0.5\%, \sigma^2 = 6.25\%$ to match the calibration in Goldstein et al. (2001), and set $l = 0.2$ and $\lambda = 0.18.$[26]. We set $\rho = 10$ to have a debt ratio of about 35%. Following Jensen and Murphy (1990), we perform the following OLS regression,[27]

$$\Delta Comp_{i,t} = \beta_c + \beta_0 \Delta S_{i,t} + \beta_1 \Delta S_{i,t-1} + \varepsilon_{i,t},$$

where $Comp_{i,t}$ includes the grants value, dividends and capital gains, and $\Delta S_{i,t}$ is the change of shareholders' wealth including dividends. We find that the mean (standard deviation) of $\beta_0$ is 0.40% (0.047%), and the mean (standard deviation) of $\beta_1$ is 0.12% (0.028%). Therefore, the estimated sensitivity is 0.51% ($0.40\% + 0.12\% \times e^{-0.04}$); this is about 15% lower than the theoretical value $(r - \mu)\,\lambda = 0.63\%$ which takes into account the continuation payoff.

[24]For instance, the authors include both the current and the lagged annual performances in their regression, assume that the change of salary and bonus are permanent, and also compute the change of probability of dismissal.

[25]For example, Hall and Liebman (1998) group the salary and bonus together with the option and stock grants, and classify the bundle as *direct compensation.*

[26]Since the (scaled) first-best firm value is $\frac{1}{r-\mu} = 28.57$, the replacement cost is about 0.7% of the firm value, and the first-best inside holding $(r - \mu)\,\lambda$ is circa 0.6% to match the median CEO pay-performance measure obtained in Hall and Liebman (1998).

[27]There are 10 years and 200 firms in each simulation (with simulating time interval 0.01, or 3.65 days), and we repeat it for 500 times. Each firm's performance is driven by an independent Brownian motion. The regression is performed on annual data.

Two reasons may exist for researchers showing little concern about this issue. First, in pay-performance regressions, Jensen and Murphy (1990) and Joskow and Rose (1994) find that the higher-order lagged performances display insignificant coefficients. Second, long-term observations of firm-CEO pairs are not easily available.[28] However, our model advocates that we focus on the *stock and option grants* when measuring the dynamic pay-performance relationship. In addition, as these grants have been growing dramatically in large companies since the late 1980s (Hall and Liebman (1998)), we expect more pronounced results from recent years.

Grants-Performance Sensitivity: An Empirical Study. To test whether CEO *future* grants provide incentives for him/her to work *now*, we carry out a *Tobit* regression and find that[29]

$$\begin{aligned} Grants^*_{i,t} &= \underset{(836.78)}{-275.049} + \underset{(0.02)}{0.0913} \times \Delta S_{i,t} + \underset{(0.0221)}{0.2076} \times \Delta S_{i,t-1} + \underset{(0.0233)}{0.1608} \times \Delta S_{i,t-2} \\ &\quad + \underset{(0.0281)}{0.2420} \times \Delta S_{i,t-3} + \underset{(0.0367)}{0.2516} \times \Delta S_{i,t-4} + \varepsilon_{i,t} \end{aligned}$$

We combine the restricted stock and option grants together as our dependent variable $Grants_{i,t}$ (in *thousands*, and $Grants_{i,t} = \max\left(Grants^*_{i,t}, 0\right)$), which is the value of total grants received by CEO $i$ at year $t$. The independent variables $\Delta S_{i,t-j}$'s are the changes of the shareholders' wealth (in *millions*) of firm $i$ at year $t-j$. We add *years served by the CEO* (not reported here) to control for possible "promised" compensations in remuneration contracts. Due to the units difference between *Grants* and $\Delta S$, the coefficients for $\Delta S$ (with standard deviation underneath)

[28] Based on a VAR analysis, Boschen and Smith (1995) study the dynamic responses of executive's performance today on their future compensation. However, the time-series regression could overlap from one CEO to another, violating the underlying assumption of a long-term agency model where the *same* agent stays in the contractual relationship.

[29] We use the ExecuComp data set in Compustat which covers S&P 500 companies from 1992 to 2004. We use only the CEO data, and all numbers are adjusted in terms of 1992 dollar. The five-consecutive-year restriction on service results in 5,040 CEO-year observations. We estimate Tobit regression since there are 1,051 observations with zero total grants. We compute $\Delta S_{i,t}$ by multiplying $TRS1YR/100$ and the company's market value in the previous year. The 4-lag structure is chosen to match the median serving years of CEOs. To calculate the CEO's tenure, we count back to 1992, or the first year (after 1992) when the manager became CEO.

measure the dollar change of the agent's grants value given a $\$1,000$ change in the company's equity value.

A number of interesting findings arise from the above regression. First, the coefficient for contemporaneous performance is dominated by those for the CEO's past performances, showing that the CEO's grants are primarily driven by his/her historical achievements. Second, in contrast to Jensen and Murphy (1990) and Joskow and Rose (1994), all coefficients are significant, and even increase with lags.[30] Their magnitudes, however, are quite small. For instance, for a discount rate $r = 4\%$, the total incentives from current and future grants are, at most, $\$0.868$ for a $\$1,000$ change of shareholders' wealth.[31] This result might be due to our simple econometric specification (see, e.g., Aggarwal and Samwick (1999)).

### 1.6. Concluding Remarks

We study optimal contracting in a GBM firm size setting. In this model, growing firm size—as the agent's positive performance—increases the agent's inside stake within the firm, and thereby alleviates the agency problem. Along the optimal path, the agent requires stronger incentives than those she would have by holding equity according to her inside stake. Such incentives can be provided by performance-based stock grants, and they implement the optimal contract when the agent is as patient as investors. In this case, if it is too costly to work all the time we further derive a new optimal contract which features two absorbing states along the optimal path: one is shirking forever, and the other is working forever.

Distinct from the existing ABM model (BS and BMPR) which only studies the case of an impatient agent, under the GBM setup we derive an optimal contract with a first-best absorbing

[30]Note that our model implies that prior lags should have larger impacts due to the discount effect. For instance, the agent who was working at $t-4$ should discount the time-$t$ compensation $\beta_4$, while $\beta_0$ compensation is in today's dollars.

[31]$0.0913+0.2076\times e^{-0.04}+0.1608\times e^{-0.08}+0.2420\times e^{-0.12}+0.2516\times e^{-0.16} = 0.868$. There is a slight overestimation due to the Tobit model structure. Note that Jensen and Murphy (1990) find a CEO pay-performance sensitivity measure (mostly driven by inside shareholdings) as $\$3.15/\$1,000$.

state for the case of equally patient agent. Also, a time-varying firm size in the GBM model highlights the connection between the agent's continuation payoff and her inside stake in the firm, which provides a better understanding of the optimal incentive provision in dynamic contracting. These interesting findings advance the current continuous-time contracting literature.

This paper initiates the first step to connect recent research on dynamic contracting with the conventional continuous-time finance literature. This line of research awaits future work; for instance, it would be interesting to incorporate systematic agency-issue-related shocks into this framework. Also, this paper enables us to draw several insights for empirical studies on CEO's pay-performance relations. Empirical results provide support to our model, which predicts that, for stock/option grants, past performance is of greater importance than contemporaneous performance. This suggests that today's executive remuneration contracts should be analyzed from a dynamic perspective.

CHAPTER 2

# Agency Problem, Firm's Valuation, and Capital Structure

## 2.1. Introduction

This paper embeds the optimal contracting problem into the cash-flow framework commonly used in the corporate security pricing literature (e.g., Goldstein, Ju, and Leland (2001)). By connecting these two literatures, we provide a general framework to study the agency impacts on the firm's valuation and optimal capital structure. Moreover, the dynamic nature of this framework allows us to calibrate our model, and in turn quantitatively assess the agency cost and its impact on firm's financing decisions.

In Section 2.2, we start by offering a general analysis in solving for the optimal contract. The analysis crucially hinges on the agent's constant-absolute-risk-aversion (CARA, or exponential) preference, which is widely used in both the optimal contracting and asset pricing literature. The results are close to that of the classic Holmstrom and Milgrom (1987). However, we allow for the firm's profitability (cash-flow) to be time-varying; and in our model the agent has intermediate consumption, and can privately save (unobserved saving). Even though dynamic contracting with private saving remains unsolved for general cases, the absence of wealth effect—due to CARA preference—simplifies the contracting problem greatly; see similar findings in Fudenburg, Holmstrom and Milgrom (1990) and the concurrent work by Williams (2006).

Based on the general analysis in Section 2.2, we then study two special cases with different cash-flow processes. Section 2.3 focuses on the square-root mean-reverting process, and Section 2.4 investigates the more widely-used geometric Brownian motion process. The mean-reversion mainly captures the temporary nature of the firm's profitability shocks, as the shocks become permanent in the geometric Brownian cash-flow setup. Therefore, from a pure calibration point of view, the mean-reverting specification is more appealing. For instance, Hennessy and Whited (2005) find that cash-flows in Compustat firms exhibit a strong mean-reverting tendency; and Sarkar and Zapatero (2003) argue that mean-reverting profitability is both theoretically sounded and empirically supported. Nevertheless, to be more consistent with the corporate security-pricing literature (e.g., Leland (1994), Goldstein, Ju, and Leland (2001), and Strebulaev (2006)) where the geometric Brownian cash-flow setup becomes the workhorse, in Section 2.4 we embed the agency issue into Leland (1994) to investigate its impact on the firm's optimal leverage decision.

In Section 2.3, the affine structure of square-root processes yields great convenience in deriving the optimal contract. Because the CARA agent requires a risk-compensation linear in the (instantaneous) variance, the sqaure-root process generates a risk-compensation linear in the cash-flow level. As a result, the solution structure stays affine, and we derive the optimal contract in close form. We then derive the agency cost, which is defined as the gap between the first-best firm value and the second-best value, normalized by the first-best value. Interestingly, in this case, the agency cost—as well as the agent's pay-performance sensitivity—is independent of firm's cash-flow level. Finally, based on the estimates of cash-flow mean-reverting intensity and pay-performance sensitivity from the empirical literature, we calibrate our model in order to gauge the magnitude of the agency cost in our framework. Under our baseline parametrization, the agency cost is around $8 \sim 15\%$.

In Section 2.4 we turn to the case of geometric Brownian cash-flow setup, which has become the workhorse for the literature of corporate security pricing and capital structure. By interpreting the cash-flow level as the firm size, we first show that, the agent's pay-performance sensitivity is decreasing with the firm size, a prediction consistent with the empirical regularity. The reason is that in the geometric Brownian case the CARA agent's risk-compensation becomes quadratic in firm size, and this matches the common wisdom that, it is risk considerations that lead CEOs in larger firms to have smaller pay-performance sensitivities.

To study the impact of agency issues on the optimal capital structure, Section 2.4 introduces debt holders into the baseline model, and revisits the classical Leland (1994). For better comparison to Leland (1994) and other related work, in this triple-party framework, we bond the agent and the equity holders together through an optimal "complete" contract, while leaving the debt contract "incomplete" to take certain exogenously specified form—specifically, only static long-term debt with constant coupon is considered, and the equity holders have the option to default when the firm profitability deteriorates.[1] In addition, we assume that equity holders and the agent design the employment contract as a best response to the capital structure.[2] Essentially, these simplifying assumptions capture the key notion that, in US corporations, managers are responsible to the shareholders (e.g., Allen, Brealey, and Myers (2006)), and their relation is far closer than the one between managers and debt holders. Moreover, as a minimum deviation from the Leland (1994), our treatment emphasizes the key agency conflicts, and makes the comparison rather transparent.

We solve for the optimal capital structure and the optimal employment contract in Section 2.4.3. An interesting interaction between the agency issue and debt-overhang problem predicts that smaller firms should take less leverage, which is consistent with the empirical regularity. In our model, consistent with the empirically observed negative relation between pay-performance sensitivities and firm size, motivating the agent is costlier in larger firms. Consequently, debt-overhang problem—which reduces the "effort" investment during financial distress—becomes

[1]In our model, the agent—once bonded with the equity holders by the optimal contract—will have perfectly aligned interest with the equity holders when dealing with the debt holders. As a result, the default policy will be independent of whether the shareholders or the manager control the bankruptcy decision. This is different from Morellec (2004) where the manager tends to keep the firm alive longer for more private benefit.

[2]This assumption can be justified by the fact that, under this framework, the long-term contract can be implemented by a sequence of short-term contracts (see Fudenburg, Holmstrom and Milgrom (1990)); therefore, equity holders can always revise the contract after the debt issuance.

more severe in smaller firms, who in turn should issue less debt in their leverage decision. In our calibrations, for small firms, the predicted optimal leverage ratios—with or without the agency issue—can have a sizeable difference (65% versus 54%). We further examine the impact of debt covenants which specify the minimum implemented effort as a means to remedy the debt-overhang problem. There, interestingly, the debt covenants elicit the equity holders' early endogenous default; and this indirect negative effect—caused by the good-intentioned debt covenants—could lead to a net dead-weight loss.

Along the literature several attempts have been made to incorporate agency issues into the corporate-security pricing setting. Leland (1998) studies the agency issue due to the endogenous choice of firm's volatility; there the agent and equity holders are treated as one party. Morellec (2004) introduces the tension between the agent and the equity holders, where the empire-building agent tends to set a lower leverage ratio for rent-maximizing purposes.[3] Cadenillas, Cvitanic and Zapatero (2004) study a different version of agency problem. They restrict the compensation contract space to be equity; since the equity payoff ties to the debt face value, in their model the capital structure becomes a direct compensation scheme. In contrast, in our model the impact of leverage decision on the compensation contract is indirect.

The key difference of our paper from the above-mentioned literature, is that we study the agency impact based on the optimal contracting approach. More importantly, even though it seems appealing to restrict the compensation contract space to be the commonly observed forms as in Cadenillas, Cvitanic and Zapatero (2004), one might wonder whether the derived impact of agency problem is sensitive to specific contract forms.[4] By adopting the optimal contracting approach, we offer an lower-bound estimator for the impact of agency problem.

[3] Another line of extension focuses on the renegotiation between equity holders and debt holders once default occurs; see Anderson and Sundaresan (1996) and Mella-Barral and Perraudin (1996).

[4] Technically speaking, in the aforementioned papers, either the volatility choice in Leland (1998), or the over-investment (observable) decision in Morellec (2004), can be easily resolved by optimal contracting. These extreme examples clearly illustrate the sensitivity of agency costs to the contracting space.

Our paper is also closely related to the continuous-time contracting literature. Williams (2006) focuses on the general hidden-state problem, and he solves an example with CARA perference. In our paper, based on a different approach, we keep the cash-flow setup general, and focus on the applications in corporate finance. DeMarzo and Fishman (2007) and DeMarzo and Sannikov (2006) solve a dynamic contracting problem with a risk-neutral agent, where the limited liability restriction is imposed. Extensions include He (2007a) who studies the optimal executive compensation in a geometric Brownian cash-flow framework. In contrast, this paper employes the framework of Holmstrom and Milgrom (1987), which not only allows for a risk-averse agent, but also easily accommodates the second state-variable to capture the firm's time-varying profitability. For one of interesting examples of extensions of Holmstrom and Milgrom (1987), Ou-Yang (2005) adapts the basic Holmstrom and Milgrom model to an asset pricing setting, and studies the pay-performance sensitivity and relative-performance in a general equilibrium framework.[5]

The rest of this paper is organized as follows. Section 2.2 derives an ODE which characterizes the optimal contracting. As applications, Section 2.3 and Section 2.4 investigates two special case of cash-flow processes: the square-root mean-reverting process, and the geometric Brownian cash-flow setup. Section 5 concludes. All proofs are in the Appendix.

## 2.2. Model and Optimal Contracting

### 2.2.1. Model

We study an infinite-horizon, continuous-time agency problem. The firm (investors) hires an agent to operate the business. The firm produces cash flows $\delta_t$ per unit of time, where $\delta_t$ follows

[5]Schattler and Sung (1993) offer a general treatment for the continuous-time contracting with CARA preference, but under the Holmstrom-Milgrom framework, i.e., a finite time horizon with lump-sum transfer.

the stochastic process

$$d\delta_t = \mu\left(\delta_t, a_t\right) dt + \sigma\left(\delta_t\right) dZ_t. \tag{2.1}$$

Here, $Z = \{Z_t, \mathcal{F}_t; 0 \leq t < \infty\}$ is a standard Brownian motion on a complete probability space $(\Omega, \mathcal{F}, \mathcal{P})$, and $\mu\left(\delta_t, a_t\right)$ and $\sigma\left(\delta_t\right)$ satisfy the regularity conditions for the existence of solutions to (2.1); see, e.g., Karatzas and Shreve (1991). Through unobservable effort $a_t \in [0, \overline{a}]$ the agent controls the cash-flow growth rate $\mu\left(\delta_t, a_t\right)$, where $\mu_a\left(\delta, a\right) \equiv \frac{\partial \mu(\delta,a)}{\partial a} > 0$ and $\mu_{aa}\left(\delta, a\right) \equiv \frac{\partial^2 \mu(\delta,a)}{\partial a^2} \leq 0$. The performances $\{\delta_t\}$ are contractible.

The agent, with a CARA (exponential) instantaneous utility and a time discounting factor $r$, maximizes his expected life-time utility

$$\mathbb{E}\left[\int_0^\infty \frac{e^{-\gamma(c_t - g(\delta_s, a_s)) - rt}}{\gamma} dt\right],$$

where $g\left(\delta, a\right)$ is the agent's monetary effort cost satisfying $g_a\left(\delta, a\right) = \frac{\partial g(\delta,a)}{\partial a} > 0$ and $g_{aa}\left(\delta, a\right) = \frac{\partial^2 g(\delta,a)}{\partial a^2} > 0$. In the special cases studied later, $g\left(\delta, a\right)$ is quadratic in $a$ and linear in $\delta$. Investors are risk-neutral with a discount rate $r$, which is also the market interest rate.

This model allows for the agent's private (unobservable) saving. The agent can borrow and save at the risk-free rate $r$ in his personal account; and the account balance, as well as the agent's actual consumption, is unobservable. As a result, in designing the optimal contract, there will be a hidden state problem, and the analysis is not tractable in general. However, without the wealth effect,[6] the CARA preference with a monetary effort cost eliminates this complicated issue. The same observation is obtained by Fudenberg, Holmstrom and Milgrom (1990) and the concurrent work by Williams (2006).

[6]The wealth effort is absent only if negative consumption is allowed for the agent, and he faces no borrowing constraint. And, in contrast to the usual arithmetic Brownian model where the output's volatility is constant, our model, according to simulation result, generates a much lower probability of a negative consumption. For a different contracting problem without negative consumption, see He (2007b).

### 2.2.2. Contracting Problem

The employment contract $\Pi = \{c; a\}$ specifies the agent's wage process $c$ and the "recommended" effort process $a$. Both elements are adapted to the filtration generated by $\delta$; in other words, they are performance-based. Note that due to private saving, the agent's actual consumption $\widehat{c}_t$ can be different from his wage $c_t$. To simplify the analysis, unless otherwise stated we assume that the implemented effort $a_t$ takes an interior solution.

As investors have access to the same saving technology, without loss of generality we focus on the incentive-compatible and no-saving contract, which is defined as follows.

**Definition 6.** *Denote the agent's actual consumption and actual effort as $\widehat{c}$ and $\widehat{\mu}$, respectively, and his private saving as $S$. A contract is incentive-compatible and no-saving, if the solution to the agent's problem*

$$
\begin{aligned}
\max_{\{\widehat{c}_t\},\{\widehat{a}_t\}} \quad & \mathbb{E}\left[\int_0^\infty -\frac{e^{-\gamma(\widehat{c}_t - g(\delta_t,\widehat{a}_t))-rt}}{\gamma}dt\right] \\
s.t. \quad & dS_t = \left(rS_t + c_t - \widehat{c}_t\right)dt \\
& d\delta_t = \mu\left(\delta_t,\widehat{a}_t\right)dt + \sigma\left(\delta_t\right)dZ_t
\end{aligned}
\tag{2.2}
$$

*is just $\{c; a\}$, where the transversality condition $\lim_{T\to\infty} \mathbb{E}e^{-rT}S_T = 0$ is imposed. Let $V(\Pi)$ be the agent's value derived from the contract $\Pi$.*

The constraints in the above problem clearly illustrate that, both the consumption policy $c$ and effort policy $a$ in the contract $\Pi = \{c; a\}$ are only "recommended." For instance, the agent can conduct some saving by setting his consumption $\widehat{c}_t$ strictly below the wage income $c_t$, or may choose an effort level $\widehat{a}_t \neq a_t$. To summarize, $\Pi$ is incentive-compatible and no-saving, if the agent, once facing the contract $\Pi$, finds it optimal to exert the recommended effort, and maintain zero savings always.

Finally, suppose that the agent has a time-0 outside option $W_0$; then investors solve the following problem

$$\begin{aligned} \max_{\Pi \text{ is incentive-compatible no-saving}} \quad & \mathbb{E}\left[\int_0^\infty e^{-rt}\left(\delta_t - c_t\right)dt\right] \\ s.t. \quad & V\left(\Pi\right) \geq W_0 \end{aligned} \tag{2.3}$$

where the second line is the agent's participation constraint. It is well-known that, under this CARA framework, the outside option $W_0$ only affects the optimal contract by a constant transfer between both parties.

### 2.2.3. Model Solution

To solve the model, we first introduce the agent's continuation value, and obtain its equilibrium evolution for the optimal contract. Then we write down the investors' Hamilton-Jacobi-Bellman (HJB) equation, and derive an Ordinary Differential Equation (ODE) that characterizes the optimal contracting.

**2.2.3.1. Agent's Continuation Value.** Following the literature, we take the agent's *continuation value* (*continuation payoff*, or *promised utility*) as the state variable. Formally, the continuation value is defined as

$$W_t = \mathbb{E}_t\left[\int_t^\infty -\frac{e^{-\gamma(c_s - g(\delta_s, a_s)) - r(s-t)}}{\gamma} ds\right]. \tag{2.4}$$

Notice that this payoff is obtained under the follow-up policies specified by the contract: he exerts effort policy $\{a_s : s \geq t\}$ recommended by the contract, and consumes exactly his future wage payments $\{c_s : s \geq t\}$. In Section 2.2.3.2, we will invoke the important fact that, these recommended follow-up policies are indeed optimal among all policies in the agent's problem (2.2).

Due to the martingale representation theorem, (2.4) implies that in equilibrium the agent's continuation value evolves as

$$dW_t = rW_t dt - u(c_t, a_t)\, dt + \beta_t(-\gamma r W_t)(d\delta_t - \mu(\delta_t, a_t)\, dt), \tag{2.5}$$

where $\{\beta\}$ is a progressively measurable process. Here, $-\gamma r W_t > 0$ is the scaling factor that facilitates the economic interpretation of $\beta_t$ in Section 2.2.3.2.

In (2.5), the agent's expected total (instantaneous) compensation change is

$$\mathbb{E}_t[dW_t + u(c_t, a_t)\, dt] = rW_t dt,$$

which is the growth rate of the agent's continuation value. This just captures the *promise-keeping* constraint in optimal contracting. On the other hand, the diffusion part of (2.5)—which involves $\beta_t$—controls the agent's working incentive. Intuitively, it is the diffusion part that directly links to the observable performance $d\delta_t$, and $\beta_t(-\gamma r W_t)$ measures the punishing-reward extent (in utility terms) in the employment contract.

Absence of Wealth Effect. Because the agent can secretly save in this model, in general, to solve the optimal contracting problem, one needs more state variables than the uni-dimensional continuation value $W$.[7] However, the CARA preference, by abstracting away from the wealth effect, makes the agent's problem invariant to his "hypothetical" hidden wealth. Consequently, we find that the continuation value $W_t$ is indeed the only state variable required in deriving the optimal contract. A similar finding is obtained in Williams (2006).

In essence, the absence of wealth effect allows us to derive the agent's deviation value (to other saving levels) only based on the agent's equilibrium value—which is just the continuation value $W$. To be specific, under the CARA preference, given any continuation value $W_t$ without

[7] See He (2007b) where two state variables are sufficient. In general, because private wealth level could take value from a continuum, we need a "state function"—as opposed to state variables—to solve the optimal contracting problem with private savings.

private saving, the agent's deviation value, by having an extra saving $S \neq 0$ at time $t$, is

$$V_t(S;\Pi) = W_t \cdot e^{-\gamma r S}. \tag{2.6}$$

Intuitively, the agent's new optimal policy is simply keeping the policy with no private saving, but consumes an extra $rS$ more (less if $S < 0$) for all future dates $s \geq t$.[8] Essentially, for a CARA preference, the agent's problem is translation-invariant to his underlying wealth level. Without CARA preference, agent's working incentives are wealth-dependent, and the deviation value representations—as simple as (2.6)—are no long available.

**2.2.3.2. Equilibrium Evolution of $W$.** Recall that in Definition 6, these recommended policies specified in $\Pi$ have to be optimal among all policies. For this to be true, we now derive the necessary and sufficient conditions for the evolution of $W$ in (2.5), based on the equation (2.6) derived above.

By the optimality of the agent's consumption-saving policy in (2.2), his marginal utility from consumption must equal his marginal value of wealth; i.e., $u_c(c_t, a_t) = \frac{\partial}{\partial S} V_t(0;\Pi)$. Therefore one necessary condition for $\Pi$ to elicit no private saving is

$$u_c(c_t, a_t) = e^{-\gamma(c_t - g(\delta_t, \widehat{a}_t))} = \frac{\partial}{\partial S} V_t(0;\Pi) = -\gamma r W_t \Rightarrow u(c_t, a_t) = r W_t,$$

where we use the functional form of $V_t(S;\Pi)$ in (2.6). Plugging this result into (2.5), we find that the instantaneous utility $u(c_t, a_t)$ just offsets $rW_t$, and (2.5) becomes

$$dW_t = \beta_t(-\gamma r W_t)(d\delta_t - \mu(\delta_t, a_t)\,dt). \tag{2.7}$$

[8] To see this, note that $W$ in (2.4) is the agent's optimal continuation value $V_t(0;\Pi)$ when $S = 0$. Now given any $S$, the agent's objective, by consuming $rS$ more permanently, is

$$e^{-\gamma r S} \max \mathbb{E}_t \left[ \int_t^\infty -\frac{e^{-\gamma(\widehat{c}_s - g(\delta_s, \widehat{a}_s)) - r(s-t)}}{\gamma} ds \right]$$

Therefore, the agent's problem is unchanged, and plugging in (2.4) we obtain the result in (2.6).

Therefore, the agent's continuation value $W_t$ follows a martingale.

Two points are note-worthy. First, $u(c_t, a_t) = rW_t$ implies that the equilibrium consumption is

$$c_t = g(\delta_t, \widehat{a}_t) - \frac{\ln \gamma r}{\gamma} - \frac{1}{\gamma} \ln(-W_t). \tag{2.8}$$

Second, because $u_c(c_t, a_t) = -\gamma r W_t$, the agent's marginal utility also follows a martingale. Not surprisingly, this is a direct implication from the agent's optimal consumption-saving policy. In contrast, in the optimal contracting without private savings as studied in Sannikov (2006), the agent's consumption directly links to the slope of the principal's value function, and it is the inverse of marginal utility that follows a martingale for better incentive provisions.[9]

Now we turn to the incentive provision to pinpoint the diffusion loading $\beta_t$ in (2.7). Intuitively, in (2.7), $\beta_t(-\gamma r W_t)$ measures the agent's continuation "utility" sensitivity with respect to the unexpected performance $d\delta_t - \mu(\delta_t, a_t)\,dt$. Now the role of the scaling factor $-\gamma r W_t$ becomes clear: since it is the agent's marginal utility $u_c$, by transforming *utilities* to *dollars*, $\beta_t$ directly measures the (monetary) compensation sensitivity with respect to his performance.

Consider the agent's effort decision. By choosing $\widehat{a}_t$, the agent will gain from his instantaneous utility $u(c_t, \widehat{a}_t)$. On the other hand, the effort $\widehat{a}_t$ sets the drift of $d\delta_t$ to be $\mu(\delta_t, \widehat{a}_t)$, and the expected change of the agent's continuation value is (recall (2.7)) $\mathbb{E}_t[dW_t(\widehat{a}_t)] = \beta_t(-\gamma r W_t)[\mu(\delta_t, \widehat{a}_t) - \mu(\delta_t, a_t)]$. Therefore, the agent is solving

$$\max_{\widehat{a}_t} \; u(c_t, \widehat{a}_t) + \beta_t(-\gamma r W_t)\,\mu(\delta_t, \widehat{a}_t).$$

[9]This important result implies that the agent's marginal utility follows a submartingale, as $1/x$ is a convex function. Therefore, the agent has incentive to save for the future, if they are allowed to. See He (2007b) for details.

Since $u_a = u_c \cdot (-g_a(\delta_t, a_t))$, and $u_c = -\gamma r W_t$, implementing $\widehat{a}_t = a_t$ requires that

$$-g_a(\delta_t, a_t) + \beta_t \mu_a(\delta_t, a_t) = 0 \Rightarrow \beta_t = \frac{g_a(\delta_t, a_t)}{\mu_a(\delta_t, a_t)}, \tag{2.9}$$

and this first-order condition is also sufficient.[10]

Equation (2.9) gives an equilibrium relation between the recommended effort $a_t$ and the incentive loading $\beta_t$. Intuitively, $\mu_a(\delta_t, a_t)$ is the agent's effort impact on the instantaneous performance; therefore $\beta_t \mu_a(\delta_t, a_t)$ gives the agent's *monetary* marginal benefit of his effort. To be incentive compatible, the marginal benefit must equal the agent's monetary marginal effort cost $g_a(\delta_t, a_t)$. And, because $g$ (or $\mu$) is convex (or concave) in $a$, the required incentive loading $\beta_t$ is increasing in $a_t$; in other words, a higher level of effort requires more incentives.

As a summary, for a contract $\Pi$ to be incentive-compatible and no-saving, the necessary and sufficient condition is that the evolution of $W$ in (2.5) takes the form

$$dW_t = \frac{g_a(\delta_t, a_t)}{\mu_a(\delta_t, a_t)} \gamma r (-W_t) \sigma(\delta_t) dZ_t, \tag{2.10}$$

where we replace the innovation term in (2.7) by $\sigma(\delta_t) dZ_t$ due to the relation (2.1).

There is one point worth noting. Equation (2.10) implies that the agent's continuation value $W$ follows an exponential (local) martingale, a familiar object studied in the asset-pricing literature. For the promise-keeping condition to hold in the optimal contract, $W$ must be indeed a martingale, and one powerful sufficient condition is the so-called Novikov condition (e.g., see Karatzas and Shreve (1991)). The verification of Novikov condition usually requires a problem-specific approach; as an example, in the Appendix we provide a proof for the case of mean-reverting sqaure-root process studied in Section 2.3.

[10]Under the current CARA with monetary effort cost structure, the agent's problem is in fact concave in both $\widehat{c}_t$ and $\widehat{a}_t$.

**2.2.3.3. Optimal Contracting.** Given the state variables $\delta$ and $W$, denote $J(\delta, W)$ as the investor's value function. The investor's HJB equation can be written as

$$rJ(\delta, W) = \max_a \left\{ \begin{array}{c} \delta - c(a;\delta) + J_\delta \cdot \mu(\delta, a) + \frac{1}{2} J_{\delta\delta} \sigma(\delta)^2 \\ J_{\delta W} \frac{g_a(\delta_t, a_t)}{\mu_a(\delta_t, a_t)} \gamma r (-W_t) \sigma(\delta_t)^2 + \frac{1}{2} J_{WW} \gamma^2 r^2 W^2 \left[ \frac{g_a(\delta_t, a_t)\sigma(\delta_t)}{\mu_a(\delta_t, a_t)} \right]^2 \end{array} \right\}, \tag{2.11}$$

where $c(a;\delta)$ takes the form in (2.8).

The absence of wealth effect, thanks to the CARA preference, leads us to guess that

$$J(\delta, W) = f(\delta) + \frac{1}{\gamma r} \ln(-\gamma r W), \tag{2.12}$$

where $f(\delta)$ captures the firm's value under the optimal contract, and $-\frac{1}{\gamma r}\ln(-\gamma r W)$ is just the agent's certainty-equivalence given his continuation value $W$. Plugging (2.8) into (2.11), and note that $J_{WW} = -\frac{1}{\gamma r}\frac{1}{W^2}$, and $J_{\delta\delta} = J_{\delta W} = 0$, we have

$$rf(\delta) = \max_a \left\{ \delta - g(\delta, a) + f'(\delta)\mu(\delta, a) + \frac{1}{2} f''(\delta)\sigma(\delta)^2 - \frac{1}{2}\gamma r \left[ \frac{g_a(\delta_t, a_t)\sigma(\delta_t)}{\mu_a(\delta_t, a_t)} \right]^2 \right\} \tag{2.13}$$

In (2.13), $f'(\delta)\mu(\delta, a)$ captures the benefit from implementing the effort $a$; note that its magnitude is endogenous as $f'(\delta)$ depends on the value function. And, similar to Holmstrom and Milgrom (1987), there are two distinct costs in implementing the effort $a$. One is the direct monetary effort cost $g(\delta, a)$, and the other is the risk-compensation

$$\frac{1}{2}\gamma r \left[ \frac{g_a(\delta_t, a_t)\sigma(\delta_t)}{\mu_a(\delta_t, a_t)} \right]^2 \tag{2.14}$$

for a risk-averse agent to bear incentives. This additional agency-related cost, as in Holmstrom and Milgrom (1987), captures the key trade-off between incentive provision and risk-sharing in the optimal contract.

The solution to (2.13), combined with (2.8) and (2.10), characterizes the optimal contracting.[11] To study the agency cost and its impact on the optimal capital structure, we apply these results to two special cases in the following sections.

## 2.3. Square-Root Mean-Reverting Process Setup

### 2.3.1. Model Specification

Suppose that $\mu(\delta_t, a_t) = (\Delta - \kappa\delta_t) + a_t\delta_t$, and $\sigma(\delta_t) = \sigma\sqrt{\delta_t}$, where $\Delta$, $\kappa$, and $\sigma$ are positive constants. Here the agent's effort impact on the firm's growth is linear in the cash-flow level; we may interpret the cash-flow level as the firm size.

As we will show shortly, the optimal effort level is constant, i.e., $a_t = a^*$. In other words, under the optimal contract,

$$d\delta_t = (\Delta - (\kappa - a^*)\,\delta_t)\,dt + \sigma\sqrt{\delta_t}dZ_t. \tag{2.15}$$

Therefore, here the firm's cash-flow follows a square-root mean-reverting (Ornstein-Uhlenbeck) process, which is often used to model the interest rate dynamics in the fixed-income literature. Later we simply call it a square-root process.

The most salient feature of (2.15) is that the firm's cash-flows (or earnings) are mean-reverting. As argued in Sarkar and Zapatero (2003), the mean-reversion in firm's profitability is both theoretically and empirically sounded. Theoretically, in a competitive economy, the project's cash-flows in the long-run should be stationary, rather than "wandering" forever. The empirical justification comes from, for instance, Fama and French (2000) and Hennessy and Whited (2005). By examining the time-series of the firm's operating income, Hennessy and

[11] In solving (2.13), we might need certain problem-specific boundary conditions. See Section 2.4 for an example of boundary conditions where the cash-flow follows a geometric Brownian motion.

Whited (2005) report an autoregressive coefficient as 0.583, which suggests a strong mean-reverting tendency.

The agent's effort cost takes the form $g(\delta, a) = \frac{\theta}{2}a^2\delta$ which is quadratic in the agent's effort. Our cost specification is linear in $\delta_t$, because the agent's effort impact is also linear in the cash-flow level $\delta_t$. Plugging the specifications of $\mu$ and $g$ into (2.14), we have $\beta_t = \frac{g_a(\delta_t, a_t)}{\mu_a(\delta_t, a_t)} = \theta a$; or, the required incentives are linear in the implemented effort. Therefore, in the optimal contract, the risk-compensation is

$$\frac{1}{2}\gamma r \left[\frac{g_a(\delta_t, a_t)\,\sigma(\delta_t)}{\mu_a(\delta_t, a_t)}\right]^2 = \frac{1}{2}\gamma r\theta^2 a^2\sigma^2\delta.$$

Now it is clear how the square-root process allows for a close-form solution for the optimal contract. Under the CARA preference, the agent's risk-compensation is the variance of the (local) risk that he takes. Therefore, given a sqaure-root process, the risk-compensation $\frac{1}{2}\gamma r\theta^2 a^2\sigma^2\delta$ is linear in $\delta$; note that a similar property renders the tractability of affine models in the fixed-income literature. This result fails in Section 2.4 where we study a geometric Brownian case where the instantaneous variance becomes quadratic in $\delta$.

### 2.3.2. Solution

Given above specifications, the solution to (2.13) is straight-forward. Since all elements in (2.13) are affine in $\delta$, we guess that $f(\delta) = A + B\delta$, and ODE (2.13) is reduced to

$$rA + rB\delta = \max_a \left\{\delta - \frac{\theta}{2}a^2\delta + B(\Delta - \kappa\delta + a\delta) - \frac{1}{2}\gamma r\theta^2 a^2\sigma^2\delta\right\}. \tag{2.16}$$

By focusing on the interior solution, the constant optimal effort $a^*$ is

$$a^* = \frac{B}{\theta(1 + \theta\gamma r\sigma^2)}. \tag{2.17}$$

Now plugging this result into (2.16), we have $A = \frac{B\Delta}{r}$, and[12]

$$B = \frac{2}{r + \kappa + \sqrt{(r+\kappa)^2 - \frac{2}{(1+\theta\gamma r\sigma^2)\theta}}}. \tag{2.18}$$

To sum up, the optimal contract implements a constant effort in (2.17). The evolutions of the agent's wage (consumption) policy follows (2.8), and the agent's continuation value is governed by

$$dW = \gamma r\left(-W_t\right)\frac{B\sigma\sqrt{\delta_t}}{1+\theta\gamma r\sigma^2}dZ_t. \tag{2.19}$$

Even though the cash-flow level is time-varying, under the CARA preference, the square-root process maintains a desirable affine structure. As a result, the contracting problem, therefore the optimal effort, is invariant to the cash-flow state $\delta$.

In the Appendix, we provide a formal verification argument for the optimality of the proposed contract.

### 2.3.3. Implications

**2.3.3.1. Firm's Valuation.** We have obtained the investors' value function

$$J\left(\delta, W\right) = \frac{B\Delta}{r} + B\delta + \frac{1}{\gamma r}\ln\left(-\gamma r W\right).$$

However, in this paper we are interested in the total value created by the firm, i.e., the frontier of the production opportunity set, which includes the agent's payoff.

Thanks to the CARA preference, in (2.12) by treating the agent's certainty equivalence $-\frac{1}{\gamma r}\ln\left(-\gamma r W\right)$ as the agent's stake (in dollars), we can measure the frontier of the production

[12]The constant $B$ solves the quadratic equation $rB = 1 + \frac{B^2}{2\theta\left(1+\theta\gamma r\sigma^2\right)} - \kappa B$. We require that $\theta\left(r+\kappa\right)^2\left(1+\theta\gamma r\sigma^2\right) - 2 > 0$ for $B$ to be real, and we take the smaller root—because one can show that the larger root implies the effort level $a^* > \kappa + r$ which results in an explosion solution.

opportunity set by $f(\delta)$, which is independent of the agent's stake inside the firm. In the current case, the firm's value is simply

$$f(\delta) = \frac{B\Delta}{r} + B\delta.$$

One way to interpret this result is that, in implementing the optimal contract, investors set up a defered-compensation fund inside the firm with balance

$$H_t = -\frac{1}{\gamma r} \ln(-\gamma r W_t);$$

and investors adjust this balance continuously according to the evolution of $W_t$ specified in (2.19). Note that we can interpret the defered-compensation balance $H_t = -\frac{1}{\gamma r} \ln(-\gamma r W_t)$ as the agent's financial wealth. By keeping the agent's stake inside the firm, the firm's (market) value becomes the total value created by the firm.[13]

**2.3.3.2. Pay-Performance Sensitivity.** In the literature, executive pay-permeance sensitivity ($PPS$) has received great attention since Jensen and Murphy (1992). The central question, which gives the measure of $PPS$, is that: "how much does the executive's wealth change when the firm's value moves by one dollar?"[14] The CARA preference allows for a natural translation from the agent's "wealth" to the certainty equivalence of his continuation value; and as discussed above, we further interpret the defered-compensation fund $H_t$ as the agent's "wealth."

[13]As in Westerfield (2006), this balance can also be interpreted as the committed separation payment if either party wants to renege in the future. Theoretically, the CARA framework cannot rule out the possibility of $H_t < 0$. We interpret this case as the agent to take a personal debt, and the debt is netted out in calculating the total firm value. One note-worthy point is that, in contrast to the constant volatility setting as in Holmstrom and Milgrom (1987), here a drop in the continuation payoff—which coincides with a drop in $\delta$—leads to a lower volatility in $W$. This implies a smaller probability of negative $H_t$, and in simulation, the probability of $\{H_t < 0\}$ is negligible. The same statement holds for the geometric Brownian cash-flow process studied in Section 2.4.
Another note-worthy point is that, in this implementation, investors conduct saving for the agent, as his wealth is kept inside the firm. Another equivalent implementation is to put $H_t$ into the agent's personal account, but allow for two-way transfers between the agent and investors according to (2.19). Then the firm's market value becomes $J(\delta, W)$. When the agent's stake is small, the difference between these two treatments are negligible.

[14]Strictly speaking, in the executive compensation literature, the pay-performance sensitivity is with respect to the shareholder's value, which should excludes the agent's non-equity stake. There are two reasons why this treatment is inessential: 1) the magnitude of $PPS$ is small ($1 \sim 5\%$); 2) Empirically, the executive' $PPS$ mainly comes from his/her inside holdings which is an equity stake.

In the current continuous-time framework, the agent's pay-performance sensitivity can be precisely measured by the response of the fund $H$, to a unit shock of the firm's value. Specifically, by neglecting all drift terms, we have

$$PPS = \frac{dH}{d\left(A + B\delta\right)} = \frac{\frac{1}{\gamma r}\frac{dW}{-W}}{B\sqrt{\delta}\sigma dZ} = \frac{1}{1+\theta\gamma r\sigma^2}. \tag{2.20}$$

Recall that Holmstrom and Milgrom (1987) derives a $PPS = \frac{1}{1+\theta\gamma\sigma^2}$; in (2.20), the agent's effective risk-aversion becomes $\gamma r$. The reason is intuitive. In our model, the agent has consumption flows, as opposed to the lump-sum consumption in Holmstrom and Milgrom (1987). As a result, the agent's effective risk-aversion becomes $\gamma r < \gamma$ due to consumption smoothing.[15]

In the empirical literature, Jensen and Murphy (1990), as the first to raise the issue, report a $PPS$ of 0.3% in their sample (1969-1983). However, Hall and Liebman (1998) find that due to the explosion in stock option issuance during 1980s and 1990s, sensitivity of compensation to firm performance have risen dramatically, and their mean $PPS$ is about 2.5%. And, Aggarwal and Samwick (1998) control for the firm's risk, and report a mean $PPS$ of 6.94% from the OLS regression.[16] In gauging the baseline parameters for our model, we will use the relation (2.20) to match the $PPS$ measure obtained from the data.

Finally, note that in this case we have a constant pay-performance sensitivity which is independent of the firm size (proxied by the cash-flow level $\delta$). However, a salient empirical regularity regarding the cross-sectional distribution of compensation-performance relationship, is that larger firms have much smaller pay-performance sensitivities for their executives. In Section 2.4, we will see that a geometric Brownian setup naturally gives rise to a heterogeneity among different sized firms, and in that model, the $PPS$ is indeed smaller for larger firms.

[15]This observation raises a critique to Haubrich (1994), who argues that a reasonable risk-aversion parameter can validate the Jenson and Murphy (1992) finding of a low PPS (0.3%). Presumably, under the intermediate-consumption setup as studied here, we have to scale up all his estimate by a factor $\frac{1}{r}$, which is far from negligible.

[16]Their median regression reports a $PPS$ ranging from 0.7% to 1.5%, indicating the presence of outliers in their sample. Same caveat applies to Hall and Liebman (1998).

**2.3.3.3. Agency Cost.** Next, we study the impact of agency problems on the firm's value. In the first-best result, the agent exerts effort without incentives, and the risk-compensation term $\frac{1}{2}\gamma r\theta^2 a^2\sigma^2\delta$ in (2.16) is annihilated. One can show that $f^{FB}(\delta) = \frac{B^{FB}\Delta}{r} + B^{FB}\delta$, with

$$B^{FB} = \frac{2}{r + \kappa + \sqrt{(r+\kappa)^2 - \frac{2}{\theta}}} > B,$$

where $B$ is defined in (2.18) as the second-best solution. It is the costly incentive due to risk-aversion that leads to the difference between $B^{FB}$ and $B$; and when the agent becomes risk-neutral $\gamma \to 0$, or the project becomes riskless $\sigma \to 0$, the first-best factor $B^{FB}$ converges to the second-best result $B$.

Due to the affine structure, our valuation results possess an appealing feature. The percentage difference between the first-best firm value $B^{FB}\left(\frac{\Delta}{r} + \delta\right)$ and the second-best one $B\left(\frac{\Delta}{r} + \delta\right)$, is independent of the cash-flow state $\delta$. Therefore, we define the agency cost $AC$ as the value difference normalized by the first-best value, i.e.,

$$AC \equiv \frac{B^{FB} - B}{B^{FB}}.$$

**2.3.3.4. Calibration and Comparative Statics.** Now we calibrate this model to quantify the agency cost $AC$. As to date almost all applications of the sqaure-root mean-reverting process are restricted to modeling the short-term interest rate, there is little reference for our parametrization. Based on Hennessy and Whited (2005) who report a first-order autoregressive

Table 2.1. Baseline Parameters for Sqaure-Root Mean-Reverting Process

| Cash-flow | | Agency Issue | |
|---|---|---|---|
| Mean-Reversion | $\kappa = 0.55$ | Risk Aversion | $\gamma = 5$ |
| Long-run Mean Factor | $\Delta = 0.55$ | Effort Cost | $\theta = 15$ |
| Volatility | $\sigma = 3$ | | |
| Others: Interest rate $r = 5\%$. | | | |

coefficient of 0.583 for Compustat firms' operating income, we set $\kappa = 0.55$ in the baseline case,[17] anticipating a small $a^*$ in equilibrium.

Table 2.1 tabulates the baseline parameters in this model. The two key guide lines are: 1) they produce Pay-Performance Sensitivity ($PPS$) within the range of empirical estimations discussed in Section 2.3.3.2; 2) on average, they generate a *return* volatility around 25%.[18] In addition, the absolute risk-aversion $\gamma = 5$ is the median value used in Haubrich (1994) who calibrate the model in Holmstrom and Milgrom (1987). Under these parameters, $a^* = 0.32\% \ll \kappa$; we set $\Delta = \kappa$ to normalize the long-run mean of $\delta$ (i.e., $\frac{\Delta}{\kappa - a^*}$) to be around 1. We provide various comparative statics in Figure 2.1 for robustness checks.

Figure 2.1 plots the agency cost $AC$ (solid line, left scale) against relevant parameters $\gamma$, $\theta$, $\sigma$ and $\kappa$. On each panel we also show the implied $PPS$ (dashed line) on the right scale. Under the baseline parameters, the predicted $PPS$ is 2.88%, and the agency cost is 10.09%.

[17] Note that given the mean-reverting intensity $\kappa$ in the continuous-time Ornstein-Uhlenbeck process, the discrete-time sample analog will produce an AR(1) coefficient $\rho = e^{-\kappa}$. In Hennessy and Whited (2005), the authors normalize the operating income (Compustat data item 13) by the firm's *book* asset (Compustat data item 6). If retained earnings inflate the book asset, this treatment tends to over-estimate the reverting speed—but the mangnitude should be small. The more relevant concern pertains to the potential survivorship bias, which also leads to an over-estimation. Since it is out of the scope of this paper to empirically estimate the cash-flow mean-reverting intensity, we will provide a wide range of comparative statics results regarding $\kappa$.

[18] Given the long-run mean of $\delta$ as 1, the firm's return volatility is approximately $Mean\left[\frac{dB\left(\frac{\Delta}{r}+\delta\right)}{B\left(\frac{\Delta}{r}+\delta\right)}\right] = Mean\left[\frac{\sqrt{\delta}\sigma}{\frac{\Delta}{r}+\delta}\right] \simeq 25\%$. Here we ignore the non-linearity issue.

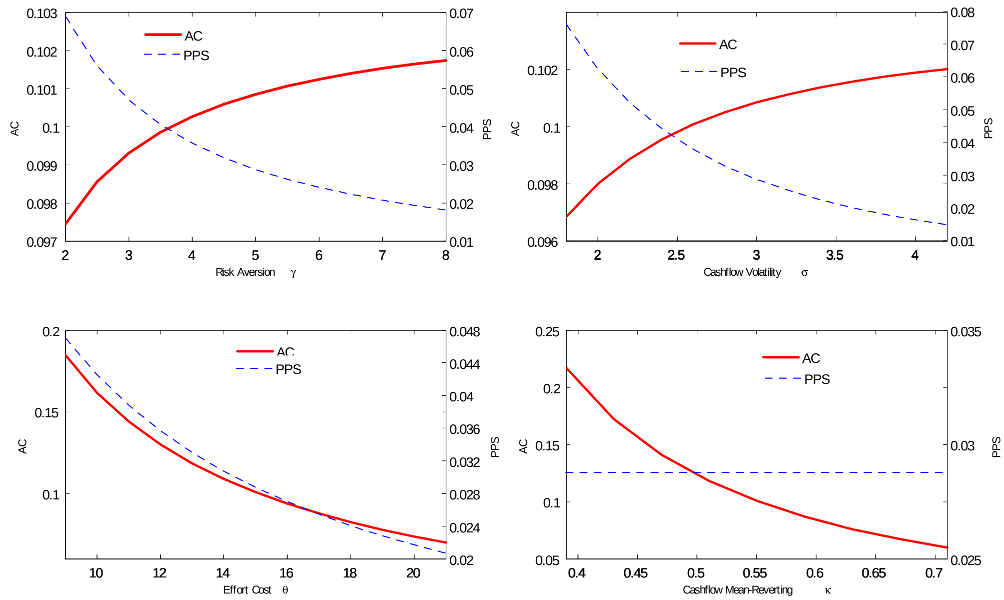


Figure 2.1. Comparative static results for the agency cost $AC$, which is measured as $\frac{B^{FB}-B}{B^{FB}}$. The baseline parameters are $r = 5\%$, $\theta = 15$, $\gamma = 5$, $\sigma = 3$, $\kappa = 0.55$, and $\Delta = 0.55$.

The top two charts show the comparative static results for variations on $\gamma$ and $\sigma$. A higher risk-aversion $\gamma$, or a higher volatility $\sigma$, leads to a more costly incentive provision. As a result, the optimal contract imposes lower incentive loadings on the agent, and the agency cost is higher. Therefore, the pay-performance sensitivity $PPS$ goes in an opposite direction with the agency cost $AC$, a prediction consistent with the direct message of Jensen and Murphy (1992): a smaller $PPS$ implies that agent's interest is less aligned, and in turn indicates a higher inefficiency. From Figure 2.1, we find that even though the change in $\gamma$ or $\sigma$ generates considerable variation in $PPS$, the agency cost measure (around $9 \sim 10\%$) remains rather stable.

The above two parameters affect the key agency-related cost, which is the risk-compensation due to costly incentives. In contrast, the case with the direct effort cost $\theta$ (the bottom-left panel) tells a different story. Because a higher $\theta$ dwarfs the incentive effectiveness, and meanwhile reduces the first-best profitability, both the pay-performance sensitivity $PPS$ and agency cost $AC$ are inversely related to the effort cost $\theta$. Intuitively, when the agent becomes less important in generating the firm's value, both the agency cost of $PPS$ should be lower. The implication is that, a lower $PPS$ measure—which means the agent's interest is less aligned with investors—does not necessarily imply that the agency cost is higher. Rather, the underlying heterogeneity matters, because the cross-sectional pattern caused by the heterogeneity in the direct effort cost $\theta$, would be different from the one due to the variations in $\gamma$ or $\sigma$, which relates to the key agency conflict. From Figure 2.1, we observe that the $AC$ measure is more sensitive to $\theta$ than in the previous two cases.

Finally, we turn to the mean-reverting intensity $\kappa$ (the bottom-right panel). First, the $PPS$ is independent of $\kappa$, as indicated by (2.20). Second, the underlying mechanism for the downward relation between $AC$ and $\kappa$, is similar to that for the effort cost parameter $\theta$. In this model, $\kappa$ measures "how temporary the cash-flow shocks are," and a higher $\kappa$ implies that the agent's effort impact is relatively "short-lived." In this sense, a larger $\kappa$, by lowering the effort benefit, is equivalent to a higher direct effort cost $\theta$. Therefore, as the agent's effort becomes less beneficial for the value improvement, a higher $\kappa$ implies a smaller agency cost. Again, the agency cost measure $AC$ is rather sensitive to $\kappa$.

Overall, the comparative static results in Figure 2.1 suggest that $8 \sim 15\%$ would be a sensible estimate for the agency cost in this model. However, since $AC$ is quite sensitive to certain parameters (e.g., $\kappa$), we await future research for better calibrations. And, a more ambitious but interesting extension would be building other aspects (e.g., investment/financing as in Hennessy and Whited (2005)) into this model.

## 2.4. Geometric Brownian Cash-Flow Setup

### 2.4.1. Model Specification

This section assumes that the firm's (after-tax) cash-flow follows a geometric Brownian motion in the first-best case. Specifically, we consider the case that $\mu(\delta, a) = (\phi + a)\delta$, and $\sigma(\delta) = \sigma\delta$, where $\phi$ and $\sigma$ are constants. Here $\phi$ is the baseline growth; by exerting effort the agent can accelerate the firm's cash-flow growth. We will interpret the firm's cash-flow level $\delta$ as the firm size. Again, here the agent's effort impact is linear in the firm size $\delta$, and as before we assume the same effort cost form $g(\delta, a) = \frac{\theta}{2}a^2\delta$.

Recall that the agent's action space is restricted to a bounded interval $[0, \overline{a}]$; the calibration in this section might call for an effort binding at $\overline{a}$ in the optimal contract. However, whether or not binding occurs, in the first-best case the model is scale-invariant to the firm size $\delta$, and the first-best effort $a^{FB}$ is always a constant. Consequently, the firm's cash-flow, as well as the firm's value, follows a geometric Brownian motion under the first-best environment.

Due to the analytical convenience of geometric Brownian setup, it has become the workhorse in the corporate security pricing literature (e.g., Goldstein, Ju, and Leland (2001), and Strebulaev (2006)).[19] Therefore, it is theoretically interesting to investigate the agency problem under this benchmark environment. Following the literature, we mainly focus on the impact of agency problem on the firm's leverage decision.

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[19]Most early work treats the value of (unlevered) firm as the state variable; for example, Kane, Marcus, and McDonald (1985) and Fischer, Heinkel, and Schwartz (1989) study the optimal leverage for exogenous bankruptcy boundary, and Leland (1994) introduces endogenous default into this literature. Recent researchers fix their attention on the geometric Brownian cash-flow setup; e.g., Goldstein, Ju, and Leland (2001), and Strebulaev (2006).
There is a natural reason to build the agency problem based upon the cash-flow setup, rather than the value setup in those early works. In a standard contracting framework, the agent controls the drift. However, it is more appropriate to let the agent to control the drift of fundamental cash flows, simply because the drift of firm's value is determined by the market, not the agent. For instance, when investors are risk-neutral as in this paper, without payout the firm's value always has a constant drift $r$ in equilibrium. This raises a critique to the framework employed by Cadenillas, Cvitanic and Zapatero (2004) who assume that the agent controls the growth of a firm's stock price.

We first characterize the optimal contract between investors and the agent, without the complication of debt investors. Then we revisit the Leland (1994) to incorporate the agency conflict into the endogenous bankruptcy model, and show that the interaction between the agency and debt-overhang problems leads smaller firms to set a lower leverage.

### 2.4.2. Optimal Contracting without Debt

Before introducing debt into this framework, we apply the results in Section 2.2 to study the optimal contract between investors and the agent. To implement effort $a_t$, we have the agent's incentive slope $\beta_t = \theta a_t$ according to (2.9). Then equation (2.13) becomes

$$rf(\delta) = \max_{a \in [0,\overline{a}]} \left\{ \delta - \frac{\theta}{2} a^2 \delta + f' \cdot (\phi + a)\,\delta + \frac{1}{2} f'' \sigma^2 \delta^2 - \frac{1}{2} \gamma r \theta^2 a^2 \sigma^2 \delta^2 \right\}. \tag{2.21}$$

Notice that, in contrast to Section 2.3, here we impose the effort constraint $a \leq \overline{a}$ explicitly; in our calibrations, the upper bound $\overline{a}$ might bind along the equilibrium path.[20] Simple calculation yields the optimal effort as[21]

$$a_t^* = \min\left( \frac{f'(\delta_t)}{\theta\left(1 + \theta\gamma r \sigma^2 \delta_t\right)}, \overline{a} \right). \tag{2.22}$$

Comparing (2.21) with (2.16), we see an important difference in the agent's risk-compensation term $\frac{1}{2}\gamma r \theta^2 a^2 \sigma^2 \delta^2$. Recall that the square-root process studied in Section 2.3 generates a risk-compensation linear in the firm size $\delta$. In contrast, in the geometric Brownian setup, because the volatility is proportional to $\delta$, the risk-compensation required by the agent becomes quadratic in $\delta$.

[20] Relative to the mean-reverting case, the geometric Brownian setup (with positive growth rate) has a much larger positve effort impact, which easily gives an exploding first-best solution under reasonable parameters.

[21] Note first that when $a$ binds at $\overline{a}$, the same incentive loading $\beta_t = \theta\overline{a}$ applies—investors can set a higher incentive loading, but it is costly to do so. Second, as intuition suggests that firm value is increasing in the cash-flow level $\delta$, one can formally show that in this model $f'$ is always positive, therefore $a^*$ never binds at zero. For a formal proof, see the Appendix.

As stated in Section 2.2.3.3, in our model the optimal contract balances incentive provision with risk-compensation. In the geometric Brownian specification, the contractual gain from incentive provision is proportional to the firm size $\delta$, while the risk-compensation cost is in the order of $\delta^2$. Therefore, when $\delta_t$ is close to zero, the risk-compensation becomes negligible, and $f(\delta)$ approaches the first-best. On the other hand, when $\delta_t$ becomes sufficiently large, the optimal contract will implement diminishing amount of effort $a \to 0$ (check (2.22)), and the firm's behavior can be described by a simple Gordon growth model with a constant growth $\phi$. Given these two boundary conditions on both ends, we can solve (2.21) numerically. For details, as well as a technical issue regarding the martingale property of $W$ in (2.10) for the geometric Brownian setup, see the Appendix.

Discussion about Firm Size and PPS. The above discussion implies that the agency issue will be heterogenous across firms with different sizes. As a result, the optimal effort policy is $\delta$-dependent, a message precisely conveyed by (2.22). Interestingly, this suggests that larger firms impose lower pay-performance sensitivities for their executives, a prediction consistent with the empirical regularity in the executive compensation literature. For instance, when the optimal effort level is interior, similar to (2.20) we have

$$PPS = \frac{dH_t}{d\left(f\left(\delta_t\right)\right)} \simeq \frac{\theta a_t \sigma \delta_t}{f' \sigma \delta_t} = \frac{\frac{f' \sigma \delta_t}{1+\theta\gamma r \sigma^2 \delta_t}}{f' \delta_t \sigma} = \frac{1}{1+\theta\gamma r \sigma^2 \delta_t}, \tag{2.23}$$

which is decreasing in $\delta$. As explained above, this result is due to the fact that as the firm grows, the risk-compensation cost is in a higher order than the incentive benefit. This exactly reflects the common wisdom that managers in larger firms have lower-powered incentive schemes due to risk considerations.[22]

[22]For instance, Murphy (1999) states that "*The inverse relation between company size and pay performance sensitivities is not surprising, since risk-averse and wealth-constrained CEOs of large firms can feasibly own only a tiny fraction of the company...the result merely underscores that increased agency problems are a cost of company size that must weighed against the benefits of expanded scale and scope.*" Notice that under an optimal contracting

One should note that the form in (2.23), and even the direction of model predictions, depend on the specific effort cost structure.[23] Because the agent's effort impact is scaled by the current firm size, it is reasonable to assume that the agent's monetary effort cost is also scaled by $\delta$. In fact, this guarantees that in the first-best scenario, the firm's cash-flow follows a geometric Brownian motion. Interestingly, one empirical paper lends certain support on the specification employed in this paper. By estimating a static agency model, Baker and Hall (2002) attempt to gauge the relation between the agent's marginal effort impact (on the firm's value), and the firm's size. They assume that the effort cost is independent of the firm size; but once accounted for this difference, their estimate suggests that the effort impact is approximately in the order of $\delta^{0.9}$, which is very close to our assumption.[24]

### 2.4.3. Optimal Capital Structure: Revisit of Leland (1994)

Now we consider the case where the equity holders of the firm, before signing the employment contract with the agent, issue a consol bond (with a constant coupon rate $C$) to take advantage of the tax shield. As in Leland (1994), equity holders can default on the debt service whenever they refuse to do so—i.e., we consider the endogenous default in evaluating the corporate debt. We simply assume that the defered-compensation fund, as a liability of the firm, is senior to the

framework, wealth-constrainedness is not an appealing justification, because empirically CEOs receive high fixed salaries in their compensation packages.

[23] To the extreme, suppose that $g(a,\delta) = \frac{\theta}{2}a^2$ which is independent of firm size, while keeping the same assumption on the effort benefit $\mu(a,\delta) = (\phi + a)\,\delta$. Then even in the second-best case, the risk-compensation in (2.14) is a constant. Now since both the direct effort cost and risk compensation are independent of $\delta$, larger firms will implement a higher effort, and in trun have a higher $PPS$. However, this is inconsistent with the data.

[24] Assume that the marginal effort impact $\eta$ is a function of firm size $\delta$; under the assumption that effort cost is independent of firm size, Baker and Hall (2002) find that the point estimate for the elasticity between marginal effort impact and the firm size is about 0.4. This result implies that $\eta \propto \delta^{0.4}$. Once we assume that the effort cost is linear in firm size, then their effort impact measure becomes $\eta/\sqrt{\delta}$. Therefore $\eta$ is approximately in the order of $\delta^{0.9}$. For details, see Baker and Hall (2002).

consol bond; therefore equity holders can fulfill the agent's continuation value at bankruptcy as a part of employment contract.[25]

Notice that relative to the standard contracting framework where the contracting relation is bilateral between investors (principal) and the agent, now we have heterogenous investors—equity holders and bond holders. To abstract from the complicated issue of partial commitments among them, essentially we employ a simple treatment as follows. First, equity holders and the agent, possessing full commitment with each other, are bonded together by an optimal *complete* contract analyzed in Section 2.2. Second, the debt contract remains *incomplete*, i.e., only static long-term debt is considered, and equity holders (and their perfectly-aligned agent when dealing with debt holders) can default when the firm profitability deteriorates.

To make the problem interesting, we also assume that the employment contract between the equity holders and the agent is an optimal response to the debt issuance. Theoretically, this assumption is consistent with the fact that, a long-term optimal contract can be implemented by a sequence short-term contracts in this framework (Fudenburg, Holmstrom and Milgrom (1990)).

These assumptions represent a minimum, but essential, departure from the Leland (1994); they allow us to derive the agency impact on the firm's capital structure in a rather stark way. In addition, they reflect the key economic rationale regarding the manager's objective in US corporations: managers are supposed to be responsible to shareholders only (Allen, Brealey, and Myers (2006)), and the relation between managers and shareholders are much closer than the one between managers (or shareholders) and bond holders. In fact, this might be the justification why most of the literature (e.g., Leland (1998) and others) treats the manager and shareholders as one single economic agent.[26]

[25] If the defered-compensation-fund balance $H_t < 0$, the agent pays the equity holder in the event of bankruptcy. Here the agent's commitment ability is important.

[26] By committing to a sepration payment of defered-compensation-fund $H$, equity holders and the agent can fully commit to each other. Future theoretical work on the "partial commitment" issue between the agent and the

**2.4.3.1. Equity's Value.** Similar to the previous analysis, we guess that the equity holders' value function is $J^E(\delta, W) = f^E(\delta) + \frac{1}{\gamma r}\ln(-\gamma r W)$, and their HJB equation becomes

$$rf^E = \max_{a\in[0,\overline{a}]}\left\{\delta - C(1-\tau) - \frac{1}{2}\theta a^2\delta + f^{E\prime}\cdot(\phi + a)\delta + \frac{1}{2}f^{E\prime\prime}\sigma^2\delta^2 - \frac{1}{2}\gamma r\theta^2 a^2\sigma^2\delta^2\right\}. \tag{2.24}$$

Comparing with (2.21) without debt, there is an additional term for the after-tax coupon payment $C(1-\tau)$ in (2.24). Similar to (2.22), the optimal effort is

$$a^* = \min\left(\frac{f^{E\prime}}{\theta(1+\theta\gamma r\sigma^2\delta)}, \overline{a}\right), \tag{2.25}$$

Plugging it into (2.24), we have an ODE to characterize the optimal contracting.

Now we specify boundary conditions for (2.24). Because equity holders can default, when $\delta$ falls to a certain level, say $\delta_B$, they refuse to continue the coupon service and declare bankruptcy. This is captured by the value-matching boundary condition

$$f^E(\delta_B) = 0, \tag{2.26}$$

and the smooth-pasting condition

$$f^{E\prime}(\delta_B) = 0. \tag{2.27}$$

Both conditions are standard in this literature (e.g., Leland (1994)).

One point is note-worthy. As discussed earlier, here we assume that the bankruptcy event of default on debt will not undermine the ability of equity holders or the agent to fulfill the agent's continuation value. Therefore, by bonding the equity holders and the agent together by an optimal contract, the above default policy maximizes $f^E(\delta)$, which is the joint (ex-post) surplus enjoyed by both the equity holders and the agent. In other words, even though there

equity holders, especially coupled with the issue of endogenous debt service, is highly appreciated. Our results here, at least, are an first-order approximation of agency impact on the debt's service and firm's leverage decision.

exist agency conflicts between the agent and equity holders, under the optimal contract they have perfectly aligned interest regarding the policy toward debt holders. As a result, the default policy will be independent of whether equity holders or the agent is in charge of the bankruptcy decision. This differs from Morellec (2004) where the agent tends to keep the firm alive longer for more private benefit.

We then turn to the boundary condition on the other end. When $\delta$ takes a sufficient large value $\overline{\delta} \to \infty$, the effort is diminishing $a \to 0$, and the bankruptcy event is negligible. Therefore,

$$f^E\left(\overline{\delta}\right) \simeq \overline{f}\left(\overline{\delta}\right) - \frac{C\left(1-\tau\right)}{r}, \tag{2.28}$$

where $\overline{f}\left(\cdot\right)$ captures the firm value under a Gordon growth model with a growth rate $\phi$ (see equation (A.5) in the Appendix). Then we can numerically solve for $f^E$, based on (2.24), (2.26), (2.27), and (2.28).

**2.4.3.2. Debt Value and Capital Structure.** Given the equity holders' value function $f^E\left(\delta\right)$ and the associated optimal effort policy $a\left(\delta\right)$ in (2.25), we can evaluate the consol bond with a promised coupon rate $C$. Since bond holders foresee the optimal contracting between equity holders and the agent, the value of the corporate debt, $D\left(\delta\right)$, must satisfy

$$rD = C + D_\delta \cdot \left(\phi + a\left(\delta\right)\right)\delta + \frac{1}{2}D_{\delta\delta}\sigma^2\delta^2,$$

with $D\left(\delta_B\right) = \left(1-\alpha\right) f\left(\delta_B\right)$ where $\alpha < 1$ is the percentage bankruptcy cost, and $D\left(\overline{\delta}\right) \to \frac{C}{r}$ when $\overline{\delta} \to \infty$. Here we simply assume that, once bankruptcy occurs, the debt holders pay the bankruptcy cost $\alpha f\left(\delta_B\right)$, and then keep running the project as an unlevered firm.[27]

Given the time-0 cash-flow $\delta_0$, the equity holders choose the coupon $C$ to maximize the total levered firm value $f^E\left(\delta_0; C\right) + D\left(\delta_0; C\right)$; they then design the optimal contract with an agent

[27] Also, the new agent's outside option is $W_0 = \frac{-1}{\gamma r}$, so $H_0 = 0$. Our result is insensitive to the treatment of unlevered firm after the bankruptcy.

who has an outside option $W_0$. In this model, because the firm's value is independent of the agent's stake, the optimal coupon $C^*$ is independent of the agent's reservation value $W_0$. We then define the firm's optimal leverage ratio as

$$LR \equiv \frac{D(\delta_0; C^*)}{f^E(\delta_0; C^*) + D(\delta_0; C^*)}.$$

In the above analysis, we implicitly assume that the debt issuance is before the optimal employment contract. As explained above, the timing becomes immaterial if equity holders and the agent can re-design the contract once the debt is issued, given that the consol bond has no covenant regarding revising the employment contract ex-post. This condition raises the interesting question about debt covenants, and in Section 2.4.3.5 we study a simple modification of our model to explore this possibility.

We parameterize the firm size as its time-0 cash-flow level $\delta_0$. In Leland (1994), the scale invariance of geometric Brownian setting implies that the leverage ratio $LR$ is independent of firm size. However, in Section 2.4.2 we have seen that the quadratic risk-compensation for a CARA agent eliminates the scale invariance in our model. In fact, in the following calibration exercises, we will mainly investigate the divergent leverage decisions for different sized firms.

**2.4.3.3. Parameterization.** Table 2.2 tabulates our baseline parametrization. Because our model features a stochastic drift process $\{\mu\}$, for better comparison, we set the volatility $\sigma =$ 24.70% which is slightly below the conventional level 25%. By simulating the model, in Table 2.3 we report the average volatility calculated from the sample path of $\{\delta\}$. The resulting average volatilities are close to the target level 25%.

We also record the average growth rate in simulation, and this measure helps us to pin down $\phi$ and $\overline{a}$. In the literature with constant coefficients, Goldstein, Ju and Leland (2001) calibrate a slightly negative $\mu$, and Leland (1998) chooses $\mu = 1\%$. From an aggregate point of view, the average $\mu$ should approximately match the long-run consumption growth 1.84%. Under the

Table 2.2. Baseline Parameters for geometric Brownian Cash-flow Setting

| Cash-flow | | Agency Issue | |
|---|---|---|---|
| Lower Bound Growth $\phi$ | $-0.5\%$ | Risk Aversion | $\gamma = 5$ |
| Effort Upper Bound $\overline{a}$ | 5% | Effort Cost | $\theta = 30$ |
| Volatility $\sigma$ | 24.7% | | |

Others: $r = 5\%$, $\alpha = 50\%$, and $\tau = 35\%$.

choice of $\phi = -0.5\%$ and $\overline{a} = 5\%$, the simulated average growth rates fit these numbers squarely across various starting firm sizes ($\delta_0$'s; see Table 2.3).

In the spirit of (2.23), the parameters for the agency problem (risk aversion $\gamma$ and effort cost $\theta$), and the starting firm size $\delta_0$, are chosen to match the pay-performance sensitivity documented in the empirical literature. Murphy (1999) finds that for large S&P500 firms, the pay-performance sensitivity is approximately 1%; for Midcap firms, it is 1.5%; and for small firms, it is 3%. Once controlling for the firm's risk, Aggarwal and Samwick (1998) report a mean $PPS$ of 6.94% from the OLS regression.[28] And, Hall and Liebman (1998) note that in their sample, "*the largest firms (with market value over $10 billion have a median PPS that is more than an order of magnitude smaller than the smallest firms (market value less than $500 million).*"

We set $\gamma = 5$ and $\theta = 30$; in our simulations, the $PPS$ measures fall within these ranges. The choices are also made in the consideration of reasonable average cash-flow growth in our simulations. The other parameters, i.e., interest rate $r = 5\%$, bankruptcy cost $\alpha = 50\%$, and tax rate $\tau = 35\%$, are typical in the literature.

**2.4.3.4. Debt-Overhang and Size-Heterogeneity.** Overall, in the presence of agency conflicts, firms will take less leverage for their optimal capital structure. The mechanism is the debt-overhang problem, where we interpret the agent's effort as a form of "investment." To see

[28]Their median regression reports PPS ranging from 0.7% to 1.5%, indicating the presence of outliers in their sample.

this, the equity holders design an employment contract to maximize the ex-post equity value, and the smooth-pasting condition implies that $f^{E\prime}(\delta)$ goes to zero as $\delta$ approaches the default boundary $\delta_B$. It implies that, once the firm is close to bankruptcy, the equity holders gain almost nothing by improving the firm's performance, and they in turn implement diminishing effort in these scenarios. As a result, in addition to the traditional bankruptcy cost, in our model the debt bears another form of cost due to debt-overhang.

We have seen that our model loses the theoretically appealing scale-invariance property. However, once we break down the scale-invariance, our model offers another explanation why small firms take less leverage relative to their large peers, a stylized fact documented in the literature (e.g., Frank and Goyal, 2005). The mechanism here is rooted in various severities of the aforementioned debt-overhang problem for different sized firms. In this model, the quadratically increasing risk-compensation (see discussion in Section 2.4.2) suggests that the agency issue is more severe in larger firms. Interestingly, this implies that, for larger firms, the potential debt-overhang problem is less of concern, because it is costly to motivate the agent even from the firm's (including debt holders) point of view. As a result, smaller firms bear more debt-overhang cost for their debt, and in response they will issue less debt to maximize the ex-ante firm value. Though each ingredient in this mechanism has been explored in the literature, to our knowledge, this is the first paper to study the interaction among agency conflicts, size-heterogeneity, and debt-overhang problems.

Table 2.3 gives the key results of our model. For each initial cash-flow level $\delta_0$, we simulate our model for 50 years, calculate the mean growth rate and the volatility of $d\delta/\delta$ along the sample path, and average both moments across 500 simulations. These two simulated moments are used to compute the Leland (1994) predictions in Figure 2.2 and Figure 2.4. We also report the sample average of pay-performance sensitivity in our simulation in Table 2.3; these numbers fit the empirical estimates discussed in Section 2.4.3.3 squarely.

Table 2.3. Optimal Capital Structure for Firms with Different Sizes

| | Initial Cashflow Level (Firm Size) $\delta_0$ | | | | |
|---|---|---|---|---|---|
| | 50 | 75 | 125 | 175 | 225 |
| Optimal Coupon $C^*$ | 40.68 | 87.42 | 111.12 | 162.44 | 220.80 |
| Default Boundary $\delta_B$ | 10.72 | 26.26 | 34.03 | 50.91 | 70.32 |
| Debt Value $D(\delta_0)$ | 712.85 | 1346.43 | 1673.05 | 2015.15 | 3202.85 |
| Firm Value $f^E(\delta_0) + D(\delta_0)$ | 1331.06 | 2358.65 | 2883.06 | 3412.09 | 5039.49 |
| Leverage Ratio % $\frac{D(\delta_0)}{D(\delta_0)+f^E(\delta_0)}$ | 53.56 | 57.08 | 58.03 | 60.37 | 63.56 |
| Average Growth % | 2.26 | 0.77 | 0.48 | 0.17 | 0.005 |
| Average Volatility % | 24.77 | 24.85 | 24.86 | 24.58 | 24.60 |
| Pay-Performance Sensitivity % | 5.13 | 2.69 | 2.18 | 1.56 | 1.22 |

The parameters are $r = 5\%$, $\sigma^2 = 6.1\%$, $\theta = 30$, $\gamma = 5$, $\phi = -0.5\%$, $\alpha = 50\%$, $\tau = 35\%$. We simulate our model for 50 years to obtain the average growth rate and volatility for $d\delta/\delta$, given the initial $\delta_0$. Pay-performance sensitivity is calculated via simulation, based on (2.23)

Figure 2.2 shows the optimal leverage ratio in our model for firms with different sizes. For better comparison, based on the sample moments from simulations (see Table 2.2), we also graph the corresponding optimal leverage ratios predicted by Leland (1994) model with constant coefficients. For small firms ($\delta_0 = 50$), the optimal leverage ratio is down from 64.8% to 53.6%. It is a considerable reduction compared to other modifications of the Leland (1994) model; for instance, by combining both the "callable" feature of the debt and upward capital restructuring together, Goldstein, Ju, and Leland (2001) only push the optimal leverage down from 49.8% to 37.14% in their baseline case. For large firms the optimal leverage ratio is close to the result under Leland (1994), because the debt-overhang problem is negligible.[29]

The above mechanism can be illustrated by the optimal effort policy implemented in a levered firm. In Figure 2.3, we plot the agent's effort $a^*$ as a function of firm's financial status $\delta$, for both the large ($\delta_0 = 225$) and small ($\delta_0 = 50$) firms. For better comparison, we adopt the same scale for both firms. For the small firm ($\delta_0 = 50$), we observe an abrupt drop of

[29]For $\delta_0 = 200$ and 225, the optimal leverage ratio is above the ratio in Leland (1994). Besides the non-linearity reasons, another explanation is that our model generates a higher unlevered firm value in bankruptcy (firm's growth is negatively related to $\delta$; see Section 2.4.2). This effectively reduces the bankruptcy cost $\alpha$, therefore leads to a more aggressive debt policy.

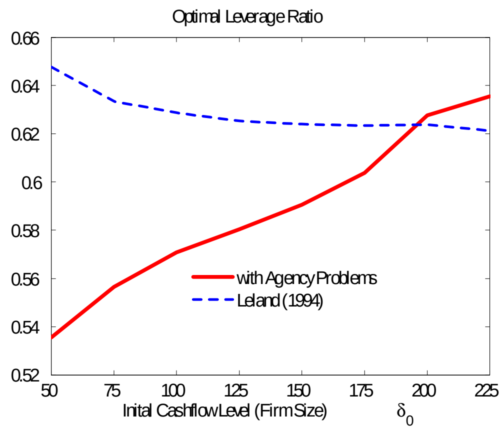


Figure 2.2. Optimal leverage ratio as a function of initial cashflow level (firm size). The parameters are $r = 5\%$, $\sigma^2 = 6.1\%$, $\theta = 30$, $\gamma = 5$, $\phi = -0.5\%$, $\alpha = 50\%$, $\tau = 35\%$. We also plot the leverage ratio according to Leland (1994), where the constant coefficients ($\mu$ and $\sigma^2$) are averages from the simulations of our model.

implemented effort when the firm is in the verge of bankruptcy. From the view of social welfare, in this situation a higher effort (or an binding effort at $\overline{a}$) is more desirable, not only because of a lower risk-compensation, but also to avoid the costly bankruptcy once $\delta$ hits $\delta_B$. However, it is not in the equity holders' interest to ask the agent to work hard. As discussed before, by the nature of smooth-pasting condition $f^{E\prime}(\delta_B) = 0$, the equity holders obtain zero marginal value from improving $\delta$ when the firm approaches the default boundary. This implies that the equity holders will implement diminishing effort when $\delta \to \delta_B$, a typical symptom of a debt-overhang problem.

In contrast, the debt-overhang problem becomes moderate for large firms. In this case, the default boundary $\delta_B = 70.32$ is quite high, and at this point the optimal effort without the complication of debt—so free of debt-overhang problem—is relatively low.[30] Therefore, the drop of $a_t^*$ when the firm approaches bankruptcy—the exact force of debt-overhang—becomes

[30] In the left panel with small firms ($\delta_0 = 50$), the implemented effort at $\delta = 70$, i.e., $a^*(70)$, is only approximately 2%. Because when small firms we have $\delta_B = 10.72 \ll 70$, the debt-overhang impact should be small for the optimal effort level $a^*(70)$ at $\delta = 70$.

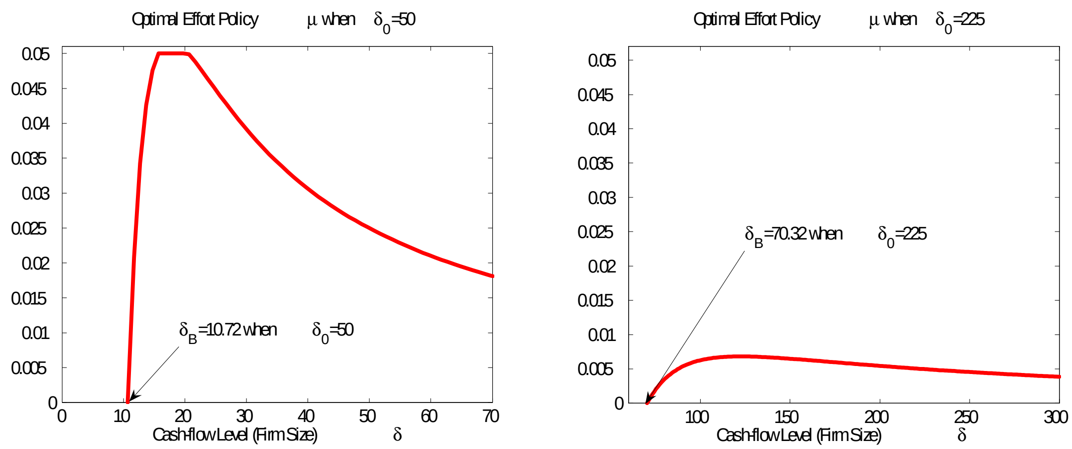


Figure 2.3. Optimal effort policy for small and large firms with optimal leverage. The parameters are $r = 5\%$, $\sigma^2 = 6.1\%$, $\theta = 30$, $\gamma = 5$, $\phi = -0.5\%$, $\alpha = 50\%$, $\tau = 35\%$. In the figure we also mark the optimal endogenous default boundary, where the equity holders optimally (ex-post) to implement the zero effort.

minuscule compared to small firms (the left panel). In other words, the relatively severe agency issue dwarfs the debt-overhang problem.

Figure 2.4 further investigates the equity holders' endogenous default policy. In the left panel, we graph the "scaled" version of default boundary $\delta_B/\delta_0$. Relative to the Leland (1994) benchmark, equity holders in our model postpone the bankruptcy. The intuition is as follows. In our model, a firm with recent unsatisfactory performances should have a lower cash-flow level, or a smaller size. But given a smaller risk-compensation, the equity holders find it cheaper to motivate the agent. By boosting the firm's current growth rate, this gives the equity holders more value to wait for future potential improvement.

Interestingly, without agency problem, the exogenous specification that "smaller firms have a higher growth rate" should predict a *higher* leverage ratio for smaller firms, because a higher growth, by reducing the likelihood of default, leads to a more aggressive debt policy. Therefore, the result here is an outcome of the interaction between agency issues and debt-overhang.

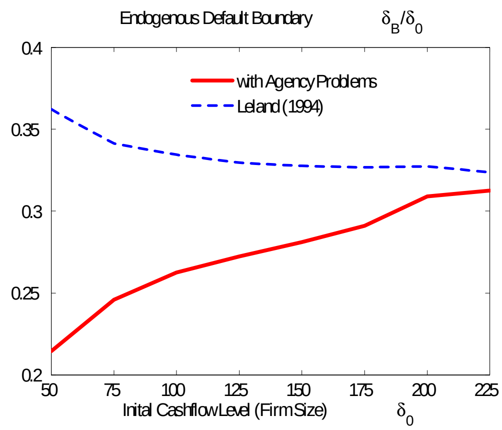


Figure 2.4. Endogenous default boundaries $\delta_B/\delta_0$ as a function of initial firm size $\delta_0$. We also plot the the optimal default ratio $\delta_B/\delta_0$ according to Leland (1994) constant coefficient model, where the coefficients are taken from the sample average in simulation. The parameters are $r = 5\%$, $\sigma^2 = 6.1\%$, $\theta = 30$, $\gamma = 5$, $\phi = -0.5\%$, $\alpha = 50\%$, $\tau = 35\%$.

**2.4.3.5. Debt Covenant.** As we emphasized before, our analysis is built upon the assumption that, the equity holders design the optimal contract to motivate the agent, and both parties can revise the contract as an optimal response to the debt position. In principal, to rectify the debt-overhang problem, one can always ask the agent to be responsible for the firm value. In reality, the widely-used debt covenants are designed to serve this role; and the theoretical ground is that, by restricting the equity holders' ex-post strategy set, they could enhance the ex-ante debt value, as well as the firm value, upon issuance. Interestingly, the following analyses suggest that it might not be always the case.

We consider the following modification in our model as a simple form of debt covenant. Suppose that the consol bond is attached with the following covenant: when the firm's financial status gets worse in the future, i.e., $\delta < \delta_0$, the implemented effort $a^*$ cannot go below certain pre-specified level $\pi > 0$. One could interpret this covenant as preventing the agent from taking certain actions, or allowing the debt holders to sit on the board once the firm's financial situation deteriorates. What is the impact of this debt covenant on the debt value and optimal capital structure?

We solve (2.24) again by imposing the condition that $a \in [\pi, \overline{a}]$ when $\delta < \delta_0$; results are reported in Table 2.4. For better comparison, we also report the corresponding valuation results under the original coupon level $C^*$ ($\pi = 0$) without debt covenants (see Table 2.3).

Compared to the results in Table 2.3, we first observe that equity holders declare bankruptcy earlier as a response to the debt covenant. When fixing the coupon level $C^*$ ($\pi = 0$), the default boundary $\delta_B$ becomes 10.74 for $\pi = 1.5\%$, and 10.75 for $\pi = 3\%$; both are above $\delta_B = 10.72$ in Table 2.2. It is not surprising, because given the hard restriction imposed by the covenant, the equity holders will simply "walk away" and stop their loss earlier.

The more surprising result is that this endogenous default policy might overturn the direct effect of a good-intentioned covenant. Under the coupon level $C^*$ ($\pi = 0$), for $\pi = 1.5\%$, both the debt and the firm experience value drops compared to the case without covenant (see Table 2.3). Note that the case of a fixed coupon corresponds to the scenario where the firm issues debt to meet certain financing needs, or the firm has reached its (exogenous) debt capacity unmodeled here. Our analysis suggests that, in these cases, given the endogenous bankruptcy policy, the good-intentioned debt covenant in fact could be value-destroying.

Once the firm adjusts the coupon upward as an optimal response to the debt covenant, there is a positive value gain from the debt covenant. Not surprisingly, the optimal leverage ratio increases, because the debt-overhang problem is less severe. The improvement on the firm value is quite small under our parameterization, and the optimal leverage ratio only increases by 1% even when $\pi = 3\%$.

Table 2.4. The Impact of Debt Covenants for Small Firms

| | Debt Covenant $\pi$ | | | |
|---|---|---|---|---|
| | 1.5% | | 3% | |
| | $C^*\,(\pi=0)$ | $C^*\,(\pi=1.5\%)$ | $C^*\,(\pi=0)$ | $C^*\,(\pi=3\%)$ |
| Coupon Level $C$ | 40.68 | 41.38 | 40.68 | 41.94 |
| Default Boundary $\delta_B$ | 10.74 | 10.97 | 10.75 | 11.18 |
| Debt Value $D\,(\delta_0)$ | 712.84 | 720.66 | 713.59 | 727.66 |
| Firm Value $f^E\,(\delta_0)$ | 1331.04 | 1331.08 | 1331.40 | 1331.51 |
| Leverage Ratio $\frac{D(\delta_0)}{D(\delta_0)+f^E(\delta_0)}\%$ | 53.56 | 54.14 | 53.60 | 54.65 |

The parameters are $r=5\%$, $\sigma^2=6.1\%$, $\theta=30$, $\gamma=5$, $\phi=-0.5\%$, $\alpha=50\%$, $\tau=35\%$, with initial cashflow level $\delta_0=50$. The debt covenant with $\pi$ requires that when $\delta<\delta_0$, the implemented $\mu^*$ cannot be below $\pi$. For better comparison, we first report the valuation results and default policy when the coupon is not updated ($C=C^*\,(\pi=0)$); then we report the results for the reoptimized coupon $C=C^*\,(\pi)$.

## 2.5. Conclusion

By generalizing the optimal contracting result to widely-used cash-flow setups in finance, this paper offers a more tractable framework to investigate the impact of agency problems in various economic contexts. The absence of wealth effect of CARA preference simplifies the optimal contracting greatly, and we characterize the optimal contract by an ODE. The solution is obtained in a closed form for the square-root cash-flow process, where we find an agency cost around $8\sim15\%$ in calibration. When we apply our analysis to the geometric Brownian cash-flow setup to revisit the Leland (1994), the interesting interaction between the agency problem and debt-overhang problem leads smaller firms to take less debt in their leverage decisions.

The relatively simple structure in this paper allows for several directions for future research. For instance, incorporating investment decisions into this framework would be desirable, as one can explore the investment distortion and its interaction with financing decisions under agency problem. Also, it is interesting to pursue the line of Ou-Yang (2005) who studies compensation under general equilibrium.

CHAPTER 3

# The Sale of Multiple Assets with Private Information

## 3.1. Introduction

Financial intermediaries manage and trade large portfolios of assets. For instance, Fannie Mae, a leading firm in the Mortgage Backed Securities (MBS) industry, issued 32 *Fannie Mae MBS* pools on November 1, 2004.[1] Meanwhile, active risk management is becoming increasingly important for financial intermediaries,[2] possibly due to the crisis in the fall of 1998.

Motivated by this fact, this paper generalizes the Leland and Pyle (1977, hereafter LP) model to study the multi-asset trading behavior of financial intermediaries, which includes bank loan sales and private equity funds. According to information theory, intermediaries suffer from a lemon's problem, and have to convey the qualities of their assets through credible signals, e.g., the retention amount. However, multiple assets lead to a scenario where the intermediary seeks to minimize overall risk by selling positively correlated assets or holding negatively correlated ones. Such risk management behavior influences the signaling incentive for each asset, rendering the equilibrium pricing rules for all assets intrinsically *interconnected*.

Based on this "interconnectedness," I derive a 2-dimensional nonlinear equilibrium pricing system in closed form. In this *cross-signaling* equilibrium, the informed agent sends a 2-dimensional vector signal—the selling fraction of each asset—to financial markets, and investors correctly price each asset by fully utilizing the 2-dimensional signal. The notion of cross-signaling hinges on the interdependence of the agent's selling incentives for different assets. Under the

[1]Data source: Fannie Mae website.
[2]See, for instance, Allen and Santomero (1997), and others.

LP framework, the agent signals an asset's quality by keeping a fraction of this asset; the larger the variance, the more credible the retention signal. Now, consider the case where the agent has two positively correlated assets. Holding more of *asset 1* is not only a credible signal of a higher quality for *asset 1*, but also a higher quality for *asset 2*. The reason is that the extra retention of asset 1 increases the agent's risk exposure to asset 2, and thus boosts her selling incentive for asset 2. Consequently, if the agent maintains the same fraction of asset 2 in equilibrium, it must be the case that her marginal benefit of holding asset 2, i.e., its quality, is higher. The larger the correlation between assets, the higher the interdependence of selling incentives, and the greater the interconnectedness of equilibrium pricing rules. Similar logic holds for the negative correlation case; in fact, due to the explicit inside-hedging motive, this case clearly illustrates the interdependence between the endogenous hedging and signaling behaviors in a multi-asset framework.

The above intuition suggests that the intermediary's equilibrium hedging activity (holding of asset 2) plays a vital role when the intermediary is signaling her asset 1's quality through retention, and this model generates several novel predictions. For instance, all else equal, when assets are positively (negatively) correlated, holding more of asset 2 leads to a higher (lower) equilibrium price for asset 1. And, the less correlated the assets are, the smaller the assets' own-price impacts—that is, the negative price response to the asset's fraction sold will be lower. This result implies that an intermediary with a more diversified underlying portfolio faces a more liquid financial market (a smaller price impact) during the asset sale. (Finally, a smaller correlation leads to a higher equilibrium payoff for the agent.)

In Section 3.5, I discuss the model's application to bank loan markets. In the context of loan syndications, a recent empirical paper by Ivashina (2007) examines the impact of information asymmetry on the equilibrium loan pricing. Ivashina proposes a portfolio-based risk measure for individual loans in transaction. In view of my theoretical model, her portfolio-based measure is much more appealing than the asset's individual risk, because as discussed above, the key determining factor for the asset's equilibrium price should be the asset's risk contribution relative to the lead bank's existing portfolio. However, the theoretical results in this paper call her identification strategy into question, and suggest that her estimate for the asymmetric-information cost is downward biased (therefore, a conservative estimate).

Directing attention to simultaneous sales in the secondary loan market, this paper generates several interesting empirical predictions. For instance, all else equal, banks with less geographically diversified lending bases will receive more favorable prices for their loan sales. Also, regarding the relation between the loan price impact and the covariance structure of concurrent loans sold by the same bank, my model suggests that the loan market will become more liquid (smaller price impacts) when loans on sale are mutually hedging (negatively correlated) assets.

The model studied in this paper can also be applied to other financial intermediaries who engage in asset sales—for instance, private equity funds who sell their multiple ventures to financial markets within a short time window. In addition, Section 3.6.1 compares different selling mechanisms available to intermediaries, where pooled sales and sequential sales are discussed. The result of "information destruction" of the pooled sale, a term coined in DeMarzo (2003), is strengthened. Also, early simultaneous sale dominates sequential sales, and the intermediary tends to accelerate the selling pace given the additional concern of "cross-signaling." This provides another possible explanation for premature IPOs in the VC industry (e.g., Barry et al. (1990)).

The rest of this paper is organized as follows. Section 2 provides a literature review. The model is presented in Section 3, and Section 4 derives separating equilibria for various cases. In Section 5, I discuss the model's application to bank loan sales. Section 6 considers extensions, and Section 7 concludes. All proofs are provided in the appendix.

### 3.2. Related Literature

This paper is based on LP (1977). In a simplified version of their model, there is a risk-averse agent with CARA utility $-e^{-r\widetilde{w}}$, where $\widetilde{w}$ is the agent's terminal wealth, and $r$ is her risk-aversion coefficient. The agent sells a fraction $\alpha \in [0,1]$ of her asset with payoff $\widetilde{x} = \mu + \varepsilon$ to risk-neutral investors, where $\varepsilon \sim \mathcal{N}\left(0, \sigma^2\right)$ ($\mathcal{N}$ indicates the normal distribution) is the payoff innovation, and $\mu \in \left[\underline{\mu}, \infty\right)$ is the asset's quality. The quality $\mu$ is the agent's only private information, and in equilibrium investors correctly price the asset based on the agent's selling fraction.

Although the first-best outcome has the agent transfer her entire asset to investors at a fair price $\mu$, in a separating equilibrium, information asymmetry leads the market to form a downward-sloping pricing function $p(\alpha)$. The Pareto-efficient equilibrium signaling schedule has the lowest type agent sell her entire asset, i.e. $p(1) = \underline{\mu}$. Given this, LP show that the equilibrium pricing function is

$$p(\alpha) = \underline{\mu} + r\sigma^2 (\alpha - \ln \alpha - 1), \text{ for } \alpha \in (0,1]. \tag{3.1}$$

Note that $p'(\alpha)$ is negative, suggesting that financial markets become illiquid (a negative price impact) due to information asymmetry, and prices are lower for assets with larger selling fractions.

Relaxing the common knowledge assumption about the asset variance $\sigma^2$, Hughes (1986) and Grinblatt and Hwang (1989) explore the 2-dimensional private information issue. In comparison, by considering a multi-asset version of LP, this paper focuses on the 2-dimensional private information pertaining to the assets' qualities, and keeps the assets' covariance matrix as common knowledge. More importantly, my model demarcates itself from Hughes (1986) and Grinblatt and Hwang (1989) in another key respect. In their models, the equilibrium variance schedule is the market *perceived* asset's variance. However, the perceived variance does not enter the agent's mean-variance objective directly; instead, the agent's risk exposure is determined by the asset's *true* variance. Therefore, in Hughes (1986) and Grinblatt and Hwang (1989), the agent has no incentives to signal her variance type to the market, and the cross-signaling incentives are absent. In contrast, in this paper, both assets' pricing functions (signaling schedules) enter the agent's payoff, and the agent will cross-signal both assets' qualities through her retention fractions of each asset.

A class of multi-asset equilibrium pricing rules has been explored in the literature (e.g., Caballe and Krishnan (1994), and Bhattacharya, Reny and Spiegel (1995)). These papers obtain a linear partially revealing equilibrium pricing system, thanks to the well-known properties of *CARA-Normality-Noise* structure.[3] Bhattacharya, Reny and Spiegel (1995) focus on the destructive inference among securities markets in an imperfect competitive setting, and show that adding new assets may eliminate trading of existing assets. In their model, the economy comprises a risk-averse informed agent, and a continuum of risk-averse, uninformed, but rational individuals. The informed agent also receives random endowments that are unknown to the market; this "noisy endowment" serves a camouflage role similar to noise traders. In their model, the agent can take any *unbounded* position in each asset, as opposed to the selling fraction between 0 and 100% assumed in this paper.[4] By an elegant convex analysis argument where *D1* refinement and unbounded action space are utilized, the authors show that, when the size of the endowment noise is smaller relative to the extent of quality uncertainty, no equilibrium pricing system—linear or nonlinear—can exist with trading. In contrast, in the *CARA-Normality*

[3]The theoretical advantage of partially revealing models, over the fully revealing one presented here, is their ability to obtain the equilibrium without certain pre-specified boundary conditions. However, by excluding "noise", I am able to analyze the equilibrium pricing system and the price impacts based only on the assets' payoff structure, without knowing the elusive characteristics of noise.

[4]The assumption of a restricted action space of selling fractions is more realistic for certain intermediaries, e.g., bank loan sales and private equity funds.

environment where their equilibrium pricing system with trading fails to exist, this paper derives a fully revealing equilibrium pricing system with trading, but without the aid of "noise."[5]

Besides the aforementioned asset-selling models based on the trade-off of risk-sharing, there exists another asset-selling literature with a risk-neutral agent that delivers similar results. DeMarzo (2005) assumes a higher discount rate for the agent, because financial intermediaries are able to use available cash proceeds, but not existing assets, to engage in profitable transactions. From a more broader view, this paper also relates to the theory of multidimensional signaling, e.g., Engers (1987), and that of multidimensional screening, e.g., Chone and Rochet (1998); the latter authors focus on maximizing total profit rather than full separation.

## 3.3. Model

### 3.3.1. Setup

Consider a risk-averse agent with CARA utility $-e^{-r\widetilde{w}}$, where $\widetilde{w}$ is the agent's terminal wealth, and $r$ is her risk-aversion coefficient. In contrast to LP, suppose that the agent has 2 assets to sell, and at $t = 0$ she simultaneously offers to sell fractions $\boldsymbol{\alpha} \equiv (\alpha_1, \alpha_2)$ of each asset to risk-neutral investors. For example, the agent may be a bank engaging in loan sales, or a private equity fund selling shares of its ventures. Each selling fraction $\alpha_i$ belongs to the interval $[0, 1]$, which precludes the agent from "purchasing" or "short selling." Both restrictions naturally fit the practice of bank loan sales or private equity funds.

At $t = 1$, the asset payoffs $\widetilde{\mathbf{x}} = \boldsymbol{\mu} + \boldsymbol{\varepsilon}$ are realized, where $\boldsymbol{\mu} = (\mu_1, \mu_2)$ is the quality vector, and the innovation $\boldsymbol{\varepsilon} = (\varepsilon_1, \varepsilon_2)$ is distributed as $\mathcal{N}(0, \Sigma)$, with the covariance matrix $\Sigma = [\sigma_{ij}]$. I denote $\sigma_i$ as the standard deviation of $\varepsilon_i$, and $\rho$ as their correlation. As in LP, $\mu_i \in \left[\underline{\mu_i}, \infty\right)$ for $i = 1, 2$ is the agent's only private information. I refer to the agent type as the asset quality vector $\boldsymbol{\mu}$.

[5]The trivial "no trade equilibrium" always exists in both models. I thank the editor Matthew Spiegel for pointing this out.

As in LP, this paper derives a separating equilibrium. Faced by a two-dimensional pricing system $\mathbf{p}(\boldsymbol{\alpha}) = \left(p^{(1)}(\boldsymbol{\alpha}), p^{(2)}(\boldsymbol{\alpha})\right)$, the agent $\boldsymbol{\mu}$ optimally chooses $\boldsymbol{\alpha}^*(\boldsymbol{\mu}) \in \mathcal{A} \equiv [0,1]^2$ to maximize her mean-variance objective (recall the CARA utility and normal distribution):

$$V(\boldsymbol{\mu}, \boldsymbol{\alpha}, \mathbf{p}(\boldsymbol{\alpha})) = (\mathbf{1} - \boldsymbol{\alpha})' \boldsymbol{\mu} + \boldsymbol{\alpha}' \mathbf{p}(\boldsymbol{\alpha}) - \frac{r}{2}(\mathbf{1} - \boldsymbol{\alpha})' \Sigma (\mathbf{1} - \boldsymbol{\alpha}). \tag{3.2}$$

In a separating equilibrium, since the market is risk-neutral and competitive, "market consistency" implies that these valuations are correct, i.e.,

$$\mathbf{p}(\boldsymbol{\alpha}^*(\boldsymbol{\mu})) = \boldsymbol{\mu}. \tag{3.3}$$

The agent sends her signals—selling fractions $\boldsymbol{\alpha}$—to investors, and the market fully utilizes these signals to correctly price both assets.

Denote the set of equilibrium strategies as $\mathcal{E} \subset \mathcal{A}$. The equilibrium strategy set $\mathcal{E}$ could be a strict subset of the action space $\mathcal{A}$, as certain selling strategies might be off-equilibrium. For simplicity, I search for equilibrium pricing rules $\mathbf{p}(\cdot)$ that are smooth (continuously differentiable) on $\mathcal{E}$. Section 3.6.3 shows that the smoothness assumption is not key to the equilibrium properties derived in this paper.

### 3.3.2. First Order Conditions (FOC) and Transport Equation

Fix any agent type $\boldsymbol{\mu}$ and her optimal selling strategies $\boldsymbol{\alpha}^*$. Maximizing (3.2) yields the *First Order Conditions (FOC)* as

$$\alpha_1^* p_i^{(1)} + \alpha_2^* p_i^{(2)} + r\left[(1 - \alpha_1^*)\sigma_{i1} + (1 - \alpha_2^*)\sigma_{i2}\right] = 0, \text{ for } i = 1, 2, \tag{FOC}$$

where I cancel $p^{(i)}(\boldsymbol{\alpha}^*)$ with $\mu_i$ by market consistency (3.3), and denote the cross partial as $p_i^{(j)} \equiv \frac{\partial p^{(j)}}{\partial \alpha_i}$. Later I show that the value function $V(\boldsymbol{\mu}, \boldsymbol{\alpha}, \mathbf{p}(\boldsymbol{\alpha}))$ is strictly concave in $\boldsymbol{\alpha}$, and therefore (FOC) is sufficient for the optimality.

In (FOC), the pricing-related term $\alpha_1^* p_i^{(1)} + \alpha_2^* p_i^{(2)}$ is the marginal benefit of retention due to price impacts, and $r\left[(1-\alpha_1^*)\sigma_{i1} + (1-\alpha_2^*)\sigma_{i2}\right]$ is the marginal cost of retention due to risk consideration. Because the covariance term (i.e., $\sigma_{12}$ or $\sigma_{21}$) contributes in the marginal cost for each asset, investors correctly ascertain that the selling incentives of these two assets are *interconnected*. Therefore, the price impacts must have non-zero cross partials, i.e., $p_i^{(j)} \neq 0$. This is exactly the interesting *cross-signaling* effect—the retention of asset $i$ affects the pricing of asset $j$.

Applying the above argument for different $\boldsymbol{\mu}$'s, with the (FOC) equations holding pointwise for each $\boldsymbol{\alpha}$, I arrive at a system of Partial Differential Equations (PDEs) for the equilibrium pricing system $\mathbf{p}(\cdot)$. Conveniently, this model exhibits an inherent symmetry which reduces this PDE system to two separate linear PDEs. In the Appendix, I show that in equilibrium, the impact of asset 1's selling amount ($\alpha_1$) on the price of asset 2 ($p^{(2)}$), is the same as the price impact of $\alpha_2$ on $p^{(1)}$; that is,

$$p_1^{(2)}(\boldsymbol{\alpha}) = p_2^{(1)}(\boldsymbol{\alpha}).$$

Plugging this symmetry result back into (FOC), I obtain a single PDE for asset $i$ where $p^{(j)}$ is no longer involved in the pricing of asset $i$ (omitting the superscript "$*$" on $\alpha$):

$$\alpha_1 p_1^{(i)} + \alpha_2 p_2^{(i)} + r\left[(1-\alpha_1)\sigma_{i1} + (1-\alpha_2)\sigma_{i2}\right] = 0 \text{ for } i = 1, 2. \tag{3.4}$$

Take asset 1; due to the cross-signaling effect, (3.4) implies that asset 1's equilibrium price, $p^{(1)}$, depends not only on its own individual variance $\sigma_{11}$, but also on its covariance $\sigma_{12}$ with asset 2. This raises the important distinction between the asset's individual variance, and its portfolio-based risk contribution. In fact, the agent's asset 1 selling incentive—which is her marginal retention cost $(1-\alpha_1)\,\sigma_{11} + (1-\alpha_2)\,\sigma_{12}$ in (3.4)—should be the incremental risk brought on by asset 1 given the agent's underlying portfolio. Therefore, the portfolio-based risk measure, rather than the asset's individual variance, is the key determining factor for the asset's equilibrium pricing rule. Empirically, this distinction has been emphasized by Ivashina (2007), who proposes a portfolio-based risk measure for an individual loan by calculating its risk contribution relative to the lead bank's existing portfolio.

In sum, the equilibrium pricing system must satisfy the PDE (3.4). Fortunately, this PDE is a *transport equation* which admits a closed-form solution. The solution's exact form depends on the boundary conditions, which are the subject of the next subsection.

### 3.3.3. Boundary Conditions ($BC$ Assumption)

To date, there are no solid theoretical foundations for boundary conditions in the multidimensional signaling literature. Similar to LP, in what constitutes the Riley outcome (Riley (1979)), I characterize the Pareto-efficient separating equilibrium with the least amount of inefficient signaling.

Recall that, in LP, the lowest type agent sells her entire asset at a fair price. I preserve this feature by imposing analogous conditions on the agents with lower bound asset $\underline{\mu_i}$. Let the first-best selling strategy of asset $i$, *given* $\alpha_j$, be

$$\alpha_i^{FB}\left(\boldsymbol{\mu}, \alpha_j\right) \equiv \underset{\alpha_i \in [0,1]}{\arg\max}\ V\left(\boldsymbol{\mu}, \left(\alpha_i, \alpha_j\right), \boldsymbol{\mu}\right),$$

where I replace $\mathbf{p}\left(\boldsymbol{\alpha}\right)$ with $\boldsymbol{\mu}$ since there is no informational problem, and the first-best incorporates the trading restrictions. In words, $\alpha_i^{FB}$ is the agent's *conditional* first-best selling amount in asset $i$ where the *conditioning* is on the selling level of asset $j$. The quadratic form of $V$ in (3.2) implies that $\alpha_i^{FB}\left(\boldsymbol{\mu}, \alpha_j\right)$ is independent of $\boldsymbol{\mu}$; or, $\alpha_i^{FB}\left(\alpha_j\right)$ is a function of the *observable*

$\alpha_j$ only. For instance, when $\sigma_{12} > 0$, we have $\alpha_1^{FB}(\alpha_2) = 1$, since the agent will sell the entire asset 1 to minimize her risk exposure, regardless of her holding position of asset 2.

To pin down the boundary pricing rules, I assume that, in equilibrium, the agents with lower bound asset $\underline{\mu_i}$ sell the conditional first-best level $\alpha_i^{FB}(\alpha_j)$ of asset $i$. Or, the market identifies the agent who engages in asset $i$'s conditional first-best selling strategy to be the type endowed with an asset $\underline{\mu_i}$. This boundary condition generalizes the one in LP, and I call this the *BC* assumption. In reference to the previous example of positive $\sigma_{12}$, *BC* implies that, regardless of $\alpha_2$, if an agent sells her entire asset 1, then investors assign a value $\underline{\mu_1}$ for this asset.

The resulting equilibrium is a natural generalization of multidimensional Riley outcome, and inherits the Pareto-efficiency property (given the trading restrictions imposed in this paper). However, due to the multidimensional structure, the *BC* assumption is not innocuous, as there are other equilibria where the boundary agents, given the equilibrium pricing system, optimally choose strategies other than their conditional first-best amount $\alpha_i^{FB}$.[6] Section 3.6.3 shows that one can construct a continuum of *Pareto-inefficient* separating equilibria where *BC* is violated. However, almost all key equilibrium properties continue to hold, suggesting that the results obtained in this paper are robust to specific boundary conditions.

## 3.4. Equilibrium Pricing System and Its Properties

In this section, I first derive the equilibrium pricing system when the assets are positively correlated; then I turn to the negative correlation case. The general properties of these equilibrium pricing systems are discussed in Section 3.4.3.

[6] In a unidimensional LP framework, this boundary condition can be justified by the belief consistency in *sequential equilibrium* (Mailath (1987)). However, the multidimensional type space makes the *BC* assumption in this paper stronger than this requirement. See Section 3.6.3 for details.

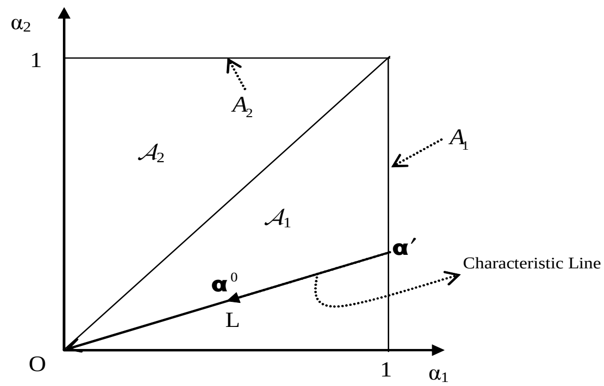


Figure 3.1. The figure for the positive correlation case. The characteristic line $L$ is a ray that emanates from the origin $\mathbf{O}$, traverses $\boldsymbol{\alpha}^0$, and then intersects with one of the boundaries $A_i \equiv \{\boldsymbol{\alpha} \in \mathcal{A} : \alpha_i = 1\}$ at $\boldsymbol{\alpha}'$. The equilibrium strategy set is $\mathcal{E} = (0,1]^2$.

### 3.4.1. Positive Correlation Case

Suppose that the assets are positively correlated, i.e., $\rho > 0$. This case corresponds to private-equity funds who specialize by investing in particular industries, or certain small local banks who face less geographically diversified lending bases.

First, I invoke $BC$ to obtain the boundary pricing rules for $p^{(i)}$. Consider $\boldsymbol{\mu} = \left(\underline{\mu_1}, \mu_2\right)$; as discussed above, I have $\alpha_1^{FB}(\alpha_2) = 1$. Intuitively, without information problems, the agent with lower bound asset 1 should discard all of this asset, because holding asset 1 always worsens the agent's risk exposure by either its own variance, or the positive covariance with asset 2. Hence, the $BC$ assumption yields

$$p^{(1)}(1, \alpha_2) = \underline{\mu_1} \text{ for } \forall \alpha_2 \in [0,1].$$

Since now the agent retains none of asset 1, I am back to the LP single-asset case for asset 2:

$$p^{(2)}(1, \alpha_2) = \underline{\mu_2} + r\sigma_{22}(\alpha_2 - \ln \alpha_2 - 1) \text{ for } \forall \alpha_2 \in (0,1].$$

By symmetry, I have $p^{(2)}(\alpha_1, 1) = \underline{\mu_2}$ for all $\alpha_1 \in (0, 1]$, and $p^{(1)}(\alpha_1, 1) = \underline{\mu_1} + r\sigma_{11}(\alpha_1 - \ln\alpha_1 - 1)$ for all $\alpha_1 \in (0, 1]$.

Given these four boundary conditions, I can solve for $\mathbf{p}(\cdot)$ using equation (3.4) in Section 3.3.2. In Figure 3.1, for any $\boldsymbol{\alpha}^0 \in \mathcal{A}$, the characteristic line, $L \equiv \left\{\boldsymbol{\alpha}(t) = \boldsymbol{\alpha}^0 \cdot t : t \geq 0\right\}$, is a ray that emanates from the origin $\mathbf{O}$, traverses $\boldsymbol{\alpha}^0$, and then intersects with one of the boundaries $A_i \equiv \{\boldsymbol{\alpha} \in \mathcal{A} : \alpha_i = 1\}$ at $\boldsymbol{\alpha}'$. Then one can obtain $p^{(i)}\left(\boldsymbol{\alpha}^0\right)$, by integrating along the ray from $\boldsymbol{\alpha}^0$ toward $\boldsymbol{\alpha}'$, plus the boundary value at $\boldsymbol{\alpha}'$.[7]

Let $\mathcal{A}_i \equiv \{\boldsymbol{\alpha} \in \mathcal{A} : 0 < \alpha_j \leq \alpha_i\}$; see Figure 3.1. The solution $\mathbf{p}(\boldsymbol{\alpha})$ is

(3.5)
$$\mathbf{p}(\boldsymbol{\alpha}) = \begin{cases} p^{(1)}(\boldsymbol{\alpha}) = \begin{cases} \underline{\mu_1} + r\sigma_{11}(\alpha_1 - \ln\alpha_1 - 1) + r\sigma_{12}\left(\alpha_2 - \ln\alpha_1 - \frac{\alpha_2}{\alpha_1}\right) & \text{if } \boldsymbol{\alpha} \in \mathcal{A}_1 \\ \underline{\mu_1} + r\sigma_{11}(\alpha_1 - \ln\alpha_1 - 1) + r\sigma_{12}(\alpha_2 - \ln\alpha_2 - 1) & \text{if } \boldsymbol{\alpha} \in \mathcal{A}_2 \end{cases} \\ p^{(2)}(\boldsymbol{\alpha}) = \begin{cases} \underline{\mu_2} + r\sigma_{22}(\alpha_2 - \ln\alpha_2 - 1) + r\sigma_{12}(\alpha_1 - \ln\alpha_1 - 1) & \text{if } \boldsymbol{\alpha} \in \mathcal{A}_1 \\ \underline{\mu_2} + r\sigma_{22}(\alpha_2 - \ln\alpha_2 - 1) + r\sigma_{12}\left(\alpha_1 - \ln\alpha_2 - \frac{\alpha_1}{\alpha_2}\right) & \text{if } \boldsymbol{\alpha} \in \mathcal{A}_2 \end{cases} \end{cases}.$$

Comparing (3.5) with the LP result, one observes that the pricing system $\mathbf{p}(\boldsymbol{\alpha})$ is the LP unidimensional pricing function (3.1), plus an additional positive term that involves the covariance $\sigma_{12}$.[8] As to be discussed shortly, this extra term is just to correct for the cross-signaling effect. The next proposition states that $\mathbf{p}$ is an equilibrium pricing system.[9]

**Proposition 7.** *Under the pricing system* $\mathbf{p}$ *in (3.5),* $V(\boldsymbol{\mu}, \boldsymbol{\alpha}, \mathbf{p}(\boldsymbol{\alpha}))$ *in (3.2) is strictly concave in* $\boldsymbol{\alpha}$. *Hence,* $\mathbf{p}$ *is an equilibrium pricing system. This equilibrium satisfies the intuitive criterion, and is Pareto-efficient relative to all (smooth) separating equilibria.*

---

[7]Along the characteristic line $L$, $\frac{dp^{(i)}(\boldsymbol{\alpha}(t))}{dt} = \frac{1}{t}\sum_{j=1}^{2}\alpha_j(t)\,p_j^{(i)} = r\sum_{j=1}^{2}\left(\alpha_j^0 - \frac{1}{t}\right)\sigma_{ij}$, which does not depend on $p^{(i)}$ itself.

[8]One can show that these terms are positive. For instance, when $\boldsymbol{\alpha} \in \mathcal{A}_1$, then $0 \leq \alpha_1 - \ln\alpha_1 - 1 \leq \alpha_2 - \ln\alpha_1 - \frac{\alpha_2}{\alpha_1} \leq \alpha_2 - \ln\alpha_2 - 1$.

[9]On the boundary $\{\boldsymbol{\alpha} \in \mathcal{A} : \alpha_i = 0\}$, $p^{(i)}$ diverges to $\infty$ (a similar result holds in the LP case). Therefore, strictly speaking, the equilibrium strategy set $\mathcal{E} = (0, 1]^2$. I can simply set the off-equilibrium pricing rule as $\mathbf{p} = \underline{\boldsymbol{\mu}} = \left(\underline{\mu_1}, \underline{\mu_2}\right)$.

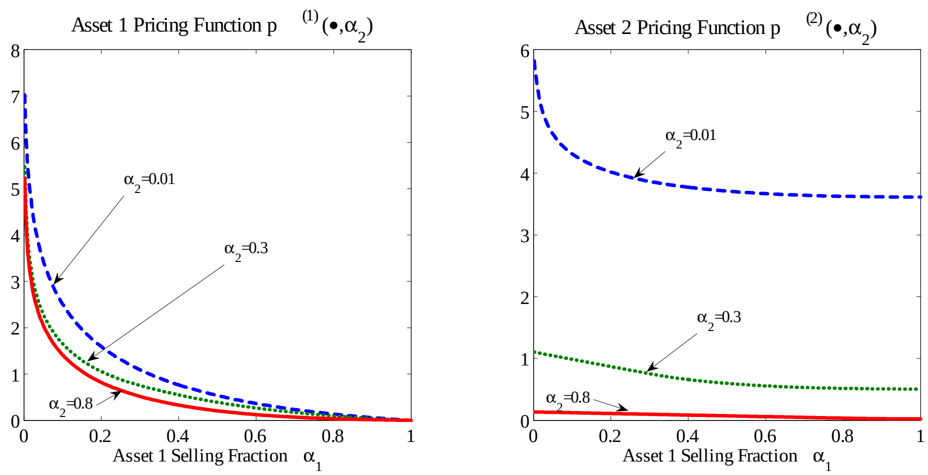


Figure 3.2. The equilibrium pricing system $\mathbf{p}(\boldsymbol{\alpha})$ when $r\sigma_{11} = r\sigma_{22} = 1, \rho = 0.5$, and $\underline{\boldsymbol{\mu}} = (0,0)$. I plot $p^{(i)}$'s against $\alpha_1$ when $\alpha_2$ takes value $0.01, 0.3$, or $0.8$. Due to symmetry, same results hold for $p^{(i)}$'s against $\alpha_2$.

The equilibrium pricing system $\mathbf{p}$ in (3.5) formally shows that when assets are correlated, one asset's pricing will depend on the transaction terms of the other asset—in other words, individual pricing rules are interconnected. Figure 3.2 plots a stylized equilibrium pricing system $\mathbf{p}(\boldsymbol{\alpha}) = \left(p^{(1)}(\boldsymbol{\alpha}), p^{(2)}(\boldsymbol{\alpha})\right)$ at different levels of $\alpha_2$. One observes a higher price for asset 1 when the agent holds more asset 2 (a smaller selling fraction $\alpha_2$). Intuitively, the fact that holding asset 1 becomes more costly—in terms of the portfolio-based risk exposure, since now more asset 2 with $\sigma_{12} > 0$ lies in the underlying portfolio—convinces the market that the marginal benefit of holding asset 1, which is just its quality, is higher. In addition, since the extra cross-signaling terms in (3.5) are positive (see footnote 8), one also observes that each pricing function in $\mathbf{p}$ is higher than the LP pricing function in (3.1); this implies that the higher the $\rho$, the stronger the cross-signaling effect.

The Ambiguous Relation between Assets' Correlation and Asset Sales. It is worth noting that the separating equilibrium stated in Proposition 7 has an interesting comparative static result, which holds in the negative correlation case as well. Fix the asset qualities, and write

$\sigma_{12}$ as $\rho\sigma_1\sigma_2$ in (3.5). The sign of $\frac{d\alpha_i^*}{d\rho}$, i.e., the effect of the correlation $\rho$ on the equilibrium selling fractions $\boldsymbol{\alpha}^*$, is ambiguous. This counter-intuitive, yet intriguing, result is unique to the two-dimensional equilibrium pricing system. The simple unidimensional LP intuition might suggest that $\frac{d\alpha_i^*}{d\rho} > 0$, because the direct effect of a higher correlation leads to a greater risk exposure, which induces the agent to sell more of both assets in equilibrium. However, the sign of $\frac{d\alpha_i^*}{d\rho}$ depends on the relative quality between these two assets.[10] Intuitively, when asset 2 is much better than asset 1, a higher $\rho$ necessarily leads to selling more of the low quality asset 1 (a larger $\alpha_1^*$). For highly correlated assets, the reduction of asset 1 position greatly lowers the marginal retention cost of the higher quality asset 2; in turn, the agent will hold more asset 2 (a smaller $\alpha_2^*$) in equilibrium.

If the current model captures the practice of bank loans sales well, then the same reason implies that relation between the assets' correlation $\rho$ and the total loan sale proceeds $\alpha_1^*\mu_1 + \alpha_2^*\mu_2$ will be ambiguous. As explained above, the direct effect, which is often termed the "*diversification hypothesis*" in the literature of bank loan sales (e.g., Demsetz (2000)), predicts that banks will engage in more loan sales when their portfolios are less diversified. However, the indirect effect shows that the above relation, under some circumstances, can potentially be reversed. Empirically, Pavel and Phillis (1987) find a positive relation between the volume of loan sales and loan portfolio concentration, suggesting that the direct effect dominates in their sample.[11]

[10]Since $\frac{d\boldsymbol{\alpha}^*}{d\rho} = -\left[\frac{\partial \mathbf{p}}{\partial \boldsymbol{\alpha}}\right]^{-1}\left[\frac{\partial \mathbf{p}}{\partial \rho}\right]$, it can be verified that if $\alpha_2^* \ll \alpha_1^*$ and $\rho$ is large, then $\frac{d\alpha_2^*}{d\rho} < 0$, i.e., the higher the $\rho$, the less the asset 2 sold by the agent.

[11]Both Pavel and Phillis (1987) and Demsetz (2000) study a *Logit* model to explain the likelihood of a bank to engage in loan sales; and Pavel and Phillis (1987) also conduct a *Tobit* estimation where the loan sale volume becomes the dependent variable. The above discussion only applies to the *Tobit* estimation. And, with more detailed data, one could explore this issue further. In particular, with data on individual loan prices and selling fractions (e.g., the data in Pennacchi and Gorton (1995)), and loan correlations (usually proxied by the geographic concentration of bank loans), one could test the following model prediction: when a bank has highly correlated loans with divergent qualities (prices), one should expect the indirect effect to prevail, i.e., the loan sale volume tends to be negatively related to the loan portfolio concentration.

### 3.4.2. Negative Correlation Case

**3.4.2.1. Inside Hedging and Outside Risk-Sharing.** Now consider the case where these two assets are negatively correlated, i.e., $\rho < 0$. From an empirical point of view, this is an equally important case—because as risk management becomes increasingly important in today's intermediaries (e.g., Allen and Santomero (1997)), intermediaries are prone to originate loans with negatively correlated payoffs. For instance, a large commercial bank, who sells its loan in an airline company, may be concurrently selling its other loans from the petroleum industry; and the performances of these two assets tend to be negatively correlated.

When assets are negatively correlated, a complication arises due to the agent's inside hedging incentive: the retention of one asset, when $\rho < 0$, could offset part of the risk associated with the other asset. This fact explicitly distinguishes this paper from LP, and underscores the underlying interdependence between endogenous hedging and signaling.

Let $\theta_i \equiv \frac{\sigma_{ij}}{\sigma_{ii}} = \frac{\sigma_j \rho}{\sigma_i} < 0$. Consider the agent $\left(\underline{\mu_1}, \mu_2\right)$ who is selling $\alpha_2$ fraction of asset 2 in equilibrium. Instead of $\alpha_1^{FB}(\alpha_2) = 1$ for the positive $\rho$ case, when $\rho < 0$, the internal hedging incentive guides the agent to set (let $a \vee b \equiv \max(a, b)$)

$$\alpha_1^{FB}(\alpha_2) = 0 \vee [1 + (1 - \alpha_2)\theta_1] \in [0, 1].$$

In other words, given asset 2's retention, $1 - \alpha_2$, the optimal hedging (holding) position for asset 1 is either $-(1 - \alpha_2)\theta_1$, or binds at 1 (she keeps the entire asset 1). I define the boundary

$$A_1^r \equiv \left\{\boldsymbol{\alpha} \in \mathcal{A} : \alpha_1 = \alpha_1^{FB}(\alpha_2)\right\},$$

anticipating that, in the spirit of $BC$, in equilibrium the agent with $\underline{\mu_1}$ will be on this boundary. Note that I simply rotate $A_1$ inside to $A_1^r$ due to the optimal hedging needs (see Figure 3.3). By

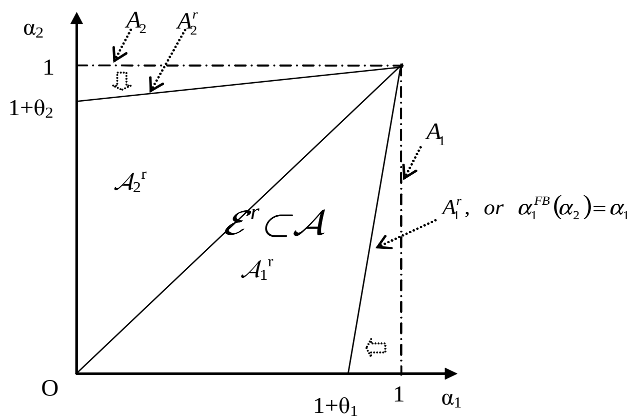


Figure 3.3. The figure for the regular case when assets are negatively correlated. The effective equilibrium strategy set $\mathcal{E}^r$ is the area between the lines $A_1^r = \left\{\boldsymbol{\alpha} \in \mathcal{A} : \alpha_1 = \alpha_1^{FB}(\alpha_2)\right\}$ and $A_2^r = \left\{\boldsymbol{\alpha} \in \mathcal{A} : \alpha_2 = \alpha_2^{FB}(\alpha_1)\right\}$.

the same token, define the boundary $A_2^r \equiv \left\{\boldsymbol{\alpha} \in \mathcal{A} : \alpha_2 = \alpha_2^{FB}(\alpha_1)\right\}$ on the other side, where $\alpha_2^{FB}(\alpha_1) = 0 \vee [1 + (1 - \alpha_1)\theta_2]$.

I classify the assets as *regular* if $A_i^r \in int\mathcal{A}_i$, or if $1 + \theta_i > 0$ for $i = 1, 2$, and *irregular* otherwise. If assets are regular, then $|\sigma_{12}|$ is dominated by $\sigma_{ii}$ for each asset, and $\alpha_i^{FB}$'s will never bind at zero. Geometrically, in the regular case, two boundaries $A_i^r$'s lie on different sides of the diagonal line (see Figure 3.3); while, in the irregular case, both boundaries fall on the same side (see Figure 3.4).

The key economic distinction between these two cases is that, in the regular case, for any asset, the *inside hedging* incentive is smaller relative to the *outside risk-sharing* motive. To see this, imagine that the agent decides to hold $\epsilon$ fraction more of asset 1. Then her optimal (marginal) internal hedging demand for asset 2 is $-\theta_2\epsilon$. If $-\theta_2 < 1$, the agent still has $(1 + \theta_2)\epsilon > 0$ fraction of asset 2 left in her hand, and tries to *sell* it to the market for better risk-sharing. However, if $-\theta_2 > 1$, the agent actually wants to *hold back* more asset 2 than $\epsilon$ for optimal internal hedging. Therefore, for asset 2, marginally the agent wants to retain more, rather than sell more, even at a fair price $\mu_2$. In this sense, when $-\theta_2 > 1$, the asset 2's inside hedging incentive dominates its outside risk-sharing motive.

**3.4.2.2. Regular Case.** I first construct the equilibrium pricing system $\mathbf{p}^r$ for the regular case. Applying the same idea as in the $\rho > 0$ case, $BC$ implies that $p^{r(1)}\left(\alpha_1^{FB}(\alpha_2), \alpha_2\right) = \underline{\mu_1}$ on $A_1^r$. A similar result holds for $p^{r(2)}$ on $A_2^r$.

Now, I derive $p^{r(1)}$on $A_2^r$ where sit the types with $\boldsymbol{\mu} = \left(\mu_1, \underline{\mu_2}\right)$. Due to the optimal hedging from asset 2 on $A_2^r$, the agent's total risk exposure is $(1-\alpha_1)^2\left(1-\rho^2\right)\sigma_{11}$, and she essentially faces a unidimensional LP problem with a variance $\left(1-\rho^2\right)\sigma_{11}$. Therefore I have

$$p^{r(1)}\left(\alpha_1, \alpha_2^{FB}(\alpha_1)\right) = \underline{\mu_1} + r\left(1-\rho^2\right)\sigma_{11}\left(\alpha_1 - \ln\alpha_1 - 1\right) \text{ for } \forall\alpha_1 \in (0,1].$$

Similarly I obtain $p^{r(2)}$ on $A_1^r$. Let $\mathcal{A}_i^r \equiv \left\{\boldsymbol{\alpha} \in \mathcal{A}_i : \alpha_i \leq \alpha_i^{FB}(\alpha_j)\right\}$, and $\mathcal{E}^r \equiv \cup_{i=1}^2 \mathcal{A}_i^r$ as the equilibrium strategy set for the regular case (see Figure 3.3). Using (3.4), I derive $\mathbf{p}^r(\boldsymbol{\alpha})$ on $\mathcal{E}^r$ as

(3.6)

$$p^{r(1)}(\boldsymbol{\alpha}) = \begin{cases} \underline{\mu_1} + r\sigma_{11}\left(\alpha_1 - \ln\frac{\alpha_1+\theta_1\alpha_2}{1+\theta_1} - 1\right) + r\sigma_{12}\left(\alpha_2 - \ln\frac{\alpha_1+\theta_1\alpha_2}{1+\theta_1} - 1\right) & \text{if } \boldsymbol{\alpha} \in \mathcal{A}_1^r \\ \underline{\mu_1} + r\sigma_{11}\left[\alpha_1 - \ln\alpha_1 - 1 - \rho^2\left(\ln\frac{\alpha_2+\theta_2\alpha_1}{1+\theta_2} - \ln\alpha_1\right)\right] + r\sigma_{12}\left(\alpha_2 - \ln\frac{\alpha_2+\theta_2\alpha_1}{1+\theta_2} - 1\right) & \text{if } \boldsymbol{\alpha} \in \mathcal{A}_2^r \end{cases}$$

$$p^{r(2)}(\boldsymbol{\alpha}) = \begin{cases} \underline{\mu_2} + r\sigma_{22}\left[\alpha_2 - \ln\alpha_2 - 1 - \rho^2\left(\ln\frac{\alpha_1+\theta_1\alpha_2}{1+\theta_1} - \ln\alpha_2\right)\right] + r\sigma_{12}\left(\alpha_1 - \ln\frac{\alpha_1+\theta_1\alpha_2}{1+\theta_1} - 1\right) & \text{if } \boldsymbol{\alpha} \in \mathcal{A}_1^r \\ \underline{\mu_2} + r\sigma_{22}\left(\alpha_2 - \ln\frac{\alpha_2+\theta_2\alpha_1}{1+\theta_2} - 1\right) + r\sigma_{12}\left(\alpha_1 - \ln\frac{\alpha_2+\theta_2\alpha_1}{1+\theta_2} - 1\right) & \text{if } \boldsymbol{\alpha} \in \mathcal{A}_2^r \end{cases}.$$

By design, the area $\mathcal{A}\backslash\mathcal{E}^r$ consists of off-equilibrium strategies where the agent sells more of asset $i$ than the conditional first-best amount $\alpha_i^{FB}(\alpha_j)$. To deter these strategies, I simply set the harshest penalty—the lower bound quality vector $\underline{\boldsymbol{\mu}} \equiv \left(\underline{\mu_1}, \underline{\mu_2}\right)$—for these strategies. The following proposition holds.

**Proposition 8.** *The pricing system* $\mathbf{p}^r$ *in (3.6) delivers a separating equilibrium when two assets are regular. The equilibrium satisfies the intuitive criterion, and is Pareto-efficient relative to all (smooth) separating equilibria.*

Relative to the LP uni-dimensional pricing function in (3.1), Proposition 10 in Section 3.4.3 will show that, *ceteris paribus*, the prices in the equilibrium pricing system (3.6) are lower. The intuition is just the same as the positive $\rho$ case discussed in Section 3.4.1, but with an opposite force: now the possibility of internal hedging dampens the agent's selling incentives, and the signal of asset retentions becomes less credible. Consequently, this leads to a lower price given the same selling fractions.

Furthermore, one can verify that, when assets are more negatively correlated (a smaller $\rho$), each asset's own-price impact $\left|\frac{\partial p^{r(i)}}{\partial \alpha_i}\right|$, which measures the market illiquidity faced by the agent, is smaller. Note that intermediaries with well-diversified underlying portfolios are endowed with assets that exhibit a smaller (more negative) $\rho$. For instance, large commercial banks tend to have geographically diversified lending bases, and skilled private equity funds with a broad investment focus will engage in advanced risk management in selecting their portfolio firms. Given this interpretation, the comparative static result regarding the relation between $\left|\frac{\partial p^{r(i)}}{\partial \alpha_i}\right|$ and $\rho$ suggests a negative relation between the "price impact" and the "portfolio diversification" for financial institutions. One immediate empirical prediction is that, large commercial banks with cross-state lending bases should have smaller price impacts in their secondary loan markets activities.

**3.4.2.3. Irregular Case.** Now I construct the equilibrium pricing system $\mathbf{p}^{ir}$ for the irregular case. Because I will show that, except for several distinct properties, the equilibrium pricing rules remain the same as in the regular case, readers may skip this subsection without hindering the reading of the rest of paper.

Without loss of generality, suppose that $1+\theta_2<0$, or $-\sigma_{12}>\sigma_{22}$. Geometrically, the boundary $A_2^r$ is also located in $\mathcal{A}_1$ (see Figure 3.4). Using (3.4), one can show that the asset 2 equilibrium pricing function $p^{ir(2)}$ achieves its *maximum* on $A_2^r$ along any characteristic line $L$ that intersects with $A_2^r$ (see Lemma 1's proof in the Appendix); therefore the $BC$ assumption for asset 2 cannot hold. This result is rooted in the fact that, in the irregular case, for asset 2 its outside risk-sharing motive is dominated by the inside hedging one (see the discussion in the end of Section 3.4.2.1), and therefore the standard LP intuition reverses.[12]

However, somewhat surprisingly, by keeping the $BC$ for asset 1 only, I find that the pricing system $\mathbf{p}^r$ defined in (3.6) still delivers a separating equilibrium for the irregular case. Take $\mathbf{p}^r$ in (3.6), and focus on $\mathcal{A}_1^r$; then there exists a curve $A_2^{ir}$, which is located between the diagonal line $\{\alpha_1=\alpha_2\}$ and $A_1^r$ (see Figure 3.4), such that $p^{r(2)}(\boldsymbol{\alpha})=\underline{\mu_2}$ for $\boldsymbol{\alpha}\in A_2^{ir}$ (see Appendix for details). Simply put, on the curve $A_2^{ir}$, $\mathbf{p}^r$ prices asset 2 at its lower bound. This suggests that, under $\mathbf{p}^r$**,** agents with the lower bound asset 2 optimally choose the selling strategies on $A_2^{ir}$, and the equilibrium strategy set $\mathcal{E}^{ir}\subset\mathcal{A}_1^r$ is the area between $A_1^r$ and $A_2^{ir}$ (the total shaded area—including both the simple-shaded area and the cross-shaded area—in Figure 3.4). Therefore, I set $\mathbf{p}^{ir}=\mathbf{p}^r$ for equilibrium selling strategies, and, as in the regular case, I design the same off-equilibrium pricing rules for determent purposes. The next proposition follows.

**Proposition 9.** *Let* $\mathbf{p}^{ir}(\boldsymbol{\alpha})=\mathbf{p}^r(\boldsymbol{\alpha})$ *for* $\boldsymbol{\alpha}\in\mathcal{E}^{ir}$*, and* $\mathbf{p}^{ir}(\boldsymbol{\alpha})=\underline{\boldsymbol{\mu}}$ *otherwise. Then,* $\mathbf{p}^{ir}$ *delivers a separating equilibrium for the irregular case. This equilibrium satisfies the intuitive criterion, and is Pareto-efficient relative to all (smooth) separating equilibria.*

[12] Recall that $BC$ requires that types with $\underline{\mu_2}$ (the lower bound) lie on $A_2^r$, contradicting to the maximum of $p^{ir(2)}$ on $A_2^r$. The reason for this result is as follows. In the irregular case, the marginal retention *cost* $r\left[(1-\alpha_1)\,\sigma_{12}+(1-\alpha_2)\,\sigma_{22}\right]$ in (3.4)—which is positive in both the positive $\rho$ and regular cases (and also in the LP unidimensional case)—turns negative, because $\sigma_{12}<0$ dominates $\sigma_{22}$. As a result, retention actually incurs some marginal *benefit* in the irregular case, which necessarily reverses the pricing effect.

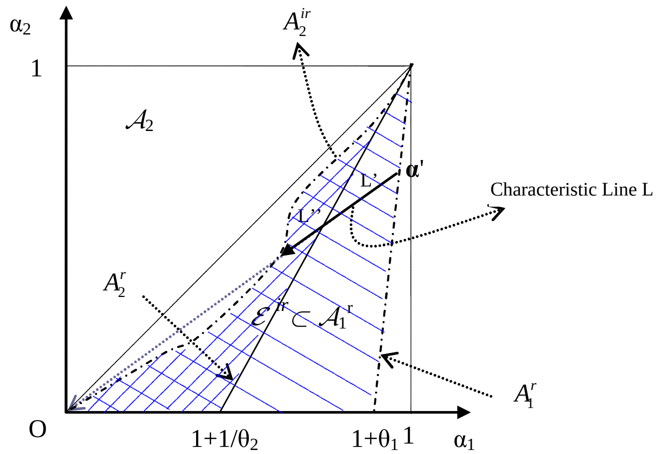


Figure 3.4. The figure for the irregular case where $\theta_2 < -1$. The equilibrium strategy set $\mathcal{E}^{ir}$ is the total shaded area (including the simple-shaded area and the cross-shaded area) between $A_1^r$ and $A_2^{ir}$. In equilibrium, agents in the cross-shaded area (between $A_2^r$ and $A_2^{ir}$) sell more asset 2 than their (conditional) first-best hedging amount.

Although $\mathbf{p}^{ir}(\cdot)$ for the irregular case is the same as $\mathbf{p}^r(\cdot)$ for equilibrium strategies, they possess distinct properties. Note that, along $A_2^{ir}$, the agent with $\underline{\mu_2}$ has $\alpha_2^* > \alpha_2^{FB}(\alpha_1)$, as $A_2^{ir}$ is above $A_2^r$. In other words, agents who lie between $A_2^{ir}$ and $A_2^r$—the cross-shaded area in Figure 3.4—"optimally" sell more of asset 2 than the conditional first-best hedging level. Therefore, for an agent lying on the cross-shaded area, retaining more of asset 2 incurs *less* risk exposure, as opposed to the original LP scenario where more retention always gives rise to *more* risk exposure. However, as to be shown in Proposition 10 in the next section, similar to LP, the own-price impact of asset 2 is *negative*, i.e., $\frac{\partial p^{ir(2)}}{\partial \alpha_2} < 0$. Therefore, this agent also receives a direct-signaling *reward* (a higher price) from the asset 2's extra retention. Then the intriguing question is, what drives the agent to sell more of asset 2 than her conditional first-best level $\alpha_2^{FB}(\alpha_1)$ in equilibrium?

The answer is to *cross-signal* her asset 1' quality. Recall that, since choosing a higher $\alpha_2 > \alpha_2^{FB}(\alpha_1)$ leads to a worse hedging position for asset 1, selling more of asset 2 is a credible signal for a higher quality of her asset 1. In fact, in this case, this cross-signaling effect could be sufficiently strong to dominate the direct-signaling effect. To be precise, the following relation holds for the area close to $A_2^{ir}$ in Figure 3.4,

$$\frac{\partial p^{ir(1)}}{\partial \alpha_2} > \left| \frac{\partial p^{ir(2)}}{\partial \alpha_2} \right| > 0, \tag{3.7}$$

which states that the asset 2 selling position has a stronger impact on the asset 1's pricing (cross-signaling) than does that of asset 2 (direct-signaling).[13] This implies that, in some circumstances, cross-signaling (as opposed to the direct-signaling) could be the leading concern during asset sales when the agent possesses multi-dimensional private information.

### 3.4.3. Properties of Equilibrium Pricing System

Proposition 10 collects the general properties of equilibrium pricing systems for all cases.

**Proposition 10.** *Let* $\mathbf{p}$ *be the equilibrium pricing system for all cases* ($\rho > 0$*, regular, and irregular).*

(1) ***(Cross-Signaling Effect)*** *When* $\rho > 0$ ($\rho < 0$)*, a larger* $\alpha_j$ *implies a lower (higher) price* $p^{(i)}$ *for asset* $i$*. And, the larger the correlation* $\rho$*, the higher the asset prices;*

(2) ***(Price Impact)*** *Each equilibrium pricing function is downward-sloping in its own selling fraction, i.e.,* $p_i^{(i)} < 0$*. Moreover, the larger* $\rho$*, the more negative (or, steeper) the own-price impact;*

(3) ***(Equilibrium Payoff)*** *The agent's equilibrium payoff is increasing in* $\boldsymbol{\mu}$*, and decreasing in* $\rho$*.*

[13]This inequality (3.7) can be easily verified in a numerical example. The intuition of a dominant cross-signaling effect is as follows: in the irregular case, the covariance $|\sigma_{12}|$ dominates the individual variance $\sigma_{22}$ (or, $|\theta_2| > 1$), and therefore the cross-signaling effect may prevail. Consequently, this result never holds in the regular case.

The cross-signaling effect constitutes one of the paper's major contributions; for a numerical example, see Figure 3.2 in Section 3.4.1. In fact, this cross-signaling property is rather straightforward if the holding position of either asset is *exogenously* given. Suppose that the agent is selling asset 1, but she holds an exogenous amount of asset 2 in her underlying portfolio. The *virtual variance* of asset 1 should be $\sigma_{11} + \sigma_{12}$; in other words, the agent's selling incentive for one asset is determined by its contribution to her portfolio variance, rather than the individual variance. As noted after equation (3.4) in Section 3.3.2, this empirically relevant distinction has been explored by Ivashina (2007) who measures a loan's risk contribution relative to the lead bank's existing portfolio, as opposed to the loan's individual risk.

Now suppose that the exogenously given asset 2 is positively correlated (i.e., $\sigma_{12} > 0$) with asset 1; then a worse hedging position gives rise to a higher virtual variance for asset 1. This higher virtual variance—by boosting the agent's selling incentives for risk-shedding purposes—makes the asset 1's retention a more credible signal for the quality of asset 1, which leads to a higher equilibrium price for asset 1. Similarly, this intuition also suggests a positive relation between the assets' correlation $\rho$ and their equilibrium prices.

This paper shows that such a positive relation between pricing and virtual variance should hold even when asset positions are *endogenously* determined in the simultaneous sale. A testable implication is that, when a bank sells its loans in the secondary markets, *ceteris paribus*, investors will attach a higher price for one loan when the bank's other assets in transaction have more correlated payoffs.

The second property depicts a cross-sectional pattern for intermediaries' price impacts. The downward-sloping pricing system echoes the information-driven illiquidity (price impact) of financial markets (LP, Kyle (1985), etc.). And, as mentioned in the discussion after Proposition 8 in Section 3.4.2.2, since the correlation $\rho$ captures the diversification of the agent's assets pool for sale, one would expect a smaller price impact (i.e., more liquidity) for financial institutions with well-diversified underlying portfolios.[14] The intuition is as follows: for an intermediary with a lower correlation $\rho$, her assets' virtual variances (relative to this specific intermediary) are smaller; then lower selling incentives leads to flatter equilibrium pricing rules, i.e., smaller price impacts, for these assets. Therefore, the model predicts that, all else equal, there should be a smaller pricing response to the fraction sold for a bank with a portfolio of (or access to) more diversified loans.

Note that Property 1 (cross-signaling effect) and Property 2 (price impact) show that, in this model, both the "price impact" and the "price" are positively related to the assets' correlation. As explained above, the underlying connection is the agent's selling incentives. On the one hand, for separation purpose, the equilibrium *price impact* has to reflect the agent's selling incentives; on the other hand, higher selling incentives make the retention signal more credible, and therefore leading to higher *prices*.

The third property is intuitive: since the agent has a mean-variance objective, higher qualities (a higher mean), or a better inside hedging opportunity (a lower variance), leads to a higher equilibrium payoff for the agent.

## 3.5. Bank Loan Markets

### 3.5.1. Background

The bank loan market includes two broad categories: 1) the syndicated loan market where a loan is originated and placed with a number of banks, and 2) the secondary market where a bank sells part of an existing loan to other institutions. The rapid development of bank loan markets

[14] Note that *less selling* can be interpreted as *more buying*. In fact, as an extension of the baseline model, He (2005) shows that when the agent has access to certain active hedging opportunities during the asset sale, the similar finding will hold—that is, the superior the intermediaries' hedging opportunities, the more liquid the assets when sold to financial markets. Therefore, this prediction is robust to the trading restrictions imposed in this paper.

has elicited extensive attention from researchers; for recent papers, see, e.g., Ivashina (2007) on loan syndications, and Gorton and Pennacchi (1995) on the secondary bank loan market.[15]

Undoubtedly, bank loan markets suffer from the *lemon's* problem studied in this paper. The following quote from Dahiya, Puri and Saunders (2001), which appears in *Bank Letter* dated 06/19/1995, illustrates this point: "*An original lender on a \$150 million Bradlees credit reportedly sold a \$5 million piece of the revolver in a hurry last week, ..., sending the message ... that the lenders most familiar with Bradlees are not comfortable with the company's situation.*"

This paper examines the bank's simultaneous loan sales on a portfolio basis. The assumption of simultaneous trading of multiple assets seems appropriate. For instance, the sample in Gorton and Pennacchi (1995) consists of 872 individual loan sales made by a single bank from 01/20/1987 to 09/01/1988. Also, according to the Dealscan dataset, *JPMorgan Chase* was the lead bank for 310 syndicated facilities within the second half of 2002. Moreover, because bank loan transactions rarely involve short positions, the trading constraint $\alpha_i \in [0, 1]$ imposed in this paper fits this application well.

Other assumptions in this model warrant further discussion. The assumption of risk-neutral buyers is standard in the loan-sale literature (e.g., Gorton and Pennacchi (1995)). This assumption can be justified on the ground that, compared to the seller bank, buyer institutions who either purchase small shares of loans for portfolio diversification purposes (e.g., Demsetz (2000)), or issue securities backed by these assets right away to other diversified institutions (e.g., MBS markets), should be far less concerned about the asset's idiosyncratic shocks. For instance, Ivashina (2007) reports that, in her loan syndication samples, "*Lead bank's average share is a substantial 27% ... Average participant share is 6%, and participants are likelier to sell or securitize their risk.*"

[15]For early studies on bank loan sales, see, e.g., Pennacchi (1988) who argues that the optimal loan sale should balance the moral hazard problem with the relative advantage in funding cost. For loan syndications, see, e.g., Simons (1993).

Also, throughout I implicitly assume that the agent's (in the current context, seller or lead banks') holding positions are publicly observable.[16] Put differently, buyers in this market know the seller or lead banks' selling incentives due to risk considerations. Though somewhat demanding for individual investors, this assumption is reasonable for the sophisticated institutional participants in this market. Following Ivashina (2007), they can at least (approximately) construct the seller bank's existing loan portfolio from Dealscan.[17]

### 3.5.2. Related Empirical Work

The empirical literature on the bank loan market provides certain indirect supporting evidence for this paper. Consistent with the agent's (here the seller or lead banks') underlying risk-diversification motive upon which this model is built, Pavel and Phillis (1987) find that loan concentration increases the likelihood of a bank's engagement in loan sales, and Demsetz (2000) shows that geographically expansive branch networks reduce the bank's secondary market activity. However, both papers study bank-level data only. In contrast, Ivashina (2007) and Gorton and Pennacchi (1995) focus on the pricing of individual loans, and find that a larger retaining share of the lead (or seller) bank indicates a higher loan quality in the transaction.[18]

In the context of loan syndications, Ivashina (2007) argues that the retention share of a lead bank has both *diversification* and *asymmetric information* effects on the equilibrium loan pricing. The *diversification* effect—without any informational concerns—simply captures the transfer from the corporate borrower to the lead bank. Under this effect, the lead bank with a

[16]Note that during loan sales or loan syndications, the seller or lead bank's identity is publicly known to the buyer institutions, which implies that its selling activity is common knowledge.

[17]Beyond the signaling framework, another piece of salient (and maybe extreme) evidence regarding the "observable asset positions" is the sizeable "fire-sale" discount for those distressed banks who cannot maintain a healthy loan loss reserve. See "*Sales in Distress*", by R. England, http://www.bai.org/bankingstrategies/2002-may-jun/sales/.

[18]In Gorton and Pennacchi (1995), the agency issue stems from the seller bank's moral hazard problem. However, its implication is the same as that of a signaling (adverse-selection) framework: a larger retention share—by inducing more effort—improves the ex-post loan quality, and therefore the buyer banks are willing to pay more ex-ante.

larger retention share will demand a higher loan spread (a lower price) for risk compensation. The more interesting *asymmetric information* effect is the pricing schedule formed by the less-informed participant banks. The underlying mechanism could be either an adverse selection story where the lead bank signals the loan's quality through retention as in LP and this paper, or a moral hazard problem where the lead bank exerts ex-post unobservable monitoring effort based on its holding fraction as in Gorton and Pennacchi (1995), or a mixture of both. Ivashina shows that, empirically, the asymmetric information effect causes the loan price (spread) to increase (decreases) with the lead bank's retention share, and she measures the asymmetric information cost as the slope (pricing impact) of this downward-sloping pricing schedule.

In Ivashina (2007)'s empirical framework, the observed spread-retention pairs are equilibrium intersection points between the asymmetric information and diversification curves described above. Therefore, as in a classical simultaneous equation framework, to trace down the asymmetric information effect, one needs to find factors that shift the diversification curve, *but without touching the asymmetric information curve.* Ivashina argues that the individual loan's risk exposure specific to the lead bank will accomplish this objective—a claim inconsistent with the theoretical results in this paper.

To obtain this portfolio-based risk exposure for individual assets, Ivashina measures the loan's risk contribution relative to the lead bank's existing portfolio, which is the marginal change of the lead bank's loss variance matrix due to holding this loan (see Ivashina (2007) for details). As discussed earlier (in Section 3.3.2 and Section 3.4.3), this treatment is especially appealing in view of this paper: note that more than the loan's individual variance, this measure is the *virtual variance* which directly determines the agent's selling incentives, and in turn the equilibrium pricing rules.

However, contrary to the Ivashina's identification assumption, my theoretical model shows that the participant banks' pricing schedule—the asymmetric information curve—does depend on the loan's risk exposure measured above, simply because the lead bank's selling incentives alter with the asset's virtual variance. Participant banks will take this into account, and the pricing schedule changes accordingly. Interestingly, according to my theoretical model, this fact causes the asymmetric information cost estimated in Ivashina (2007) to be downward biased, a fact that strengthens her empirical findings.[19]

### 3.5.3. Future Empirical Work

The direct application of this paper lies in secondary loan sales. There is a fast growing empirical literature on the secondary market for bank loans. For instance, to study the specialness of banks in the presence of a secondary market for bank loans, Gande and Saunders (2005) use Loan Pricing Corporation/Loan Syndications and Trading Association (LSTA/LPC) Mark-to-Market Pricing dataset, which contains daily bid-ask price quotes aggregated across dealers. This newly available dataset, combined with other detailed loan transaction and bank-level asset-holding data, potentially opens the door to investigating a variety of interesting empirical questions. For instance, it would be desirable to revisit Gorton and Pennacchi (1995), and identify the downward sloping pricing schedule based on a larger and more representative sample.[20]

[19]Take the retention-spread plot used in Ivashina (2007) where retention is on the $x$-axis, and consider the situation where the loan is riskier (specific to the lead bank). The *diversification* curve shifts up (the lead bank needs more compensation from the corporate borrower, therefore a higher spread). Now, because the retention signal becomes more credible given a worse hedging position (see Figure 3.2 in this paper), the *asymmetric information* curve shifts down (note that I use spread instead of price here). Therefore, the new equilibrium point, relative to the one without shifting the *asymmetric information* curve as assumed in Ivashina (2007), has both a smaller retention fraction and a lower spread. This implies that, the slope (which is negative, but consider its absolute value here) in Ivashina (2007) is downward-biased relative to the true slope for a given asymmetric information curve (i.e., fixing the asset's virtual variance). Of course, to address this issue fully one needs a more complicated model, as the pricing schedule required by participant banks, in a signaling framework, should interact with the *diversification* curve. I leave this possibility for future research.

[20]Gorton and Pennacchi (1995) only study a sample of 872 loan sales conducted by a single bank. Beyond this issue, relative to loan syndications, two facts about secondary loan sales can facilitate a cleaner empirical identification for "asymmetric information effect," which causes a downward sloping pricing schedule. First, the corporate borrower no longer plays any role in secondary loan sales, therefore the confounding "diversification effect" in Ivashina (2007) (which captures the transfer from the borrower to the lead bank in a loan syndication) is absent in this market. Second, in contrast to the loan origination where usually the corporate borrower only has a limited number of relationship lenders, in the secondary market, the seller bank faces a more competitive group of buyers, and the price should be relatively more "fair."

By studying the loan pricing in a multi-asset framework, this paper suggests other testable implications. For instance, using Ivashina's portfolio-based risk contribution measurement mentioned above, one can carry out a potentially interesting test: holding the selling fraction constant, will the market attach a higher price for the loan whose virtual variance is higher, just as suggested by Figure 3.2 in Section 3.4.1? Notice that a story with noncompetitive buyers—combined with a bargaining mechanism—can potentially lead to an opposite answer, as the seller bank will be left with a worse outside option in this scenario.

More importantly, this paper highlights simultaneous multiple loan transactions in the secondary market. As suggested by Section 3.4.3, there are several empirical predictions regarding the relation between asset retention positions and their transaction prices. For instance, suppose that a holding bank sells two positively correlated assets, then it would be interesting to identify the cross-signaling effect as illustrated in Figure 3.2—that is, the pricing of loan 1 should respond positively when the bank sells less of its second loan. Also, as the retention signals become more credible for more correlated assets, the model predicts that, *ceteris paribus*, transaction prices will be higher if the loans in simultaneous sales are more correlated.

Future research can also investigate the *price impact* in the secondary bank loan market. My model predicts that, the price impact of loan sales becomes larger, either when the bank faces an under-diversified existing portfolio, or when the loans on transaction are more positively correlated. To be more specific, in explaining the loan prices, the coefficient on the interaction between the loan's selling share and its virtual variance should be positive. If it is indeed the case, then these empirical findings might help deepen our understanding of the potentially time-varying liquidity in this market.[21] For instance, an interesting follow-up question would be, how

[21]For data availability issue, there are few academic studies on this topic, even though some anecdotal evidence suggests so (e.g., "*Sales in Distress,*" http://www.bai.org/bankingstrategies/2002-may-jun/sales/). Nevertheless, note that the time-varying liquidity in the equity market is well-documented; then because the secondary bank loan market requires more specialty, one would expect at least the same result for this market, if not stronger.

much the variability of the liquidity in the secondary bank loan market can be explained by the movement of loan correlations over time?

It is worth noting that, one can raise similar empirical questions for the asset selling behaviors of private equity funds who sell their portfolio firms in financial markets. In addition, the model studied in this paper could also be applied to financially troubled hedge funds who conduct forced sales in a short time window; one such example would be the recent meltdown of hedge funds under *Bear Stearns* due to the subprime-mortgage crisis during the summer of 2007.

## 3.6. Extensions

In this section, I first compare different selling mechanisms available to intermediaries, then move on to discuss the model's extension to general $n$-asset cases. Finally I show that the resulting equilibrium properties obtained in Section 3.4.3 are robust to the specific Boundary Conditions imposed in this paper.

### 3.6.1. Selling Mechanisms

**3.6.1.1. Pooled Sale.** In the previous 2-asset case, aside from selling separately the agent has another option: to package both assets and sell them as a single portfolio. Which selling strategy is optimal? DeMarzo (2003) finds that a separate sale always dominates pooling when $\rho = 0$, and he labels this the "*information destruction*" effect of pooling.[22] In this section, I show that this property remains true even if $\rho \neq 0$. Therefore, information destruction by pooling is significantly more general than what has been acknowledged in the current literature.

[22]The information destruction effect of pooling under the multi-asset LP case with $\rho = 0$ is presented in the Appendix of DeMarzo (2003). In its final version published in *Review Financial Studies* 18 (2005), this analysis is taken out.

If the agent decides to sell as a pool, the equilibrium selling fraction $\alpha$ is characterized by (3.1) with $\underline{\mu} = \underline{\mu_1} + \underline{\mu_2}$ and $\sigma^2 = \sigma_{11} + 2\sigma_{12} + \sigma_{22}$, and her equilibrium payoff is

$$\mu_1 + \mu_2 - \frac{r}{2}\left[(1-\alpha)^2(\sigma_{11} + 2\sigma_{12} + \sigma_{22})\right].$$

To verify the "*information destruction*" effect, it amounts to showing that the agent will face a larger risk exposure by pooling her assets before the sale.

**Proposition 11.** *A separate sale dominates a pooled sale. The two mechanisms are equivalent if and only if* $\frac{\mu_1 - \underline{\mu_1}}{\sigma_{11}+\sigma_{12}} = \frac{\mu_2 - \underline{\mu_2}}{\sigma_{22}+\sigma_{12}}$ *in the positive correlation and regular case.*

The above result is somewhat surprising since, for a risk-averse agent, it appears that pooling negatively correlated assets together has a "risk-diversification" benefit, a term used in DeMarzo (2003). However, the discussion in Section 3.4.2.1 suggests that the risk-diversification benefit from negatively correlated assets is present even without pooling. In fact, by pooling assets before the sale, the agent simply loses the flexibility to take different positions on each asset. The principal implication is that, in DeMarzo (2003), the "risk-diversification effect" which favors pooling assets (and then tranching) is fundamentally different from the diversification benefit in the standard portfolio theory where a risk-averse preference is assumed. Rather, as in Diamond (1984) where the bank takes all individual loans and issues deposits to outside investors, the *diversification* in DeMarzo (2003) plays a similar beneficial role where the per-asset likelihood of a tail-event shrinks as the number of assets increases (recall that both DeMarzo (2003) and Diamond (1984) have debt as the optimal contract).

Of course, pooling assets may prove optimal for other reasons. Suppose that, before shedding the assets, the agent could exert certain unobservable monitoring effort to improve asset qualities. By the envelope theorem, the impact of asset $i$'s quality on the agent's equilibrium payoff is simply $\frac{dV^*}{d\mu_i} = 1 - \alpha_i$. Since the asset retention $(1 - \alpha_i)$ is small when asset quality is low, the agent has little incentive to monitor low quality assets, even though it might be socially optimal to exert a significant amount of effort on them. For instance, in the scenario of bank loans, equity holders of the borrower firm might gain substantially from the bank's effective monitoring.

**3.6.1.2. Sequential Sales.** Another possible selling mechanism is sequential sales. For illustrative purposes, I discuss the case where the agent has two assets with positive correlation; this case resembles a private equity specialty fund who sells multiple portfolio firms within the same industry. Clearly a separating equilibrium requires the agent's commitment on her selling fraction of the second sale; otherwise, when selling asset 1, the unidimensional signal cannot separate multidimensional types.

Under commitment, He (2005) considers the scenario where the payoff of the first asset ($\widetilde{x}_1$) is realized *before* the sale of the second asset (otherwise it is equivalent to a simultaneous sale), and derives a separating equilibrium in closed form. The main complication is that investors will update asset 2's quality given asset 1's innovation, due to the positive correlation. Interestingly, when comparing the agent's payoff from sequential sales with the one obtained from simultaneous sales *before the realization of* $\widetilde{x}_1$, He (2005) finds that the simultaneous sale always dominates the sequential sale *if the delay does not improve the expected quality of asset 2 in the latter case.* The reason is that, the agent will be exposed to more risk when the market updates asset 2's quality due to new information about $\varepsilon_1$. Therefore the agent tends to accelerate her selling pace given the additional concern of cross-signaling, and this finding provides another possible explanation for premature IPOs in the VC industry (e.g., Barry et al. (1990)).

### 3.6.2. General $n$-Asset Cases

By induction, one can construct the Pareto-efficient separating equilibrium for the general $n$-asset case. However, the tractability of construction crucially depends on the structure of the covariance matrix $\Sigma$.

When all assets are positively correlated, i.e., $\sigma_{ij} \geq 0$ for all $i$, $j$, the equilibrium pricing system $\mathbf{p}$ is available in closed form. The tractability in this case stems from the simple equilibrium strategies for the *boundary* agents: when there is no internal hedging motive for any asset, $BC$ implies that the agents with asset $\underline{\mu_i}$ will set $\alpha_i = 1$ and then retain none of this asset. This reduces an $n$-asset problem to an $n-1$-asset problem, and, in the Appendix, I derive a closed-form $n$-dimensional equilibrium pricing system. Though a bit more cumbersome, this system shares the same properties with the 2-dimensional model (the pricing system in (3.5)) derived in Section 3.4.1.

When assets have an arbitrary covariance matrix, internal hedging motives across different assets—coupled with trading restrictions—induce complicated equilibrium strategies for boundary agents. As a result, the *boundary* pricing rules are quite involved. In a 3-asset example studied in the Appendix, asset 1 and asset 3 are independent; however, asset 2 is positively (negatively) correlated with asset 1 (asset 3). Since asset 2 serves opposite roles relative to the other two assets, I show that, in equilibrium, the agents endowed with $\underline{\mu_2}$ sit on a kinked surface in $[0,1]^3$, due to the asset 2's conditional first-best hedging strategy given trading restrictions. This fact renders the non-tractability of the general covariance matrix case.

However, by an induction procedure, there are no conceptual difficulties in deriving the Pareto-efficient separating equilibrium for the higher dimensional case. Similar to the derivation in Section 3.4.1, the induction step involves constructing the $m$-dimensional pricing systems (where $m = 0, 1, \cdots, n$) for boundary types who possess $n-m$ assets with the lower bound qualities (note that there are $\frac{n!}{m!(n-m)!}$ pricing systems to be solved for). Here, the $BC$ assumption, which ensures that there is no private information in deriving $\mathbf{p}$ for these boundary types, is pivotal. Clearly, $m = 0$ is the degenerate case (the pricing system is simply $\underline{\boldsymbol{\mu}}$ for the first-best selling strategies). Once given the collection of pricing systems for $m \geq 0$, which might be quite complicated due to the optimal hedging positions from those $n-m$ boundary assets (see the previous 3-asset example), the construction of pricing systems for $m+1$ is merely a line integration in some appropriate $m+1$-dimensional space in $(0,1]^n$, based on a transport equation similar to (3.4) which involves $m+1$ unknown pricing functions. In the Appendix, interested readers can find the construction details for the equilibrium pricing system for the 3-asset case mentioned above. Nevertheless, the key economic insights delivered in the 2-asset case will continue to hold.

### 3.6.3. Boundary Conditions and Pareto-inefficient Separating Equilibria

As discussed in Section 3.3.3, the Boundary Condition assumption $BC$ is designed to pick out the Pareto-efficient Riley outcome. Unlike the unidimensional LP case (see footnote 6), under the current multidimensional signaling framework, one can construct a continuum of Pareto-inefficient separating equilibria that satisfy both the belief consistency requirement and the intuitive criterion. In the Appendix, I give such an example with $\rho > 0$; in this inefficient equilibrium, the agents endowed with $\underline{\mu_1}$ optimally choose $\alpha_1 < 1$ (as opposed to $\alpha_1^{FB} = 1$ in the equilibrium derived in Section 3.4.1), because selling the entire asset 1 incurs a substantial penalty imposed by the asset 2 pricing rule. However, almost all key properties established in Proposition 10 continue to hold in these equilibria.

This Pareto-inefficient equilibrium represents a cross-disciplining mechanism. Compared to this Pareto-inefficient equilibrium, our Pareto-efficient $BC$ assumption essentially rules out the cross-disciplining from the second asset submarket to the selling pattern of asset $\underline{\mu_1}$ on the first submarket. Therefore, from this view, the $BC$ assumption is reasonable if competitive investors cannot coordinate to give rise to a sophisticated cross-deterring scheme.

There is another interesting fact about this inefficient equilibrium. In its equilibrium pricing system, the individual pricing functions are non-smooth (i.e., not continuously differentiable on the equilibrium strategy set $\mathcal{E}$).[23] Therefore, the "smoothness" assumption about individual pricing functions is not crucial for the key equilibrium properties derived in this paper.

Admittedly, these Pareto-inefficient examples are far from constituting solid theoretical grounds for equilibrium refinement in the multidimensional signaling game. However, they advance our understanding of the multiplicity of separating equilibria. In addition, the results here demonstrate that, the key properties derived under the Pareto-efficient equilibrium are in fact robust within a wide class of equilibria. As this paper mainly focuses on the intermediaries' equilibrium trading behaviors, and the associated price and price impacts, this subsection serves as a robustness check for these results.

### 3.7. Conclusion

I study the signaling and hedging behaviors of financial intermediaries under a multi-asset environment. By generalizing LP's result, this paper develops a multidimensional equilibrium pricing system for correlated assets, and offers a framework to analyze the interdependence between asset selling and risk management for financial intermediaries. By acknowledging the interconnectedness between any single asset and the existing portfolio of the informed owner, I derive the interesting cross-signaling patterns that arise in equilibrium. In short, the agent's holding position of asset $i$ reveals information about her asset $j$'s quality.

The resulting equilibrium pricing system allows one to examine the cross-sectional difference in asset prices and price impacts faced by heterogenous financial firms, and provides several interesting testable empirical predictions. For instance, in the presence of information asymmetry, financial firms with under-diversified portfolios should have larger price impacts, and therefore suffer greater illiquidity problems when selling their assets. This prediction is robust to a wide

[23]However, the agent's payoff function $V$, as a composite function of the entire system $\mathbf{p}$, continues to behave smoothly.

range of equilibria, as well as the possibility of active hedging performed by the agent (see He (2005)).

The novel transport equation technique developed in this paper could be potentially applied to other CARA-Normality models in finance. Similar to LP, I derive the Pareto-efficient separating equilibrium (a multidimensional version of Riley outcome), and discuss the empirical predictions based on the resulting equilibrium pricing system where the individual pricing functions are interconnected. I also construct a continuum of Pareto-inefficient separating equilibria that satisfy both intuitive criterion and belief consistency. Hopefully, these results can shed some light on general theories about multidimensional signaling games in the future work, especially on the theoretical grounds for boundary conditions.

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APPENDIX A

# Appendices

## A.1. Appendix for Chapter 1

### A.1.1. Proof for Proposition 1 in Section 1.3.1

Given any contract $\Pi = \{\{U\}, \tau\}$, define the process $V_t \equiv \mathbb{E}_t \left[\int_0^\tau e^{-\gamma s} dU_s\right]$ for $t \in [0, \tau)$ as the value process of the agent's discounted wages. Under condition (1.1), $\{V_t : 0 \leq t < \tau\}$ forms a square-integrable martingale until $\tau$. According to the *Martingale Representation Theorem*, there exists a progressively measurable process $\left\{\sigma_t^W : 0 \leq t < \tau\right\}$ s.t. $V_t = V_0 + \int_0^t e^{-\gamma s} \delta_s \sigma_s^W dZ_s$ for $\forall t \in [0, \tau)$. Hence under $\{a_t = \mu : 0 \leq t < \tau\}$ we have $V_t = V_0 + \int_0^t \frac{e^{-\gamma s}\delta_s \sigma_s^W}{\sigma} \left(\frac{d\delta_s}{\delta_s} - \mu ds\right)$ for $\forall t \in [0, \tau)$. Now since $W_t = \mathbb{E}_t \left[\int_t^\tau e^{-\gamma(s-t)} dU_s\right]$, we have $V_t = \int_0^t e^{-\gamma s} dU_s + e^{-\gamma t} W_t$. By taking derivative on both sides, we obtain $W$'s evolution.

We show that $\Pi \in \mathbb{IC}$ if and only if $\sigma_t^W \geq \lambda\sigma$ a.e.. Consider any effort policy $a = \{a_t \in \{0, \mu\} : 0 \leq t < \tau\}$. For $t < \tau$ her associated value process is $V_t(a) = V_0 + \int_0^t \frac{e^{-\gamma s}\delta_s \sigma_s^W}{\sigma} \left(\frac{d\delta_s}{\delta_s}(a) - \mu ds\right) + \int_0^t e^{-\gamma s} \lambda \delta_s (\mu - a_s) ds$. We have,

$$dV_t(a) = e^{-\gamma t}\delta_t \left(\frac{\sigma_t^W}{\sigma} - \lambda\right)(a_t - \mu) dt + e^{-\gamma t}\delta_t \sigma_t^W dZ_t.$$

If $\sigma_t^W \geq \lambda\sigma$, then it has a non-positive drift, and is a martingale if $\{a_t = \mu : 0 \leq t < \tau\}$. If there is a positive probability event that $\sigma_t^W < \lambda\sigma$ during $[0, \tau)$, the agent will deviate to $a_t = 0$, and $\{a_t = \mu : 0 \leq t < \tau\}$ is suboptimal. Therefore $\Pi \in \mathbb{IC}$ iff $\sigma_t^W \geq \lambda\sigma$ a.e. Q.E.D.

### A.1.2. From HJB Equation to Optimality Equation (Section 1.3.2.1)

Recall the evolutions of two state variables $d\delta_t = \mu\delta_t dt + \sigma\delta_t dZ_t$ and $dW_t = \gamma W_t dt - dU_t + \lambda\sigma\delta_t dZ_t$. Therefore $b(\delta_t, W_t)$ must satisfy the following Hamilton-Jacobi-Bellman equation,

$$rbdt = \sup_{dU_t \geq 0} \{\delta dt - dU_t + b_1 \mu\delta dt + b_2 (\gamma W dt - dU_t) + \frac{1}{2}\left(\sigma^2\delta^2 b_{11} + 2\lambda\sigma^2\delta^2 b_{12} + \lambda^2\sigma^2\delta^2 b_{22}\right) dt\};$$

where $b_i$ and $b_{ij}$ denote the $1^{st}$- and $2^{nd}$-order partial derivatives, respectively. Immediately we see that the optimal wage policy satisfies $dU_t = 0$ when $b_2 > -1$. The optimality equation is derived by utilizing $b(\delta, W) = \delta c(k)$ where $k = \delta/W$, hence $b_2 = c'(k), b_1 = c(k) - kc'(k)$, and $\delta b_{11} = -\delta k b_{12} = \delta k^2 b_{22} = k^2 c''(k)$.

**Lemma 12.** *Suppose that $k_t$ evolves according to $dk_t = \beta k_t dt + (\lambda - k_t)\sigma dZ_t - du_t$, and stops at $\tau$ when $k_t$ hits 0, where $u_t$ is a non-decreasing process that reflects $k_t$ at $\overline{k}$. Let $\theta \in \mathbb{R}$, and $g : [0, \overline{k}] \to \mathbb{R}$ is a bounded function. Then the function $F \in \mathbb{C}^2 : [0, \overline{k}] \to \mathbb{R}$ solves the $2^{nd}$-order ODE,*

$$rF(k) = g(k) + \beta k F'(k) + \frac{1}{2}(\lambda - k)^2 \sigma^2 F''(k), \tag{A.1}$$

*with boundary conditions $F(0) = L$ and $F'\left(\overline{k}\right) = -\theta$, if and only if it satisfies*

$$F(k_0) = \mathbb{E}^{k=k_0}\left[\int_0^\tau e^{-rt} g(k_t)\, dt - \theta \int_0^\tau e^{-rt} du_t + e^{-r\tau} L\right].$$

*If $k_t$ evolves according to $dk_t = \beta k_t dt + (\lambda - k_t)\sigma dZ_t - du_t + k^* dN_t$, where $dN_t \equiv \mathbf{1}_{\{k_t=0\}}$ regenerates $k_t$ back to $k^*$, then a function $F \in \mathbb{C}^2 : [0, \overline{k}] \to \mathbb{R}$ solves the $2^{nd}$-order ODE in (A.1) with boundary conditions $F(k^*) - F(0) = l$ and $F'\left(\overline{k}\right) = -\theta$, if and only if it satisfies*

$$F(k_0) = \mathbb{E}^{k=k_0}\left[\int_0^\infty e^{-rt} g(k_t)\, dt - \theta \int_0^\infty e^{-rt} du_t - l \int_0^\infty e^{-rt} dN_t\right].$$

**Proof.** The proof is similar to DS lemma D. The result with jumps is a simple extension. □

### A.1.3. A Lemma for the Homogenous Version of (1.4)

The following lemma is repeatedly used in our later proofs.

**Lemma 13.** *Suppose* $f(\cdot) \in \mathbb{C}^2\left[0, \overline{k}\right]$ *where* $\overline{k} \leq \lambda$, *and it satisfies,*

$$(r-\mu) f(k) = (\gamma - \mu) k f'(k) + \frac{1}{2}(\lambda - k)^2 \sigma^2 f''(k).$$

*We have the following results:*

*1. For* $k_1 \in (0, \lambda)$, *if* $f(k_1) < 0$ *and* $f'(k_1) \geq 0$, *then* $f(k) < 0, f'(k) > 0$ *and* $f''(k) < 0$ *for* $k \in [0, k_1)$.

*2. If* $0 \leq k_1 < k_2 \leq \lambda$, *and* $f(k_1) = f(k_2) = 0$, *then* $f(k) = 0$ *for all* $k \in [0, \lambda]$.

*3. If* $0 \leq k_1 < k_2 \leq \lambda$, *and* $f(k_1) < 0$ *but* $f(k_2) = 0$, *then* $f(k) < 0$, $f'(k) > 0$ *and* $f''(k) < 0$ *for* $k \in [0, k_2]$.

**Proof.** 1) Let us show $f'(k) > 0$ for $k \in [0, k_1)$ first. Note that $f'(k_1 - \epsilon) > 0$ for some small $\epsilon > 0$ (because even if $f'(k_1) = 0$, $f''(k_1) = \frac{2(r-\mu)}{(\lambda-k_1)^2\sigma^2} f(k_1) < 0$). Suppose that $f' < 0$ for some points on $[0, k_1]$; then $x \equiv \sup\{k \in [0, k_1) : f'(k) \leq 0\} < k_1$ is well-defined, and $f'(x) = 0$, $f(x) < 0$ and $f'(x+\epsilon) > 0$ for some small $\epsilon > 0$. In words $x$ is the local minimum points closest (from left) to $k_1$. But then $\frac{1}{2}(\lambda - x)^2 \sigma^2 f''(x) = (r-\mu) f(x) < 0$, contradicting with $f'(x+\epsilon) > 0$. Therefore $f$ is increasing on $[0, k_1)$, which implies $f(k) < 0$ for $k \in [0, k_1]$. Finally, suppose that $f'' \geq 0$ for some $k$, then define $y \equiv \sup\{k \in [0, k_1] : f''(k) \geq 0\}$, and $f''(y) = 0$. If $y = 0$, then $f(0) = 0$, contradiction; if $y > 0$, then $f'(y) = \frac{(r-\mu) f(y)}{(\gamma-\mu) y} < 0$, contradiction.

2) It is sufficient to consider the case $0 < k_1 < k_2 < \lambda$. W.l.o.g, suppose there exists $x \in (k_1, k_2)$ such that $f(x) < 0$, and let $y \equiv \inf\{k \in [x, k_2] : f(k) \geq 0\}$ (which could be $k_2$). According to the intermediate value theorem, there exists $z \in (x, y)$ such that $f(z) < 0$ and

$f'(z) > 0$. Result 1) implies that $f(k_1) < 0$, contradiction. Therefore we have $f(k) = 0$ for $k \in [k_1, k_2]$. Furthermore, on $[0, k_1]$ given the initial condition $f(k_1) = 0$ and $f'(k_1) = 0$ the solution $f = 0$ is unique. Similarly, for $k \in \left[k_2, \lambda - \frac{1}{n}\right]$ we have $f = 0$ for $n = 1, 2, \cdots$. Invoking continuity, we have $f(\lambda) = 0$.

3) Similar arguments in 2) and the result in 1) show that $f(k) < 0$ for all $k \in (k_1, k_2)$. Again, the intermediate value theorem shows that there exists $x \in (0, \lambda)$ such that $f(x) < 0$ and $f'(x) > 0$, delivering our claim by 1). Q.E.D. □

### A.1.4. Proof of Proposition 2 in Section 1.3.3.1

We first show that $\overline{k} \neq \lambda$. Suppose $\overline{k} = \lambda$, so $c(\lambda) = \frac{1}{r-\mu} - \frac{\gamma-\mu}{r-\mu}\lambda$. Taylor expansion gives us $c(\lambda - \varepsilon) = c(\lambda) + \varepsilon + \frac{1}{2}c''(\theta_1)\varepsilon^2$ where $\theta_1 \in (\lambda - \varepsilon, \lambda)$, and (Taylor expansion for $c'(\lambda - \varepsilon)$),

$$(r-\mu)\,c(\lambda-\varepsilon) = 1 + (\gamma-\mu)(\lambda-\varepsilon)\left(-1 - c''(\theta_2)\,\varepsilon\right) + \frac{\varepsilon^2\sigma^2}{2}c''(\lambda-\varepsilon),$$

where $\theta_2 \in (\lambda - \varepsilon, \lambda)$. It implies that,

$$r - \gamma = -\frac{r-\mu}{2}c''(\theta_1)\,\varepsilon - c''(\theta_2)(\gamma-\mu)(\lambda-\varepsilon) + \frac{\varepsilon\sigma^2}{2}c''(\lambda-\varepsilon).$$

When $\varepsilon \to 0$, $c''(\theta_i) \to 0$ for both $\theta_i$'s and $c''(\lambda - \varepsilon) \to 0$ due to $c \in \mathbb{C}^2$, $RHS$ goes to 0, inconsistent with $r - \gamma < 0$. Notice that this argument does not involve the information about $c'''(\lambda)$, which might not exist due to the singularity of $2^{nd}$-order term in (1.4).

Now we show that $c''(k) < 0$ for $\forall k \in \left[0, \overline{k}\right)$. Suppose not. When $k = \overline{k} \neq \lambda$, $\frac{1}{2}\left(\lambda - \overline{k}\right)^2 \sigma^2 c'''\left(\overline{k}\right) = \gamma - r > 0$ implies that $c''\left(\overline{k} - \epsilon\right) < 0$ for some small $\epsilon > 0$. (Note $c'''\left(\overline{k}\right)$ always exists if $\overline{k} \neq \lambda$.) Let $x = \sup\left\{k \in \left[0, \overline{k}\right) : c''(k) \geq 0\right\}$; continuity implies $c''(x) = 0$ and $c''(k) < 0$ for $k \in \left(x, \overline{k}\right)$. We have $c(x) = \frac{1}{r-\mu} + \frac{\gamma-\mu}{r-\mu}xc'(x)$. Because $c(x) < \frac{1}{r-\mu}$, $c'(x) < 0$. Hence

$\frac{1}{2}(\lambda - x)^2 \sigma^2 c'''(x) = (r-\gamma) c'(x) > 0$, which implies that $c''(x+\epsilon) > 0$, contradiction. Therefore $c(k)$ is strictly concave on $\left[0, \overline{k}\right]$. Now suppose $\overline{k} > \lambda$; strict concavity implies that $c(\lambda) < c\left(\overline{k}\right) - \left(\lambda - \overline{k}\right) = \frac{1}{r-\mu} - \frac{\gamma - r}{r-\mu}\overline{k} - \lambda < \frac{1}{r-\mu} - \frac{\gamma-\mu}{r-\mu}\lambda$. But we know that $c(\lambda) \geq \frac{1}{r-\mu} - \frac{\gamma-\mu}{r-\mu}\lambda$, simply because it can be achieved by granting $\alpha^* = (\gamma - \mu)\lambda$ shares of stock and the agent is working forever. Therefore we have $\overline{k} < \lambda$.

Existence follows from the probabilistic representation. Now we show uniqueness. Take $\overline{k} \in [0, \lambda)$; use initial condition $c\left(\overline{k}\right) = \frac{1}{r-\mu} - \frac{\gamma-\mu}{r-\mu}\overline{k}$ and $c'\left(\overline{k}\right) = -1$, $c(\cdot)$ is unique on $\left[0, \overline{k}\right]$, and the solutions $c\left(\cdot; \overline{k}\right)$ is continuous in $\overline{k}$. We want to show that $c\left(0; \overline{k}\right)$ is strictly increasing in $\overline{k}$. Suppose that $c(\cdot; k_1)$ and $c(; k_2)$ solves (1.4) while taking $\overline{k}_1 < \overline{k}_2$ as upper boundaries respectively, and define $f(k) \equiv c\left(k; \overline{k}_2\right) - c\left(k; \overline{k}_1\right)$ on $\left[0, \overline{k}_1\right]$. We have $f\left(\overline{k}_1\right) < 0$ and $f'\left(\overline{k}_1\right) > 0$. According to lemma 13, $f(k) < 0$ for $k \in \left[0, \overline{k}_1\right]$, which implies $f(0) < 0$. Therefore $c\left(0; \overline{k}\right)$ is increasing in $\overline{k}$, and as a result there is a unique $\overline{k}$ s.t. $c\left(0; \overline{k}\right) = L$. Q.E.D.

### A.1.5. Proof of Proposition 3 in Section 1.3.3.2

Suppose $\overline{k} > \lambda$. Given $\overline{k}$, $c'\left(\overline{k}\right) = -1$ and $c''\left(\overline{k}\right) = 0$ the only solution to (1.4) on $\left(\lambda, \overline{k}\right]$ is $c^{fb}(k) = \frac{1}{r-\mu} - k$. It implies that $\overline{k} - \epsilon$ can serve the same role as $\overline{k}$ satisfying (1.5) and (1.6). Similarly, if $\overline{k} < \lambda$, on $\left[0, \overline{k}\right]$ the solution is uniquely determined as $c(k) = c^{fb}(k) = \frac{1}{r-\mu} - k$; then $c(0) = \frac{1}{r-\mu}$, contradicting with (1.7). If $c''(\cdot) \geq 0$ for some point on $[0, \lambda)$, then we can pick the closest one to $\lambda$ (call it $x < \lambda$), with $c''(x) = 0$ and $c'(x) > -1$. But it immediately implies $c(k) > c^{fb}(k)$, contradiction. We conclude that $c''(k) < 0$ for $\forall k \in [0, \lambda)$. Existence and uniqueness follow by the same argument as in proof of Proposition 2. Q.E.D.

### A.1.6. Proof of Theorem 4 in Section 1.3.4

Under any incentive compatible contract, for the auxiliary gain process we have

$$dG_t(\Pi) = \mu_G(t)\,dt + e^{-rt}\delta_t\sigma\left[c(k_t) - k_t c'(k_t) + c'(k_t)\frac{\sigma_t^W}{\sigma}\right]dZ_t,$$

where $\mu_G(t) \le 0$. Let $\varphi_t \equiv e^{-rt}\delta_t\sigma\left[c(k_t) - k_t c'(k_t) + c'(k_t)\frac{\sigma_t^W}{\sigma}\right]$. Recall $c(k) = c(\overline{k}) + k - \overline{k}$ for $k > \overline{k}$, which says $c'(k)$ and $c(k) - c'(k)k$ are bounded. Combining with the condition (1.1) and the related argument in the proof for Proposition 1, we conclude that $\mathbb{E}\left[\int_0^T \varphi_t dZ_t\right] = 0$ for $\forall T > 0$. And, under $\Pi$ the investors' expected payoff is

$$\widetilde{G}(\Pi) \equiv \mathbb{E}\left[\int_0^\tau e^{-rs}\delta_s ds - \int_0^\tau e^{-rs}dU_s + e^{-r\tau}L\delta_\tau\right],$$

where each integral, even if $\tau = \infty$ (where $e^{-r\tau_n}L\delta_{\tau_n} = 0$), is well-defined since they are monotone. Moreover, since $\mathbb{E}\left[\int_0^\tau e^{-rs}\delta_s ds + e^{-r\tau}L\delta_\tau\right] < \frac{\delta_0}{r-\mu} < \infty$, the payoff $\widetilde{G}$ is well-defined. Then, given any $t < \infty$,

$$\begin{aligned}
&\widetilde{G}(\Pi) = \mathbb{E}\left[G_\tau(\Pi)\right] \\
=\;& \mathbb{E}\left[G_{t\wedge\tau}\left(\widetilde{\Pi}\right) + \mathbf{1}_{t\le\tau}\left[\int_t^\tau e^{-rs}\left(\delta_s ds - dU_s\right)\right] + e^{-r\tau}b(\delta_\tau, 0) - e^{-rt}b(\delta_t, W_t)\right] \\
=\;& \mathbb{E}\left[G_{t\wedge\tau}\left(\widetilde{\Pi}\right)\right] + e^{-rt}\mathbb{E}\left\{\left[\int_t^\tau e^{-r(s-t)}\left(\delta_s ds - dU_s\right) + e^{-r(\tau-t)}c(0)\,\delta_\tau - b(\delta_t, W_t)\right]\mathbf{1}_{t\le\tau}\right\} \\
\le\;& G_0 + e^{-rt}\mathbb{E}\left[\int_t^\infty e^{-r(s-t)}\delta_s ds\right].
\end{aligned}$$

The first term of third inequality follows from the negative drift of $dG_t(\Pi)$ and martingale property of $\int_0^{t\wedge\tau}\varphi_s dZ_s$, and the second term is the first-best without any payment and termination (note that $dU$ and $b(\delta, W)$ are positive, and $c(0) < \frac{1}{r-\mu}$). But since $e^{-rt}\mathbb{E}\left[\int_t^\infty e^{-r(s-t)}\delta_s ds\right] = \frac{\delta_0 e^{-(r-\mu)t}}{r-\mu} \to 0$ as $t \to \infty$, we have $\widetilde{G} \le G_0$ for all $\Pi \in \mathbb{IC}$. On the other hand, under the optimal contract $\Pi^*$ the investors' payoff $\widetilde{G}(\Pi^*)$ achieves $G_0$ because the above weak inequality holds in equality when $t \to \infty$. Q.E.D.

### A.1.7. Proofs for Comparative Static Results in Section 1.4.1

We only provide the lemma for the replacement case. The liquidation case is immediate (see DS).

**Lemma 14.** *For $\theta \in \{r, \gamma, \mu, l, \lambda, \sigma^2\}$, denote by $c_\theta(k)$ the scaled value function for that parameter value. We have*

$$\frac{\partial c_\theta(k)}{\partial \theta} = \mathbb{E}^{k_0=k}\left\{\int_0^\infty e^{-(r-\mu)t}\left[\left(-\frac{\partial(r-\mu)}{\partial\theta}c_\theta(k_t) + \frac{\partial(\gamma-\mu)}{\partial\theta}k_t c_\theta'(k_t) + \frac{1}{2}\frac{\partial\left(\sigma^2(\lambda-k_t)^2\right)}{\partial\theta}c_\theta''(k_t)\right)dt - \frac{\partial l}{\partial\theta}dN_t\right]\right\}.$$

**Proof.** The proof is similar to DS Lemma F. Given a policy $\mathbb{P} \equiv (\overline{k}, k^*)$ which simply sends out cash at $\overline{k}$ and replaces a new agent back to $k^*$, the investors' payoff $c_\theta(k; \mathbb{P})$ must solve the ODE

$$(r-\mu)c_\theta(k;\mathbb{P}) = 1 + (\gamma-\mu)kc_\theta'(k;\mathbb{P}) + \frac{1}{2}(\lambda-k)^2\sigma^2 c_\theta''(k;\mathbb{P}), \tag{A.2}$$

with boundary conditions $c_\theta'(\overline{k};\mathbb{P}) = -1$ and $c_\theta(k^*;\mathbb{P}) - c_\theta(0;\mathbb{P}) = l$. Note that both conditions are independent of $\mathbb{P}$. It follows that $\frac{\partial}{\partial\theta}c_\theta'(\overline{k};\mathbb{P}) = 0$ and $\frac{\partial}{\partial\theta}[c_\theta(k^*;\mathbb{P}) - c_\theta(0;\mathbb{P})] = \frac{\partial l}{\partial\theta}$ for any feasible $\mathbb{P}$ (so does the optimal policy). Denote $\mathbb{P}(\theta)$ as the optimal policy under $\theta$; then by definition $c_\theta(k) = c_\theta(k;\mathbb{P}(\theta))$. Differentiate both sides of (A.2) with respect to $\theta$ and evaluate at $\mathbb{P} = \mathbb{P}(\theta)$:

$$\begin{aligned}(r-\mu)\frac{\partial c_\theta(k)}{\partial\theta} &= -\frac{\partial(r-\mu)}{\partial\theta}c_\theta(k_t) + \frac{\partial(\gamma-\mu)}{\partial\theta}k_t c_\theta'(k_t) + \frac{1}{2}\frac{\partial\left(\sigma^2(\lambda-k_t)^2\right)}{\partial\theta}c_\theta''(k_t) \\ &\quad + (\gamma-\mu)k\frac{d}{dk}\left[\frac{\partial c_\theta(k)}{\partial\theta}\right] + \frac{\sigma^2(\lambda-k_t)^2}{2}\frac{d^2}{dk^2}\left[\frac{\partial c_\theta(k)}{\partial\theta}\right],\end{aligned}$$

with boundary conditions $\frac{d}{dk}\left[\frac{\partial c_\theta(k)}{\partial\theta}\right] = \frac{\partial}{\partial\theta}c'_\theta\left(\overline{k}\right) = 0$ and $\frac{\partial c_\theta(k^*)}{\partial\theta} - \frac{\partial c_\theta(0)}{\partial\theta} = \frac{\partial l}{\partial\theta}$ evaluated at the optimal policy $\mathbb{P}(\theta)$. According to lemma 12 we get the stated result. Q.E.D. □

We now show the signs for three terms inside $\{\cdot\}$ without derivatives of $d(k)$'s. First, $\partial c(k)/\partial\mu = d_1(k) - d_2(k)$, and $d_1(k) - d_2(k) = \mathbb{E}^{k_0=k}\left[\int_0^\infty e^{-(r-\mu)t}\left(c(k_t) - k_t c'(k_t)\right)dt\right] > 0$ since $c(k) - kc'(k) > 0$ (its derivative is $-kc''(k) > 0$ and $c(0) > 0$). Second, $\partial\overline{k}/\partial\gamma \propto -\left[\overline{k} + (r-\mu)\,d_2\left(\overline{k}\right)\right]$, while

$$
\begin{aligned}
(r-\mu)\,d_2\left(\overline{k}\right) &= \mathbb{E}^{k_0=k}\left[\int_0^\infty e^{-(r-\mu)t}(r-\mu)\,k_t c'(k_t)\,dt\right] \\
&> \mathbb{E}^{k_0=k}\left[\int_0^\infty e^{-(r-\mu)t}(r-\mu)\left(-\overline{k}\right)dt\right] = -\overline{k}.
\end{aligned}
$$

Third, for $\partial\overline{k}/\partial\mu \propto \overline{k} + c\left(\overline{k}\right) - (r-\mu)\left(d_1\left(\overline{k}\right) - d_2\left(\overline{k}\right)\right)$, notice that $c(k) - kc'(k)$ is increasing in $k$; then applying the same argument we obtain the result $\partial\overline{k}/\partial\mu > 0$.

For those $\partial k^*/\partial\theta$'s listed in the third column, we have to invoke the following lemma.

**Lemma 15.** *Let $\beta \geq r > 0$. Suppose $dk_t = \beta k_t dt + (\lambda - k_t)\,\sigma dZ_t - du_t$ where $du_t$ reflects $k_t$ at $\overline{k}$, and $dN_t$ regenerates the system back to $k^*$ once $k_t$ hits $0$. We have:*

*1) Let $Q(k) = r\mathbb{E}^{k_0=k}\left[\int_0^\infty e^{-rt}f(k_t)\,dt\right]$, where $\overline{k} \leq \lambda$. Suppose a smooth function $f$ satisfies $f(k) < f(k^*)$ for $\forall k \in [0, k^*)$ and $f'(k) > 0$ for $\forall k \in \left[k^*, \overline{k}\right]$; then we have $Q'(k) > 0$ for $\forall k \in \left[k^*, \overline{k}\right)$.*

*2) Let $Q(k) = r\mathbb{E}^{k_0=k}\left[\int_0^\infty e^{-rt}dN_t\right]$, then $Q(k) > 0$ is decreasing and convex. $Q(\lambda) = 0$ when $\overline{k} = \lambda$. Similar results hold for $Q(k) = r\mathbb{E}^{k_0=k}\left[e^{-r\tau}\right]$, if $dk_t = \beta k_t dt + (\lambda - k_t)\,\sigma dZ_t - du_t$ and stops at $0$ at time $\tau$.*

*3) For the normalized future termination cost $Q(k) = r\mathbb{E}^{k_0=k}\left[\int_0^\infty e^{-rt}dN_t\right]$, index the solution $Q\left(\cdot;\overline{k}\right)$ by policy $\overline{k}$. Using subscript indicates the partial derivative, we have $Q_2\left(\overline{k};\overline{k}\right) \equiv$*

$\frac{\partial}{\partial \overline{k}} Q\left(\overline{k};\overline{k}\right) < 0$ *for* $\overline{k} < \lambda$ *and* $Q_2(\lambda;\lambda) = 0$. *This implies that the marginal cost of reducing the cash-payment barrier from* $\overline{k} = \lambda$ *is zero, and positive when* $\overline{k} < \lambda$.

**Proof.** For 1) we use two facts. First, $Q(k)$ is the time-discounting average of $f(k_t)$ under the measure induced by $k_0 = k$. Second, for $k_1 > k_2$ where $k_1 \geq k^*$ it must be true that $f(k_1) > f(k_2)$. According to lemma 12, $Q$ must solve the following $2^{nd}$-order ODE

$$rQ(k) = rf(k) + \beta k Q'(k) + \frac{(\lambda - k)^2 \sigma^2}{2} Q''(k), \tag{A.3}$$

with boundary conditions $Q(0) = Q(k^*)$ and $Q'\left(\overline{k}\right) = 0$. We first show $Q'\left(\overline{k} - \epsilon\right) > 0$ for small $\epsilon > 0$. To see this, if $\overline{k} < \lambda$, then since $rQ\left(\overline{k}\right) = rf\left(\overline{k}\right) + \frac{\left(\lambda - \overline{k}\right)^2 \sigma^2}{2} Q''\left(\overline{k}\right)$ and $f\left(\overline{k}\right)$ is the maximum, we have $Q''\left(\overline{k}\right) < 0$. If $\overline{k} = \lambda$, $Q(\lambda) = f(\lambda)$ (as $\lambda$ is the absorbing state) reaches the unique maximum, therefore $Q'(\lambda - \epsilon) < 0$. Suppose $Q'(x) < 0$ for some $x \in \left[k^*, \overline{k}\right)$, then there must be a point $y > x$ such that $Q'(y) = 0$, $Q''(y) > 0$, and $Q$ is decreasing on $(x, y)$. ($y$ is the locally minimum point closest to $x$). We know then $rQ(y) = rf(y) + \frac{(\lambda - k)^2 \sigma^2}{2} Q''(y)$, so $Q(y) > f(y)$. Now focus on $[0, y]$ which contains $k^*$. We claim that $Q(\cdot)$ must be convex on $[0, y]$. To see this, suppose we can find a reflecting point (closest to $y$) $z < y$ satisfying $Q''(z) = 0$, $Q'(z) < 0$, and it must be the case that $Q(z) > Q(y)$. However, we have $rQ(z) = rf(z) + \beta k Q'(z) < rf(z)$. Combining with the result $Q(y) > f(y)$, we have $f(z) > f(y)$ with $z < y$ and $y \geq k^*$, contradiction. But if $Q(\cdot)$ is convex on $[0, y]$, then $Q' < 0$ on $(x, y)$ implies that $Q' < 0$ on $(0, y)$, contradicting to the boundary condition $Q(0) = Q(k^*)$. Hence the original counter-factual assumption of the existence of $x$ s.t. $Q'(x) < 0$ does not hold, and the conclusion follows.

For 2), it is the extreme case of 1) (with $k^* = 0$ and $f$ as a Dirac delta function with the support $\{0\}$), and the results directly follow from lemma 13. When $\overline{k} = \lambda$, $Q(\lambda) = 0$ as $k = \lambda$ is absorbing state and the probability to return to $[0, \lambda)$ is zero.

For 3), let $P\left(k;\overline{k}\right) \equiv Q_2\left(k;\overline{k}\right)$. Note that it is different from differentiating w.r.t parameter as we do in lemma 14: we are now differentiating w.r.t policy. We still have $P\left(k;\overline{k}\right)$ solves $rP\left(k;\overline{k}\right) = \beta k P_1\left(k;\overline{k}\right) + \frac{(\lambda-k)^2\sigma^2}{2}P_{11}\left(k;\overline{k}\right)$, with condition $P\left(k^*;\overline{k}\right) - P\left(0;\overline{k}\right) = 0$, where $P_i$ and $P_{ij}$ denote partial derivatives (similarly for $Q_i$ and $Q_{ij}$). To use lemma 12, we have to pin down $P_1\left(\overline{k};\overline{k}\right) = Q_{12}\left(\overline{k};\overline{k}\right)$. In fact, since $Q_1\left(\overline{k};\overline{k}\right) = 0$ for all $\overline{k}$, $0 = \frac{d}{d\overline{k}}Q_1\left(\overline{k};\overline{k}\right) = Q_{11}\left(\overline{k};\overline{k}\right) + Q_{12}\left(\overline{k};\overline{k}\right)$. Hence invoking the result in lemma 12, we find that $P\left(\overline{k};\overline{k}\right) = -Q_{11}\left(\overline{k};\overline{k}\right)\mathbb{E}^{k_0=\overline{k}}\left\{\int_0^\infty e^{-rt}du_t\right\} \leq 0$ since $Q$ is convex in $k$ (note that $\mathbb{E}^{k_0=\overline{k}}\left\{\int_0^\infty e^{-rt}du_t\right\} > 0$). When $\overline{k} < \lambda$, because using $Q\left(\overline{k},\overline{k}\right) > 0$ and $Q_1\left(\overline{k},\overline{k}\right) = 0$, we find that (A.3) yields $Q_{11}\left(\overline{k};\overline{k}\right) > 0$.

When $\overline{k} = \lambda$, (A.3) (with $f = 0$) is an ODE with an essential (irregular) singularity at $\lambda$. Our goal is to show that when $\overline{k} = \lambda$, $Q''(\lambda) = 0$. We first show that $Q''(\lambda-) = 0$, as the main concern is the explosion of $Q''$ near singularity. First, notice that $Q''(k)$ must be bounded in the vicinity of $\lambda$. Otherwise, since $Q'' \geq 0$, we can always find a point $\lambda-\varepsilon$ such that $Q'''(\lambda-\varepsilon) > 0$, $Q''(\lambda-\varepsilon) > B$ where $\varepsilon$ small enough and $B$ large enough. Then differentiating (A.3) at $\lambda-\varepsilon$, we observe that the term involving $Q''$ is greater than $\left(\beta(\lambda-\varepsilon) - \varepsilon\sigma^2\right)B > 0$ and $(r-\beta)Q'$ is bounded (note the fact that $Q(0) - Q(k^*) = 1$ and $Q$ is convex implies $(r-\beta)Q'(\lambda-\varepsilon) < (r-\beta)Q'(k^*) < \frac{\beta-r}{k^*}$), and a contradiction follows. Now the mean-value-theorem argument similar to the proof of Proposition 2) shows that $Q''(\lambda-) = 0$. Finally, since it is easy to show $Q'(\lambda-) = 0$, $Q''(\lambda) = \lim\limits_{h\to 0}\frac{Q'(\lambda)-Q'(\lambda-h)}{h} = \lim\limits_{h\to 0}\frac{-Q'(\lambda-h)}{h} = \lim\limits_{h\to 0} -\frac{rQ(\lambda-h)-\frac{h^2}{2}Q''(\lambda-h)}{h\beta(\lambda-h)} = -\frac{rQ'(\lambda)}{\beta\lambda} = 0$ where we use (A.3) with $f = 0$, and the fact that $Q(\lambda) = Q'(\lambda) = 0$. Q.E.D. □

Now we can apply this lemma to show our claims. Note that $kc'(k)$ is positive for $k \in [0,k^*]$ and reaches 0 when $k = k^*$; moreover, it is decreasing for $k \in \left[k^*,\overline{k}\right]$. Hence $\partial k^*/\partial\gamma \propto d_2'(k^*) < 0$. For $\partial k^*/\partial\mu \propto d_2'(k^*) - d_1'(k^*) > 0$, it is sufficient to see that $c(k) - kc'(k)$ in fact is increasing on $\left[0,\overline{k}\right]$. Finally, $\partial k^*/\partial l \propto -d_{rp}'(k^*) > 0$ follows immediately from 2) of the above lemma.

### A.1.8. Appendix for Section 1.4.2.2

Based on the policy proposed in the main text, we construct the scaled value function with shirking $c^S(\cdot)$ as follows (see Figure 1.4). Starting from $\left(\frac{\phi}{r}, \frac{1}{r}\right)$ above $c(\cdot)$, we extend $c^S(k)$ to the right according to, $(r-\mu)\,c^S(k) = 1 + (\gamma-\mu)\,kc^{S\prime}(k) + \frac{1}{2}(\lambda-k)^2\sigma^2 c^{S\prime\prime}(k)$; to do this we just pick the appropriate value for $c_+^{S\prime}\left(\frac{\phi}{r}\right)$ so that $c^S(\lambda)$ lands at $\frac{1}{r-\mu}-\lambda$. Comparing to $c(\cdot)$ one can show that $c_+^{S\prime}\left(\frac{\phi}{r}\right) < c'\left(\frac{\phi}{r}\right)$. Similarly we extend $c^S(k)$ to the left of $\frac{\phi}{r}$. The next lemma states that we have $c_-^{S\prime}\left(\frac{\phi}{r}\right) > c'\left(\frac{\phi}{r}\right)$ to meet the termination boundary condition.

**Lemma 16.** $c_-^{S\prime}\left(\frac{\phi}{r}\right) > c'\left(\frac{\phi}{r}\right) > c_+^{S\prime}\left(\frac{\phi}{r}\right)$.

**Proof.** We consider the replacement case only (liquidation case is easier). Suppose $c_-^{S\prime}\left(\frac{\phi}{r}\right) \leq c'\left(\frac{\phi}{r}\right)$. Since $\frac{1}{r} = c^S\left(\frac{\phi}{r}\right) > c\left(\frac{\phi}{r}\right)$, according to lemma 13 we know that $c^S(0) > c(0)$ for all $k \in \left[0, \frac{\phi}{r}\right]$. Moreover, we have $c^S\left(k^{*S}\right) - c^S(0) < c(k^*) - c(0)$. To see this, suppose that $k^{*S} \leq \frac{\phi}{r}$; then $c^S\left(k^{*S}\right) - c^S(0) < c\left(k^{*S}\right) - c(0) < c(k^*) - c(0)$. If $k^{*S} > \frac{\phi}{r}$ which implies that $k^* > \frac{\phi}{r}$ as well (since $c'\left(\frac{\phi}{r}\right) > c_+^{S\prime}\left(\frac{\phi}{r}\right) > 0$), we have $c^S\left(k^{*S}\right) - c^S\left(\frac{\phi}{r}\right) < c\left(k^{*S}\right) - c\left(\frac{\phi}{r}\right)$ (using lemma 13 part 3) and $c^S\left(\frac{\phi}{r}\right) - c^S(0) < c\left(\frac{\phi}{r}\right) - c(0)$, hence $c^S\left(k^{*S}\right) - c^S(0) < c\left(k^{*S}\right) - c(0) \leq c(k^*) - c(0)$. In conclusion, when $c_-^{S\prime}\left(\frac{\phi}{r}\right) \leq c'\left(\frac{\phi}{r}\right)$ the resulting function $c^S(\cdot)$ fails to meet the termination condition. Since a large $c_-^{S\prime}\left(\frac{\phi}{r}\right)$ could deliver an arbitrarily small $c^S(0)$, we conclude that $c_-^{S\prime}\left(\frac{\phi}{r}\right) > c'\left(\frac{\phi}{r}\right) > c_+^{S\prime}\left(\frac{\phi}{r}\right)$. Also note that this proof shows that this result holds even if liquidation is optimal for $c$ while replacement is optimal for $c^S$, because $c^S\left(k^{*S}\right) - c^S(0) < c(k^*) - c(0) \leq l$ given $c_-^{S\prime}\left(\frac{\phi}{r}\right) \leq c'\left(\frac{\phi}{r}\right)$. Q.E.D. $\square$

Now we prove the Proposition 5. Construct the auxiliary gain process $G$ as in (1.12) where $b(\delta, W) = \delta c^S(k)$. Note that the scaled value function $c^S(\cdot)$ with shirking has a kink at $\frac{\phi}{r}$, but it is still strictly concave over the whole domain $[0, \lambda]$. It follows that $\varphi^S(k) \equiv c^S(k) - \left(k - \frac{\phi}{r}\right) c^{S\prime}(k)$ for $k \in [0, \lambda]$ is still quasiconvex in $k$, and the minimum takes place at $\varphi^S\left(\frac{\phi}{r}\right) =$

$\frac{1}{r}$. For $k \in \left[0, \frac{\phi}{r}\right) \cup \left(\frac{\phi}{r}, \lambda\right]$ similar argument as in Theorem 4 shows that the above policy is optimal if we are trying to implement $a = \mu$. Furthermore, because we now have $\varphi^S(k) > \varphi^S\left(\frac{\phi}{r}\right) = \frac{1}{r}$ always, it is never optimal to induce shirking before $k$ touches $\frac{\phi}{r}$.

When $k = \frac{\phi}{r}$, $c^S(\cdot)$ is kinked at $\frac{\phi}{r}$. We want to show that it is optimal to set $\sigma^W\left(\frac{\phi}{r}\right) = \frac{\phi\sigma}{r}$, and hence $k_t$ stays at the constant $\frac{\phi}{r}$ (no diffusion). To show this, extend $c^S\left(\frac{\phi}{r}\right)$ (from the right) to the left $\frac{\phi}{r} - \epsilon$ according to (1.4), and denote the curve as $\widetilde{c}^S(k)$; it is strictly larger than $c^S(k)$ for $k < \frac{\phi}{r}$. Suppose that we want to implement $\mu$ instead of 0 at $k = \frac{\phi}{r}$, and $k_t$ has a diffusion $\left(\lambda - \frac{\phi}{r}\right)\sigma > 0$. If our scaled value function is $\widetilde{c}^S(\cdot)$, then the drift for the gain process $G$ is 0. But since our actual scaled value function $c^S(\cdot)$ has a concave kink at $\frac{\phi}{r}$, the drift under $c^S(\cdot)$ is negative (more formally, in the generalized Ito's Lemma there is an additional negative local time term which aims to correct for the $2^{nd}$-order impact on the kink—see Karatzas and Shreve (1991), page 215). Hence, inducing working makes $G$ a supermartingale at $k = \frac{\phi}{r}$, while shirking delivers a constant payoff $\frac{1}{r}$. Therefore it is optimal to implement shirking at $\frac{\phi}{r}$. Q.E.D.

## A.2. Appendix for Chapter 2

### A.2.1. Appendix for Section 2.3.2

In this Appendix, we verify that the optimal contract derived in Section 2.3.2 is indeed optimal. First, we show that under certain parameterization conditions specified in Lemma 17 (see below), the agent's policy stated in Section 2.2.3.2 is optimal. Once writing down the agent's gain process

$$G_t^A = \int_0^t \frac{e^{-\gamma(c_s - g(a_s)) - rs}}{\gamma} ds + e^{-rt} W_t,$$

there are three steps in the standard verification procedure in dynamic programming. First, it is easy to show that the agent's instantaneous control ($c$ and $a$) are well-defined concave problem, therefore FOC's are sufficient. It ensures that under the policy specified in Section

2.2.3.2, the agent's gain $dG_t^A$ is driftless. Second, we need to show that the accumulated gain process $\int_0^T e^{-rt} dW_t$ is indeed a martingale, which requires $W_t$ to be a martingale. Finally the third step is to ensure the transversality condition $\mathbb{E}\left[\lim_{T\to\infty} e^{-rT} W_T\right] = 0$.

Under the optimal contract, $W$ evolves as

$$\frac{dW}{W} = -\gamma r \theta a^* \sigma \sqrt{\delta_t} dZ_t.$$

according to (2.19), where $a^* = \frac{B}{\theta(1+\theta\gamma r\sigma^2)}$, and $B$ is defined in (2.18). With $W_0 < 0$, the solution to the above SDE is

$$W_t = W_0 \exp\left(-\int_0^t \frac{1}{2}\gamma^2 r^2 \theta^2 a^{*2} \sigma^2 \delta_s ds - \int_0^t \gamma r \theta a^* \sigma \sqrt{\delta_t} dZ_s\right) < 0.$$

Interestingly, $-W$ is an exponential martingale, a well-studied object in the Asset Pricing literature. For $-W$ to be a martingale, a well-known sufficient condition is so-called Novikov condition. The following Lemma formalizes this idea, and gives a necessary and sufficient condition for the Novikov condition.

**Lemma 17.** *Suppose $W$ follows (2.19). Then $W$ is a martingale if $(\kappa - a^*)^2 \geq 2M^2\sigma^2$, where $M^2 = \frac{1}{2}\gamma^2 r^2 \theta^2 a^{*2} \sigma^2$.*

**Proof.** Denote $k = \kappa - a^*$; then in equilibrium

$$d\delta_t = (\Delta - k\delta_t)\, dt + \sigma\sqrt{\delta_t} dZ_t,$$

The standard Novikov condition on $W$ requires that

$$\mathbb{E}\left[\exp \int_0^T M^2 \delta_t dt\right] < \infty$$

for all $T > 0$, where $M^2 = \frac{1}{2}\gamma^2 r^2 \theta^2 a^{*2} \sigma^2$. Borrowing the techniques from fixed-wealth literature, we guess that

$$\mathbb{E}_t \left[ \exp \int_t^T M^2 \delta_s ds \right] = \exp\left(\rho\left(T-t\right) + \eta\left(T-t\right)\delta_t\right).$$

It is easy to verify that

$$\begin{aligned} \eta'(t) &= \frac{\sigma^2}{2}\eta^2(t) - k\eta(t) + M^2 \\ \rho'(t) &= \eta(t)\Delta \end{aligned}$$

with initial value $\rho(0) = \eta(0) = 0$. When

$$k^2 > 2M^2\sigma^2$$

Denote $\lambda = \sqrt{k^2 - 2M^2\sigma^2} < k$, we have the solution

$$\eta(t) = \frac{2M^2\left(e^{\lambda t} - 1\right)}{(\lambda + k)e^{\lambda t} - (k - \lambda)}$$

and clearly both $\rho(T) = \int_0^T \eta(t)\Delta dt$ and $\eta(T)$ are bounded. When $k^2 = 2M^2\sigma^2$, $\eta(t) = \frac{kt}{\sigma^2\left(t + \frac{2}{k}\right)}$, and the same result holds. However, when $k^2 < 2M^2\sigma^2$, $\beta(t)$ blows up for some $T' < \infty$, so Novikov condition fails. In fact, setting $\lambda = \sqrt{2M^2\sigma^2 - k^2}$, we find the exact solution is

$$\eta(t) = \frac{\left(k^2 + \lambda^2\right)\sin\frac{\lambda t}{2}}{\sigma^2\left(k\sin\frac{\lambda t}{2} + \lambda\cos\frac{\lambda t}{2}\right)},$$

which blows up when $\tan\frac{\lambda t}{2} = -\frac{\lambda}{k}$. Notice that it is still possible to have $\{-W_t : 0 \leq t < \infty\}$ to be martingale as Novikov is only a sufficient condition. $\square$

Once we show that $W$ is a martingale,

$$\begin{aligned}\mathbb{E}\int_0^T e^{-rt}dW_t &= \mathbb{E}\left[e^{-rT}W_T - W_0 + \int_0^T rW_te^{-rt}dt\right] \\ &= W_0\left[e^{-rT} - 1 + \int_0^T re^{-rt}dt\right] = 0,\end{aligned}$$

so the second step holds.[1] And the third step transversality follows trivially, as $\mathbb{E}W_T = W_0$.

Now we verify the optimality of investors' policy. Define the investors' gain process as

$$G_t^{\mathbb{I}} = \int_0^t e^{-rs}\left(\delta_s - c_s\right)ds + e^{-rt}J\left(\delta_t, W_t\right)$$

where $J\left(\delta_t, W_t\right) = \frac{B\Delta}{r} + B\delta_t + \frac{\ln(-\gamma r W_t)}{\gamma r}$. In the spirit of HJB equation, under any employment contract we have

$$dG_t^{\mathbb{I}} = a_{G^{\mathbb{I}}}dt + e^{-rt}\left(B - \beta_t\right)\sigma\sqrt{\delta}dZ_t.$$

Due to construction, $a_{G^{\mathbb{I}}}$ is nonpositive, and zero under the optimal contract. we impose a usual square-integrable condition on feasible employment contract through the volatility $\beta_t$:

$$\mathbb{E}\left[\int_0^T e^{-rt}\beta_t^2\sigma^2\delta_t dt\right] < \infty \text{ for } \forall T \tag{A.4}$$

Notice that in the optimal contract, $\beta_t$ is the constant $\theta a^*$; therefore the optimal contract satisfies condition (A.4), and $\mathbb{E}\int_0^T e^{-rt}dG_t^{\mathbb{I}} = 0$ for all $T$.

The last step is to check the transversality condition $\mathbb{E}\left[\lim_{T\to\infty} e^{-rT}J\left(\delta_T, W_T\right)\right] = 0$. Under the optimal contract, since $\delta$ is stationary, we only have to check the term associated with $W_T$. But

$$\frac{\ln\left(-W_T\right)}{\gamma r} = Cons. - \int_0^T \frac{1}{2}\gamma^2 r^2\theta^2 a^{*2}\sigma^2\delta_t dt + \int_0^T \gamma r\theta a^*\sigma\sqrt{\delta_t}dZ_t,$$

[1] Note that if $-W_t$ is a supermartingale (as $-W_t$ is positive local martingale), the similar argument shows that $\mathbb{E}\int_0^T e^{-rt}dW_t > 0$. This implies that the assigned continuation payoff $W_0$ is lower than the agent's actual payoff, violating the promise-keeping condition.

and again the second diffusion term is a martingale under the conditions stated in Lemma. Because $\mathbb{E}\left[\int_0^T \delta_t dt\right]$ is in the order of $T$ when $T \to \infty$ as the long-run mean of $\delta_t$ is $\frac{\Delta}{\kappa - a^*}$, transversality condition holds for the optimal contract. For general policies, notice that $J\left(\delta_T, W_T\right) \leq \frac{B^{FB}\Delta}{r} + B^{FB}\delta_T + \frac{\ln(-\gamma r W_T)}{\gamma r}$ which is the first-best result. Because $\delta$ is mean-reverting, the sufficient condition is $\mathbb{E}\left[\lim_{T\to\infty} e^{-rT}\frac{\ln(-\gamma r W_T)}{\gamma r}\right] \leq 0$, which is the condition that we impose in addition to (A.4) for any feasible contract $\Pi$.

### A.2.2. Appendix for Section 2.4.2

Similar to the previous case, we can solve for the firm value $f\left(\delta\right)$ without debt. Because the risk-compensation term $\frac{1}{2}\gamma r\theta^2 a^2\sigma^2\delta^2$ is in the order of $\delta^2$, when $\delta \to 0$, the agent becomes as if risk-neutral. As a result, the optimal effort level tends to be the first-best, or binding at the upper bound $\overline{a}$. In all parameterization we considered here, the optimal effort binds at $\overline{a}$ when $\delta$ is sufficiently small. On the other hand, when $\delta$ is sufficiently large, the exploding agency cost mandates diminishing effort (to zero), and $f'$ approaches to $\frac{1}{r-\phi}$ as the Gordon growth model. Using (2.21) and (2.22) when $\delta \to \infty$,

$$f\left(\delta\right) \simeq \overline{f}\left(\delta\right) \equiv \frac{1}{r-\phi}\delta + \frac{1}{2\left(r-\phi\right)^2\theta^2\gamma r\sigma^2}. \tag{A.5}$$

This pins down the upper boundary condition for $f$.

There is one technical issue under this geometric Brownian motion setup. To ensure that the agent's problem is well-behaved, one requires that the agent's continuation value defined in (2.10) is indeed a martingale. However, if $\delta$ follows an unbounded geometric Brownian motion, the Novikov condition easily fails and the promise keeping condition might be violated. To circumvent this technical issue, we simply assume that when the firm's $\delta$ reaches a sufficiently large $\overline{\delta}$, equity holders simply sell the firm at a price $\overline{f}\left(\overline{\delta}\right)$, and the employment contract terminates once this upper threshold is reached. This simplifying assumption corresponds to the exit option of M&A in reality, and has no impact on the static capital structure decision at date 0.

We then show that under the boundary conditions specified later, $f'$ is always positive in 2.21, therefore $\mu^*$ never binds at zero in this problem. To see this, clearly $f'(0) > 0$ (even with zero effort the value is positive), and when $\delta \to \infty$, $f = \overline{f} = \frac{1}{r-\phi}\delta + \frac{1}{2(r-\phi)^2\theta^2\gamma r\sigma^2}$ which is increasing in $\delta$. Now suppose that there exists $\delta_1 > 0$ such that $f'(\delta_1) = 0$ and $f''(\delta_1) < 0$, therefore $rf(\delta_1) = \delta_1 + \frac{1}{2}f''(\delta_1)\sigma^2\delta_1^2 < \delta_1$. But there must exists $\delta_2 > \delta_1$ such that $f$ is deceasing in $[\delta_1, \delta_2]$, and $f$ becomes increasing again after $\delta_2$. Since now $f'(\delta_2) = 0$ and $f''(\delta_2) < 0$, $rf(\delta_2) > \delta_2 > \delta_1$, contradiction with $f(\delta_1) < f(\delta_2)$.

## A.3. Appendix for Chapter 3

Throughout the Appendix I denote $\Xi \equiv [\mu_1, \infty) \times [\mu_2, \infty)$ as the agent's type space, and denote $\partial_i\Xi \equiv \left\{\boldsymbol{\mu} \in \Xi : \mu_i = \underline{\mu_i}\right\}$ as the set of $i^{th}$ boundary types. Similar notations are used for the $n$-asset case. Also, $V(\boldsymbol{\mu}, \boldsymbol{\alpha}, \mathbf{p}(\boldsymbol{\alpha}))$ is often simply written as $V(\boldsymbol{\alpha})$.

### A.3.1. Appendix for Section 3.3.2

Recall that $\mathcal{E}$ is the set of equilibrium selling strategies. I am interested in the behavior of $\mathbf{p}$ on $int\mathcal{E}$, which is the interior part of $\mathcal{E}$. Fix an agent type $\boldsymbol{\mu}$ first. I denote $V(\boldsymbol{\mu}, \boldsymbol{\alpha}, \mathbf{p}(\boldsymbol{\alpha}))$ as $V(\boldsymbol{\alpha}; \boldsymbol{\mu})$ without the risk of confusion. The gradient of (3.2) evaluated at $\boldsymbol{\alpha}$ is,

$$V_i(\boldsymbol{\alpha}; \boldsymbol{\mu}) \equiv \frac{\partial V(\boldsymbol{\alpha}; \boldsymbol{\mu})}{\partial \alpha_i} = p^{(i)} - \mu_i + \sum_{j=1}^{2} \alpha_j p_i^{(j)} + r \sum_{j=1}^{2} (1 - \alpha_j)\sigma_{ij},\ i = 1, 2.$$

(FOC) essentially evaluates $V_i(\boldsymbol{\alpha}; \boldsymbol{\mu})$ at the agent's optimal strategy $\boldsymbol{\alpha}^*$, which should be 0 due to its optimality. Now consider $V_i(\widehat{\boldsymbol{\alpha}}; \boldsymbol{\mu})$ for $\forall \widehat{\boldsymbol{\alpha}} \in int\mathcal{E}$. Since there exists some $\widehat{\boldsymbol{\mu}} \neq \boldsymbol{\mu}$ who takes $\widehat{\boldsymbol{\alpha}}$, the agent $\widehat{\boldsymbol{\mu}}$'s FOC requires that $\sum_{j=1}^{2} \widehat{\alpha}_j p_i^{(j)}(\widehat{\boldsymbol{\alpha}}) + r \sum_{j=1}^{2} (1 - \widehat{\alpha}_j)\sigma_{ij} = 0$. Therefore I have

$V_i(\widehat{\boldsymbol{\alpha}}) = p^{(i)}(\widehat{\boldsymbol{\alpha}}) - \mu_i$ for all $\widehat{\boldsymbol{\alpha}} \in int\mathcal{E}$. Differentiating this equation once more yields that

$$H(\boldsymbol{\alpha}) \equiv \frac{\partial^2 V}{\partial \boldsymbol{\alpha} \partial \boldsymbol{\alpha}'} = \frac{d\mathbf{p}(\boldsymbol{\alpha})}{d\boldsymbol{\alpha}'};$$

or, the Hessian matrix of the agent's value function $V$ equals the Jacobian matrix of the equilibrium pricing system $\mathbf{p}$, a useful result to prove the concavity of $V$ later on. Finally, the symmetry of Hessian matrix implies the symmetry result in the main text. In fact, this symmetry property holds for the $n$-asset case, i.e., $p_i^{(j)} = p_j^{(i)}$ for all $i \neq j$.

### A.3.2. Proof of Proposition 7

It is easy to check $\mathbf{p} \in \mathbf{C}^1$, i.e., continuously differentiable despite different functional forms on $\mathcal{A}_i$'s. I first show the strict concavity of $V$. Since $\frac{\partial^2 V(\boldsymbol{\alpha})}{\partial \boldsymbol{\alpha} \partial \boldsymbol{\alpha}'} = \frac{d\mathbf{p}(\boldsymbol{\alpha})}{d\boldsymbol{\alpha}'}$, it is easy to check that on $\mathcal{A}_1$,

$$H = r \begin{bmatrix} \sigma_{11}\left(1 - \frac{1}{\alpha_1}\right) - \sigma_{12}\frac{\alpha_1 - \alpha_2}{\alpha_1^2} & \sigma_{12}\left(1 - \frac{1}{\alpha_1}\right) \\ \sigma_{12}\left(1 - \frac{1}{\alpha_1}\right) & \sigma_{22}\left(1 - \frac{1}{\alpha_2}\right) \end{bmatrix},$$

which is negative definite (except at the point $(1,1)$; but it has no bite on the strict concavity of $V$). Similar results hold for $\mathcal{A}_2$. Therefore $V$ is strictly concave on $\mathcal{A}_i$'s. Because $\mathbf{p} \in \mathbf{C}^1$, and $V \in \mathbf{C}^1$, which implies that on any line crossing the diagonal, $V$ has strictly decreasing derivatives. Therefore $V$ is strictly concave on the entire domain $\mathcal{A}$. Furthermore, it implies that each agent $\boldsymbol{\mu} = \mathbf{p}(\boldsymbol{\alpha})$ has a unique optimal selling strategy $\boldsymbol{\alpha}$, therefore if $\boldsymbol{\mu} \neq \boldsymbol{\mu}'$ then their equilibrium strategies have to be different (otherwise market consistency is violated). Hence $\mathbf{p}$ is an equilibrium pricing system for a separating equilibrium.

To verify the intuitive criterion for this equilibrium, I use the definition in "*Game Theory*" by Fudernberg and Tirole (1991), page 448. For simplicity, I imagine a Bertrand-type competition between $m$ identical risk neutral investors ($m \geq 2$), and only consider investors' equilibrium response given any possible belief system $\boldsymbol{\psi}$ as in page 469 of Mas-Colell et al. (1995). Given any

$0 < \boldsymbol{\alpha} \in \mathcal{A}$, the best response from any investor is $BR(\Xi, \boldsymbol{\alpha}) = \Xi$, simply because $\mathbb{E}[\boldsymbol{\mu} \,|\boldsymbol{\psi}(\boldsymbol{\alpha})]$ could be any element in $\Xi$ if all reasonable beliefs $\boldsymbol{\psi}$ are allowed. Since $\mathbf{p} \in BR(\Xi, \boldsymbol{\alpha}) \in \Xi$ could approach $+\infty$ in any entry, $J(\boldsymbol{\alpha}) \equiv \left\{ \boldsymbol{\mu} : V^*(\boldsymbol{\mu}) > \sup\limits_{\mathbf{p} \in BR(\Xi, \boldsymbol{\alpha})} V(\boldsymbol{\mu}, \boldsymbol{\alpha}, \mathbf{p}) \right\} = \varnothing$ for any $\boldsymbol{\alpha}$. As a result, $BR(\Xi \backslash J(\boldsymbol{\alpha}), \boldsymbol{\alpha}) = \Xi$; then for any $\boldsymbol{\mu}'$ and her equilibrium strategy $\boldsymbol{\alpha}'$, $V^*(\boldsymbol{\mu}') = V(\boldsymbol{\mu}', \boldsymbol{\alpha}', \boldsymbol{\mu}') \geq \min\limits_{\mathbf{p} \in \Xi} V(\boldsymbol{\mu}', \boldsymbol{\alpha}', \mathbf{p})$. Finally, Pareto-efficiency follows from the fact that now the one-dimensional types in $\partial_i \Xi$ behave as if their $i^{th}$ asset is observable. Applying the same discretization method in the Corollary 1 of Grinblatt and Hwang (1989), it is not difficult to show that under this uni-dimensional type space and multi-dimensional signaling space setup, the standard LP result—which is preserved in this equilibrium—is the Pareto dominant separating schedule. In other words, this equilibrium is Pareto-efficient for $\partial_i \Xi$ relative to all separating equilibria. Because of smoothness, once the boundary conditions are determined the transport PDE leads to the pricing system obtained in the text. Q.E.D.

### A.3.3. Proof of Proposition 8

Consider $\mathcal{A}_1^r$; it is not difficult to verify that the Hessian matrix is

$$H = r \begin{bmatrix} \sigma_{11}\left(1 - \frac{1+\theta_1}{\alpha_1 + \theta_1 \alpha_2}\right) & \sigma_{12}\left(1 - \frac{1+\theta_1}{\alpha_1 + \theta_1 \alpha_2}\right) \\ \sigma_{12}\left(1 - \frac{1+\theta_1}{\alpha_1 + \theta_1 \alpha_2}\right) & \sigma_{22}\left(1 - \frac{1-\rho^2}{\alpha_2} - \frac{\rho^2(1+\theta_1)}{\alpha_1 + \theta_1 \alpha_2}\right) \end{bmatrix},$$

and note that $1 - \frac{1-\rho^2}{\alpha_2} - \rho^2 \frac{1+\theta_1}{\alpha_1 + \theta_1 \alpha_2} < 1 - \frac{1+\theta_1}{\alpha_1 + \theta_1 \alpha_2} < 0$ on $int\mathcal{A}_1^r$. This shows that $H$ is negative definite on $int\mathcal{A}_1^r$, and similar results hold for $int\mathcal{A}_2^r$. To show that $V(\boldsymbol{\alpha})$ is strictly concave on $\mathcal{E}^r$, one could use the argument in Proposition 7; hence I have a separating equilibrium. Similar arguments in Proposition 7 show that the equilibrium satisfies the intuitive criterion, and it is Pareto-efficient relative to other (smooth) separating equilibria. Q.E.D.

### A.3.4. Appendix for Section 3.4.2.3

Consider any characteristic line $L \in \mathcal{A}_1^r$. Along $L$ toward the origin $\mathbf{O}$, the proof of Lemma 18 (see below) shows that $p^{r(2)}$ is greater than $\underline{\mu_2}$ initially, but finally drops below $\underline{\mu_2}$ when it is sufficiently close to $\mathbf{O}$ (see Figure 3.4). Therefore I define the curve $A_2^{ir} \equiv \left\{\boldsymbol{\alpha} \in \mathcal{A}_1^r : p^{r(2)}(\boldsymbol{\alpha}) = \underline{\mu_2}\right\}$ as the new boundary for $\mathcal{E}^{ir}$, and in equilibrium all types in $\partial\Xi_2$ lie on this curve. The following lemma states the properties of this curve. When $\theta_2 = -1$, $A_2^{ir} = A_2^r$ is just the diagonal line $\{\boldsymbol{\alpha} \in \mathcal{A} : \alpha_1 = \alpha_2\}$.

**Lemma 18.** *$A_2^{ir}$, with $(1,1)$ and $(0,0)$ as its upper- and lower- ending points respectively, lies between $A_2^r$ and the diagonal line $\{\boldsymbol{\alpha} \in \mathcal{A} : \alpha_1 = \alpha_2\}$.*

**Proof.** First of all, $(1,1) \in A_2^{ir}$; later on I ignore this upper ending point. Given any $L \in \mathcal{A}_1^r$, decompose $L$ into $L' \cup L''$ where $L'$ ($L''$) is on the right (left) hand side of $A_2^r$ (see Figure 3.4). Rewrite the transport equation satisfied by $p^{r(2)}$ as,

$$-\left(\alpha_1 p_1^{r(2)} + \alpha_2 p_2^{r(2)}\right) = r\sigma_{22}\left[\alpha_2^{FB}(\alpha_1) - \alpha_2\right],$$

which describes the (scaled) marginal increment of $p^{r(2)}$ along $L$ toward $\mathbf{O}$. Since on $L'$ ($L''$) I have $\alpha_2^{FB}(\alpha_1) - \alpha_2 \geq (<) 0$, $p^{r(2)}$ increases first on $L'$ then decreases on $L''$, and achieves maximum on $A_2^r$. This fact implies that on $A_2^r$ I have $p^{r(2)} > \underline{\mu_2}$. To show this, note that on $A_1^r$, $p^{r(2)}(\boldsymbol{\alpha}) = \underline{\mu_2} + r\left(1-\rho^2\right)\sigma_{22}(\alpha_2 - \ln\alpha_2 - 1) > \underline{\mu_2}$, and for any point $\boldsymbol{\alpha}' \in A_2^r$ I can find a transport path from a particular point on $A_1^r$. Because $p^{r(2)}$ achieves maximum on $A_2^r$, the claim follows. Next, it is not difficult to see that along the diagonal, $p^{r(2)}(\alpha,\alpha) = \underline{\mu_2} + r\sigma_{22}(1+\theta_2)(\alpha - \ln\alpha - 1) \leq \underline{\mu_2}$ since $1+\theta_2 \leq 0$. According to the continuity of $p^{r(2)}(\boldsymbol{\alpha})$, $A_2^{ir}$ must be between $A_2^r$ and the diagonal line. Finally I show that $\lim_{y\to 0^+} \alpha_1(y) = 0$, or $(0,0)$ is the lower-ending point of $A_2^{ir}$. It suffices to show that the *limsup* is 0. For a sequence $\{y_n\} \to 0$, let the *limsup* of $\alpha_1(y_n)$ be $\overline{\alpha}_1$. Suppose $\overline{\alpha}_1 > 0$; I can choose subsequence $\{y_{n'}\}$ so that

$\alpha_1(y_{n'}) \geq \overline{\alpha}_1/2 > 0$ for all $n'$. But $p^{r(2)}(\alpha_1(y_{n'}), y_{n'}) \geq p^{r(2)}(\overline{\alpha}_1/2, y_{n'})$, and the latter could be arbitrarily large for $y_{n'}$ close enough to 0. This contradicts the equality $p^{r(2)}(\alpha_1(y_{n'}), y_{n'}) = \underline{\mu_2}$. Q.E.D. □

### A.3.5. Proof of Proposition 9

Note that given $\mathbf{p}^{ir}(\cdot)$, the optimal strategy of any type of agent must lie on $\mathcal{E}^{ir}$. Once I restrict the agent's selling strategy to be within $\mathcal{E}^{ir} \subset \mathcal{A}_1^r$, $V(\boldsymbol{\alpha})$ is strictly concave on $\mathcal{E}^{ir}$, and her optimal selling strategy is unique and truth-telling by the results obtained in the regular case in Proposition 8 (recall that $V(\boldsymbol{\alpha})$ is strictly concave on $\mathcal{A}_1^r$). Therefore $\mathbf{p}^{ir}(\cdot)$ delivers a separating equilibrium for the irregular case. The verification of intuitive criterion and Pareto-efficiency follows similarly as the argument in Proposition 7. Q.E.D.

### A.3.6. Proof of Proposition 10

The first and second claims follow from a direct calculation of $\mathbf{p}(\cdot)$. The proof of $\frac{\partial p^{(i)}}{\partial \rho} > 0$ needs a bit explanation. The least obvious case is $\frac{\partial p^{r(1)}}{\partial \rho}$ on $\mathcal{A}_2^r$ when $\rho < 0$. For simplicity, setting $r\sigma_{11} = r\sigma_{22} = 1$, then on $\mathcal{A}_2^r$ I have

$$\frac{\partial p^{r(1)}}{\partial \rho} = \rho(1-\rho)\left(\frac{\alpha_1}{\alpha_2+\rho\alpha_1} - \frac{1}{1+\rho}\right) - 2\rho \ln \frac{\alpha_2+\rho\alpha_1}{(1+\rho)\alpha_1} - \ln \frac{\alpha_2+\rho\alpha_1}{1+\rho} + \alpha_2 - 1;$$

but the second term is positive when $\rho < 0$ on $\mathcal{A}_2^r$, and once substituting $\beta = \frac{\alpha_2+\rho\alpha_1}{1+\rho}$ into the above equation, using $\beta - \ln\beta + 1$ and $\alpha_2 + \rho\alpha_1 \leq 1+\rho$ one can show that $\frac{dp^{r(1)}}{d\rho} > 0$. The third result follows from the envelope theorem. I need some extra work for $\frac{\partial V}{\partial \rho}$. When $\rho > 0$, one can check that when $\boldsymbol{\alpha} \in \mathcal{A}_1$ (i.e., $\alpha_1 \geq \alpha_2$),

$$\begin{aligned} \frac{\partial V}{\partial \rho} &= \alpha_1 \frac{\partial p^{(1)}}{\partial \rho} + \alpha_2 \frac{\partial p^{(2)}}{\partial \rho} - r\sigma_1\sigma_2(1-\alpha_1)(1-\alpha_2) \\ &= -(\alpha_1+\alpha_2)\ln\alpha_1 - (1+\alpha_1)(1-\alpha_2) \end{aligned}$$

which is increasing in $\alpha_2$. Therefore to verify $\frac{\partial V}{\partial \rho} < 0$ it suffices to show $m(\alpha_1) \equiv -2\alpha_1 \ln \alpha_1 + \alpha_1^2 - 1 < 0$ for $\alpha_1 \in (0,1)$. But it is easy to check that $m(1) = 0$, and $m'(\alpha_1) = 2(\alpha_1 - \ln \alpha_1 - 1) > 0$, which implies the claim I need. The results for $\boldsymbol{\alpha} \in \mathcal{A}_2$ and negative correlation case follows similarly (when $\rho < 0$, I need a trick similar to proof for Proposition 11: consider the change of $\frac{\partial V}{\partial \rho}$ along the lines parallel to $A_1^r$). Q.E.D.

### A.3.7. Proof of Proposition 11

It is easier to start with the case $\rho < 0$. Suppose the assets are regular, and denote the equilibrium selling strategy under separate sale is $(\alpha_1, \alpha_2)$. I can view $\alpha$ as a function of $(\alpha_1, \alpha_2)$ implicitly defined by the market consistency condition:

$$\alpha(\boldsymbol{\alpha}) - \ln \alpha(\boldsymbol{\alpha}) - 1 = \frac{p^{r(1)}(\boldsymbol{\alpha}) - \underline{\mu_1} + p^{r(2)}(\boldsymbol{\alpha}) - \underline{\mu_2}}{r(\sigma_{11} + 2\sigma_{12} + \sigma_{22})}.$$

It is clear that when $\alpha_1 = \alpha_2$, which holds if $\frac{\mu_1 - \underline{\mu_1}}{\sigma_{11} + \sigma_{12}} = \frac{\mu_2 - \underline{\mu_2}}{\sigma_{22} + \sigma_{12}}$, $\alpha$ has to equal to them too; hence I have the equivalence between separate sale and pooled sale. Now consider the domain $\mathcal{A}_1^r$. Define

$$Q(\boldsymbol{\alpha}) \equiv (1 - \alpha(\boldsymbol{\alpha}))^2 - \frac{(1-\alpha_1)^2 \sigma_{11} + 2(1-\alpha_1)(1-\alpha_2)\sigma_{12} + (1-\alpha_2)^2 \sigma_{22}}{(\sigma_{11} + 2\sigma_{12} + \sigma_{22})};$$

since $Q(\boldsymbol{\alpha}) = 0$ when $\alpha_1 = \alpha_2$, and now $\alpha_2 < \alpha_1$, it is sufficient to show that $Q_1(\boldsymbol{\alpha}) \equiv \frac{\partial Q}{\partial \alpha_1} > 0$. Direct calculation and the implicit function theorem yields ($Q_1(\boldsymbol{\alpha}) = M(\boldsymbol{\alpha}) \cdot \left[1 - \frac{(1+\theta_1)\alpha(\boldsymbol{\alpha})}{\alpha_1 + \theta_1 \alpha_2}\right]$ where $M(\boldsymbol{\alpha})$ is positive)

$$sign(Q_1(\boldsymbol{\alpha})) = sign\left(\frac{\alpha_1 + \theta_1 \alpha_2}{1 + \theta_1} - \alpha(\boldsymbol{\alpha})\right).$$

Let $\delta^{-}(\boldsymbol{\alpha}) \equiv \frac{\alpha_1+\theta_1\alpha_2}{1+\theta_1}$; to show $\delta^{-}(\boldsymbol{\alpha}) > \alpha(\boldsymbol{\alpha})$, it suffices to show (note that $x - \ln x - 1$ is strictly decreasing):

$$\delta^{-}(\boldsymbol{\alpha}) - \ln\delta^{-}(\boldsymbol{\alpha}) - 1 > \frac{p^{r(1)}(\boldsymbol{\alpha}) - \underline{\mu_1} + p^{r(2)}(\boldsymbol{\alpha}) - \underline{\mu_2}}{r(\sigma_{11} + 2\sigma_{12} + \sigma_{22})}.$$

Plug in $\mathbf{p}^r$ and rearrange, I need to show that $\sigma_{22}\left(1-\rho^2\right)\left[\ln\frac{\delta^{-}(\boldsymbol{\alpha})}{\alpha_2} - \frac{\alpha_1-\alpha_2}{1+\theta_1}\right] \geq 0$; but this holds trivially since for $\ln\frac{\delta^{-}(\boldsymbol{\alpha})}{\alpha_2} - \frac{\alpha_1-\alpha_2}{1+\theta_1}$, when $\alpha_1 = \alpha_2$ it is 0 and its derivative w.r.t $\alpha_1$ is $\frac{1}{\alpha_1+\theta_1\alpha_2} - \frac{1}{1+\theta_1} > 0$ for $\boldsymbol{\alpha} \in int\mathcal{A}_1^r$. I can carry out the same argument for $\mathcal{A}_2^r$; this concludes the proof for the regular case with $\rho < 0$.

If the assets are irregular, pick the corresponding domain and apply the argument above. Note that in the equilibrium of separate sales I cannot have equal fractions, hence separate sale always dominates pooled sale.

The positive correlation case is more involved. Replace $\mathbf{p}^r$ by $\mathbf{p}$ and define $Q(\boldsymbol{\alpha})$ as before. Consider $\mathcal{A}_1$ first, but I show $Q_2(\boldsymbol{\alpha}) < 0$ instead of $Q_1(\boldsymbol{\alpha}) > 0$. I have

$$sign\left(Q_2(\boldsymbol{\alpha})\right) = \frac{1-\alpha_2+\theta_2(1-\alpha_1)}{\frac{1}{\alpha_2} - 1 + \theta_2\left(\frac{1}{\alpha_1} - 1\right)} - \alpha(\boldsymbol{\alpha}).$$

Let $\delta^{+}(\boldsymbol{\alpha}) \equiv \frac{1-\alpha_2+\theta_2(1-\alpha_1)}{\frac{1}{\alpha_2}-1+\theta_2\left(\frac{1}{\alpha_1}-1\right)}$, then it suffices to show $\delta^{+}(\boldsymbol{\alpha}) < \alpha(\boldsymbol{\alpha})$. Let $\gamma = \frac{\sigma_{11}}{\sigma_{22}}$; the strategy it to show that (note that $\alpha_2 - \ln\alpha_2 - 1 \geq \alpha_2 - \ln\alpha_1 - \frac{\alpha_2}{\alpha_1}$)

$$\begin{aligned}\delta^{+}(\boldsymbol{\alpha}) - \ln\delta^{+}(\boldsymbol{\alpha}) - 1 &> \frac{(\gamma+\theta_2)(\alpha_1 - \ln\alpha_1 - 1) + (\theta_2+1)(\alpha_2 - \ln\alpha_2 - 1)}{\gamma + 2\theta_2 + 1} \\ &\geq \frac{(\gamma+\theta_2)(\alpha_1 - \ln\alpha_1 - 1) + \theta_2\left(\alpha_2 - \ln\alpha_1 - \frac{\alpha_2}{\alpha_1}\right) + (\alpha_2 - \ln\alpha_2 - 1)}{\gamma + 2\theta_2 + 1}.\end{aligned}$$

Note also that the second line is just $\alpha(\boldsymbol{\alpha}) - \ln\alpha(\boldsymbol{\alpha}) - 1$ on $\mathcal{A}_1$. Define

$$R(\boldsymbol{\alpha}) \equiv \delta^{+}(\boldsymbol{\alpha}) - \ln\delta^{+}(\boldsymbol{\alpha}) - 1 - \frac{(\gamma+\theta_2)(\alpha_1 - \ln\alpha_1 - 1) + (\theta_2+1)(\alpha_2 - \ln\alpha_2 - 1)}{\gamma + 2\theta_2 + 1};$$

I want to show $R(\boldsymbol{\alpha}) > 0$ for $\alpha_2 < \alpha_1$. (note that $R(\boldsymbol{\alpha}) = 0$ when $\alpha_1 = \alpha_2$.)

Let $b \equiv \frac{\alpha_2 + \theta_2 \alpha_1}{1+\theta_2} \in (0,1)$, and I can define a parameterized line $\boldsymbol{\alpha}(t)$ for $t \geq 0$ :

$$\begin{cases} \alpha_1(t) = b + t \\ \alpha_2(t) = b - \theta_2 t \end{cases}$$

It starts from $(b,b)$ and reaches $\boldsymbol{\alpha}$ when $t = \frac{\alpha_1 - \alpha_2}{\theta_2 + 1} > 0$. Now, $R(\boldsymbol{\alpha}) = R(\boldsymbol{\alpha}) - R(b,b) = \int_0^{\frac{\alpha_1-\alpha_2}{\theta_2+1}} \frac{dR(\boldsymbol{\alpha}(t))}{dt} dt$, so it is sufficient to show that $\frac{dR(\boldsymbol{\alpha}(t))}{dt} = R_1(\boldsymbol{\alpha}(t)) - \theta_2 R_2(\boldsymbol{\alpha}(t)) > 0$. Tedious calculations yields

$$\begin{aligned} \frac{dR(\boldsymbol{\alpha}(t))}{dt} &= \frac{\theta_2\left(\alpha_2 + \frac{1}{\alpha_2} - 2 + \theta_2\left(\alpha_1 + \frac{1}{\alpha_1} - 2\right)\right)\left(\frac{1}{\alpha_2^2} - \frac{1}{\alpha_1^2}\right)}{\left(\frac{1}{\alpha_2} - 1 + \theta_2\left(\frac{1}{\alpha_1} - 1\right)\right)^2} \\ &\quad + \frac{\theta_2\left(\frac{1}{\alpha_1} - \frac{1}{\alpha_2}\right) + \gamma\left(\frac{1}{\alpha_1} - 1\right) - \theta_2^2\left(\frac{1}{\alpha_2} - 1\right)}{\gamma + 2\theta_2 + 1} \\ &> \theta_2\left(\frac{1}{\alpha_2} - \frac{1}{\alpha_1}\right)\left(\frac{\left(\alpha_2 + \frac{1}{\alpha_2} - 2 + \theta_2\left(\alpha_1 + \frac{1}{\alpha_1} - 2\right)\right)\left(\frac{1}{\alpha_1} + \frac{1}{\alpha_2}\right)}{\left(\frac{1}{\alpha_2} - 1 + \theta_2\left(\frac{1}{\alpha_1} - 1\right)\right)^2} - \frac{\theta_2 + 1}{\gamma + 2\theta_2 + 1}\right) \end{aligned}$$

Hence it amounts to showing that the second bracket is positive. To show this, let $u \equiv \frac{1}{\alpha_1} - 1 \geq 0, v \equiv \frac{1}{\alpha_2} - 1 \geq 0$, and note that $\gamma^2 > \theta_2^2 = \frac{\sigma_{11\rho^2}}{\sigma_{22}}$, therefore I have

$$\begin{aligned} &\left(v^2 + \frac{u+1}{v+1}v^2 + \theta_2 \frac{u+1}{v+1}u^2 + \theta_2 u^2\right)(\gamma + 2\theta_2 + 1) - (\theta_2 + 1)(v + \theta_2 u)^2 \\ \geq\ & \theta_2 \frac{u+1}{v+1}u^2 + \theta_2 u^2 + 2\theta_2 \frac{u+1}{v+1}v^2 + \gamma u^2 + \gamma\theta_2 \frac{u+1}{v+1}u^2 > 0 \end{aligned}$$

In sum, when $\boldsymbol{\alpha} \in \mathcal{A}_1$, $R(\boldsymbol{\alpha}) > 0$; then $\delta^+(\boldsymbol{\alpha}) < \alpha(\boldsymbol{\alpha})$, or $Q_2(\boldsymbol{\alpha}) < 0$; finally $Q(\boldsymbol{\alpha}) > 0$, which implies the dominance of separate sale. Similarly I can show that it holds for $\boldsymbol{\alpha} \in \mathcal{A}_2$. Q.E.D.

### A.3.8. Appendix for Section 3.6.2

Equilibrium Pricing System for the Positively Correlated $n$-Asset Case. I can solve the $n$-asset case recursively. Let $\mathbf{n} \equiv \{1, 2, \cdots, n\}$. When $\sigma_{ij} \geq 0$ for $\forall i, j \in \mathbf{n}$, $BC$ implies that the $i^{th}$ boundary-type agent will always set $\alpha_i = 1$. Once the agent keeps zero inventory for asset $i$, I essentially reduce an $n$-dimensional problem to an $n-1$-dimensional one. By induction, I have a simple formula for the $n$-dimensional equilibrium pricing schedule

$$p^{(i)}(\boldsymbol{\alpha}) = \underline{\mu_i} + r \sum_{j \in \overline{\mathcal{I}}_i(\boldsymbol{\alpha})} \sigma_{ij} (\alpha_j - \ln \alpha_j - 1) + r \sum_{j \in \underline{\mathcal{I}}_i(\boldsymbol{\alpha})} \sigma_{ij} \left( \alpha_j - \ln \alpha_i - \frac{\alpha_j}{\alpha_i} \right),$$

where $\overline{\mathcal{I}}_i(\boldsymbol{\alpha}) \equiv \{k \in \mathbf{n} : \alpha_k \geq \alpha_i\}$, and $\underline{\mathcal{I}}_i(\boldsymbol{\alpha}) \equiv \mathbf{n} \backslash \overline{\mathcal{I}}_i(\boldsymbol{\alpha})$. One can check that $p^{(i)}$ has continuous derivatives despite the different functional forms across various regions. Finally, similar to the proof for 2-asset case, I can show that $V(\boldsymbol{\mu}, \boldsymbol{\alpha}, \mathbf{p}(\boldsymbol{\alpha}))$ is strictly concave on $(0, 1]^n$, and therefore $\mathbf{p}(\cdot)$ is a separating equilibrium. For details, see He (2005).

Example of Equilibrium Construction for a 3-Asset Case with a General Covariance Matrix. This example illustrates how to construct equilibrium pricing system under a general variance structure. For simplicity I assume $\underline{\boldsymbol{\mu}} = (0, 0, 0)$, and $r = 1$. I consider the covariance matrix,

$$\begin{bmatrix} 1 & \frac{1}{2} & 0 \\ \frac{1}{2} & 1 & -\frac{1}{2} \\ 0 & -\frac{1}{2} & 1 \end{bmatrix};$$

as shown later this case demonstrates all main issues in finding the equilibrium pricing system. In this example, asset 1 and asset 2 are positively correlated, while asset 2 and asset 3 negatively correlated. Asset 1 and asset 3, however, are independent. The new insight comes from the asset 2, because it serves two opposite roles relative to the other two assets.

I need to find pricing formulae for boundaries of $\mathcal{E} \subset (0,1]^3$ (i.e., the surfaces where the boundary agents sit) by invoking *BC*. First consider $\partial\Xi_1$; because asset 1 has non-negative covariances with both assets, $\alpha_1^{FB}(\alpha_2, \alpha_3) = 1$ always. Therefore $\partial\Xi_1$ types still employ the strategy with $\{\boldsymbol{\alpha} : \alpha_1 = 1\}$, and the pricing system for asset 2 and 3 (negatively correlated) is simply the system (3.6) obtained in Section 3.4.2.2, with $\rho = -\frac{1}{2}$. Of course $p^{(1)}(\boldsymbol{\alpha}) = 0$ for $\{\boldsymbol{\alpha} : \alpha_1 = 1\}$.

Now consider $\partial\Xi_3$. Given $\alpha_1$ and $\alpha_2$, the first-best selling amount is $\alpha_3^{FB}(\alpha_1, \alpha_2) = \frac{1}{2} + \frac{\alpha_2}{2} \in [0,1]$. Therefore the type-3 boundary for $\mathcal{E}$ is a plane $\left\{\boldsymbol{\alpha} : \left(\alpha_1, \alpha_2, \frac{1}{2} + \frac{\alpha_2}{2}\right)\right\}$. On this plane, the problem is a two-asset case with a positive correlation; but the covariance matrix between these two assets becomes $\begin{bmatrix} 1 & 1/2 \\ 1/2 & 3/4 \end{bmatrix}$ due to the optimal hedging from asset 3. Therefore the 3-dimensional pricing system on this plane is as in (3.5), and $p^{(3)}(\boldsymbol{\alpha}) = 0$.

Finally I consider $\partial\Xi_2$; because the retention of asset 2 is beneficial for asset 3 but harmful in the view of asset 1, this becomes the most intriguing case which involves a kinked surface on the boundary. Given $\alpha_1$ and $\alpha_2$, the unconstrained optimal solution is $\alpha_2^{FB}(\alpha_1, \alpha_3) = 1 + \frac{\alpha_3 - \alpha_1}{2}$. However, since I require $\alpha_2^{FB} \in [0,1]$, as a result

$$\alpha_2^{FB}(\alpha_1, \alpha_3) = \begin{cases} 1 & \text{if } \alpha_3 > \alpha_1 \\ 1 + \frac{\alpha_3 - \alpha_1}{2} & \text{if } \alpha_3 \leq \alpha_1 \end{cases}.$$

Intuitively, when the agent has a large position on asset 1 which pushes her to sell asset 2, the short-sale constraint might bind. On the other hand, if she retains a fair amount of asset 3 which calls for hedging from asset 2, she is free to do so. This complication delivers a kinked surface for the type-2 boundary $\left\{\boldsymbol{\alpha} : \left(\alpha_1, \alpha_2^{FB}(\alpha_1, \alpha_3), \alpha_3\right)\right\}$. When $\alpha_3 > \alpha_1$, the agent sells her entire asset 2; and because asset 1 and 3 are independent, I have the LP results for both assets. When $\alpha_3 \leq \alpha_1$, these two assets have an effective variance matrix $\begin{bmatrix} 3/4 & 1/4 \\ 1/4 & 3/4 \end{bmatrix}$ (asset 3 needs retention from asset 1 which has positive correlation with asset 2, therefore these two independent assets look as if they are positively correlated), and the pricing system is the one as in (3.5).

Once these boundary pricing systems are ready, the last step is to obtain the interior function value by applying the transport equation technique. Nevertheless, the tedious line integration cannot bring any new economic insight.

### A.3.9. Appendix for Section 3.6.3

Constructing a Pareto-inefficient Separating Equilibrium. For illustrative purpose, consider the case $n = 2$ and $\rho > 0$, with $r\sigma_{11} = r\sigma_{22} = 1$, and $\underline{\boldsymbol{\mu}} = (0, 0)$. I try to construct an equilibrium schedule where agents with $\underline{\mu_1}$ optimally choose selling strategies $\alpha_1 < 1$. Specifically, for any small $k > 0$, define the potential boundary $B^k$ as:

$$B^k \ni \boldsymbol{\alpha} = \begin{cases} \left(1 - \frac{x}{2}, 1 - x\right) & \text{for } 0 \leq x < 2k \\ \left(1 - 2k + \frac{x}{2}, 1 - x\right) & \text{for } 2k \leq x < 4k \\ (1, 1 - x) & \text{for } 4k \leq x < 1 \end{cases} .$$

In words, $B^k$ has a triangle dent (call the triangle $\overline{\mathcal{B}}^k$, which is the shaded area in Figure A.1; the slope 2 for the dent is inessential) relative to $A_1$. The set of equilibrium strategies, $\mathcal{E}^k$, will be $(0, 1]^2 \setminus \overline{\mathcal{B}}^k$ which includes $B^k$ as its boundary.

Now I derive the equilibrium pricing system $\mathbf{q} = \left(q^{(1)}, q^{(2)}\right)$ for this case. I restrict the analysis on $\mathcal{A}_1$ where $\alpha_1 \geq \alpha_2$ (for $\mathcal{A}_2$ the pricing system $\mathbf{q}$ is just the one in (3.5)). Consider

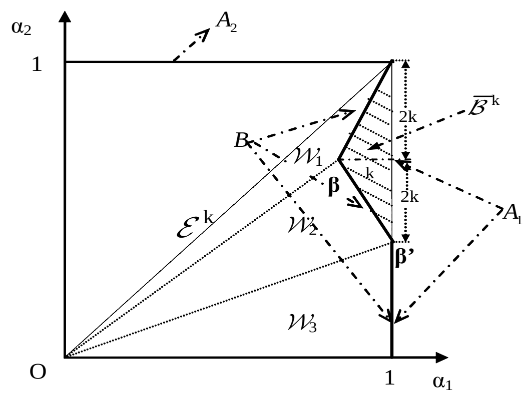


Figure A.1. An example of inefficient separating equilibria which violate the $BC$ assumption. In this equilibrium, $\partial\Xi_1$ types lie on the dented line $B$ instead of $A_1 = \{\boldsymbol{\alpha} \in \mathcal{A} : \alpha_1 = 1\}$, because of the penalty from the asset 2 pricing. As a result, $\overline{\mathcal{B}}^k$ is off-equilibrium.

those types in $\partial\Xi_1$; for their asset 2 pricing, the unidimensional analysis yields,

$$q^{(2)}(\boldsymbol{\alpha}) = \begin{cases} \frac{5+4\rho}{4}\left(\alpha_2 - \ln\alpha_2 - 1\right) & \text{for } 1-2k < \alpha_2 \leq 1 \\ q^{(2)}(\boldsymbol{\beta}) + \frac{5-4\rho}{4}\left(\alpha_2 - \ln\frac{\alpha_2}{1-2k} - 1 + 2k\right) & \text{for } 1-4k < \alpha_2 \leq 1-2k \\ \quad - (2\rho - 1)\, k \ln\frac{\alpha_2}{1-2k} & \\ q^{(2)}(\boldsymbol{\beta}') + \alpha_2 - \ln\frac{\alpha_2}{1-4k} - 1 + 4k & \text{for } 0 < \alpha_2 \leq 1-4k \end{cases}$$

for $\boldsymbol{\alpha} \in B$. Of course $q^{(1)}(\boldsymbol{\alpha}) = 0$. As shown in Figure A.1, I denote three regions—generated by two boundary characteristic lines—as $\mathcal{W}_m$ for $m = 1, 2, 3$ on $\mathcal{A}_1 \backslash \overline{\mathcal{B}}^k$. Using the same transport equation technique, I obtain,

$$\begin{cases} q^{(1)}(\boldsymbol{\alpha}) = -(\rho\alpha_2 + \alpha_1)\left(\frac{1}{2\alpha_1 - \alpha_2} - 1\right) - (1+\rho)\ln(2\alpha_1 - \alpha_2) \\ q^{(2)}(\boldsymbol{\alpha}) = \frac{5+4\rho}{4}\left(\frac{\alpha_2}{2\alpha_1 - \alpha_2} - \ln\frac{\alpha_2}{2\alpha_1 - \alpha_2} - 1\right) - (\rho\alpha_1 + \alpha_2)\left(\frac{1}{2\alpha_1 - \alpha_2} - 1\right) - (1+\rho)\ln(2\alpha_1 - \alpha_2) \end{cases}$$

for $\boldsymbol{\alpha} \in \mathcal{W}_1$, and

$$\begin{cases} q^{(1)}(\boldsymbol{\alpha}) = -(\rho\alpha_2+\alpha_1)\left(\frac{3-4k}{2\alpha_1+\alpha_2}-1\right)+(1+\rho)\ln\frac{3-4k}{2\alpha_1+\alpha_2} \\ q^{(2)}(\boldsymbol{\alpha}) = \begin{array}{c} q^{(2)}(\boldsymbol{\beta})+\frac{5-4\rho}{4}\left(\frac{\alpha_2(3-4k)}{2\alpha_1+\alpha_2}-\ln\frac{\alpha_2(3-4k)}{2\alpha_1+\alpha_2}-1+2k+\ln(1-2k)\right) \\ -(2\rho-1)k\ln\frac{\alpha_2(3-4k)}{(2\alpha_1+\alpha_2)(1-2k)}-(\rho\alpha_1+\alpha_2)\left(\frac{3-4k}{2\alpha_1+\alpha_2}-1\right)+(1+\rho)\ln\frac{3-4k}{2\alpha_1+\alpha_2} \end{array} \end{cases}$$

for $\boldsymbol{\alpha} \in \mathcal{W}_2$, and

$$\begin{cases} q^{(1)}(\boldsymbol{\alpha}) = -(\rho\alpha_2+\alpha_1)\left(\frac{1}{\alpha_1}-1\right)-(1+\rho)\ln\alpha_1 \\ q^{(2)}(\boldsymbol{\alpha}) = q^{(2)}(\boldsymbol{\beta}')-\ln\frac{\alpha_2/\alpha_1}{1-4k}+4k+(\rho\alpha_1+\alpha_2)-\rho-(1+\rho)\ln\alpha_1 \end{cases}$$

for $\boldsymbol{\alpha} \in \mathcal{W}_3$. For $\boldsymbol{\alpha} \in \mathcal{A}_2$ the pricing system $\mathbf{q}$ is the same as in (3.5), and $\mathbf{q}(\boldsymbol{\alpha}) = (0,0)$ for $\boldsymbol{\alpha} \in \overline{\mathcal{B}}^k$.

One can check that $\mathbf{q} \notin \mathbb{C}^1\left(\mathcal{E}^k\right)$ (along the boundary between $\mathcal{W}_i$'s, the individual pricing functions in $\mathbf{q}$ have kinks); however, $V$, as a composite function of $\mathbf{q}$, is in fact smooth (both kinks from $q^{(i)}$'s cancel each other). One can check that $H = \frac{\mathcal{D}\mathbf{q}}{\mathcal{D}\boldsymbol{\alpha}}$ is negative-definite over all domains. Therefore if the agent is restricted in the domain $\mathcal{E}^k$, $\mathbf{q}$ constitutes an equilibrium pricing system.

To show that the agent will not deviate to $\overline{\mathcal{B}}^k$, it suffices to consider the boundary types $(0,\mu) \in \partial\Xi_1$. The reason is that the interior types (with a better asset 1) have less incentives to sell more. Formally, for any interior type $\boldsymbol{\mu} = (\mu_1, \mu_2)$ who chooses $\boldsymbol{\alpha}$, and its corresponding boundary type $\boldsymbol{\mu}' = (0, \mu_2)$ who chooses $\boldsymbol{\alpha}'$, if the deviating point is $\boldsymbol{\gamma}$ which necessarily has $\gamma_1 = 1$, then it is easy to check that $V(\boldsymbol{\mu},\boldsymbol{\gamma}) - V(\boldsymbol{\mu},\boldsymbol{\alpha}') < V(\boldsymbol{\mu}',\boldsymbol{\gamma}) - V(\boldsymbol{\mu}',\boldsymbol{\alpha}')$. But because $V(\boldsymbol{\mu},\boldsymbol{\alpha}) > V(\boldsymbol{\mu},\boldsymbol{\alpha}')$, we have $V(\boldsymbol{\mu},\boldsymbol{\gamma}) - V(\boldsymbol{\mu},\boldsymbol{\alpha}) < V(\boldsymbol{\mu}',\boldsymbol{\gamma}) - V(\boldsymbol{\mu}',\boldsymbol{\alpha}')$, i.e., a smaller deviation gain.

Now, for a sufficiently small $k$, I only have to consider $\mu$ close to 0. Generally, in these separating equilibria, the type $(0,\mu)$'s value is in the order of $\mu^{\frac{3}{2}}$ for small $\mu$. (To see this, if the agent is on $\mathcal{W}_1$, then $V = \mu - \frac{5+4\rho}{8}\left(1-\alpha\left(\mu\right)\right)^2$ and $\mu = \frac{5+4\rho}{4}\left[\alpha\left(\mu\right) - \ln\alpha\left(\mu\right) - 1\right]$. Now view $\mu\left(x\right)$ as a function of $x \equiv 1-\alpha \geq 0$, and the Taylor expansion around $\mu\left(0\right) = 0$ yields the result. If the agent is on $\mathcal{W}_2$ the value incremental is in the same order.) However, since the value from deviating (to $\left(1,\gamma\right)$) for a small $\mu$ (she solves $\max_{\gamma\in[1-4k,1]}\left(1-\gamma\right)\mu - \frac{1}{2}\left(1-\gamma\right)^2$) is at most in the order of $\mu^2$, for any sufficiently small $k$ the resulting pricing system is indeed a separating equilibrium.

In this equilibrium, those types in $\partial\Xi_1$ hold certain positive positions of asset 1, due to a substantial penalty imposed by $p^{(2)}\left(\cdot\right)$. The resulting continuum of equilibria (indexed by $k$) are less efficient by design. One can verify that the agent's value is decreasing in $k$ (by the envelope theorem, $\frac{dV}{dk} = \frac{\partial V}{\partial k} = \sum\alpha_i\frac{\partial q^{(i)}}{\partial k}$), and the resulting equilibrium preserves the properties in Proposition 10—except the property 1 when $\boldsymbol{\alpha}$ is on the top region of $\mathcal{W}_1$. The reason that this property fails is rather mechanical. It is clear that by design, when $\boldsymbol{\alpha}$ is on the top region of $\mathcal{W}_1$, a smaller $\alpha_2$ (keeping $\alpha_1$ constant) makes the selling position closer to the boundary $B^k$—and therefore a lower $p^{(1)}$. However, since $\mathcal{W}_1$ is small when $k$ is small, this exception is of little consequence.