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Group Actions via Interval Exchange Transformations

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ABSTRACT

Group Actions via Interval Exchange Transformations

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This dissertation addresses the structure of the group of interval exchange transformations. The two primary topics considered are:

a) the classification of interval exchange actions for certain groups; and
b) properties of the interval exchange group which are reflected in the dynamics of interval exchange maps.

In Chapter 3 a classification is given for the continuous interval exchange actions of the group of real numbers. The interval exchange group is endowed with a natural topological group structure, with respect to which any continuous one-parameter action must factor through a toral action generated by disjointly supported rotation groups.

In Chapter 4 the asymptotics are classified for the number of discontinuities exhibited by the iterates of an interval exchange. It is seen that the number of discontinuities is either bounded or exhibits linear growth; no intermediate growth rates are possible. It is further shown that any map with bounded discontinuity growth is essentially an element of a toral rotation action.

In Chapter 5, the dichotomy in discontinuity growth is used to prove that no finitely generated subgroup of the interval exchange group contains a distortion element. Consequently, no group having a distortion element can act faithfully via interval exchanges.

Chapter 6 contains a complete classification of centralizers in the interval exchange group. The structure of an element's centralizer is controlled by three types of dynamic behavior: the existence of periodic points, minimal sets with bounded discontinuity growth, and minimal sets with linear discontinuity growth.

The classification of centralizers is used in Chapter 7 to compute the automorphism group of the interval exchange group. Since automorphisms preserve the group structure of centralizers, they also preserve the associated dynamics. Consequently, an automorphism must be induced under conjugation by a map on the circle. It is then seen that the group of outer automorphisms is generated by an order-two orientation reversing map on the circle.

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CHAPTER 1

Introduction

An interval exchange transformation is an invertible transformation on the circle $\mathbb{T}^1 = \mathbb{R}/\mathbb{Z}$ that acts as a finite piecewise translation on subintervals. In short, an interval exchange is defined by partitioning $\mathbb{T}^1$ into finitely many subintervals and rearranging the placement of these subintervals in a non-overlapping, orientation-preserving manner. By convention, interval exchanges are assumed to be right-continuous; thus, a translation applied to an open interval $(a, b) \subseteq \mathbb{T}^1$ extends to a translation on $[a, b)$.

The dynamics of interval exchange transformations were first studied in depth during the late 1970's by Keane [**4**] [**5**], who formulated the conjecture that almost every interval exchange is uniquely ergodic. This conjecture was proven independently by Masur [**9**] and Veech [**11**] in 1982. More recently, Avila and Forni [**2**] have shown that almost all interval exchanges are weakly mixing. A good introduction to the dynamics of interval exchanges and their connection to translation surfaces may be found in Viana [**12**].

To give a precise definition and to introduce notation, consider the following construction, which defines an interval exchange as a transformation on the interval $[0, 1)$. Let $\pi \in \Sigma_n$ be a permutation, and let λ be a vector in the unit simplex

$$\Lambda_n := \left\{ \lambda = (\lambda_1, \ldots, \lambda_n) : \lambda_i > 0, \sum \lambda_i = 1 \right\} \subseteq \mathbb{R}^n.$$

The vector λ induces a partition of $[0,1)$ into intervals

$$I_j = \left[\beta_{j-1} := \sum_{i=1}^{i=j-1} \lambda_i,\ \beta_j := \sum_{i=1}^{i=j} \lambda_i\right),\quad 1 \le j \le n. \tag{1.1}$$

Let $f_{(\pi,\lambda)}$ be the interval exchange which translates each I_j so that the ordering of these intervals within $[0,1)$ is permuted according to π.

More precisely, the amount by which $f_{(\pi,\lambda)}$ translates each interval I_j is calculated in the following way. Define the linear map $\Omega_\pi : \mathbb{R}^n \to \mathbb{R}^n$ by

$$\Omega_\pi(\lambda)_j \ = \sum_{i:\,\pi(i)<\pi(j)} \lambda_i \ - \sum_{i:\,i<j} \lambda_i. \tag{1.2}$$

For $\lambda \in \Lambda_n$, the number $\Omega_\pi(\lambda)_j$ represents the translation applied to I_j under the reordering induced by π. Observe from Figure 1.1 that the term $\left(-\sum_{i<j} \lambda_i\right)$ in (1.2) corresponds to translating the interval I_j from its original position to the left end of $[0,1)$, and the term $\left(\sum_{\pi(i)<\pi(j)} \lambda_i\right)$ corresponds to translating it from the left end to its image under the map $f_{(\pi,\lambda)}$. The upper half of Figure 1.1 illustrates the initial partition induced by λ, and the bottom half depicts the images of these interval under $f_{(\pi,\lambda)}$.

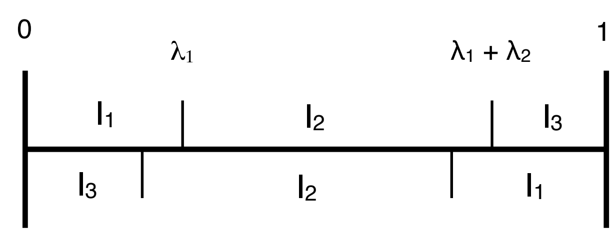

Figure 1.1. An interval exchange transformation: $\pi = (1,3)$

Given π and λ, the vector $\omega = \Omega_\pi(\lambda)$ is called the *translation vector* of $f_{(\pi,\lambda)}$. The map $f_{(\pi,\lambda)}$ may now be expressed by the formula

$$f_{(\pi,\lambda)}(x) = x + \omega_j, \quad \text{if } x \in I_j. \tag{1.3}$$

In addition, for any interval exchange f, define the translation function $\omega_f : [0,1) \to \mathbb{R}$ by the formula

$$\omega_f(x) = f(x) - x, \tag{1.4}$$

where f is considered as a transformation $[0,1) \to [0,1)$.

A general topic in dynamical systems is the study of group actions. Their dynamics generalize the dynamical behavior of a single invertible map. From a group-theoretic standpoint, one goal is to gain some understanding of the subgroup structure within a large group of transformations. This may be achieved by investigating the interplay between the algebraic structure of subgroups and the dynamics of the maps through which they are realized. Group actions have been studied extensively in the context of homeomorphisms and diffeomorphisms on the circle or other higher-dimensional manifolds. A good introduction to continuous group actions on the circle is found in Ghys [**3**].

As was mentioned above, the dynamics of individual interval exchange transformations have been well studied, but relatively little attention has been given to the study of groups which act by interval exchanges. Let $\mathcal{E}$ represent the space of all interval exchanges on $\mathbb{T}^1$, which forms a group under the composition of maps. If G is a group, then an *interval exchange action* of G, also called an *action of G via interval exchanges*, is a group homomorphism

$$G \to \mathcal{E}.$$

An action is *faithful* if the above homomorphism has no kernel. It generally suffices to consider faithful actions, since it is always possible to replace G with its quotient by the action's kernel. One basic question to consider is:

Question 1.1. *Given a group G, do there exist faithful interval exchange actions of G? If so, can they be classified in any way?*

A faithful interval exchange action of a group G is an embedding of G into $\mathcal{E}$, so a faithful action may be identified with its image in $\mathcal{E}$. Thus, the previous question may be rephrased as:

Question 1.2. *Given a group G, does $\mathcal{E}$ contain subgroups isomorphic to G, and can they be classified?*

Such a classification is discussed in Chapter 3, in the case where G is the group $\mathbb{R}$ of real numbers and we restrict to continuous actions. In order to do so, a natural topology is defined on $\mathcal{E}$, giving it a topological group structure. A ready source of examples of continuous $\mathbb{R}$-actions are the one-parameter subgroups of rotation actions. A rotation action is an action of $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$ whose image is generated by rotations supported on disjoint subintervals; see Figure 1.2 for an example, and see Section 3.1 for a complete definition. The map in Figure 1.2 is invariant on the intervals $[0, 1/2)$ and $[1/2, 1)$, and it rotates these subintervals by $\frac{1}{2}s$ and $\frac{1}{2}t$, respectively.

The only continuous actions of $\mathbb{R}$ are one-parameter subgroups of rotation actions.

Theorem 1.3. *Up to conjugacy in $\mathcal{E}$, any continuous homomorphism $\mathbb{R} \to \mathcal{E}$ is a one-parameter rotation action.*

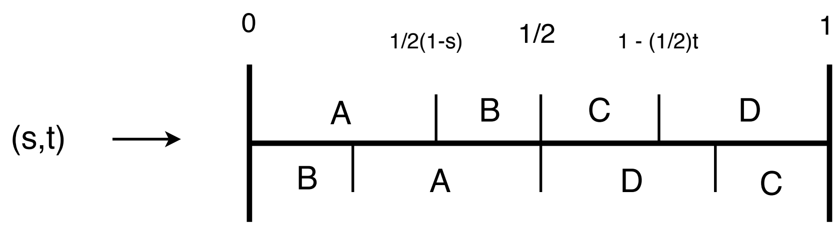

Figure 1.2. A rotation action of $\mathbb{T}^2$

In contrast to $\mathbb{R}$, there are groups which do not have any faithful interval exchange actions, namely those groups which possess *distortion elements.* Let G be a finitely generated group, and let $S = \{g_1, \dots, g_n\}$ be a set of generators. An element $f \in G$ is a *distortion element* if f has infinite order and

$$\liminf_{n\to\infty} \frac{|f^n|_S}{n} = 0,$$

where $|\cdot|_S$ denotes the minimal word length in terms of the generators and their inverses. For example, the central elements of the discrete Heisenberg group are distortion elements. The main result of Chapter 5 is the following.

Theorem 1.4. *Distortion elements do not exist in $\mathcal{E}$.*

In short, this theorem states that no finitely generated subgroup of $\mathcal{E}$ can have a distortion element. As a consequence, it is shown that a large class of higher-rank lattices in semisimple Lie groups do not act faithfully via interval exchanges.

A key idea in the proof of the previous theorem is an analysis of the growth rate exhibited by $d(f^n)$, the number of discontinuities of f^n. As discussed in Chapter 4, there is a fundamental dichotomy for the asymptotic behavior of $d(f^n)$. Given an interval exchange f, either $d(f^n)$ exhibits linear growth with respect to n, or $d(f^n)$ is bounded independently of n. By extending a result of Li [**6**], which gives a criterion for when an interval exchange is conjugate to an irrational rotation, the following result is shown.

Theorem 1.5. *Let f be an infinite-order map such that $d(f^n)$ is bounded. Then some f^k, $k \geq 1$, is conjugate to a product of disjointly supported irrational restricted rotations.*

The fundamental dichotomy for discontinuity growth is useful for investigating other aspects of the group structure of $\mathcal{E}$, and this dichotomy is reflected in a very strong algebraic manner. Given an interval exchange $f \in \mathcal{E}$, the *centralizer* of f in $\mathcal{E}$ is the subgroup of $\mathcal{E}$ containing all interval exchanges which commute with f:

$$C(f) = C_{\mathcal{E}}(f) := \{g \in \mathcal{E} : fg = gf\}.$$

A complete classification of centralizers in $\mathcal{E}$ is given in chapter 6. This classification relies on analyzing three cases:

(i) f has periodic points;

(ii) f is minimal and has bounded discontinuity growth; and

(iii) f is minimal and has linear discontinuity growth.

Minimality here refers to the property that every orbit of f is dense in $\mathbb{T}^1$. The periodic case is characterized by the fact that $C(f)$ contains a subgroup isomorphic to the entire group $\mathcal{E}$. In the bounded growth case, $C(f)$ is virtually abelian and contains a continuously embedded copy of $\mathbb{R}/\mathbb{Z}$. Finally, if f is minimal and has linear discontinuity growth, the centralizer $C(f)$ is virtually cyclic. Thus, the three basic dynamical behaviors exhibited by interval exchanges are reflected in the algebraic structure of their centralizers.

The classification of centralizers in Chapter 6 is used in Chapter 7 to compute the automorphism group of $\mathcal{E}$. Intuitively, one would hope that all automorphisms of $\mathcal{E}$ reflect the underlying geometric action on $\mathbb{T}^1$. This is the case, and it is shown that any automorphism of $\mathcal{E}$ is induced from conjugation by some transformation on the circle.

Theorem 1.6. $Aut(\mathcal{E}) \cong \mathcal{E} \rtimes \langle \Psi_T \rangle$.

The factor $\mathcal{E}$ is the group of inner automorphisms, and the factor $\langle \Psi_T \rangle$ is generated by an order-two automorphism which is essentially conjugation by the map $T : x \mapsto -x$.

CHAPTER 2

General Results

This chapter discusses the general dynamic behavior of interval exchange transformations, and a normal dynamical form is described. In addition, a useful invariant on $\mathcal{E}$ is introduced.

Let $\mathcal{P}$ denote the set algebra which consists of all finite unions of half-open subintervals $[a, b) \subseteq \mathbb{T}^1 = \mathbb{R}/\mathbb{Z}$. For any $f \in \mathcal{E}$, several dynamically defined f-invariant sets are members of the algebra $\mathcal{P}$. For $x \in \mathbb{T}^1$, let $\mathcal{O}_f(x)$ denote the *orbit of x under f*:

$$\mathcal{O}_f(x) = \{f^n(x)\}_{n \in \mathbb{Z}}.$$

The set Per(f) of *periodic points for f* is the set of all points x such that $\mathcal{O}_f(x)$ is finite, since a point has a finite orbit if and only if $f^n(x) = x$ for some $n \geq 1$.

Lemma 2.1. *For any $f \in \mathcal{E}$, Per(f) $\in \mathcal{P}$.*

Proof. It will first be shown that Fix(f), the set of fixed points for f, is a member of $\mathcal{P}$. Consider the translation function ω_f, as defined in (1.4). This function is right continuous and piecewise constant, so for any y, the pre-image $\omega_f^{-1}\{y\}$ is a member of $\mathcal{P}$. In particular, Fix(f) $= \omega_f^{-1}\{0\}$ is in $\mathcal{P}$.

Using the fact that Fix(f) $\in \mathcal{P}$ for any interval exchange, it can be seen that for any $k \in \mathbb{N}$,

$$\mathrm{Per}_k(f) := \{x : |\mathcal{O}_f(x)| = k\} \in \mathcal{P}.$$

In particular, note that

$$\operatorname{Fix}(f^k) = \coprod_{d|k} \operatorname{Per}_d(f).$$

It immediately follows that $\operatorname{Per}_k(f) \in \mathcal{P}$ if k is prime, and it follows for any $k \in \mathbb{N}$ by induction on the number of prime factors of k.

To prove that $\operatorname{Per}(f) = \coprod_{k\in\mathbb{N}} \operatorname{Per}_k(f) \in \mathcal{P}$, it suffices to show that this disjoint union has only finitely many nonempty elements. If $\operatorname{Per}_k(f)$ is nonempty for some k, let p be a left boundary point of $\operatorname{Per}_k(f)$. If f were continuous at each of the k points in the orbit $\mathcal{O}_f(p)$, then p would be an interior point of $\operatorname{Per}_k(f)$. Thus $\mathcal{O}_f(p)$ must contain a discontinuity point of f. Since the sets $\operatorname{Per}_k(f)$ are pairwise disjoint and each nonempty member of this family contains at least one discontinuity of f, it follows that $\operatorname{Per}_k(f)$ is nonempty for only finitely many values of k. $\square$

All interval exchange maps preserve Lebesgue measure on $\mathbb{T}^1$. By Poincaré recurrence, the first return map of f to a subinterval J is well-defined. In fact, it can be shown that this first return map is also an interval exchange map.

Lemma 2.2 (Viana [**12**] and Veech [**10**]). *Fix $f \in \mathcal{E}$. Given any subinterval $J = [a, b)$, there exists a partition $\{J_j : 1 \le j \le k\}$ of J and integers $n_1, \ldots, n_k \ge 1$, such that*

(1) $f^i(J_j) \cap J = \emptyset$ for all $0 < i < n_j$ and $1 \le j \le k$;

(2) each $f^{n_j}|_{J_j}$ is a translation from J_j to some subinterval of J;

(3) the subintervals $f^{n_j}(J_j), 1 \le j \le k$ are pairwise disjoint.

Corollary 2.3 ([**12**]). *The union $\widehat{J}$ of all forward iterates of J is an f-invariant element of $\mathcal{P}$.*

Proof. [**12**] From the lemma,

$$\widehat{J} = \bigcup_{j=1}^{k} \bigcup_{i=0}^{n_j-1} f^i(J_j) \in \mathcal{P}. \ \square$$

Proposition 2.4. *Let $f \in \mathcal{E}$ and suppose $x \in \mathbb{T}^1$ is such that the orbit $\mathcal{O}(x) := \mathcal{O}_f(x)$ is infinite. Let J be the right-closure of $\mathcal{O}(x)$:*

$$J := \{y \in \mathbb{T}^1 : [y, y+\epsilon) \cap \mathcal{O}(x) \neq \emptyset, \forall\, \epsilon > 0\}.$$

Then J is a member of $\mathcal{P}$.

Proof. The set J is f-invariant, since f is right-continuous. In addition, J contains no periodic points: if $y \in \mathrm{Per}(f)$, then by Lemma 2.1, $[y, y+\epsilon) \subseteq \mathrm{Per(f)}$ for some $\epsilon > 0$. Since all points in $\mathcal{O}(x)$ have infinite orbits, $[y, y+\epsilon) \cap \mathcal{O}(x)$ is empty, and it follows that $y \notin J$.

Let $A = \mathbb{T}^1 \setminus (J \cup \mathrm{Per}(f))$; it suffices to show $A \in \mathcal{P}$. Consider a point $y \in A$ (if A is empty then $J = \mathbb{T}^1 \setminus \mathrm{Per}(f) \in \mathcal{P}$). Then there exists some $\epsilon > 0$ such that $[y, y+\epsilon)$ contains no points in $\mathcal{O}(x)$, and consequently $[y, y+\epsilon) \subseteq A$. By Corollary 2.3 and the invariance of A, the point y is contained in an f-invariant set $B \in \mathcal{P}$, such that $B \subseteq A$.

Moreover, the set B must contain a discontinuity point of f. Since all points in B have infinite orbits under f, there must be a left boundary point $p \in B$ which is mapped by f to an interior point of B. Since points immediately to the left of p are not in B, they cannot be mapped into the interior of B, which implies that f is discontinuous at p.

It is now possible to construct A as a finite union of members of $\mathcal{P}$. Pick any $y_1 \in A$ and construct an f-invariant $B_1 \in \mathcal{P}$ as above, such that $y_1 \in B_1 \subseteq A$. If $B_1 = A$, then we're done. If not, choose some $y_2 \in A \setminus B_1$. Since $B_1 \in \mathcal{P}$, there is some $[y_2, y_2 + \epsilon)$ contained in $A \setminus B_1$, and it is possible to construct an f-invariant set $B_2 \in \mathcal{P}$, disjoint from B_1, such that $y_2 \in B_2 \subseteq A$. Seemingly, this process will continue as long as the B_k do not cover all of A. However, the process must terminate after a finite number of steps, since each B_k must contain a discontinuity point of f. Thus A is a finite union of the sets B_k, which implies that $A \in \mathcal{P}$. $\square$

Corollary 2.5. *Suppose that $\mathcal{O}(x)$ is infinite, and let J be its right-closure. If $y \in J$, then $\mathcal{O}(y)$ is infinite and dense in J. Thus, $f|_J$ is minimal.*

Proof. It has already been shown that J contains no periodic points. The right-closure K of $\mathcal{O}(y)$ is an f-invariant member of $\mathcal{P}$, and it is contained in J since $\mathcal{O}(y)$ is a subset of J. Since $\mathcal{O}(x)$ is dense in J, it follows from f-invariance that K coincides with J. $\square$

From the above results, it can be seen that for any $f \in \mathcal{E}$, there is a finite partition of $\mathbb{T}^1$,

$$\mathbb{T}^1 = \left(\coprod_{i=1}^{k} J_i \right) \amalg \operatorname{Per}(f),$$

where each J_i is an f-invariant member of $\mathcal{P}$, such that $f|_{J_i}$ is minimal. The sets J_i, as well as the restricted mappings $f|_{J_i}$, are called the *minimal components* of f. An interval exchange f is said to be in *normal form* if its minimal components are intervals $J_i = [a_{i-1}, a_i)$, where $0 = a_0 < a_1 < \cdots < a_k \leq 1$. Note that after replacing f by a conjugate in $\mathcal{E}$, it may be assumed that f is in normal form.

There is a generalization of the notion of rotation number which is defined for interval exchange transformations. Let $f = f_{(\pi,\lambda)} \in \mathcal{E}$ be defined by a partition vector $\lambda = (\lambda_1, \ldots, \lambda_k)$ and a translation vector $\omega = (\omega_1, \ldots, \omega_k) = \Omega_\pi(\lambda)$, as described in Chapter 1. The *scissors invariant* of f, denoted $\phi(f)$, is defined by

$$\phi(f) = \sum_{i=1}^{k} \lambda_i \otimes_{\mathbb{Q}} \omega_i \in \mathbb{R} \otimes_{\mathbb{Q}} \mathbb{R}.$$

Lemma 2.6 (Arnoux [**1**]). *$\phi : \mathcal{E} \to \mathbb{R} \otimes_{\mathbb{Q}} \mathbb{R}$ is a group homomorphism.*

Proof. The translation function $\omega_f(x) = f(x) - x$ is piecewise constant, so an equivalent definition of ϕ is

$$\phi(f) = \int_0^1 (1 \otimes_{\mathbb{Q}} \omega_f(x))\ d\mu(x).$$

This formulation shows that ϕ is independent of the choice of λ and ω. Moreover, since all interval exchanges preserve the Lebesgue measure μ,

$$\begin{aligned}
\phi(gf) &= \int_0^1 (1 \otimes_{\mathbb{Q}} \omega_{gf}(x))\ d\mu(x) = \int_0^1 [1 \otimes_{\mathbb{Q}} (\omega_g(fx) + \omega_f(x))]\ d\mu(x) \\
&= \int_0^1 (1 \otimes_{\mathbb{Q}} \omega_g(x))\ d(f_*\mu)(x) + \int_0^1 (1 \otimes_{\mathbb{Q}} \omega_f(x))\ d\mu(x) \\
&= \phi(g) + \phi(f). \quad \square
\end{aligned}$$

To motivate the claim that ϕ may be viewed as a type of rotation number, consider the scissors invariant of a rotation r_α. As illustrated in Figure 2.1, $\lambda = (1 - \alpha, \alpha)$ and

$\omega = (\alpha, \alpha - 1)$ for this map. Thus,

$$\phi(r_\alpha) = (1-\alpha) \otimes \alpha + \alpha \otimes (\alpha - 1) = 1 \otimes \alpha - \alpha \otimes 1 = 1 \wedge_{\mathbb{Q}} \alpha.$$

In particular, $\phi(r_\alpha) = 0$ if $\alpha \in \mathbb{Q}$, so one may think of ϕ as detecting the rotation contributed by the infinite-order dynamics of the map.

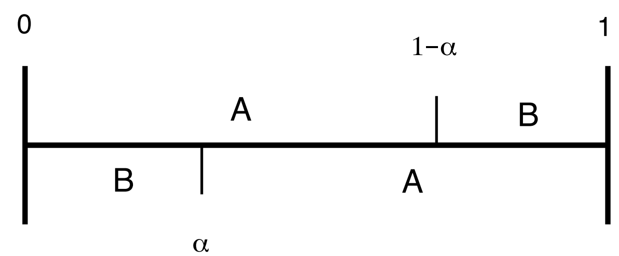

Figure 2.1. The rotation r_α

Given real numbers α and β, such that $0 \le \alpha \le \beta \le 1$, let $r_{\alpha,\beta}$ denote the *restricted rotation by* α *on the interval* $[0, \beta)$. Precisely, $r_{\alpha,\beta}$ is defined by the permutation $\pi = (1,2) \in \Sigma_3$ and the partition vector $\lambda = (\beta - \alpha, \alpha, 1 - \beta)$. See Figure 2.2 for a diagram of $r_{\alpha,\beta}$. The map $r_{\alpha,\beta}$ has translation vector $\omega = (\alpha, \alpha - \beta, 0)$, and a short calculation shows that $\phi(r_{\alpha,\beta}) = \beta \wedge_{\mathbb{Q}} \alpha$.

Lemma 2.7 (Arnoux [**1**])**.** *$\mathcal{E}$ is generated by the collection of restricted rotations.*

Corollary 2.8. *The homomorphism ϕ takes values in the alternating algebra $\mathbb{R} \wedge_{\mathbb{Q}} \mathbb{R}$.*

Proof of Lemma 2.7. Given an interval exchange f, it is possible to construct a finite sequence of restricted rotations which successively reverse the translations induced by f

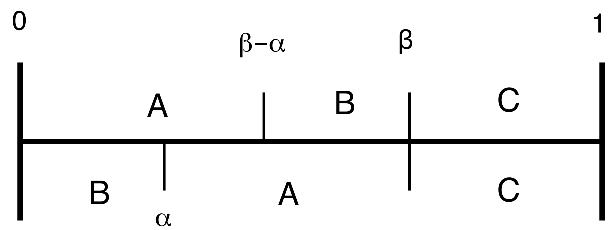

Figure 2.2. The restricted rotation $r_{\alpha,\beta}$

on its partition intervals. Suppose f is defined by $\lambda = (\lambda_1, \ldots, \lambda_k)$ and $\omega = (\omega_1, \ldots, \omega_k)$, and let $I_j = [\beta_{j-1}, \beta_j)$ be the associated partition intervals, as defined in (1.1). Since I_k is the right-most partition interval, $\omega_k \leq 0$. Let $\alpha_k = -\omega_k$, and consider the composition $f_k = r_{\alpha_k} f$. By its construction, f_k fixes the inteval I_k. Moreover, all other partition intervals I_j are mapped by f to subintervals of one of the two partition intervals of r_{α_k}, and thus the map f_k acts by translating the same collection of partition intervals $\{I_j\}$.

In general, for $1 \leq j \leq k-2$, recursively define

$$\alpha_{k-j} = -\left(\omega_{k-j} + \alpha_k + \alpha_{k-1} + \cdots + \alpha_{k-j+1}\right),$$

and let

$$f_{k-j} = r_{\alpha_{k-j},\beta_{k-j}} \circ r_{\alpha_{k-j+1},\beta_{k-j+1}} \circ \cdots \circ r_{\alpha_k} \circ f.$$

In short, α_{k-j} is defined to be the opposite of the necessarily nonpositive translation that f_{k-j+1} applies to I_{k-j}. By induction it is seen that f_{k-j} fixes $I_{k-j}, \ldots, I_k$. Thus, the map f_2 is the identity, which exhibits the original map f as a product of restricted rotations. $\square$

CHAPTER 3

Continuous Interval Exchange Actions

In order to consider continuous actions of $\mathbb{R}$, the group $\mathcal{E}$ is given a metric space structure. For $x = \tilde{x} + \mathbb{Z}$ and $y = \tilde{y} + \mathbb{Z}$ in $\mathbb{T}^1$, let

$$\rho_{\mathbb{T}^1}(x, y) = \min_{k,j \in \mathbb{Z}} \{|(\tilde{x} + k) - (\tilde{y} + j)|\}.$$

The function $\rho_{\mathbb{T}^1}$ is a multiple of the metric induced by the standard embedding of the circle into $\mathbb{R}^2$. Given two interval exchanges f and g, the distance between them is defined as

$$\rho(f, g) = \int_{\mathbb{T}^1} \rho_{\mathbb{T}^1}(fx, gx) d\mu(x),$$

where μ is Lebesgue measure . This function is essentially the L^1 distance between functions on $\mathbb{T}^1$ which take values in $\mathbb{T}^1$, so it is not surprising that

Lemma 3.1. *The function ρ is a metric on $\mathcal{E}$.*

Proof. $\rho(f, g) \geq 0$ for any $f, g \in \mathcal{E}$, since the function $\rho_{\mathbb{T}^1}$ is nonnegative. If $\rho(f, g) = 0$, then it follows that $\rho_{\mathbb{T}^1}(fx, gx) = 0$, for μ-a.e. $x \in \mathbb{T}$. However, since f and g are interval exchanges, $\rho_{\mathbb{T}^1}(fx, gx)$ is piecewise constant. Thus $\rho_{\mathbb{T}^1}(fx, gx) = 0$ for all x, which implies $f = g$.

The function ρ is symmetric since the metric $\rho_{\mathbb{T}^1}$ is symmetric. The triangle inequality follows similarly:

$$\rho(f,h) = \int \rho_{\mathbb{T}^1}(f,h) \leq \int \rho_{\mathbb{T}^1}(f,g) + \int \rho_{\mathbb{T}^1}(g,h) = \rho(f,g) + \rho(g,h). \quad \square$$

The next issue is to show that the group operations of $\mathcal{E}$ are continuous under this metric.

Proposition 3.2. *The metric space $(\mathcal{E}, \rho)$ is a topological group.*

Proof. Suppose there are convergent sequences $f_n \to f$ and $g_n \to g$. It will be shown that the products $f_n g_n$ converge to fg. Given $\epsilon > 0$, choose some $\delta_0 < \min\{\epsilon^2, \frac{1}{100}\}$. Since f is a finite piecewise isometry, it is possible to choose $\delta_1 < \delta_0$ such that

$$\mu\left(\left\{x \in \mathbb{T}^1 : f \text{ is a translation on } \left(x - \sqrt{\delta_1}, x + \sqrt{\delta_1}\right)\right\}\right) > 1 - \sqrt{\delta_0}.$$

Let the set in the previous expression be denoted by C.

By convergence,

$$\rho(f_n, f) < \delta_0 \quad \text{and} \quad \rho(g_n, g) < \delta_1 \tag{3.1}$$

hold for all sufficiently large n. Fix one such n, and define

$$A = \left\{x \in \mathbb{T} : \rho_{\mathbb{T}^1}(f_n x, fx) > \sqrt{\delta_0}\right\},$$

$$B = \left\{x \in \mathbb{T} : \rho_{\mathbb{T}^1}(g_n x, gx) > \sqrt{\delta_1}\right\}.$$

Then $\mu(A) \leq \sqrt{\delta_0}$ and $\mu(B) \leq \sqrt{\delta_1}$, since otherwise (3.1) would be contradicted.

An estimate of $\rho(f_n g_n, fg)$ is achieved by estimating the measure of the set

$$D = \left\{ x \in \mathbb{T}^1 : \rho_{\mathbb{T}^1}(f_n g_n x, fgx) > 2\sqrt{\delta_0} \right\}.$$

Since

$$\rho_{\mathbb{T}^1}(f_n g_n x, fgx) \leq \rho_{\mathbb{T}^1}(f_n g_n x, fg_n x) + \rho_{\mathbb{T}^1}(fg_n x, fgx),$$

it follows that if $\rho_{\mathbb{T}^1}(f_n g_n x, fgx) > 2\sqrt{\delta_0}$, then either

$$\rho_{\mathbb{T}^1}(f_n g_n x, fg_n x) > \sqrt{\delta_0}, \text{ or} \tag{3.2}$$

$$\rho_{\mathbb{T}^1}(fg_n x, fgx) > \sqrt{\delta_0}. \tag{3.3}$$

The condition (3.2) is satisfied only for $g_n x \in A \Leftrightarrow x \in g_n^{-1}(A)$, and $\mu(g_n^{-1}(A)) \leq \sqrt{\delta_0}$, since g_n preserves μ. The condition (3.3) fails if $gx \in C$ and $\rho_{\mathbb{T}^1}(gx, g_n x) \leq \sqrt{\delta_1}$; i.e., if $x \in g^{-1}(C)$ and $x \in \mathbb{T}^1 \setminus B$. Thus the set on which (3.3) holds is contained in $B \cup (\mathbb{T}^1 \setminus g^{-1}(C))$, and

$$\mu\left(B \cup (\mathbb{T}^1 \setminus g^{-1}(C))\right) < \sqrt{\delta_1} + \sqrt{\delta_0} < 2\sqrt{\delta_0}.$$

Thus, $\mu(D) < 3\sqrt{\delta_0}$; consequently, using the fact that $\rho_{\mathbb{T}^1} \leq 1$,

$$\begin{aligned}\rho(f_n g_n, fg) &= \int_D \rho_{\mathbb{T}^1}(f_n g_n x, fgx) \; + \; \int_{\mathbb{T}^1 \setminus D} \rho_{\mathbb{T}^1}(f_n g_n x, fgx) \\ &< 3\sqrt{\delta_0} + 2\sqrt{\delta_0} < 5\epsilon.\end{aligned}$$

This estimate holds for all sufficiently large n, and thus composition in $\mathcal{E}$ is continuous with respect to the metric ρ.

It remains to show that inversion in $\mathcal{E}$ is continuous. To see this, note that the metric ρ is invariant under *right* translation in the group, since all interval exchange transformations preserve Lebesgue measure. Thus,

$$\rho(f, id) = \rho(id, f^{-1}).$$

Consequently, if $f_n \to id$, then $f_n^{-1} \to id$. Thus inversion is continuous at the identity. In general, if $f_n \to f$, the continuity of composition implies that $f^{-1} f_n \to id$. But then $f_n^{-1} f \to id$, and applying the continuity of composition again yields $f_n^{-1} \to f^{-1}$, as desired. □

Remark: The metric ρ is invariant under conjugation by a rotation $r = r_\alpha$ of $\mathbb{T}^1$. Using the right-invariance of ρ for any interval exchange and the fact that $\rho_{\mathbb{T}^1}$ is rotation invariant,

$$\rho(r^{-1} f r, r^{-1} g r) = \rho(r^{-1} f, r^{-1} g) =$$

$$\int \rho_{\mathbb{T}^1}(r^{-1} f, r^{-1} g) = \int \rho_{\mathbb{T}^1}(f, g) = \rho(f, g).$$

Discussing interval exchanges as being defined on the interval $[0, 1)$ amounts to choosing a base point of $\mathbb{T}^1$; consequently, the choice of base point has no effect on the structure of $\mathcal{E}$ as a metric space.

Recall the definition in Chapter 1 of the interval exchange $f_{(\pi,\lambda)}$, where $\pi \in \Sigma_n$ and $\lambda \in \Lambda_n$. One initial drawback of these coordinates is that they are not unique. For instance, the data

$$\pi = (1,2) \in \Sigma_2, \lambda = (1-\alpha, \alpha), \text{ and}$$

$$\pi' = (1,2,3) \in \Sigma_3, \lambda' = (1-\alpha-\beta, \beta, \alpha)$$

both define the rotation r_α; see Figure 3.1.

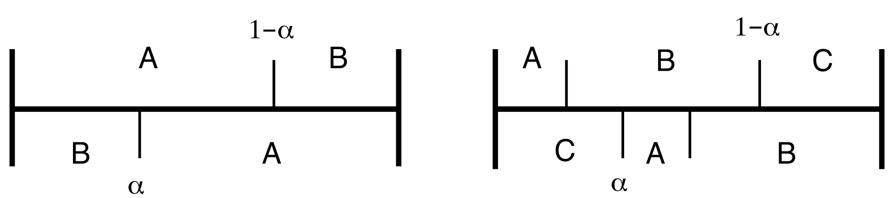

Figure 3.1. Multiple choices of π and λ defining the same map

In order to have unique coordinates associated to an interval exchange, a restriction is made on the permutations that are used.

Definition: A permutation $\pi \in \Sigma_n$ is said to be *unpartitioned* if

$$\pi(j+1) \neq \pi(j) + 1, \text{ for all } j \text{ such that } 1 \leq j \leq n-1.$$

π is said to be *partitioned* if equality holds above for some j.

Observe that if π is partitioned (suppose $\pi(j+1) = \pi(j)+1$), then, for any $\lambda \in \Lambda_n$, the intervals I_j and I_{j+1} are adjacent and in the same order before and after the application of the map $f_{(\pi,\lambda)}$. Thus, $f_{(\pi,\lambda)}$ restricts to a translation on $I_j \cup I_{j+1}$, and $f_{(\pi,\lambda)}$ is continuous at β_j. In fact,

Lemma 3.3. *$\pi \in \Sigma_n$ is unpartitioned if and only if, for any $\lambda \in \Lambda_n$, the interval exchange $f_{(\pi,\lambda)}$ is discontinuous, as a map $[0,1) \to [0,1)$, at precisely each of $\beta_1, \ldots, \beta_{n-1}$.*

Proof. It remains to consider the situation when π is unpartitioned. If $f = f_{(\pi,\lambda)}$ is continuous at β_j, then both I_j and I_{j+1} are translated the same distance by f. This is only the case if $\pi(j+1) = \pi(j) + 1$, which is impossible if π is unpartitioned. Thus f is discontinuous at each β_j, $1 \leq j \leq n-1$, and f has no further discontinuities, since f restricts to a translation on each I_j. $\square$

Remark: The remainder of this chapter is concerned with the structure of $\mathcal{E}$ in terms of the coordinates $(\pi \in \Sigma_n, \lambda \in \Lambda_n)$. Consequently, it is more natural here to consider an interval exchange f as being a map $[0,1) \to [0,1)$. In particular, for the remainder of this chapter, the discontinuities of f refer to discontinuities of $f : [0,1) \to [0,1)$.

Having restricted to unpartitioned permutations, it follows that

Proposition 3.4. *For any interval exchange $f \in \mathcal{E}$, there exists a unique $n \in \mathbb{N}$, a unique unpartitioned permutation $\pi \in \Sigma_n$, and a unique $\lambda \in \Lambda_n$, such that $f = f_{(\pi,\lambda)}$.*

Proof. Let

$$0 < \beta_1 < \beta_2 < \ldots < \beta_{n-1} < 1$$

be the finite set of points in $(0,1)$ at which f is discontinuous as a map $[0,1) \to [0,1)$; this defines n. Setting $\beta_0 = 0$ and $\beta_n = 1$, define $\lambda \in \Lambda_n$ by

$$\lambda_j = \beta_j - \beta_{j-1}, \quad j = 1, \ldots, n.$$

The permutation π is defined as follows. For some $i \in \{1, \ldots, n\}$, the point β_{i-1} is mapped to zero by f. Let $i = j_1$ be the index for which this happens, and define $\pi(j_1) = 1$. Next, one of the remaining β_{i-1}, $i \in \{1, \ldots, \widehat{j_1}, \ldots, n\}$, must be mapped to λ_{j_1}. Call this index j_2, and define $\pi(j_2) = 2$. Continuing in this manner defines the permutation π; i.e., π describes the reordering of the points $\beta_{i-1}, 1 \leq i \leq n$, induced by the map f. By construction $f = f_{(\pi,\lambda)}$, and π is unpartitioned by Lemma 3.3, since f is discontinuous at precisely $\beta_1, \ldots, \beta_{n-1}$.

The uniqueness of $(\pi \in \Sigma_n, \lambda \in \Lambda_n)$ follows since they were constructed using intrinsic features of the transformation f. More precisely, suppose $(\pi' \in \Sigma_{n'}, \lambda' \in \Lambda_{n'})$ also induce the map f. The permutations π and π' are both unpartitioned, so it follows that $n = n'$, since both numbers are counting the number of points at which f is discontinuous. These points of discontinuity also determine the lengths of the intervals which are permuted by f, and thus $\lambda = \lambda'$. Finally, the above characterization of π as the reordering of the left endpoints of the partition induced by λ implies $\pi = \pi'$. $\square$

Given some unpartitioned $\pi \in \Sigma_n$, the mapping $\lambda \mapsto f_{(\pi,\lambda)}$ defined on Λ_n may be extended to a mapping

$$\Gamma_\pi : \overline{\Lambda}_n \to \mathcal{E},$$

where $\overline{\Lambda}_n$ is the closed simplex

$$\overline{\Lambda}_n = \left\{ \lambda = (\lambda_1, \ldots, \lambda_n) : \lambda_i \geq 0, \sum \lambda_i = 1 \right\}.$$

The space $\overline{\Lambda}_n$ is given the subspace topology induced by $\overline{\Lambda}_n \subseteq \mathbb{R}^n$. Points on the boundary of $\overline{\Lambda}_n$ map under Γ_π to an exchange of n intervals where some of the intervals are empty. Consequently, the transformation $\lambda \mapsto \Gamma_\pi(\lambda) = f_{(\pi,\lambda)}$ is well-defined by (1.1)-(1.3) for all $\lambda \in \overline{\Lambda}_n$, provided that a degenerate interval $[\beta, \beta)$ is interpreted as the empty set. As π ranges over all unpartitioned permutations, the maps Γ_π produce a reasonable system of coordinates on the metric space $(\mathcal{E}, \rho)$. In particular,

Proposition 3.5. *For any unpartitioned permutation π, the map Γ_π is continuous, and the restriction $\Gamma_\pi|_{\Lambda_n}$ is a homeomorphism onto its image. Moreover, for any $f \neq id$ in $\mathcal{E}$, there is a unique unpartitioned π, such that f is the image of an interior point of $\overline{\Lambda}_n$ under Γ_π.*

Proof. The last assertion is just a restatement of Proposition 3.4. To show the continuity of $\Gamma_\pi : \overline{\Lambda}_n \to \mathcal{E}$, suppose that $\lambda^{(n)} \to \lambda$ in $\overline{\Lambda}_N$, and let $f^{(n)}$ and f denote $\Gamma_\pi(\lambda^{(n)})$ and $\Gamma_\pi(\lambda)$, respectively. Given some $\epsilon > 0$, for all sufficiently large n,

$$\left| \lambda_j - \lambda_j^{(n)} \right| < \frac{\epsilon}{N}, \quad j = 1, \ldots, N.$$

Then, comparing the difference between boundary points of the partition intervals of $\Gamma_\pi(\lambda^{(n)})$ and $\Gamma_\pi(\lambda)$, we have

$$\left| \beta_j - \beta_j^{(n)} \right| = \left| \sum_{k=1}^{j} \lambda_k - \sum_{k=1}^{j} \lambda_k^{(n)} \right| \leq \sum_{k=1}^{j} \left| \lambda_k - \lambda_k^{(n)} \right| < \epsilon.$$

Thus, for sufficiently large n, the partition intervals I_j and $I_j^{(n)}$ overlap up to a set of small measure. That is,

$$\mu\left(I_j \setminus I_j^{(n)}\right) < 2\epsilon.$$

Next, observe that the translation vectors $\omega^{(n)} = \Omega_\pi(\lambda^{(n)})$ converge to $\omega = \Omega_\pi(\lambda)$, since the map Ω_π is linear. Thus, for all sufficiently large n,

$$\left|\omega_j - \omega_j^{(n)}\right| < \epsilon.$$

Therefore,

$$\rho(f, f^n) = \sum_{j=1}^{N} \int_{I_j} \rho_{\mathbb{T}^1}(fx, f^{(n)}x)\, d\mu(x) =$$

$$\sum_{j=1}^{N} \left(\int_{I_j \cap I_j^{(n)}} \rho_{\mathbb{T}^1}(fx, f^{(n)}x)\, d\mu(x) + \int_{I_j \setminus I_j^{(n)}} \rho_{\mathbb{T}^1}(fx, f^{(n)}x)\, d\mu(x) \right) =$$

$$\sum_{j=1}^{N} \int_{I_j \cap I_j^{(n)}} \rho_{\mathbb{T}^1}(x + w_j, x + w_j^{(n)})\, d\mu(x) + \sum_{j=1}^{N} \int_{I_j \setminus I_j^{(n)}} \rho_{\mathbb{T}^1}(fx, f^{(n)}x)\, d\mu(x).$$

The first term in this last expression is bounded by ϵ, since $\rho_{\mathbb{T}^1}(x+w_j, x+w_j^{(n)}) < \epsilon$ on the sets $I_j \cap I_j^{(n)}$. The second term is bounded by $N\epsilon$, since $\rho_{\mathbb{T}^1} \leq 1/2$ and $\mu(I_j \setminus I_j^{(n)}) < 2\epsilon$. Thus $\rho(f, f^{(n)}) < (N+1)\epsilon$ for all sufficiently large n, which proves that Γ_π is continuous.

It remains to show that Γ_π is a homeomorphism when restricted to Λ_n. It has already been established that this restriction is injective, so it suffices to show that

$$\Gamma_\pi^{-1} : \Gamma_\pi(\Lambda_n) \to \Lambda_n$$

is continuous. Let $f = f_{(\pi,\lambda)} \in \Gamma_\pi(\Lambda_n)$, and let $f_n = f_{(\pi,\lambda^{(n)})}$ be a sequence in $\Gamma_\pi(\Lambda_n)$ which converges to f. By the compactness of $\overline{\Lambda}_n$, the sequence $\lambda^{(n)}$ must have a limit point $v \in \overline{\Lambda}_n$. Let $\lambda^{(n_i)}$ be a subsequence which converges to v. The map $\Gamma_\pi : \overline{\Lambda}_n \to \mathcal{E}$ is continuous, so f_{n_i} converges to $f_{(\pi,v)}$. Thus $f_{(\pi,v)} = f_{(\pi,\lambda)}$, and it follows that $v = \lambda$ by Proposition 3.4. Moreover, it follows that $v = \lambda$ is the unique limit point of $\lambda^{(n)}$. In other words, $\lambda^{(n)}$ converges to λ in Λ_n, which implies Γ_π^{-1} is continuous on the image $\Gamma_\pi(\Lambda_n)$. $\square$

3.1. Classification of interval exchange $\mathbb{R}$-actions

Having given $\mathcal{E}$ a topological group structure, it is possible to classify the continuous group homomorphisms $\mathbb{R} \to \mathcal{E}$. As an initial example of such an action, consider the following actions of $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$. For a given $\lambda \in \Lambda_n$, define the points β_j and the intervals $I_j = [\beta_{j-1}, \beta_j)$ as before. Given some $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{T}^n$, associate the following interval exchange:

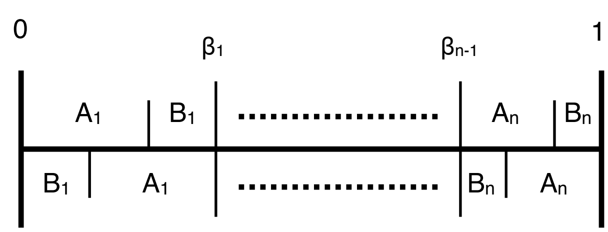

Figure 3.2. The rotation map $f_{\alpha,\lambda}$

$$f_{\alpha,\lambda} : x \mapsto \begin{cases} x + \lambda_j \widetilde{\alpha}_j, & x \in [\beta_{j-1},\, \beta_j - \lambda_j \widetilde{\alpha}_j) \\ x + \lambda_j \widetilde{\alpha}_j - \lambda_j, & x \in [\beta_j - \lambda_j \widetilde{\alpha}_j,\, \beta_j), \end{cases}$$

where $\widetilde{\alpha}_j \in \mathbb{R}$ is the unique representative of α_j in the interval $[0, 1)$; see Figure 3.2.

The map $f_{\alpha,\lambda}$ restricts to a rotation by $\lambda_j \widetilde{\alpha}_j$ on each interval I_j. Since disjoint rotations commute, for any fixed λ the mapping

$$\alpha \mapsto f_{\alpha,\lambda}$$

is an injective homomorphism $\mathbb{T}^n \to \mathcal{E}$. One may define an interval exchange action of $\mathbb{R}$ by restricting any such action of a torus to a one-parameter subgroup. Such an action is called a *one-parameter rotation action*, since $\mathbb{R}$ acts on the circle by rotating a collection of invaraint subintervals. Note that under a one-parameter rotation action, all orbits are periodic; however, a one-parameter rotation action can be faithful if the periods of two invariant subintervals are not rationally related.

Rotation actions are essentially the only possible one-parameter actions.

Theorem 1.3. *Up to conjugacy in $\mathcal{E}$, any continuous homomorphism $\mathbb{R} \to \mathcal{E}$ is a one-parameter rotation action.*

If a one-parameter subgroup f_t in $\mathcal{E}$ is conjugate to the image of a rotation action, then essentially the maps f_t still act by rotations on a collection of invariant subintervals. These subintervals may now be split into finitely many pieces and reordered within $\mathbb{T}^1$, but this is the only effect that conjugacy in $\mathcal{E}$ can have upon a rotation action. To make a precise statement, let Fix denote the set of global fixed points for a given one-parameter subgroup f_t.

Lemma 3.6. *A one-parameter subgroup f_t of $\mathcal{E}$ is conjugate to a rotation action if and only if $\underline{Fix} \in \mathcal{P}$ and for all but finitely many $x \in [0,1)$, there exists $\alpha_x \in \mathbb{R}$ and $\epsilon_x > 0$, such that*

$$f_t(x) = x + t\alpha_x, \quad \textit{if } |t| < \epsilon_x.$$

In short, if the orbit of all but finitely many points are locally those of a rotation action, then the action is globally a rotation action.

Proof. It is easy to see that if f_t is conjugate to a rotation action, then the action of f_t satisfies the local condition stated in the lemma. Conversely, suppose $\underline{\mathrm{Fix}} \in \mathcal{P}$ and f_t is locally a rotation at all but finitely many points. Let $0 = x_0 < x_1 < x_2 < \cdots < x_n = 1$ be the exceptional points, including all boundary points of Fix. Over all $x \in (x_{i-1}, x_i)$ the rotation speed α_x must be constant, since by definition it is locally constant. It remains to consider the behavior of f_t at the exceptional points.

Consider the interval $I_j = [x_{j-1}, x_j)$ of length λ_j. By replacing t with $-t$, it may be assumed that α_j, the constant rotation speed on the interior points of I_j, is nonnegative. The maps f_t are all right-continuous at x_{j-1}, and any interval $(x_{j-1}, x_{j-1}+\delta)$, $\delta \ll \lambda_j$, is translated a distance of $t\alpha_j$ under f_t, for sufficiently small nonnegative t. It follows that $f_t(x_{j-1}) = x_{j-1} + t\alpha_j$, for sufficiently small nonnegative t. Moreover, the group f_t acts (locally) on all of (x_{j-1}, x_j) by translation by $t\alpha_j$, and so

$$f_t(x_{j-1}) = x_{j-1} + t\alpha_j, \quad \text{for } 0 \le t < \frac{\lambda_j}{\alpha_j}.$$

Consider what happens for $t = \lambda_j/\alpha_j$. First, suppose the interval I_j is f_t-invariant. If $y = f_{(\lambda_j/\alpha_j)}(x_{j-1})$ is in the interior of I_j, then $[y, x_j)$ is a periodic orbit properly contained in the orbit of x_{j-1}, which is impossible. Thus, $f_{(\lambda_j/\alpha_j)}(x_{j-1}) = x_{j-1}$, and the action of f_t on I_j is globally a rotation action.

In general, if I_j is not invariant, suppose that y is in the interior of some I_k, with $k \neq j$, since $y \in I_j$ would imply invariance. For small $t < 0$, $f_t(y)$ is in I_k, since the f_t locally act as a rotation on the interior of I_k. However, $y = f_{(\lambda_j/\alpha_j)}(x_{j-1})$, and it is also the case that $f_t(y)$ is in I_j for small $t < 0$, which is a contradiction. Thus $f_{(\lambda_j/\alpha_j)}(x_{j-1})$ must be some other exceptional point x_{k-1}. The transformations f_t all preserve Lebesgue measure, and by right-continuity $f_t(x_{k-1}) = x_{k-1} + t\alpha_k$ for small $t \geq 0$ or small $t \leq 0$. Thus, if $f_{(\lambda_j/\alpha_j)}(x_{j-1}) = x_{k-1}$, then $\alpha_j = \alpha_k$. Consequently, the orbit of x_{j-1} is a finite union of intervals I_k, each of which has the same rotation speed. After applying a suitable conjugacy, each invariant collection of these intervals may be reassembled into a single invariant subinterval on which the conjugate action is a rotation action. $\square$

It is possible to improve on the previous lemma's recharacterization of rotation actions. In particular, the condition of a point x having an orbit locally given by a rotation action may be weakened to the condition that $t \mapsto f_t(x)$ is continuous for t in a neighborhood of zero. If x satisfies this weaker condition, it is said to have a *locally continuous orbit* under f_t.

Proposition 3.7. *A one-parameter subgroup f_t is a rotation action if and only if $\underline{Fix} \in \mathcal{P}$ and for all but finitely many $x \in [0, 1)$, the function $\mathbb{R} \to [0, 1)$ defined by*

$$t \mapsto f_t(x)$$

is continuous in some open neighborhood around $t = 0$.

Proof. By applying the previous lemma, it suffices to show that if x has a locally continuous orbit, then there exists α_x, such that $f_t(x) = x + t\alpha_x$ for t in a neighborhood of zero. If $f_t(x) = x$ for all t in a neighborhood, then x is a global fixed point of the action, and $\alpha_x = 0$ will suffice.

Suppose that x is not a global fixed point, and assume that the orbit $t \mapsto f_t(x)$ is continuous for $t \in [-\epsilon, \epsilon]$. By reducing ϵ if necessary, it may be assumed that the function $t \mapsto f_t(x)$ is one-to-one on $[-\epsilon, \epsilon]$. To see this, suppose that

$$f_{t_1}(x) = f_{t_2}(x), \quad \text{for } -\epsilon \leq t_1 < t_2 \leq \epsilon.$$

Then $f_{t_2 - t_1}(x) = x$, and it follows that x has a periodic f_t-orbit, with period less than 2ϵ. However, if x is periodic under f_t with period less than 2ϵ for arbitrarily small ϵ, then it follows that x is a global fixed point, since x has a locally continuous orbit. It has been assumed that x is not a global fixed point, and consequently $t \mapsto f_t(x)$ is one-to-one on $[-\epsilon, \epsilon]$, for some suitably small choice of ϵ.

By reversing the parameter t, it may be assumed that $t \mapsto f_t(x)$ is increasing on $[-\epsilon, \epsilon]$. Define $\alpha \neq 0$ to satisfy

$$f_\epsilon(x) = x + \epsilon\alpha.$$

Next, for any $n \in \mathbb{N}$, consider the increasing sequence of points

$$x, f_{\frac{\epsilon}{n}}(x), f_{\frac{2\epsilon}{n}}(x), \ldots, f_{\epsilon}(x).$$

Since all f_t preserve Lebesgue measure,

$$\mu\left(\left[f_{\frac{(j-1)\epsilon}{n}}(x),\ f_{\frac{j\epsilon}{n}}(x)\right)\right) = \mu\left(\left[f_{\frac{(k-1)\epsilon}{n}}(x),\ f_{\frac{k\epsilon}{n}}(x)\right)\right),$$

for all $0 \leq j, k \leq n$. Thus

$$f_{\frac{j\epsilon}{n}}(x) = x + \alpha\left(\frac{j\epsilon}{n}\right), \quad j = 1, \ldots, n.$$

Thus, $f_t(x) = x + t\alpha$ for a dense set of $t \in [0, \epsilon]$, and by continuity of the orbit this holds at all $t \in [0, \epsilon]$. A similar argument shows $f_t(x) = x + t\alpha'$ for all $t \in [-\epsilon, 0]$. Finally, $\alpha = \alpha'$ is a consequence of the fact that the f_t preserve Lebesgue measure. $\square$

Therefore, to prove Theorem 1.3 it suffices to prove the following:

Proposition 3.8. *If f_t is a continuous one-parameter subgroup of $\mathcal{E}$, then all but finitely many $x \in [0, 1)$ have locally continuous orbits and $\underline{Fix} \in \mathcal{P}$.*

Define the function $\delta : \mathcal{E} \to \mathbb{N}$ by

$$\delta(f) = (\text{card}\,\{x \in (0, 1) : f \text{ is discontinuous at } x\}) + 1.$$

Thus, $\delta(f)$ returns the number of discontinuities of f, considered as a map $[0, 1) \to [0, 1)$, where 0 is counted as a discontinuity. Observe that if $\delta(f) = n$, then f is in the image of the interior of the $(n-1)$-dimensional simplex $\overline{\Lambda}_n$ under the parametrization Γ_π, for some unique unpartitioned $\pi \in \Sigma_n$. The simplex $\overline{\Lambda}_n$ is compact, the number of unpartitioned permutations in Σ_k for $k \leq n$ is finite, and the parametrizations Γ_π are continuous, so the set

$$K_n = \{f \in \mathcal{E} : \delta(f) \leq n\}$$

is compact. Therefore, if $\delta(f) = n$,

$$\rho(f, K_{n-1}) > 0.$$

Consequently, for all g in some neighborhood of f, $\delta(g) \geq \delta(f)$. In other words, the function δ is lower semicontinuous.

Lemma 3.9. *For any continuous one-parameter subgroup f_t, the function $t \mapsto \delta(f_t)$ is bounded on any compact subset of $\mathbb{R}$.*

Proof. Since f_t is a one-parameter subgroup, $f_{s+t} = f_s \circ f_t$ for all $s, t \in \mathbb{R}$. Consequently,

$$\delta(f_{s+t}) \leq \delta(f_s) + \delta(f_t), \quad s, t \in \mathbb{R}. \tag{3.4}$$

This inequality records the fact that a composition of two interval exchange maps cannot have more discontinuities than occur over both of its factors. From this inequality, it also follows that

$$\delta(f_{s+t}) \geq |\delta(f_s) - \delta(f_t)|, \quad s, t \in \mathbb{R}. \tag{3.5}$$

By (3.4), if $\delta(f_t)$ is bounded for $t \in [-\epsilon, \epsilon]$, then $\delta(f_t)$ is bounded on all compact subsets. Thus, if $\delta(f_t)$ is unbounded on some compact subset, then $\delta(f_t)$ is unbounded in any neighborhood of zero. In fact, the inequality (3.5) further implies that $\delta(f_t)$ is unbounded in any neighborhood of any $t \in \mathbb{R}$.

This local unboundedness and the semicontinuity of δ cannot coexist. To derive a contradiction, suppose that $\delta(f_t)$ is unbounded in any neighborhood of any t. Let

$$A_n = \{t \in \mathbb{R} : \delta(f_t) \leq n\}.$$

By the lower semicontinuity of δ, the sets A_n are closed, and their complements

$$B_n = \{t \in \mathbb{R} : \delta(t) > n\}$$

are open. If δ is locally unbounded at every point, each set B_n is dense in $\mathbb{R}$. However,

$$\bigcap B_n = \{t \in \mathbb{R} : \delta(f_t) > n, \text{ for all } n \in \mathbb{N}\} = \varnothing,$$

which is a contradiction by the Baire Category Theorem. Thus, $\delta(f_t)$ must be bounded on any compact subset of $\mathbb{R}$. $\square$

Proof of Proposition 3.8. Applying Lemma 3.9, let

$$n = \max\{\delta(f_t) : t \in [-1, 1]\}.$$

By the lower semicontinuity of δ, the set $\{t \in [-1, 1] : \delta(f_t) = n\}$ is relatively open in $[-1, 1]$. Therefore, there exists some $t_0 \in (-1, 1)$ and $\epsilon > 0$, such that $\delta(f_t) = n$

for $t \in (t_0 - \epsilon, t_0 + \epsilon)$. Let $\pi \in \Sigma_n$ be the unique unpartitioned permutation such that $f_{t_0} \in \Gamma_\pi(\Lambda_n)$. By Proposition 3.4, the sets $\Gamma_\sigma(\Lambda_n)$ are pairwise disjoint, for σ ranging over Σ'_n, the unpartitioned permutations in Σ_n. The sets $\Gamma_\sigma(\overline{\Lambda}_n)$ are compact and do not contain f_{t_0} if $\sigma \neq \pi$, so

$$\rho(f_{t_0}, \Gamma_\sigma(\overline{\Lambda}_n) > 0),$$

for all $\sigma \neq \pi$ in Σ'_n. Consequently, after possibly replacing ϵ by a smaller value, it follows that $f_t \in \Gamma_\pi(\Lambda_n)$, for all $t \in (t_0 - \epsilon, t_0 + \epsilon)$.

In this situation, it can be seen that the paths

$$t \mapsto f_t(x)$$

are continuous in a neighborhood of t_0 for all but finitely many points, namely the discontinuity points of f_{t_0}. For $t \in (t_0 - \epsilon, t_0 + \epsilon)$ let $\lambda^{(t)} \in \Lambda_n$ be such that

$$f_t = \Gamma_\pi \left(\lambda^{(t)}\right),$$

where the $\lambda^{(t)}$ vary continuously in Λ_n. Thus, if x is an interior point of the interval I_j induced by f_{t_0}, then

$$t \mapsto f_t(x) = x + \Omega_\pi \left(\lambda^{(t)}\right)_j$$

is continuous in a neighborhood of t_0. Since f_{-t_0} is continuous at all but a finite number of points, the path $t \mapsto f_t(x)$ is continous in a neighborhood of zero for all but finitely many points.

It remains to consider the set of global fixed points for f_t. As before, define $\beta_j^{(t)}$ in terms of $\lambda^{(t)}$ and let

$$I_j^{(t)} = \left[\beta_{j-1}^{(t)}, \beta_j^{(t)}\right).$$

Suppose the interior of $I_j^{(t_0)}$ contains a global fixed point x. Then for all t in some $(t_0 - \epsilon, t_0 + \epsilon)$, the point x is located in the interval $I_j^{(t)}$. Thus, for each t in $(t_0 - \epsilon, t_0 + \epsilon)$, the interval $I_j^{(t)}$ is fixed by f_t. In addition, the intervals $I_{j-1}^{(t)}$ and $I_{j+1}^{(t)}$ cannot be fixed by f_t, since otherwise π would be partitioned. As a result, the boundary points $\beta_{j-1}^{(t)}$ and $\beta_j^{(t)}$ must be constant over $t \in (t_0 - \epsilon, t_0 + \epsilon)$, since otherwise there would be points fixed by f_t for t in some nonempty, proper open set of $\mathbb{R}$, which is impossible. Thus, the set $\underline{\text{Fix}}$ of global fixed points for f_t is a finite union of intervals $I_j^{(t_0)}$, which implies that $\underline{\text{Fix}}$ is a member of $\mathcal{P}$. $\square$

CHAPTER 4

Discontinuity Growth

We now investigate the growth rate of the number of discontinuities for iterates f^n of an interval exchange f. In the context of this and all following chapters, consider an interval exchange as being defined on the circle $\mathbb{T}^1$; let $d(f)$ denote the number of discontinuities of $f : \mathbb{T}^1 \to \mathbb{T}^1$.

For a map $f \in \mathcal{E}$, let $\widetilde{D}(f)$ denote the set of points at which f is discontinuous. Let $D(f)$ be those discontinuities of f which are not periodic points:

$$D(f) = \widetilde{D}(f) \setminus \mathrm{Per}(f).$$

In other words, $D(f)$ is the set of discontinuity points of f which have infinite f-orbits. If f is an infinite-order map where $\widetilde{D}(f)$ is nonempty, then $D(f)$ is also nonempty. To see this, first note that $\widetilde{D}(f) = D(f)$ if $\mathrm{Per}(f)$ is empty. Otherwise, the set $\mathbb{T}^1 \setminus \mathrm{Per}(f)$ of points with nonperiodic orbits is a member of $\mathcal{P}$ and a proper subset of $\mathbb{T}^1$. Some left boundary point of this set is mapped into its interior by f, since there are only finitely many such boundary points. This point is necessarily an infinite-order discontinuity point, which shows $D(f)$ is nonempty.

If $x \in D(f)$, a simple but key observation is the following: both the forward and backward orbits of x eventually consist entirely of points at which f is continuous. This is a consequence of the fact that $D(f)$ is a finite set of points with nonperiodic orbits.

Moreover, for each $x \in D(f)$, there is some $k \geq 0$, such that $f^{-k}(x)$ is the last point of $D(f)$ encountered in the backward orbit of x; in particular, f is continuous at all negative iterates $f^{-n}(x)$, such that $n > k$.

Definition: A point $x \in D(f)$ is a *fundamental discontinuity* (of f) if f is continuous at all negative iterates of x:

$$\{f^{-i}(x)\}_{i=1}^{\infty} \subseteq \mathbb{T}^1 \setminus \widetilde{D}(f).$$

Thus, any discontinuity in $D(f)$ is either fundamental or a forward iterate of a fundamental discontinuity. In particular, the set of fundamental discontinuities is nonempty whenever f has points with infinite orbits. Fundamental discontinuities of f are so-named because they completely control the asymptotics of $d(f^n)$. To state this connection precisely, additional notation is needed. For any interval exchange $f \in \mathcal{E}$, let f_- denote the left-continuous form of f. That is,

$$f_-(x) = \begin{cases} \lim_{y \to x^-} f(y), & \text{if } f \text{ is discontinuous at } x; \\ f(x), & \text{otherwise.} \end{cases}$$

Similarly, $f_+ = f$ is used to denote the original right-continuous map when the distinction between f_- and f_+ is to be emphasized. Observe that $(f_-)^n = (f^n)_-$ and $(f_+)^n = (f^n)_+$ for all integers n; such compositions are thus denoted f_-^n and f_+^n without ambiguity.

For any $x \in \mathbb{T}^1$ and any interval exchange f, let $f(x^+)$ denote the right-hand limit of f at x:

$$f(x^+) = \lim_{y \to x^+} f(y).$$

Similary, $f(x^-)$ denotes the left-hand limit of f at x. The sets

$$\{f^n(x^+)\}_{n=0}^{\infty} \quad \text{and} \quad \{f^n(x^-)\}_{n=0}^{\infty}$$

are respectively called the *right* and *left (forward) orbits* of x.

It is sensible to refer to these sets as orbits. Since $f = f_+$ is right-continous,

$$f^n(x^+) = \lim_{y \to x^+} f^n(y) = f_+^n(x).$$

Similary,

$$f^n(x^-) = \lim_{y \to x^-} f^n(y) = f_-^n(x).$$

In essence, the right orbit of x is the orbit of x under the convention that f is right-continuous, and the left orbit of x is its orbit under the opposite convention.

Let $x \in D(f)$ be a fundamental discontinuity. By the definition of $D(f)$, the right orbit $\{f^n(x^+)\}$ is nonperiodic. Since x is fundamental, f is continuous at all points in the negative orbit of x. Thus, the left and right orbits coincide for negative iterates of f, and it follows that the left orbit of x is also nonperiodic. Therefore, since the set $\widetilde{D}(f)$ is finite and the left and right forward orbits of x are nonperiodic, both of these forward orbits eventually consist entirely of points at which f is continuous.

In light of this observation, let n_0 be the smallest positive integer such that, for every fundamental discontinuity x, f is continuous at $f^n(x^+)$ and $f^n(x^-)$ for all $n \geq n_0$. This integer n_0 is called the *stabilization time* of f. For any fundamental $x \in D(f)$, one of two things may happen at the stabilization time:

$$\text{either } f^{n_0}(x^+) = f^{n_0}(x^-), \quad \text{or} \quad f^{n_0}(x^+) \neq f^{n_0}(x^-).$$

In the first of these cases, $f^n(x^+) = f^n(x^-)$ for all $n \geq n_0$, since f is always continuous at these points. In this situation it is said that x is an *eventually resolving* fundamental discontinuity. Similarly, in the second case $f^n(x^+) \neq f^n(x^-)$ for all $n \geq n_0$. In this situation it is said that x is a *nonresolving* fundamental discontinuity.

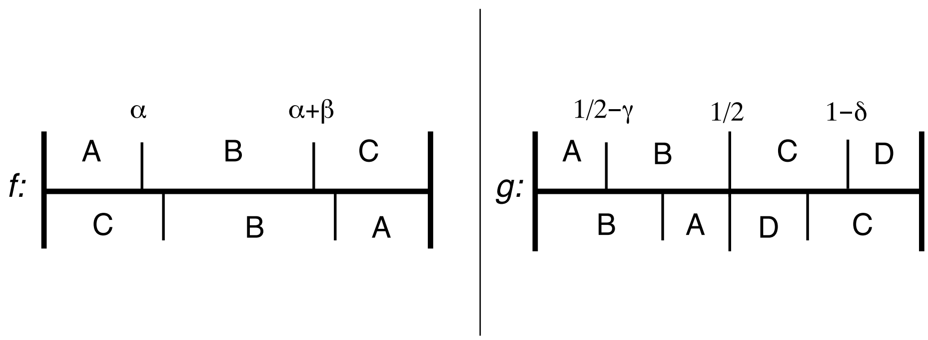

Figure 4.1. Nonresolving and resolving fundamental discontinuities

To illustrate these two types of fundamental discontinuity, consider the following examples. Let $f = f_{(\pi,\lambda)}$ be the 3-interval exchange represented by $\pi = (1,3) \in \Sigma_3$ and $\lambda = (\alpha, \beta, 1-\alpha-\beta)$, where α, β, and 1 are rationally independent; see Figure 4.1. The map f has fundamental discontinuities at α and $\alpha+\beta$. Since $f(\alpha+\beta) = 0$, the stabilization time of f is two, and it may be checked that both fundamental discontinuities are nonresolving. Upon computing large iterates of f, it is observed that $d(f^n)$ seems to grow on the order of $2n$.

Next, let $g = g_{(\pi',\lambda')}$ be defined by $\pi' = (1,2)(3,4) \in \Sigma_4$ and $\lambda' = \left(\frac{1}{2} - \gamma, \gamma, \frac{1}{2} - \delta, \delta\right)$, where γ and δ are both irrational numbers in $[0, 1/2)$; see Figure 4.1. The map g is a product of two restricted irrational rotations. It has fundamental discontinuities at $\frac{1}{2} - \gamma$ and $1 - \delta$, both of which are eventually resolving. For instance,

$$g^2\left(\left(\frac{1}{2} - \gamma\right)^+\right) = g(0^+) = \gamma,$$

$$g^2\left(\left(\frac{1}{2} - \gamma\right)^-\right) = g\left(\frac{1}{2}^-\right) = \gamma,$$

and g is continuous at γ and all of its forward iterates. All iterates g^n are products of two infinite-order rotations, which implies that $d(g^n) = 4$ for all $n \geq 1$.

Proposition 4.1. *For any infinite-order $f \in \mathcal{E}$, exactly one of the following holds:*

a) *All fundamental discontinuities of f are eventually resolving, in which case $d(f^n)$ is bounded independently of n.*

b) *The map f has at least one nonresolving fundamental discontinuity, in which case $d(f^n)$ has linear growth.*

Proof. It remains only to show that the fundamental discontinuities of f completely control the asymptotics of $d(f^n)$. First, suppose that $x \in D(f)$ is not a fundamental discontinuity, in which case some negative iterate $y = f^{-j}(x)$, such that $j \geq 1$, is a fundamental discontinuity. The right and left orbits of x are only relevant to the continuity status of finitely many preimages of x. In particular, the f^n-continuity status of a point p is determined by the right and left orbits of the first discontinuity which p meets in its forward f-orbit. For instance, if k is such that $j < k < n$, then the right and left orbits of $f^{-k}(x)$ are determined by those of the fundamental discontinuity $y = f^{-j}(x)$:

$$f^n((f^{-k}x)^+) = f_+^n(f^{-k}x) = f_+^{n-k+j}(f^{-j}x) = f^{n-k+j}((f^{-j}x)^+) = f^{n-k+j}(y^+),$$

and similarly, since $f_- = f$ at points where f is continuous,

$$f^n((f^{-k}x)^-) = f_-^n(f^{-k}x) = f_-^{n-k+j}(f^{-j}x) = f^{n-k+j}(y^-).$$

Consequently, the number of points whose f^n-continuity status is determined by a non-fundamental discontinuity of f is bounded independently of n. Likewise, there is a uniform bound to the number of points whose f^n-continuity status is determined by the periodic discontinuities of f.

In general, the discontinuities of f^n are contained in the set

$$\bigcup_{i=0}^{n-1} f^{-i}\left(\widetilde{D}(f)\right).$$

In other words, a point x is a discontinuity of f^n only if the forward f-orbit of x reaches $\widetilde{D}(f)$ in fewer than n iterates. Consequently, to show that $d(f^n)$ is bounded when all fundamental discontinuities of f are eventually resolving, it suffices to show that the number of f^n-discontinuities in the set $\{x, f^{-1}(x), \ldots, f^{-(n-1)}(x)\}$ is bounded independently of n, for each eventually resolving fundamental discontinuity x.

Suppose x is an eventually resolving fundamental discontinuity of f, and let n_0 be the stabilization time of f. Then,

$$f^n(x^+) = f^n(x^-)$$

for all $n \geq n_0$. Thus, for any $n \geq n_0$, and for all k such that $0 \leq k \leq (n - n_0)$, f^n is continuous at the point $f^{-k}(x)$. To see this, note that f is continuous at all forward iterates of $f^{-k}(x)$ until its orbit reaches x. Thus, the right and left orbits of $f^{-k}(x)$ are determined by the right and left orbits of x. In fact,

$$f^n((f^{-k}x)^+) = f^{n-k}(x^+) = f^{n-k}(x^-) = f^n((f^{-k}x)^-),$$

where the middle equality holds because $n - k \geq n_0$. It follows that for all $n \geq n_0$, $\{x, f^{-1}(x), \ldots, f^{-(n-1)}(x)\}$ contains at most n_0 discontinuities of f^n. Therefore, $d(f^n)$ is bounded if all fundamental discontinuities of f are eventually resolving.

Alternately, suppose that x is a nonresolving fundamental discontinuity. Then

$$f^n(x^+) \neq f^n(x^-)$$

for all $n \geq n_0$. By an argument similar to the one above, it follows that f^n is discontinuous at $f^{-k}(x)$, for all k such that $0 \leq k \leq (n - n_0)$. Thus, for any $n \geq n_0$, f^n has at least $n - n_0$ discontinuities in the set $\{x, f^{-1}(x), \ldots, f^{-(n-1)}(x)\}$. Since n_0 is fixed relative to n, this implies that $d(f^n)$ has linear growth. Consequently, the presence of at least one nonresolving fundamental discontinuity implies linear growth of $d(f^n)$. $\square$

This proposition raises the question of what may be said about an infinite-order interval exchange f for which $d(f^n)$ is bounded. An example of such a map is an irrational rotation, or, more generally, any map which is conjugate to an irrational rotation. A result of Li [**6**] asserts that with some additional conditions, these are the only such examples.

Recall from Chapter 3 the function $\delta : \mathcal{E} \to \mathbb{N}$,

$$\delta(f) = (\text{card}\,\{x \in (0,1)|\, f : [0,1) \to [0,1) \text{ is discontinuous at } x\}) + 1,$$

which counts the number of intervals exchanged when $f = f_{(\pi,\lambda)}$ is uniquely represented by a length vector $\lambda \in \mathbb{R}^{\delta(f)}$ and an unpartitioned permutation $\pi \in \Sigma_{\delta(f)}$.

Theorem (Li [**6**])**.** *An interval exchange map f is conjugate to an irrational rotation if and only if the following hold:*

(i) *$\delta(f^n)$ is bounded by some positive integer N,*

(ii) *f^n is minimal for all $n \in \mathbb{N}$, and*

(iii) *There are integers $k > 0$ and $M \geq 2^{N^3+3N^2}$ such that $\widetilde{f} = f^k$ satisfies $\delta(\widetilde{f}) = \delta(\widetilde{f}^2) = \cdots = \delta(\widetilde{f}^M)$.*

The quantity $\delta(f)$ is essentially counting the discontinuities of f as a map on $[0,1)$, so generally $\delta(f)$ does not equal $d(f)$. Moreover, these two quantities do not differ by the same constant for all $f \in \mathcal{E}$. For example, $\delta(r_\alpha) = 2$ and $d(r_\alpha) = 0$, while $\delta(r_{\alpha,\beta}) = d(r_{\alpha,\beta}) = 3$, if $\beta < 1$.

To state a version of Li's theorem in terms of $d(f)$, a closer comparison of these functions is needed. The function $\delta(f)$ always counts $0 \in [0,1)$ as a discontinuity, but 0 may be a continuity point of $f : \mathbb{T}^1 \to \mathbb{T}^1$. If $f = f_{(\pi,\lambda)}$, then $f : \mathbb{T}^1 \to \mathbb{T}^1$ is continuous at 0 if and only if I_1 and I_n, the first and last partition intervals induced by λ, are mapped adjacent to each other with their order reversed. Symbolically,

$$0 \notin \widetilde{D}(f) \iff \pi(n) + 1 = \pi(1).$$

In addition, the point $f^{-1}(0)$ is always counted as a discontinuity by δ, but it may also be a continuity point when f is considered on the circle. Suppose $f^{-1}(0)$ is the boundary point between the partition intervals I_{k-1} and I_k. Then $f : \mathbb{T}^1 \to \mathbb{T}^1$ is continuous at $f^{-1}(0)$ if and only if I_{k-1} and I_k are mapped to the right and left ends of the interval, respectively. That is,

$$f^{-1}(0) \notin \widetilde{D}(f) \iff \pi(k-1) = n \text{ and } \pi(k) = 1.$$

Thus, for a given f it may be the case that $f : \mathbb{T}^1 \to \mathbb{T}^1$ is continuous at either one or both of the points 0 and $f^{-1}(0)$, which means that $d(f)$ is, respectively, one or two less than $\delta(f)$. This discrepancy is of no account in considering the boundedness of $d(f^n)$ or $\delta(f^n)$, but it presents problems for the condition (iii) in Li's theorem, where it is required that $\delta(f^k)$ is constant over many values of k. Conceivably, one might observe $d(f^k)$ to be constant over a large range of k, while $\delta(f^k)$ is changing quite frequently.

This difficulty may be overcome by a good choice of the base point on $\mathbb{T}^1$. Recall that presenting an interval exchange as defined on $[0, 1)$ amounts to specifying a base point 0 at which to cut the circle. Choosing a new base point amounts to conjugation by a rotation; since the conclusion of Li's theorem is up to conjugacy, there is no loss in changing the base point.

Given any infinite-order $f \in \mathcal{E}$, select a nonperiodic point p such that $f : \mathbb{T}^1 \to \mathbb{T}^1$ is continuous at all points on the orbit $\mathcal{O}_f(p)$. Then for any $n \in \mathbb{N}$, f^n is continuous at p and $f^{-n}(p)$. Choosing this point p to be the base point 0, it follows that $d(f^n) = \delta(f^n) - 2$ for all $n \in \mathbb{N}$. Thus, up to conjugacy, showing that $d(f^k)$ is constant is equivalent to showing that $\delta(f^k)$ is constant.

Theorem (Alternate Version of Li's Theorem)**.** *An interval exchange map f is conjugate to an irrational rotation if and only if the following hold:*

(i) *$d(f^n)$ is bounded by some integer N,*

(ii) *f^n is minimal for all $n \in \mathbb{N}$, and*

(iii) *after redefining the base point (conjugating by a rotation) so that f is continuous on the orbit of 0, there are integers $k > 0$ and $M \geq 2^{N^3+3N^2}$ such that $\widetilde{f} = f^k$ satisfies $d(\widetilde{f}) = d(\widetilde{f}^2) = \cdots = d(\widetilde{f}^M)$.*

It is natural to consider to what extent this result holds if it is only assumed that $d(f^n)$ is bounded. It is possible to find examples for which condition (ii) fails when $d(f^n)$ is bounded. See Figure 4.2 for an example where f is minimal and f^2 is not. This complication may be removed by replacing f with a higher iterate.

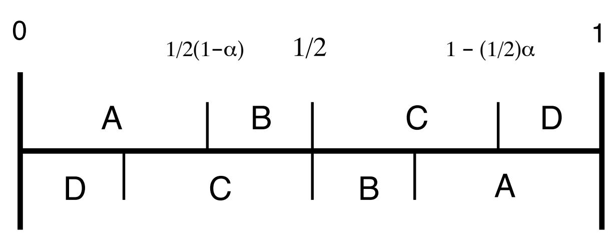

Figure 4.2. A minimal map with a non-minimal square

Lemma 4.2. *Suppose that f is minimal and $d(f^n)$ is bounded. Then for some $k \in \mathbb{N}$, all iterates f^{nk} are minimal when restricted to each minimal component of f^k.*

Proof. Suppose that no such integer k exists. Then f is minimal, but for some $k_1 = m_1 > 1$, f^{m_1} has multiple minimal components. Suppose that this integer k_1 has been chosen to be as small as possible. Since f and f^{m_1} commute, f permutes the minimal components of f^{m_1}. This permutation induced by f is transitive since f is minimal, and it must be of order m_1, by the choice of m_1. Thus f^{m_1} has exactly m_1 minimal components, denoted by $J_{1,1}, \ldots, J_{1,m_1}$.

It has been assumed that no power f^k is minimal for all iterates when restricted to any of its minimal components. Thus, there exists a smallest integer $k_2 > 1$ such that f^{m_2}, where $m_2 = k_1 k_2$, is not minimal when restricted to some minimal component of f^{m_1}. Suppose this component is $J_{1,1}$. The map f^{m_1} permutes the minimal components of f^{m_2} which are contained in $J_{1,1}$; f^{m_1} acts minimally on $J_{1,1}$, and so this permutation must be transitive and have order k_2. Additionally, the original map f permutes the minimal components of f^{m_2}; since it also transitively permutes the minimal components of f^{m_1}, it follows that f^{m_2} must have k_2 minimal components in each one of the $J_{1,j}$. Thus f^{m_2} has exactly $k_2 k_1 = m_2$ minimal components.

By the assumption that no k satisfies the conclusion of the lemma, this process may continue indefinitely. In particular, there are sequences of integers $k_i > 1$ and $m_i = \Pi_{j=1}^{i} k_j$, such that f^{m_i} has exactly m_i minimal components.

To arrive at a contradiction with the hypothesis that $d(f^n)$ is bounded, observe that if a map g has $m > 1$ minimal components $J_1, \ldots, J_m$, then it must have at least m discontinuities. To see this, consider a left-boundary point x_i of J_i. Since some iterate of x_i will eventually fall in the interior of J_i, it follows that the orbit of each x_i must contain a discontinuity of g. Since these orbits are distinct, the map must have at least

m discontinuities. Thus, it is impossible for f^n to have an arbitrarily large number of minimal components if $d(f^n)$ is bounded. $\square$

Lemma 4.3. *Suppose f has infinite order and $d(f^n)$ is bounded. Then for some $N \in \mathbb{N}$, $d(f^{nN})$ is constant over all $n \in \mathbb{N}$.*

Proof. By initially replacing f with an iterate, it may be assumed that $\mathrm{Per}(f) = \mathrm{Fix}(f)$. Let $A = \{x_1, \ldots, x_k\}$ be the fundamental discontinuities of f. Since $d(f^n)$ is bounded, each x_i is eventually resolving. All other non-fixed discontinuities are found in the forward orbits of the fundamental discontinuities. Choose an integer $N_1 > 0$ such that any point of $D(f)$ may be reached from A by at most N_1 iterates of f. Such an N_1 exists since the set $D(f)$ is finite.

Choose N_2 such that the right and left orbits of all discontinuities in $D(f)$ are stabilized after N_2 iterates of f. In the situation where a non-fundamental discontinuity $x \in D(f)$ is fixed from the left (i.e., $f(x^-) = x$), it is the case that $f^n(x^+) \neq f^n(x^-)$ for all $n \geq 1$, since the right orbit of x is nonperiodic. Otherwise, both the right and left forward orbits of any $x \in D(f)$ eventually consist entirely of continuity points of f. Thus, the notion of stabilization time is well-defined for all $x \in D(f)$, not just the fundamental discontinuities.

Finally, choose $N > N_1 + N_2$. It will be shown that $d(f^{kN})$ is constant over all $k \in \mathbb{N}$. Since $\mathrm{Per}(f) = \mathrm{Fix}(f)$, the set of fixed discontinuities is identical for all nonzero iterates of f. Thus, it suffices to only consider the set $D(f^N)$ of non-fixed discontinuities of f^N; any such point must be of the form $f^{-i}(x)$, where $x \in D(f)$ and $0 \leq i < N$. The non-fixed discontinuities of f are contained in the set

$$\bigcup_{i=0}^{N_1} f^i(A).$$

It follows that the non-fixed discontinuities of f^N are contained in the set

$$\bigcup_{i=-(N-1)}^{N_1} f^i(A).$$

Let

$$P = D(f^N) \cap \left(\bigcup_{i=1}^{N_1} f^i(A) \right), \quad Q = D(f^N) \cap \left(\bigcup_{i=-(N-1)}^{0} f^i(A) \right).$$

Consider a point $x \in P$. Since this is a discontinuity of f^N, the forward f-orbit of x must encounter a discontinuity of f whose right and left orbits control the continuity status of x. Since x is in P, this controlling discontinuity is non-fundamental, and it must be encountered within N_1 iterates of f. Since $N > N_1 + N_2$, the inequality between $f^N(x^+)$ and $f^N(x^-)$ occurs at a place where the right and left orbits of the controlling discontinuity have already stabilized. Thus, the right and left orbits of the controlling discontinuity are nonresolving, and it follows that x is a discontinuity of f^n, for all $n \geq N$. In particular, x is a discontinuity for all f^{kN}. Similarly, if a point in $\bigcup_{i=1}^{N_1} f^i(A)$ is a point of continuity for f^N, it must be a point of continuity for all f^{kN}.

Next, consider a point $x \in Q$. This point is a discontinuity of f^N whose f-orbit is controlled by a fundamental discontinuity x_i of f. Observe that under f^{kN}, the image of x is contained in

$$\bigcup_{i=(k-1)N+1}^{kN} f^i(A).$$

Consequently, if $k \geq 2$, the right and left orbits of x (which are controlled by the right and left orbits of the fundamental discontinuity x_i) have resolved once f^{kN} iterates have been applied to x. Thus x, as well as all other points in $\bigcup_{i=-(N-1)}^{0} f^i(A)$, are continuity points for f^{kN}, $k \geq 2$. In general, the f^{kN}-continuity status of any point in $\bigcup_{i=-(kN-1)}^{0} f^i(A)$ is controlled by the right and left orbits of a fundamental discontinuity. Since these orbits all resolve within N iterates, it follows that

$$D(f^{kN}) \cap \left(\bigcup_{i=-(kN-1)}^{0} f^i(A) \right) = f^{-(k-1)N}(Q).$$

The previous two paragraphs have shown that

$$D(f^{kN}) = P \cup f^{-(k-1)N}(Q).$$

This union is always disjoint, so the size of $D(f^{kN})$ is constant over all $k \in \mathbb{N}$. Since

$$d(f^{kN}) = \left|D(f^{kN})\right| + \left|\{\text{fixed discontinuities of } f^{kN}\}\right|,$$

and the second term in this sum is constant over all iterates of f, it follows that $d(f^{kN})$ is constant over all $k \in \mathbb{N}$, as desired. $\square$

Theorem 1.5. *Let f be an infinite-order map such that $d(f^n)$ is bounded. Then some f^k, $k \geq 1$, is conjugate to a product of disjointly supported irrational restricted rotations.*

Proof. Since f may be replaced with a power of itself, it may be assumed that $\mathrm{Per}(f) = \mathrm{Fix}(f)$. By applying Lemma 4.2 to the restriction of f on each of its minimal components, there is some k such that any f^{nk} is minimal when restricted to any minimal component $J_1, \ldots, J_m$ of f^k. Since the result is up to conjugacy in $\mathcal{E}$, it may be assumed that f^k is in normal form.

Consider the restriction f_j of f^k to its minimal component J_j. This map f_j is considered as being defined on $\mathbb{T}^1$; imagine that the subinterval J_j has been cut out and glued together into a circle. It suffices to show that f_j is conjugate to an irrational rotation. The function $d(f_j^n)$ is bounded, and by construction f_j^n is minimal for all $n > 0$. Moreover, by Lemma 4.3, there exists N_j such that $d(f_j^{nN_j})$ is constant for all n. Consequently, the alternate version of Li's theorem applies to the restricted map f_j, and so this map is conjugate to an irrational rotation. $\square$

CHAPTER 5

Distortion Elements in $\mathcal{E}$

The results concerning the rate of discontinuity growth may be applied to the question of whether distortion elements exist in the group $\mathcal{E}$.

Definition: Let G be a finitely generated group with a chosen set of generators $S = \{g_1, \dots, g_n\}$. For any $g \in G$, let $|g|_S$ denote the word length of g in terms of the generators; i.e., $|g|_S$ is the length of the shortest word in the generators and their inverses that represents g. An element $f \in G$ is a *distortion element* in G if f has infinite order and

$$\liminf_{n \to \infty} \frac{|f^n|_S}{n} = 0.$$

If G is not finitely generated, $f \in G$ is said to be a distortion element in G if it is a distortion element in some finitely generated subgroup.

As an example, consider the Heisenberg group H, which may be presented by generators a, b, c, subject to the commutator relations

$$[a, b] = c, \quad [a, c] = [b, c] = \mathrm{id}. \tag{5.1}$$

More concretely, the group H is isomorphic to the group of strictly upper triangular matrices with integer coefficients. The generators in the above presentation may be represented by

$$a = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad b = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \quad c = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The element c is a distortion element in H. By the relations (5.1) it may be computed that

$$[a^n, b^m] = c^{nm},$$

and consequently $|c^N|_S$ is on the order of $\sqrt{N}$.

Theorem 1.4. *Distortion elements do not exist in $\mathcal{E}$.*

Proof. By Proposition 4.1, the iterates of any interval exchange transformation have linear or bounded discontinuity growth. If an interval exchange f has linear discontinuity growth, then it cannot be a distortion element. To see this, suppose $S = \{g_1, \ldots, g_k\}$ generate a subgroup of $\mathcal{E}$ in which f is a distortion element, and suppose there is a constant $C > 0$ such that $d(f^n) \geq Cn$ for all sufficiently large n. Let

$$M = \max_i \{d(g_i)\}.$$

Then

$$d(f^n) \leq M|f^n|_S,$$

since f^n may be expressed as a composition of $|f^n|_S$ elements from the set of generators. Since f is a distortion element,

$$\liminf_{n\to\infty} \frac{|f^n|_S}{n} = 0.$$

Thus, for infinitely many n,

$$|f^n|_S \leq \frac{Cn}{2M},$$

and

$$d(f^n) \leq M|f^n|_S \leq \frac{C}{2}n.$$

This contradicts the assumption that $d(f^n) \geq Cn$ for all sufficiently large n, and it follows that an interval exchange which has linear discontinuity growth cannot be a distortion element.

Suppose now that $f \in \mathcal{E}$ is a distortion element, meaning f is a distortion element in some finitely generated subgroup

$$G = \langle g_1, \ldots, g_n \rangle < \mathcal{E}.$$

By the previous paragraph, f must have bounded discontinuity growth. By Theorem 1.5, after conjugation and replacing f by an iterate it may be assumed that f is a product of disjointly supported infinite-order rotations. Suppose that one of these rotations is the restricted rotation $r_{\alpha,\beta}$. Assume first that $\alpha \notin \mathbb{Q}$. The case where α is rational and β is necessarily irrational will be addressed later.

Let V be the $\mathbb{Q}$-vector subspace of $\mathbb{R}/\mathbb{Q}$ which is generated by the set of distances that an element of G may translate a point of $\mathbb{T}^1$. The space V is a finite-dimensional $\mathbb{Q}$-vector space, since it is generated by the finite set of translations induced by the generators $g_1, \ldots, g_n$.

Fix a basis for V which includes α (precisely, the class $[\alpha] \in \mathbb{R}/\mathbb{Q}$), and let

$$P_\alpha : V \to \mathbb{Q}$$

be the linear map which returns the α-coordinate of a vector with respect to this basis. For a point $p \in \mathbb{T}^1$ define a function $\phi_{\alpha,p} : G \to \mathbb{Q}$ by

$$\phi_{\alpha,p}(g) = P_\alpha(g(p) - p)$$

Roughly, $\phi_{\alpha,p}(g)$ measures the α-component by which g translates p. Since the distortion element f rotates by α on the interval $[0, \beta)$ and β is not a rational multiple of α, it follows that

$$\phi_{\alpha,0}(f^n) = n, \text{ for all } n \in \mathbb{Z}.$$

Now consider the generators $g_1, \ldots, g_n$. Each one of these maps only induces finitely many distinct translations (i.e., the components of ω_{g_i}), and consequently there is a constant $M > 0$ such that

$$|\phi_{\alpha,p}(g_i)| \leq M, \quad 1 \leq i \leq n, \quad p \in \mathbb{T}^1.$$

Thus, for any $g \in G$,

$$|\phi_{\alpha,0}(g)| \leq M|g|_S.$$

Since $\liminf |f^n|_S/n = 0$,

$$|f^n|_S \leq \frac{n}{2M},$$

for infinitely many sufficiently large n. But then, for such n,

$$n = \phi_{\alpha,0}(f^n) \leq M|f^n|_S \leq \frac{n}{2},$$

which is a contradiction. Thus, f cannot be a distortion element in $\mathcal{E}$.

Suppose we are in the case where f is a product of infinite-order rotations, but all of these rotations are by some $\alpha_i \in \mathbb{Q}$ (mod $\beta_i \notin \mathbb{Q}$). The argument above fails in this case because the map $\phi_{\alpha,p}$ cannot be defined when α is rational. However, a similar argument can be made by tracking the contribution from the irrational number β. Choose a new basis for V which contains β, and consider the map $\phi_{\beta,0}$. The rotation by α mod β on $[0, \beta)$ will contribute $(-1)\beta$ for every loop the iterated rotation makes around this interval. Thus, there exists some constant $C > 0$, (for instance, any $C > \beta/\alpha$), such that

$$|\phi_{\beta,0}(f^n)| \geq \frac{n}{C},$$

for all sufficiently large n. It is still the case that there is a constant $M > 0$ such that

$$\phi_{\beta,p}(g_i) \leq M, \quad 1 \leq i \leq n, \quad p \in \mathbb{T}^1.$$

Consequently, a similar contradiction may be reached in the case where $\alpha \in \mathbb{Q}$, and it follows that no distortion elements exist in $\mathcal{E}$. $\square$

As a consequence of Theorem 1.4, it can be shown that a large class of lattices in higher-rank semisimple Lie groups do not act faithfully via interval exchanges.

Theorem (Lubotzky-Mozes-Raghunathan [**7**])**.** *Suppose Γ is a non-uniform irreducible lattice in a semisimple Lie group $\mathcal{G}$ with $\mathbb{R}$-rank ≥ 2. Suppose further that $\mathcal{G}$ is connected, with finite center and no nontrivial compact factors. Then Γ has distortion elements, in fact, elements whose word length growth is at most logarithmic.*

If Γ is any lattice satisfying the conditions of the previous theorem, then for any homomorphism $\Gamma \to \mathcal{E}$, all distortion elements in Γ must map to finite-order interval exchanges. Consequently, an infinite-order element maps to a finite-order one, and the action is not faithful. In fact, a much stronger conclusion can be made regarding such actions. A group is called *almost simple* if its normal subgroups are all either finite or finite-index.

Theorem (Margulis [**8**])**.** *Suppose* Γ *is an irreducible lattice in a semisimple Lie group with* $\mathbb{R}$*-rank* ≥ 2*. Then* Γ *is almost simple.*

Consequently, if Γ is a lattice which satisfies the conditions of the theorem of Lubotzky, Mozes, and Raghunathan, then any homomorphism $\Gamma \to \mathcal{E}$ must have finite image. The existence of a distortion element implies that the kernel of the action is infinite. Since Γ is almost simple, this implies that the kernel actually has finite index. A concrete example of lattices which satisfy the conditions of the above theorems are the lattices $\mathrm{SL}(n, \mathbb{Z})$, such that $n \geq 3$. Thus, any interval exchange action of one of these matrix groups factors through a finite group.

CHAPTER 6

Classification of Centralizers in $\mathcal{E}$

For an interval exchange f, let $C(f)$ denote the centralizer of f in the group $\mathcal{E}$:

$$C(f) = C_{\mathcal{E}}(f) := \{g \in \mathcal{E} : fg = gf\}.$$

If f is a minimal transformation, then the structure of $C(f)$ is primarily determined by the discontinuity growth of f. In considering the situation where $d(f^n)$ is bounded, the first case to consider is when f is an irrational rotation. Let R denote the subgroup consisting of the rotations $\{r_\alpha : \alpha \in \mathbb{R}/\mathbb{Z}\}$.

Lemma 6.1. *If $f = r_\alpha$ is an irrational rotation, then $C(r_\alpha)$ is the rotation group R.*

See Li [**6**] for an alternate proof.

Proof. It is clear that $R \subseteq C(r_\alpha)$, so given $g \in C(r_\alpha)$, it suffices to show that g must be a rotation. For a point $x \in \mathbb{T}^1$, consider the translation number of the commutator $r_\alpha^{-1} g^{-1} r_\alpha g$ at x. Performing all calculations mod 1, we have

$$\begin{aligned}
\omega_{r_\alpha^{-1} g^{-1} r_\alpha g}(x) &= \omega_g(x) + \omega_{r_\alpha^{-1} g^{-1} r_\alpha}(g(x)) = \omega_g(x) + \alpha + \omega_{r_\alpha^{-1} g^{-1}}((r_\alpha g)(x)) \\
&= \omega_g(x) + \alpha + \omega_{g^{-1}}((r_\alpha g)(x)) + \omega_{r_\alpha^{-1}}((g^{-1} r_\alpha g)(x)) \\
&= \omega_g(x) + \alpha + \omega_{g^{-1}}((r_\alpha g)(x)) - \alpha \\
&= \omega_g(x) - \omega_g((g^{-1} r_\alpha g)(x)) = \omega_g(x) - \omega_g(r_\alpha(x)).
\end{aligned}$$

Since $r_\alpha^{-1} g^{-1} r_\alpha g = id$, it follows that

$$\omega_g(x) \equiv \omega_g(r_\alpha(x)) \pmod 1.$$

This calculation holds for any point in $\mathbb{T}^1$, from which it follows that ω_g is constant (mod 1) on the entire orbit $\{r_{n\alpha}(x)\}$. This orbit is dense, and specifying ω_g on a dense set will specify it everywhere, since it is a right-locally constant function. Thus ω_g is a constant (mod 1), which implies that g is a rotation. $\square$

In general, if f is minimal and $d(f^n)$ is bounded, by Theorem 1.5 some power f^k is conjugate to a product of disjoint irrational rotations. Suppose that k is chosen to be as small as possible, and let J_i, $1 \leq i \leq l$, denote the minimal components of f^k. Replace f by a conjugate so that f^k is in normal form: the J_i are assumed to be intervals, and let r_i denote the restricted rotation supported on J_i induced by f^k. Since f is minimal and commutes with f^k, f transitively permutes the J_i and induces conjugacies between all of the r_i. Consequently, the J_i are all intervals of length $1/l$, and each r_i rotates by the same proportion of $1/l$. Let R_i denote the full rotation group supported on the interval J_i. Then f^k is an element of the diagonal subgroup of

$$R_1 \times \cdots \times R_l,$$

and it follows from Lemma 6.1 that

$$C(f^k) = (R_1 \times \cdots \times R_l) \rtimes \Sigma_l,$$

where Σ_l is the embedding of the symmetric group which permutes the J_i by translation.

Proposition 6.2. *If f is minimal and $d(f^n)$ is bounded, then $C(f)$ is virtually abelian and contains a rotation subgroup.*

Proof. Continue with the notation of the above paragraphs. $C(f)$ is a subgroup of $C(f^k) \cong (\mathbb{R}/\mathbb{Z})^l \rtimes \Sigma_l$, and so it is a virtually abelian group. Also, $f \in C(f^k)$ implies that f has the form

$$f = r_1 \cdots r_l \sigma,$$

where $r_i \in R_i$ and σ is a permutation of the J_i by translation. In particular, f commutes with the diagonal subgroup in $R_1 \times \cdots \times R_l$, and so $C(f)$ contains a continuously embedded copy of $\mathbb{R}/\mathbb{Z}$. $\square$

Suppose now that f is minimal and $d(f^n)$ exhibits linear growth. The discontinuity structure of f and its powers is significantly more complicated than the bounded case. Any map g which commutes with f must preserve this structure, and so one would expect that the centralizer of f should be significantly smaller than in the bounded discontinuity situation. This is in fact the case:

Proposition 6.3. *If f is minimal and $d(f^n)$ has linear growth, then $C(f)$ is virtually cyclic.*

Let $D = D(f)$ be the discontinuity set of f and let $D_{NR} = \{x_1, \ldots, x_k\}$ be the set of nonresolving fundamental discontinuities of f (see Chapter 4 to recall the definition). By Proposition 4.1, linear growth of $d(f^n)$ is equivalent to D_{NR} being nonempty. Let n_0 be the *symmetric stabilization time* for f: n_0 is the minimal positive integer such that f is continuous at $f^i(x)$, for all $|i| \geq n_0$ and all $x \in D$. The following lemma states that if a sufficiently long piece of f-orbit contains enough discontinuity points for a large power of f, then the f-orbit must contain a nonresolving fundamental discontinuity of f.

Lemma 6.4. *Suppose f is minimal and has symmetric stabilization time n_0. Let $M > 3n_0$, and suppose that for some $y \in \mathbb{T}^1$ the set*

$$B = \{y, f^{-1}(y), \dots, f^{-M+n_0}(y)\}$$

contains strictly more that $2n_0+1$ discontinuities of f^M. Then some $f^k(y)$, with $|k| \leq M$, is a nonresolving fundamental discontinuity of f.

Proof. For the convenience of notation, let y_m denote $f^m(y)$. Let $j \in \mathbb{N}$ be the smallest positive integer such that f^M is discontinuous at y_{-j}. Since f^M is discontinuous at y_{-j}, this point has a controlling f-discontinuity at $y_{\tilde{k}} \in D(f)$, where $-j \leq \tilde{k} \leq -j + M - 1$. Consequently, there must be a fundamental discontinuity of f at some y_k, where $-j-n_0 \leq k \leq -j + M - 1$.

Since B contains more than $2n_0$ f^M-discontinuities at points y_{-i} with $i > j$, there are more than n_0 f^M-discontinuities whose status is controlled by y_k. In particular, at least one of the f^M-discontinuities in B is induced by the stabilized behavior of y_k, which implies that y_k is a nonresolving fundamental discontinuity of f. $\square$

Proof of Proposition 6.3. Suppose that $gf = fg$. Consider some integer N such that

$$N \gg n_0 \text{ and } N \gg d(g).$$

(For instance, assume $N > 100n_0$ and $N - n_0 > 100d(g)$.) Let $x \in D_{NR}$. Since x is a nonresolving fundamental discontinuity of f, the set

$$A = \{x, f^{-1}x, f^{-2}x, \ldots, f^{-(N-n_0)}x\}$$

consists entirely of discontinuity points of f^N.

Since f and g commute, $f^N = g^{-1}f^Ng$, so $g^{-1}f^Ng$ is discontinuous at all points of A. Consider how this composition acts upon the set A :

$$A = \{x, f^{-1}x, \ldots, f^{-(N-n_0)}x\}$$

$$\downarrow g$$

$$g(A) = \{gx, f^{-1}(gx), \ldots, f^{-(N-n_0)}(gx)\}$$

$$\downarrow f^N$$

$$f^Ng(A) = \{f^N(gx), f^{N-1}(gx), \ldots, f^{n_0}(gx)\}$$

$$\downarrow g^{-1}$$

$$f^N(A) = \{f^Nx, f^{N-1}x, \ldots, f^{n_0}x\}$$

Since the cardinality of the set A is significantly larger than $d(g)$, g acts continuously on most points of A in the first stage of the above composition. Similarly, g^{-1} acts continuously on most points of $f^Ng(A)$ in the third stage. However, since $g^{-1}f^Ng$ is discontinuous at all points of A, it follows that f^N is discontinuous at most of the points in

$$\{gx, f^{-1}(gx), \ldots, f^{-(N-n_0)}(gx)\}.$$

By Lemma 6.4, it follows that some f-iterate of $g(x)$ must be a member of D_{NR}. (To make the estimates above precise using the initial assumptions on N, note that $|A| = N - n_0 \geq 100d(g)$, from which it follows that f^N is discontinuous for at least $0.98|A|$ of the points in $g(A)$. This, together with the assumption that $N > 100n_0$, is more than sufficient to satisfy the conditions of Lemma 6.4.)

The argument in the preceding paragraphs shows that $g \in C(f)$ must permute the f-orbits of the points in $D_{NR} = \{x_1, \ldots, x_k\}$. In particular, there is some integer i and some $x_j \in D_{NR}$ such that

$$g(x_1) = f^i(x_j).$$

This relation determines g on the entire f-orbit of x_1 :

$$g(f^n x_1) = f^n(g x_1) = f^{n+i} x_j.$$

Since the orbit $\mathcal{O}_f(x_1)$ is dense, the relation $g(x_1) = f^k(x_j)$ fully determines g.

For each j such that $1 \leq j \leq k$, let h_j denote the unique interval exchange in $C(f)$ such that

$$h_j(x_1) = x_j,$$

if such a map exists. Then, if $g \in C(f)$ satisfies $g(x_1) = f^k(x_j)$, it follows that $g = f^k h_j$. In particular, $\{h_i\}$ is a set of representatives for the finite quotient group $C(f)/\langle f \rangle$. Thus, $C(f)$ is a virtually cyclic group. $\square$

Now let $f \in \mathcal{E}$ be a finite-order map. For simplicity, fix $n \geq 2$ and assume that f is a transitive permutation of the intervals

$$I_i = \left[\frac{i-1}{n}, \frac{i}{n}\right), \quad 1 \leq i \leq n,$$

such that f maps each I_i by translation. After conjugation it may be assumed that f is rotation by $\frac{1}{n}$.

Define the *support* of an interval exchange f as the complement of its set of fixed points. Note that the support of an interval exchange is always a set in the algebra $\mathcal{P}$. For any nonempty $A \in \mathcal{P}$, let $\mathcal{E}_A$ denote the subgroup consisting of all interval exchanges whose support is contained in A. It follows that $\mathcal{E}_A \cong \mathcal{E}$; a suitable conjugacy gives an isomorphism between $\mathcal{E}_A$ and $\mathcal{E}_I$ for an interval I, and there is an isomorphism $\mathcal{E}_I \cong \mathcal{E}$ induced by rescaling I to be the interval $[0, 1)$.

Consider the following subgroups in the centralizer $C(r_{1/n})$. First, let $\mathcal{E}^n_\Delta$ represent the subgroup of maps in $C(r_{1/n})$ which preserve the intervals I_i:

$$\mathcal{E}^n_\Delta = \{g \in C(r_{1/n}) : g(I_i) = I_i, \text{ for } 1 \leq i \leq n\}.$$

Note that $\mathcal{E}^n_\Delta$ is the diagonal subgroup of the product

$$\mathcal{E}_{I_1} \times \cdots \times \mathcal{E}_{I_n},$$

as induced by the natural isomorphisms $\mathcal{E} \cong \mathcal{E}_{I_i}$. In particular, $\mathcal{E}^n_\Delta \cong \mathcal{E}$.

Next, let P_n denote the subgroup of maps in $C(r_{1/n})$ which preserve $r_{1/n}$-orbits:

$$P_n = \left\{ g \in C(r_{1/n}) : \forall x \in \mathbb{T}^1, \exists k \in \mathbb{Z}, \text{ such that } g(x) = x + \frac{k}{n} \pmod 1 \right\}.$$

Fix $g \in P_n$, and consider a point $x = x_1 \in I_1$. Let

$$x_i = x_1 + \frac{i-1}{n},\ 2 \leq i \leq n,$$

denote the other points in the $r_{1/n}$-orbit of x, and let $\sigma_{g,x} \in \Sigma_n$ denote the permutation that g induces on $\{x_i\}$:

$$g(x_i) = x_{\sigma_{g,x}(i)}.$$

Since $\sigma_{g,x}$ commutes with the permutation $r : i \mapsto i + 1 \pmod n$ induced by the rotation $r_{1/n}$, it follows that $\sigma_{g,x}$ must be a power of r. Thus, the transformation g is described by a right-continuous (and hence piecewise constant) map

$$\sigma_g : I_1 \to \langle r \rangle \cong \mathbb{Z}/n\mathbb{Z}$$

Conversely, any such right-continuous map $I_1 \to \mathbb{Z}/n\mathbb{Z}$ with only finitely many discontinuities defines a map in P_n. Thus, P_n is isomorphic to the abelian group of right-continuous functions $I_1 \to \mathbb{Z}/n\mathbb{Z}$ having finitely many discontinuities, where the group operation is by addition of functions.

Proposition 6.5. $C(r_{1/n}) = P_n \rtimes \mathcal{E}^n_\Delta$.

Proof. First, suppose $g \in P_n \cap \mathcal{E}^n_\Delta$. Then g preserves the intervals I_i, which implies that $\sigma_{g,x} = id$ for all $x \in I_1$. Thus $g = id$, and the subgroups P_n and $\mathcal{E}^n_\Delta$ have trivial intersection.

Next, suppose g is an arbitrary element of $C(r_{1/n})$. Construct $h \in P_n$ as follows. For $x = x_1 \in I_1$, define $\{x_i\}$ as before and let $\sigma_{h,x}$ be the permutation such that

$$g(x_i) \in I_{\sigma_{h,x}(i)}.$$

Observe that $\sigma_{h,x} \in \Sigma_n$ is well-defined since g maps the $r_{1/n}$-orbit $\{x_i\}$ to another $r_{1/n}$-orbit; there is exactly one of the $g(x_i)$ in each interval I_j. Since g commutes with $r_{1/n}$, the permutation $\sigma_{h,x}$ is a power of the permutation r. Moreover, the function $x \mapsto \sigma_{h,x} \in \mathbb{Z}/n\mathbb{Z}[r]$ is right-continuous and has finitely many discontinuities since g has these properties. From its construction, it follows that gh^{-1} preserves each interval I_i, and so $gh^{-1} \in \mathcal{E}^n_\Delta$. Thus $C(r_{1/n}) = P_n \cdot \mathcal{E}^n_\Delta$.

It remains to show that P_n is a normal subgroup of $C(r_{1/n})$. Let $g \in P_n$ and let $h \in \mathcal{E}^n_\Delta$. If $\{x_i\}$ is an $r_{1/n}$-orbit, then h maps it to some other $r_{1/n}$-orbit $\{y_i\}$, g permutes the orbit $\{y_i\}$, and h^{-1} maps $\{y_i\}$ back to $\{x_i\}$. Thus $h^{-1}gh$ is invariant on $r_{1/n}$-orbits, which implies that $h^{-1}gh \in P_n$. Consequently, $P_n \trianglelefteq C(r_{1/n})$. $\square$

Corollary 6.6. *For $n \geq 2$, let $G_n = P_n \rtimes \mathcal{E}^n_\Delta$ denote the centralizer of the rotation $r_{1/n}$, and let $G_1 = \mathcal{E}$. If f is any finite-order map, then $C(f)$ is isomorphic to a finite direct product of the G_i.*

Proof. Decompose $\mathbb{T}^1$ into finitely many nonempty $I_j = \mathrm{Per}_j(f)$. After replacing f by a conjugate, it may be assumed that the I_j are intervals on which f acts by a finite-order rotation. The I_j are invariant under all $g \in C(f)$ and $C(f) \cap \mathcal{E}_{I_j}$, the subgroup of maps in $C(f)$ with support in I_j is isomorphic to G_j. $\square$

By combining the particular cases considered above in light of the normal dynamical form of an interval exchange f, a general characterization of $C(f)$ can be given. Let $J_1, \ldots, J_k$ be the minimal components of f, let $A = \mathrm{Per}(f) \setminus \mathrm{Fix}(f)$ and let $B = \mathrm{Fix}(f)$. Assume that all of these sets are intervals. Let f_i be the map defined by

$$f_i(x) = \begin{cases} f(x), & \text{if } x \in J_i \\ x, & \text{otherwise.} \end{cases}$$

Let $g \in C(f)$. The sets A and B are both g-invariant, but g may permute the minimal components J_i. If g maps J_i onto J_j, then g induces a conjugacy between f_i and f_j. After replacing f_j by a conjugate in $\mathcal{E}_{I_j}$, it may be assumed that

$$f_i = \tau_{ij} f_j \tau_{ij}$$

where τ_{ij} is the order-two map which interchanges J_i and J_j by translation and fixes all other points. Replace f by a further conjugate so that $f_i = \tau_{ij} f_j \tau_{ij}$ holds for all pairs $i \neq j$ such that f_i and f_j are conjugate, and let F be the group generated by all such τ_{ij}. Note that F is isomorphic to a direct product of symmetric groups, since the relation $i \sim j \Leftrightarrow$ (f_i is conjugate to f_j) is an equivalence relation on $\{1, \ldots, k\}$. Let

$C_i = C_{\mathcal{E}_{J_i}}(f) = C(f) \cap \mathcal{E}_{J_i}$ denote the subgroup of maps in $C(f)$ with support in J_i, and let $C_A = C(f) \cap \mathcal{E}_A$.

Proposition 6.7. *For f with normal dynamical form as denoted above,*

$$C(f) \cong \left(\left(\prod_{i=1}^{k} C_i \right) \rtimes F \right) \times C_A \times \mathcal{E}_B,$$

where each C_i is either an infinite virtually cyclic group or isomorphic to a subgroup of $(\mathbb{R}/\mathbb{Z})^n \rtimes \Sigma_n$ containing the diagonal in $(\mathbb{R}/\mathbb{Z})^n$.

Proof. It is clear that

$$C(f) \cong C_{\cup J_i}(f) \times C_A \times \mathcal{E}_B,$$

since these are disjoint and non-conjugate f-invariant sets which cover $\mathbb{T}^1$. The verification that

$$C_{\cup J_i}(f) \cong \left(\prod_{i=1}^{k} C_i \right) \rtimes F$$

is similar to the proof of Proposition 6.5. $\square$

Corollary 6.8. *For any $f \in \mathcal{E}$, Per(f) is nonempty if and only if $C(f)$ contains a subgroup isomorphic to $\mathcal{E}$.*

Proof. It remains to show that if $\mathrm{Per}(f)$ is empty, then no subgroup of $C(f)$ is isomorphic to $\mathcal{E}$. By the proposition,

$$C(f) = \prod_{i=1}^{k} C_i \rtimes F,$$

where each C_i is either virtually cyclic or isomorphic to a subgroup of $(\mathbb{R}/\mathbb{Z})^n \rtimes \Sigma_n$ containing the diagonal. If C_i is virtually cyclic, then for every infinite-order $g \in C_i$, some nontrivial power g^n is central in $\prod C_i$. Similary, if $g = r_1 \cdots r_n \sigma$ is an infinite-order element in $(\mathbb{R}/\mathbb{Z})^n \rtimes \Sigma_n$, then for any other infinite-order element h in this group, some nontrivial powers g^n and h^m commute, since suitable powers of each map yield elements in $(\mathbb{R}/\mathbb{Z})^n$. Thus, given any two infinite-order $g, h \in C(f)$, there are nontrivial powers of these maps which commute. This property does not hold for the group $\mathcal{E}$. For instance, consider an irrational rotation r_α and any infinite-order map $f \in \mathcal{E}$ which is not a rotation. No nontrivial powers of r_α and f commute. Thus, it is not possible to exhibit $\mathcal{E}$ as a subgroup of $C(f)$ when f has no periodic points. $\square$

Corollary 6.9. *For any nonidentity $f \in \mathcal{E}$, the index $[\mathcal{E} : C(f)]$ is infinite.*

Proof. From the structure of $C(f)$ given in the proposition, it suffices to consider the following three cases:

Case 1: f is periodic. It suffices to consider the case $f = r_{1/n}$. For any $h \in \mathcal{E}$, define

$$\alpha(h) = \min_{x \neq y \colon x = y + j/n} \{\rho_{\mathbb{T}^1}(h(x), h(y))\}.$$

The function α measures how close together h is able to map a pair of points in the same $r_{1/n}$-orbit. Since transformations in $C(r_{1/n})$ preserve the collection of sets $\{x + i/n\}$, the function α is constant on cosets $h \cdot C(f)$. The function α takes all values in $(0, 1/n)$, and thus the index of $C(f)$ in $\mathcal{E}$ is (uncountably) infinite.

Case 2: f is minimal, with linear discontinuity growth. By Proposition 6.3, $C(f)$ is virtually cyclic. In particular, it is countable, which implies that $C(f)$ has (uncountably) infinite index in $\mathcal{E}$.

Case 3: f is minimal, with bounded discontinuity growth. By the proof of Proposition 6.2, there is a uniform bound for the number of discontinuities possessed by maps in $C(f)$. Consequently, there is a bound on the number of discontinuities possessed by maps in a given coset $C(f) \cdot h$. Thus $C(f)$ must have infinite index, since otherwise there would be a uniform bound on discontinuities for the entire group $\mathcal{E}$. $\square$

CHAPTER 7

Automorphisms of $\mathcal{E}$

We now investigate the structure of the automorphism group $\mathrm{Aut}(\mathcal{E})$. One ready source of examples is the group of inner automorphisms. Since $\mathcal{E}$ has trivial center, $\mathrm{Inn}(\mathcal{E}) \cong \mathcal{E}$. Another example of an automorphism is induced by switching the orientation of the circle $\mathbb{T}^1$, as illustrated in Figure 7.1. More precisely, let $T : \mathbb{T}^1 \to \mathbb{T}^1$ be defined by $T(x) = -x$. For any $f \in \mathcal{E}$, $T^{-1}fT$ is still an invertible piecewise translation, but it is now continuous from the left. Let Ψ_T be the automorphism of $\mathcal{E}$ defined by conjugation by T followed by the natural isomorphism from the group of left-continuous interval exchanges to the right-continuous interval exchange group $\mathcal{E}$. Note that Ψ_T is not an inner automorphism, since it switches the sign of the scissors invariant:

$$\phi(\Psi_T(r_\alpha)) = \phi(r_{-\alpha}) = -(1 \wedge \alpha).$$

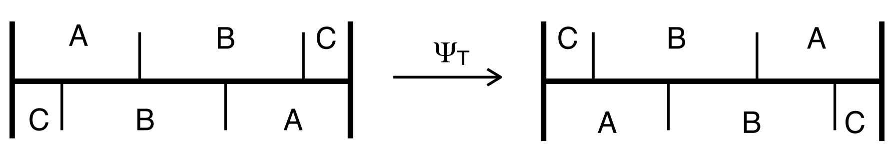

Figure 7.1. The action of the automorphism Ψ_T

The inner automorphisms and the automorphism Ψ_T act by conjugation on $\mathcal{E}$, and these automorphisms generate $\mathrm{Aut}(\mathcal{E})$.

Theorem 1.6. *$Aut(\mathcal{E}) \cong \mathcal{E} \rtimes \langle \Psi_T \rangle$.*

The proof of Theorem 1.6 is based on observing that an arbitrary $\Psi \in \mathrm{Aut}(\mathcal{E})$ preserves the structure of $\mathcal{P}$.

Lemma 7.1. *An interval exchange f is conjugate to an irrational rotation r_α if and only if the following conditions hold:*

(1) $C(f) \cong \mathbb{R}/\mathbb{Z}$;

(2) if $g \in C(f)$ has infinite order, then $C(g) = C(f)$.

Proof. By Lemma 6.1, conditions (1) and (2) hold if $f = r_\alpha$ is an irrational rotation. In addition, conditions (1) and (2) are both preserved under conjugacy, so they hold for any f which is conjugate to an irrational rotation.

Conversely, assume that f satisfies (1) and (2), and consider the normal dynamical form of f. By Corollary 6.8, $\mathrm{Per}(f)$ is empty since no subgroup of $\mathbb{R}/\mathbb{Z}$ is isomorphic to $\mathcal{E}$. Next, it will be shown that f^n is minimal for all $n \geq 1$. To reach a contradiction, suppose that f^n has at least two minimal components, and denote them by J_i. Let g be the map which is equal to f on J_1 and fixes all other points. Then g has infinite order and commutes with f, so $C(g) \cong \mathbb{R}/\mathbb{Z}$ by condition (2). However, g has fixed points, and so $C(g)$ contains a subgroup isomorphic to $\mathcal{E}$, which is a contradiction. Thus, f^n is minimal for all $n \geq 1$.

Furthermore, f must have bounded discontinuity growth. If not, then $C(f)$ would be virtually cyclic by Proposition 6.3, which is not the case for $\mathbb{R}/\mathbb{Z}$. Consequently, some power f^k must be conjugate to an irrational rotation, by Theorem 1.5. Since $C(f) = C(f^k)$, it follows that f is also conjugate to an irrational rotation. $\square$

Let $R < \mathcal{E}$ denote the group of circle rotations $\{r_\alpha : \alpha \in \mathbb{R}/\mathbb{Z}\}$. For any $f \in \mathcal{E}$, let Φ_f denote conjugation by f^{-1}; i.e., $\Phi_f(g) = fgf^{-1}$.

Corollary 7.2. *For any $\Psi \in Aut(\mathcal{E})$, Ψ maps the rotation group R to a conjugate. That is, there exists $g \in \mathcal{E}$ such that $\Psi(R) = gRg^{-1}$.*

Proof. Since conditions (1) and (2) in Lemma 7.1 are purely group theoretic, they are preserved by any automorphism Ψ. Fix an irrational rotation r_α. By the Lemma, $\Psi(r_\alpha)$ is conjugate to an irrational rotation. In particular, there is some $g \in \mathcal{E}$ and some irrational $\beta \in \mathbb{R}/\mathbb{Z}$ such that

$$\Psi(r_\alpha) = \Phi_g(r_\beta).$$

Then

$$\Psi(R) = \Psi(C(r_\alpha)) = C(\Psi(r_\alpha)) = C(\Phi_g(r_\beta)) = gRg^{-1}. \ \square$$

A similar result hold for maps which are conjugate to an infinite-order restricted rotation $r_{\alpha,\beta}$.

Lemma 7.3. *An infinite-order interval exchange f is conjugate to an infinite-order restricted rotation $r_{\alpha,\beta}$ with $\beta < 1$ if and only if the following hold:*

(1) $C(f) = \mathcal{E}_* \times H$, where $\mathcal{E}_* \cong \mathcal{E}$, $H \cong \mathbb{R}/\mathbb{Z}$, and $f \in H$;

(2) if $g \in H$ has infinite order, then $C(g) = C(f)$;

(3) for $h \in C(f)$, if the index $[C(f) : C(h) \cap C(f)]$ is finite and

$C(h) \supsetneq C(h) \cap C(f)$, then h is a finite-order element of H.

Proof. Suppose that $f = r_{\alpha,\beta}$ with $\beta < 1$ and α/β irrational. Let $I = [\beta, 1)$. Then

$$C(r_{\alpha,\beta}) = \mathcal{E}_I \times R_\beta,$$

where $R_\beta \cong \mathbb{R}/\mathbb{Z}$ is the group of all restricted rotations $r_{\gamma,\beta}$ on $[0, \beta)$. In particular, any other infinite-order element of R_β has the same centralizer as $r_{\alpha,\beta}$. Thus, $r_{\alpha,\beta}$ satisfies conditions (1) and (2).

To verify condition (3) for $f = r_{\alpha,\beta}$, take $h \in C(r_{\alpha,\beta})$ and write $h = h_I r_{\gamma,\beta}$, where $h_I \in \mathcal{E}_I$ and $r_{\gamma,\beta} \in R_\beta$. Assume that $C(h)$ satisfies the hypotheses of condition (3). Note that

$$C(h) \cap C(r_{\alpha,\beta}) = C_{\mathcal{E}_I}(h_I) \times R_\beta.$$

By Proposition 6.9, the index $[\mathcal{E}_I : C_{\mathcal{E}_I}(h_I)]$ is infinite unless h_I is the identity. However,

$$[C(r_{\alpha,\beta}) : C(h) \cap C(r_{\alpha,\beta})] = [\mathcal{E}_I : C_{\mathcal{E}_I}(h_I)]$$

is finite by assumption, so $h = r_{\gamma,\beta}$. It has also been assumed that

$$C(h) \supsetneqq C(h) \cap C(r_{\alpha,\beta}) = C(r_{\alpha,\beta}),$$

and this is possible only if the rotation $h = r_{\gamma,\beta}$ is rational.

Finally, observe that conditions (1)-(3) are all preserved by conjugation in $\mathcal{E}$. Consequently, they hold for any conjugate of $r_{\alpha,\beta}$.

Conversely, suppose that f is an infinite-order interval exchange satisfying conditions (1)-(3). It will first be shown that $\mathrm{Fix}(f)$ is nonempty and $\mathrm{Per}(f) = \mathrm{Fix}(f)$. Since $C(f)$ contains a subgroup isomorphic to $\mathcal{E}$, $\mathrm{Per}(f)$ is nonempty by Corollary 6.8. The map f

cannot have periodic points of arbitrarily large period, so $\mathrm{Fix}(f^k) = \mathrm{Per}(f^k) = \mathrm{Per}(f)$ for some power k. Let A denote $\mathrm{Per}(f)$. Since f^k fixes A, $\mathcal{E}_A < C(f^k)$. By condition (2), $C(f^k) = C(f)$, and it follows that f fixes all points in A. Similarly, all infinite-order $g \in H$ must fix the set A, and consequently all maps in H must fix A. In other words, H is contained in $\mathcal{E}_B$, where $B = \mathbb{T}^1 \setminus A$.

Next, it will be shown that f has only a single minimal component. As in the proof of Lemma 7.1, suppose f has minimal components J_i, $1 \leq i \leq k$, for some $k \geq 2$. Let h be the map which equals f on the component J_1 and fixes all other points. Then h has infinite order and commutes with f. In terms of their normal forms,

$$\begin{aligned} C(f) =& \left(\left(\prod_{i=1}^{k} C_i \right) \rtimes F \right) \times \mathcal{E}_{\mathcal{A}}, \\ C(h) =& C_1 \times \mathcal{E}_{A \cup J_2 \cup \cdots \cup J_k}, \end{aligned}$$

where $C_i = C(f) \cap \mathcal{E}_{J_i}$ and F is a finite group which permutes the J_i. In particular, $C(h) \cap C(f)$ contains $(\prod C_i) \times \mathcal{E}_A$, which has finite index in $C(f)$. In addition, $C(h)$ strictly contains $C(h) \cap C(f)$ since h has a larger fixed point set than f. Thus, condition (3) implies that h must have finite order, which is a contradiction. A similar argument may be applied to any infinite-order $g \in H$; consequently, all such maps have a single minimal component, namely B.

Consider the natural isomorphism

$$\mathcal{E} \to \mathcal{E}_B.$$

Let $\widetilde{f}$ denote the preimage of $f \in \mathcal{E}_B$, and let $\widetilde{H}$ denote the preimage of H. Then all infinite-order $\widetilde{g} \in \widetilde{H}$ are minimal, and

$$C(\widetilde{g}) = C(\widetilde{f}) > \widetilde{H},$$

which implies that all infinite-order $\widetilde{g}$ have bounded discontinuity growth. As in the proof of Lemma 7.1, it follows that all $\widetilde{g} \in \widetilde{H}$ are simultaneously conjugate to irrational rotations. Back in the group $\mathcal{E}_B$, this implies that H is conjugate to a group of restricted rotations. $\square$

Corollary 7.4. *For any $\Psi \in Aut(\mathcal{E})$ and any f which is conjugate to a restricted rotation, $\Psi(f)$ is also conjugate to a restricted rotation.*

Proof. By Lemma 7.3, all maps conjugate to restricted rotations are characterized by conditions (1)-(3). As these conditions are all purely group theoretic, they are preserved by the automorphism Ψ. $\square$

Proposition 7.5. *For any $\Psi \in Aut(\mathcal{E})$ and any nonempty $A \in \mathcal{P}$, there is a (necessarily unique) $B \in \mathcal{P}$ such that $\Psi(\mathcal{E}_A) = \mathcal{E}_B$.*

Proof. It suffices to consider $A \in \mathcal{P}$ to be a proper subset of $\mathbb{T}^1$. Let $g \in \mathcal{E}$ be a map with support equal to $\mathbb{T}^1 \setminus A$ which is conjugate to an infinite-order restricted rotation. By Corollary 7.4, $\Psi(g)$ is also conjugate to a restricted rotation. Let $B = \mathrm{Fix}(\Psi(g))$.

It will first be shown that $\Psi(\mathcal{E}_A) \subseteq \mathcal{E}_B$. By Lemma 2.7, $\mathcal{E}_A$ is generated by the infinite-order restricted rotations with support contained in A. Two infinite-order restricted rotations g and h commute if and only if one of the following holds:

(a) their supports coincide and they are simultaneously conjugate to elements in some R_β; or

(b) their supports are disjoint.

These two conditions can be detected group-theoretically: condition (a) implies that $C(g) = C(h)$, while condition (b) implies $C(g) \neq C(h)$. In particular, each condition is preserved by any automorphism of $\mathcal{E}$.

Any restricted rotation with support contained in A commutes with g and has support disjoint from that of g. Consequently, all restricted rotations in $\mathcal{E}_A$ must map under Ψ to restricted rotations with support in $B = \mathrm{Fix}(\Psi(g))$. These maps generate $\Psi(\mathcal{E}_A)$, and it follows that $\Psi(\mathcal{E}_A) \subseteq \mathcal{E}_{\mathcal{B}}$.

Similary, under Ψ^{-1} all restricted rotations with support in B are mapped to restricted rotations which commute with g and have support disjoint from that of g. Therefore, $\Psi^{-1}(\mathcal{E}_B) \subseteq \mathcal{E}_A$, and it follows that $\Psi(\mathcal{E}_A) = \mathcal{E}_B$. $\square$.

7.1. Definition and properties of $\widetilde{\Psi}$

Given an automorphism $\Psi \in \mathrm{Aut}(\mathcal{E})$, Proposition 7.5 implies that Ψ induces a transformation

$$\widetilde{\Psi} : \mathcal{P} \to \mathcal{P},$$

defined by the relation

$$\Psi(\mathcal{E}_A) = \mathcal{E}_{\widetilde{\Psi}(A)}, \quad A \in \mathcal{P}.$$

In particular, $\widetilde{\Psi}(\mathbb{T}^1) = \mathbb{T}^1$ and $\widetilde{\Psi}(\emptyset) = \emptyset$, for all $\Psi \in \mathrm{Aut}(\mathcal{E})$. An interval exchange $f \in \mathcal{E}$ also induces a transformation $\widetilde{f} : \mathcal{P} \to \mathcal{P}$, defined by $\widetilde{f}(A) = f(A)$.

Proposition 7.6. *For all $\Psi \in Aut(\mathcal{E})$, the transformation $\widetilde{\Psi} : \mathcal{P} \to \mathcal{P}$ has the following properties:*

(1) $\widetilde{\Psi}$ preserves complements: $\widetilde{\Psi}(\mathbb{T}^1 \setminus A) = \mathbb{T}^1 \setminus \widetilde{\Psi}(A)$, $\forall A \in \mathcal{P}$.

(2) $\widetilde{\Psi}$ preserves inclusion: if $A \subseteq B$, then $\widetilde{\Psi}(A) \subseteq \widetilde{\Psi}(B)$.

(3) $\widetilde{\Psi}$ preserves finite unions: $\widetilde{\Psi}(A \cup B) = \widetilde{\Psi}(A) \cup \widetilde{\Psi}(B)$.

(4) $\widetilde{\Psi}$ preserves finite intersections.

(5) For any $f \in \mathcal{E}$, $\widetilde{\Psi(f)} = \widetilde{\Psi} \widetilde{f} \widetilde{\Psi}^{-1}$.

(6) The Lebesgue measure $\mu : \mathcal{P} \to [0, 1]$ is $\widetilde{\Psi}$-invariant; $\mu(\widetilde{\Psi}(A)) = \mu(A)$.

Proof. (1): Suppose A and B are complements in $\mathcal{P}$: they are disjoint and $A \cup B = \mathbb{T}^1$. Then the centralizer in $\mathcal{E}$ of $\mathcal{E}_A$ is $\mathcal{E}_B$, and vice versa. This same relation will hold for $\Psi(\mathcal{E}_A) = \mathcal{E}_{\widetilde{\Psi}(A)}$ and $\Psi(\mathcal{E}_B) = \mathcal{E}_{\widetilde{\Psi}(B)}$, which implies that $\widetilde{\Psi}(A)$ and $\widetilde{\Psi}(B)$ are complements.

(2): Observe that

$$A \subseteq B \Rightarrow \mathcal{E}_A \leq \mathcal{E}_B \Rightarrow \Psi(\mathcal{E}_A) \leq \Psi(\mathcal{E}_B) \Rightarrow$$

$$\mathcal{E}_{\widetilde{\Psi}(A)} \leq \mathcal{E}_{\widetilde{\Psi}(B)} \Rightarrow \widetilde{\Psi}(A) \subseteq \widetilde{\Psi}(B).$$

(3): The sets A and B are both subsets of $A \cup B$, so by (2), $\widetilde{\Psi}(A) \subseteq \widetilde{\Psi}(A \cup B)$ and $\widetilde{\Psi}(B) \subseteq \widetilde{\Psi}(A \cup B)$. Conversely, suppose that $\widetilde{\Psi}(A \cup B) \nsubseteq \widetilde{\Psi}(A) \cup \widetilde{\Psi}(B)$. To derive a contradiction, let

$$C = \widetilde{\Psi}(A \cup B) \setminus \left(\widetilde{\Psi}(A) \cup \widetilde{\Psi}(B)\right).$$

Then $C \in \mathcal{P}$ is nonempty, and there exists a non-identity interval exchange $f \in \mathcal{E}_C \leq \mathcal{E}_{\widetilde{\Psi}(A\cup B)}$. The map f centralizes both $\mathcal{E}_{\widetilde{\Psi}(A)}$ and $\mathcal{E}_{\widetilde{\Psi}(B)}$, so the map $\Psi^{-1}(f)$ centralizes $\mathcal{E}_A$ and $\mathcal{E}_B$. This implies $\Psi^{-1}(f)$ has support disjoint from both A and B. However, this is impossible since $\Psi^{-1}(f)$ is in $\mathcal{E}_{A\cup B}$. Thus, $\widetilde{\Psi}(A\cup B) \subseteq \widetilde{\Psi}(A) \cup \widetilde{\Psi}(B)$.

(4): This follows from (1) and (3) by DeMorgan's Law.

(5): Recall $\Phi_f \in \mathrm{Aut}(\mathcal{E})$ denotes conjugation by f^{-1}. In particular, if f maps the set A to the set B ($\widetilde{f}(A) = B$), then Φ_f induces an isomorphism from $\mathcal{E}_A$ to $\mathcal{E}_B$.

For any $g \in \mathcal{E}$,

$$\Psi\Phi_f\Psi^{-1}(g) = \Psi(f(\Psi^{-1}g)f^{-1}) = \Psi(f) \circ g \circ \Psi(f)^{-1}.$$

Thus $\Psi\Phi_f\Psi^{-1} = \Phi_{(\Psi f)}$; i.e., the following diagram commutes:

$$\begin{tikzcd}
\mathcal{E}_A \arrow[r, "\Phi_f"] \arrow[d, "\Psi"'] & \mathcal{E}_B \arrow[d, "\Psi"] \\
\mathcal{E}_{\widetilde{\Psi}(A)} \arrow[r, "\Phi_{(\Psi f)}"'] & \mathcal{E}_{\widetilde{\Psi}(B)}
\end{tikzcd}$$

Consequently, $\widetilde{\Psi(f)} = \widetilde{\Psi}\widetilde{f}\widetilde{\Psi}^{-1}$.

(6): It will first be shown that if $A, B \in \mathcal{P}$ are disjoint and $\mu(A) = \mu(B)$, then $\widetilde{\Psi}(A)$ are $\widetilde{\Psi}(B)$ are disjoint and $\mu(\widetilde{\Psi}(A)) = \mu(\widetilde{\Psi}(B))$. For disjoint A and B with equal measure, let $f \in \mathcal{E}$ be any interval exchange such that $f(A) = B$. Then f is a conjugacy between the subgroups $\mathcal{E}_A$ and $\mathcal{E}_B$:

$$\mathcal{E}_B = f\mathcal{E}_A f^{-1}.$$

Conversely, suppose A and B are disjoint sets in $\mathcal{P}$ such that there exists some map f which satisfies $\mathcal{E}_B = f\mathcal{E}_A f^{-1}$. Then it can be shown that $f(A) = B$. For any $x \in A$, pick some $g \in \mathcal{E}_A$ such that $y = g(x) \neq x$. Let $h = fgf^{-1}$. Since f conjugates g to h, f maps the non-fixed points of g to the non-fixed points of h. Since $h \in \mathcal{E}_B$, it follows that $f(x) \in B$. Thus $f(A) \subseteq B$, and symmetrically $f^{-1}(B) \subseteq A$, which implies $f(A) = B$. In particular, when A and B are disjoint with $\mathcal{E}_A$ and $\mathcal{E}_B$ conjugate, it must be the case that $\mu(A) = \mu(B)$.

Consider the action of $\widetilde{\Psi}$ on disjoint A and B with $\mu(A) = \mu(B)$. Let f be a map which induces a conjugacy $\mathcal{E}_B = f\mathcal{E}_A f^{-1}$. By (5), $\widetilde{\Psi f}$ maps $\widetilde{\Psi}(A)$ to $\widetilde{\Psi}(B)$. As a result, $\mu(\widetilde{\Psi}(A)) = \mu(\widetilde{\Psi}(B))$, which proves the initial claim.

To prove that $\mu(\widetilde{\Psi}(A)) = \mu(A)$ for any $A \in \mathcal{P}$, assume first that $\mu(A)$ is rational. Since any $\widetilde{\Psi}$ preserves finite disjoint unions by (3) and (4), it may be further assumed that $\mu(A) = 1/n$. Lebesgue measure is invariant under any conjugacy Φ_f, so it finally suffices to consider the case $A = [0, 1/n)$. Each of the intervals

$$A_i = \left[\frac{i-1}{n}, \frac{i}{n}\right), \quad 2 \leq i \leq n,$$

has the same measure as $A = A_1$ and is disjoint from it. Thus

$$\mu(\widetilde{\Psi}(A_i)) = \mu(\widetilde{\Psi}(A)), \quad 2 \leq i \leq n.$$

Since the sets $\widetilde{\Psi}(A_i)$ are also pairwise disjoint and cover $\mathbb{T}^1$, it follows that $\mu(\widetilde{\Psi}(A_i)) = 1/n$. Consequently, $\widetilde{\Psi}$ preserves the measure of sets with rational measure. In general, the set A may be approximated by an increasing family of sets in $\mathcal{P}$ having rational measure. □

7.2. Proof of Theorem 1.6

Let Ψ be an arbitrary automorphism of $\mathcal{E}$. It will be seen that $\Psi \in \langle \mathrm{Inn}(\mathcal{E}), \Psi_T \rangle$ by showing that the identity may be reached by successively replacing Ψ with a composition of Ψ and some automorphism in $\langle \mathrm{Inn}(\mathcal{E}), \Psi_T \rangle$.

To begin, by Corollary 7.2, Ψ maps the rotation group R to a conjugate $\Phi_g(R)$, for some $g \in \mathcal{E}$. Replacing Ψ by $\Phi_g^{-1} \circ \Psi$, it may now be assumed that R is invariant under Ψ:

$$\Psi(R) = R.$$

Let $\Psi_R : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ denote the restriction $\Psi|_R$, where $r_\alpha \mapsto \alpha$ is the natural identification of R and $\mathbb{R}/\mathbb{Z}$.

Lemma 7.7. Ψ_R *is continuous (w.r.t. the standard topology on* $\mathbb{R}/\mathbb{Z}$*).*

Proof. It suffices to show that Ψ_R is continuous at $0 \in \mathbb{R}/\mathbb{Z}$. Suppose that $\alpha_n \to 0$. Then for any nonempty $A \in \mathcal{P}$, there exists a constant $M_A > 0$ such that

$$A \cap r_{\alpha_n}(A) \neq \emptyset, \ \forall\, n \geq M_A.$$

Conversely, this condition characterizes sequences in $\mathbb{R}/\mathbb{Z}$ which converge to 0. In particular, given some sequence α_n, suppose that there exists a constant M_A as above for every nonempty $A \in \mathcal{P}$. For any $\epsilon > 0$, let $A_\epsilon = [0, \epsilon)$. Then

$$A_\epsilon \cap r_{\alpha_n}(A_\epsilon) \neq \emptyset, \ \forall\, n \geq M_{A_\epsilon},$$

which implies that $|\alpha_n| < \epsilon$ for all $n \geq M_{A_\epsilon}$. Thus, $\alpha_n \to 0$.

Assuming again that $\alpha_n \to 0$, define $\beta_n = \Psi_R(\alpha_n)$, so $r_{\beta_n} = \Psi(r_{\alpha_n})$. Let $B \in \mathcal{P}$ be nonempty, and let $A = \widetilde{\Psi}^{-1}(B)$. Then by Proposition 7.6, part (5),

$$r_{\beta_n}(B) = \widetilde{\Psi}(r_{\alpha_n}(A)).$$

Consequently, $A \cap r_{\alpha_n}(A) \neq \emptyset$ if and only if $B \cap r_{\beta_n}(B) \neq \emptyset$. Therefore, if $\alpha_n \to 0$, then there exists M_B (namely, the M_A associated with α_n), such that

$$B \cup r_{\beta_n}(B) \neq \emptyset, \ \forall\, n \geq M_B.$$

From the above characterization of sequences converging to zero, it follows that Ψ_R is continuous at zero. $\square$

The only continuous automorphisms of $\mathbb{R}/\mathbb{Z}$ are the identity and $x \mapsto -x$. Note that the restriction to R of the orientation reversing automorphism Ψ_T is the second of these automorphisms. Subsequently, after replacing Ψ by $\Psi \circ \Psi_T$ if Ψ_R is not the identity, it may be assumed that $\Psi_R = id$.

Lemma 7.8. *If $\Psi \in Aut(\mathcal{E})$ fixes the rotation group R, then $\widetilde{\Psi}$ maps any interval in $\mathcal{P}$ to another interval.*

Proof. Since any rotation will preserve intervals in $\mathcal{P}$, it suffices to consider $I_a = [0, a)$. Then there exists some $\epsilon > 0$, such that for any $\alpha \in (-\epsilon, \epsilon)$,

$$\mu(I_a \cap r_\alpha(I_a)) = a - |\alpha|.$$

Therefore, since $\widetilde{\Psi} \circ \widetilde{r_\alpha} = \widetilde{\Psi r_\alpha} \circ \widetilde{\Psi} = \widetilde{r_\alpha} \circ \widetilde{\Psi}$,

$$\mu(\widetilde{\Psi}(I_a) \cap r_\alpha(\widetilde{\Psi}(I_a))) = a - |\alpha|,$$

for $\alpha \in (-\epsilon, \epsilon)$.

Suppose that $\widetilde{\Psi}(I_a)$ has $k \geq 1$ components:

$$\widetilde{\Psi}(I_a) = A_1 \cup \cdots \cup A_k,$$

where the A_i are pairwise disjoint intervals. Since the A_i are disjoint, there is some $\delta > 0$ such that

$$r_\beta(A_i) \cap A_j = \emptyset, \quad \text{for all } |\beta| < \delta \text{ and } i \neq j.$$

Consequently, if $|\beta| < \delta$, then

$$\mu(\widetilde{\Psi}(I_a) \cap r_\beta(\widetilde{\Psi}(I_a))) = a - k|\beta|.$$

It follows that $k = 1$; i.e., $\widetilde{\Psi}(I_a)$ must be an interval. $\square$

Continue with the assumption that Ψ_R is the identity. By the previous lemma, $\widetilde{\Psi}$ maps the interval $I_a = [0, a)$ to some translate of I_a. After composing Ψ with a suitable

Φ_{r_β}, it may be assumed that Ψ_R is the identity and $\widetilde{\Psi}(I_a) = I_a$. Since

$$\widetilde{\Psi} \circ \widetilde{r_\beta} = \widetilde{r_\beta} \circ \widetilde{\Psi},$$

for all $\beta \in \mathbb{R}/\mathbb{Z}$, it follows that $\widetilde{\Psi}$ fixes any translate $\widetilde{r_\beta}(I_a) = [\beta, a+\beta)$. Thus, for any β, $0 < \beta < a$, $\widetilde{\Psi}$ fixes the intersection

$$I_a \cap r_\beta(I_a) = [\beta, a).$$

Thus $\widetilde{\Psi}$ fixes all translates of arbitrarily small intervals, which implies that $\widetilde{\Psi}$ is the identity on $\mathcal{P}$. Consequently, for any $f \in \mathcal{E}$, $\Psi(f)$ acts on the sets in $\mathcal{P}$ identically to the way f does, which implies Ψ is the identity. It has been shown that any $\Psi \in \mathrm{Aut}(\mathcal{E})$ is in the group $\langle \mathrm{Inn}(\mathcal{E}), \Psi_T \rangle$, and the proof of Theorem 1.6 is complete. $\square$

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