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ABSTRACT

Dynamic Games in Industrial Organization

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This dissertation formulates, solves and estimates dynamic stochastic games to answer various questions that arise in Industrial Organization. First chapter is a "theoretic" investigation of learning-by-doing and organizational forgetting that shows them to be distinct economic forces whose interplay gives rise to aggressive pricing behavior, market dominance, and multiple equilibria. The homotopy method is used to identify multiple equilibria in a systematic fashion. Second chapter is an empirical study that structurally estimates a dynamic model of the drug development process in the pharmaceutical industry, and uses counterfactual experiments to evaluate the effects of various policy interventions aimed at increasing the introduction rate of new drugs within a specific therapeutic area. I find that policies affecting the earlier stages of the drug development are less effective than those aimed at the later stages; this phenomenon stems from strategic response to policy-induced changes, and would not be observable outside a game framework. Final chapter introduces the homotopy path-following as a new method of exploring parameter space and identifying multiple equilibria in dynamic games; we believe that wider adoption of this method will substantially increase the quality of research in the field.

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Introduction

This dissertation formulates, solves and estimates dynamic stochastic games in order to answer various questions that arise in Industrial Organization. Ericson & Pakes (1995) introduced dynamic stochastic games to the field and defined the solution concept as Markov Perfect equilibrium, setting up a modeling framework that is particularly effective in studies of strategic interaction in the oligopoly setting where firms have to continuously invest into building up and maintaining the competitive advantage. Dynamic games have been used[1] to study such economic features as capacity and quality investment, entry and exit, advertising, network effects, R&D.

The dynamic stochastic games with economically relevant level of detail often prove analytically intractable, so both empirical and theoretic studies rely on numeric solutions obtained using a variety of computational methods. This dissertation presents an example of a theoretic study, an empirical one and a detailed description of a new computational method.

Chapter 1 is a "theoretic" study of learning-by-doing and organizational forgetting that have been shown to be important in a variety of industrial settings. This coauthored work provides a general model of dynamic competition that accounts for these economic fundamentals and shows how they shape industry structure and dynamics. Results of Cabral & Riordan (1994) regarding dominance properties of firms' pricing behavior no longer hold in

[1]See Doraszelski & Pakes (2007) and the references therein.

this more general setting. We show that organizational forgetting does not simply negate learning-by-doing, but instead these are two distinct economic forces. In particular, a model with both learning-by-doing and organizational forgetting can give rise to aggressive pricing behavior, market dominance, and multiple equilibria, whereas a model with learning-by-doing alone cannot. We use a novel computational approach (homotopy path-following) to explore the impact of learning and forgetting in the presence of multiple equilibria.

Chapter 2 is an empirical study that structurally estimates a dynamic model of drug development process in the pharmaceutical industry, and uses counterfactual experiments to evaluate the effects of various policy interventions aimed at increasing the introduction rate of new drugs within a specific therapeutic area. Advanced computational methods – such as continuous time and polynomial approximations – overcome the modeling difficulties that limited previous studies to one or two stages of development process, and allow the model to describe the progress of drugs through all three phases of clinical trials and FDA approval. The primary result is that policies affecting the earlier stages of drug development are less effective than those aimed at the later stages. The reason for this finding is the strategic response to policy-induced changes in the number of drugs under development and on the market.

Chapter 3 is contributes to the methodological toolbox of the field. This coauthored work provides a step-by-step guide to solving dynamic stochastic games using the homotopy method. The homotopy method offers a systematic approach to exploring the correspondence that maps parameters into equilibria; it is especially useful in games that have multiple equilibria. As discussed in the introduction to that chapter, multiplicity is an important issue in theoretical, empirical and policy work. We discuss the theory of the homotopy method and its implementation and present two detailed examples of dynamic stochastic

games that are solved using this method. The set of codes that we provide online enable a researcher to apply homotopy method to their own model.

CHAPTER 1

Learning-by-Doing, Organizational Forgetting, and Industry Dynamics (joint with David Besanko, Ulrich Doraszelski and Mark Satterthwaite)

1.1. Introduction

Empirical studies provide ample evidence that the marginal cost of production decreases with cumulative experience in a variety of industrial settings (see, e.g., Wright 1936, Hirsch 1952, DeJong 1957, Alchian 1963, Levy 1965, Kilbridge 1962, Hirschmann 1964, Preston & Keachie 1964, Baloff 1971, Dudley 1972, Zimmerman 1982, Lieberman 1984, Gruber 1992, Irwin & Klenow 1994, Jarmin 1994, Pisano 1994, Bohn 1995, Hatch & Mowery 1998, Thompson 2001, Thornton & Thompson 2001). This fall in marginal cost is known as learning-by-doing. More recent empirical studies also suggest that organizations can forget the know-how gained through learning-by-doing due to labor turnover, periods of inactivity, and failure to institutionalize tacit knowledge (see, e.g., Argote, Beckman & Epple 1990, Darr, Argote & Epple 1995, Benkard 2000, Shafer, Nembhard & Uzumeri 2001, Thompson 2003). Organizational forgetting has been largely ignored by the theoretical literature.[1] This is especially troubling because Benkard (2004) shows that organizational forgetting is essential to explain the dynamics in the market for wide-bodied airframes in the 1970s and 1980s.

In this paper we provide a general model of dynamic competition based on the Markov-perfect equilibrium framework of Ericson & Pakes (1995). We show how the economic fundamentals of learning-by-doing and organizational forgetting interact to determine the structure and dynamics of an industry. Closest in spirit to our model is the Cabral & Riordan (1994) model with learning-by-doing alone. We build on Cabral & Riordan's (1994) seminal paper and other existing models of learning-by-doing by accounting for organizational

[1] An exception is Lewis & Yildirim (2005) who study the role of organizational forgetting in the context of a multi-period procurement auction in which a single buyer faces switching costs.

forgetting.[2] This seemingly small change has surprisingly large effects. Dynamic competition with learning-by-doing and organizational forgetting is akin to racing down an upward moving escalator. As long as a firm makes sales sufficiently frequently so that the gain in know-how from learning-by-doing outstrips the loss in know-how from organizational forgetting, it moves down its learning curve and its marginal cost decreases. However, if sales slow down or come to a halt, then the firm slides back up its learning curve and its marginal cost increases. This cannot happen in a model with learning-by-doing alone. Due to this qualitative difference, adding organizational forgetting to a model of learning-by-doing leads to a rich array of pricing behaviors and industry dynamics that the existing literature neither imagined nor explained.

It is often said that learning-by-doing promotes market dominance because it gives a more experienced firm the ability to profitably underprice its less experienced rival. As Dasgupta & Stiglitz (1988) put it

> ... firm-specific learning encourages the growth of industrial concentration. To be specific, one expects that strong learning possibilities, coupled with vigorous competition among rivals, ensures that history matters ... in the sense that if a given firm enjoys some initial advantages over its rivals it can, by undercutting them, capitalise on these advantages in such a way that the advantages accumulate over time, rendering rivals incapable of offering effective competition in the long run ... (p. 247)

But if learning-by-doing can be "undone" by organizational forgetting, this raises the question whether organizational forgetting is an antidote to market dominance for two reasons.

[2]Prior to the infinite-horizon price-setting model of Cabral & Riordan (1994), the literature has studied learning-by-doing using finite-horizon quantity-setting models (e.g., Spence 1981, Fudenberg & Tirole 1983, Ghemawat & Spence 1985, Ross 1986, Dasgupta & Stiglitz 1988, Cabral & Riordan 1997).

First, to the extent that the leader has more to forget than the follower, organizational forgetting should work to equalize differences between firms. Second, because organizational forgetting makes improvements in competitive position from learning-by-doing more transitory, it should make firms more reluctant to invest in the acquisition of know-how through price cuts in the first place. We reach the opposite conclusion: organizational forgetting tends to make firms more instead of less aggressive. This aggressive pricing behavior, in turn, puts the industry on a path towards market dominance.

In the absence of organizational forgetting, the price that a firm sets reflects two goals. First, by winning a sale, the firm moves down its learning curve. This is the advantage-building motive. Second, the firm prevents its rival from moving down its learning curve. This is the advantage-defending motive. In the presence of organizational forgetting, bidirectional movements through the state space are possible, and this opens up new strategic possibilities for firms that work to enhance the advantage-building and advantage-defending motives. By winning a sale, a firm makes itself less vulnerable to future losses from organizational forgetting, thus enhancing the advantage-building motive. It also makes its rival more vulnerable to future losses from organizational forgetting, thus enhancing the advantage-defending motive. Because these additional benefits are achieved by winning a sale, organizational forgetting creates strong incentives to cut prices. It is thus a source of aggressive pricing behavior.

While the existing literature has mainly focused on the dominance properties of firms' pricing behavior, we find that these properties are neither necessary nor sufficient for economically meaningful market dominance in our more general setting. We therefore go beyond the existing literature and directly examine the industry dynamics implied by firms' pricing

behavior. We find that organizational forgetting is a source of—and not an antidote to—market dominance. If organizational forgetting is sufficiently weak, then asymmetries may arise but they cannot persist. If organizational forgetting is sufficiently strong, then asymmetries cannot arise in the first place because organizational forgetting stifles investment in learning-by-doing altogether. By contrast, for intermediate degrees of organizational forgetting, asymmetries arise and persist. Even extreme asymmetries akin to near-monopoly are possible. This is because organizational forgetting predisposes the leader to defend its position aggressively against imminent and distant threats. This more than offsets the increased vulnerability to organizational forgetting as the stock of know-how grows and therefore makes the leadership position more secure than it would have been in the absence of organizational forgetting.

Organizational forgetting is also a source of multiple equilibria. If the inflow of know-how into the industry due to learning-by-doing is substantially smaller than the outflow of know-how due to organizational forgetting, then it is virtually impossible that both firms reach the bottom of their learning curves. Conversely, if the inflow is substantially greater than the outflow, then it is virtually inevitable that they do. An extreme example is the Cabral & Riordan (1994) model with learning-by-doing alone. In both cases, the primitives of the model tie down the equilibrium. This is no longer the case if the inflow roughly balances the outflow, setting the stage for multiple equilibria. If firms believe that they cannot profitably coexist at the bottom of their learning curves and that instead one firm comes to dominate the market, then both firms cut their prices in the hope of acquiring a competitive advantage early on and maintaining it throughout. This aggressive pricing behavior, in turn, leads to market dominance. However, if firms believe that they can profitably coexist, then neither firm cuts its price, thereby ensuring that the anticipated

symmetric industry structure actually emerges. Consequently, in addition to the degree of organizational forgetting, the equilibrium by itself is an important determinant of pricing behavior and industry dynamics.

In our model multiple equilibria do not arise because of the specification of the primitives. In fact, we are able to show that multiple equilibria arise from firms' expectations regarding the value of continued play. In this sense multiplicity is rooted in the dynamics of the model. Our finding of multiplicity is important for two reasons. First, to our knowledge, all applications of Ericson & Pakes's (1995) framework have found a single equilibrium. It indeed is often held that "nonuniqueness does not seem to be a problem" in this setting (Pakes & McGuire 1994, p. 570). It is therefore striking that we obtain up to nine equilibria for some parameterizations. Second, being able to pinpoint the driving force behind multiple equilibria is a first step towards tackling the multiplicity problem that plagues the estimation of dynamic stochastic games and inhibits the use of counterfactuals in policy analysis (see Ackerberg, Benkard, Berry & Pakes (2005) and Pakes (2008) for a discussion of the issue).

In sum, learning-by-doing and organizational forgetting are distinct economic forces. Organizational forgetting, in particular, does not simply negate learning-by-doing. The unique role played by organizational forgetting comes about because it makes bidirectional movements through the state space possible. As a consequence, a model with both learning-by-doing and organizational forgetting can give rise to aggressive pricing behavior, market dominance, and multiple equilibria, whereas a model with learning-by-doing alone cannot.

We also make two methodological contributions. First, we point out a weakness of the major tool for computing equilibria in the literature following Ericson & Pakes (1995). Specifically, we prove that our dynamic stochastic game has equilibria that cannot be computed by the Pakes & McGuire (1994) algorithm. Roughly speaking, in the presence of multiple

equilibria, "in between" two equilibria that can be computed by the Pakes & McGuire (1994) algorithm, there is one equilibrium that cannot. This severely limits the ability of the Pakes & McGuire (1994) algorithm to provide a reasonably complete picture of the set of solutions to the model.

Second, we propose a homotopy or path-following algorithm. The algorithm traces out the equilibrium correspondence by varying the degree of organizational forgetting and allows us to compute equilibria that cannot be computed by the Pakes & McGuire (1994) algorithm. We find that the equilibrium correspondence contains a unique path that starts at the equilibrium of the model with learning-by-doing alone. Whenever this path bends back on itself and then forward again, there are multiple equilibria. In addition, the equilibrium correspondence may contain (one or more) loops that cause additional multiplicity. To our knowledge, our paper is the first to describe in detail the structure of the set of equilibria of a dynamic stochastic game in the tradition of Ericson & Pakes (1995).

The organization of the remainder of the paper is as follows. Sections 1.2 and 1.3 describe the model specification and our computational strategy. Section 1.4 provides an overview of the equilibrium correspondence. Section 1.5 analyzes industry dynamics and Section 1.6 characterizes the pricing behavior that drives it. Section 1.7 describes how organizational forgetting can lead to multiple equilibria. Section 1.8 undertakes a number of robustness checks. Section 1.9 summarizes and concludes. All proofs are relegated to the Appendix.

Throughout the paper we distinguish between propositions which are established through formal proofs and results. A result either establishes a possibility through a numerical example or summarizes a regularity through a systematic exploration of the parameter space.

1.2. Model

For expositional clarity we focus on the basic model of an industry with two firms and neither entry nor exit. The general model is outlined in the Online Appendix.

Firms and states. We consider a discrete-time, infinite-horizon dynamic stochastic game of complete information played by two firms. Firm $n \in \{1, 2\}$ is described by its state $e_n \in \{1, \ldots, M\}$. A firm's state indicates its cumulative experience or stock of know-how. By making a sale, a firm can add to its stock of know-how. Following Cabral & Riordan (1994), we take a period to be just long enough for a firm to make a sale.[3] In contrast to Cabral & Riordan (1994), however, we incorporate organizational forgetting in our model as suggested by the empirical studies of Argote et al. (1990), Darr et al. (1995), Benkard (2000), Shafer et al. (2001), and Thompson (2003). Accordingly, the evolution of firm n's stock of know-how is governed by the law of motion

$$e'_n = e_n + q_n - f_n,$$

where e'_n and e_n is firm n's stock of know-how in the subsequent and current period, respectively, the random variable $q_n \in \{0, 1\}$ indicates whether firm n makes a sale, and the random variable $f_n \in \{0, 1\}$ represents organizational forgetting. If $q_n = 1$, the firm gains a unit of know-how through learning-by-doing, while it loses a unit of know-how through organizational forgetting if $f_n = 1$.

At any point in time, the industry is characterized by a vector of firms' states $\mathbf{e} = (e_1, e_2) \in \{1, \ldots, M\}^2$. We refer to $\mathbf{e}$ as the state of the industry. We use $\mathbf{e}^{[2]}$ to denote the vector (e_2, e_1) found by interchanging the stocks of know-how of firms 1 and 2.

[3] A sale may involve a single unit or a batch of units (e.g., 100 aircraft or 10,000 memory chips) that are sold to a single buyer.

Learning-by-doing. Firm n's marginal cost of production $c(e_n)$ depends on its stock of know-how e_n through a learning curve

$$c(e_n) = \begin{cases} \kappa e_n^{\eta} & \text{if} \quad 1 \leq e_n < m, \\ \kappa m^{\eta} & \text{if} \quad m \leq e_n \leq M, \end{cases}$$

where $\eta = \log_2 \rho$ for a progress ratio of $\rho \in (0, 1]$. Marginal cost decreases by $100(1 - \rho)$ percent as the stock of know-how doubles, so that a lower progress ratio implies a steeper learning curve. The marginal cost of production at the top of the learning curve, $c(1)$, is $\kappa > 0$ and, in line with Cabral & Riordan (1994), m represents the stock of know-how at which a firm reaches the bottom of its learning curve.[4]

Organizational forgetting. We let $\Delta(e_n) = \Pr(f_n = 1)$ denote the probability that firm n loses a unit of know-how through organizational forgetting. We assume that this probability is nondecreasing in the firm's experience level. This has several advantages. First, experimental evidence in the management literature suggests that forgetting by individuals is an increasing function of the current stock of learned knowledge (Bailey 1989). Second, a direct implication of $\Delta(\cdot)$ being increasing is that the expected stock of know-how in the absence of further learning is a decreasing convex function of time.[5] This phenomenon, known in the psychology literature as Jost's second law, is consistent with experimental evidence on forgetting by individuals (Wixted & Ebbesen 1991). Third, in the capital-stock model employed in empirical work on organizational forgetting the amount of depreciation is assumed to be proportional to the stock of know-how. Hence, the additional know-how needed

[4]While Cabral & Riordan (1994) formally consider the state space to be infinite (i.e., $M = \infty$ in our notation), they make the additional assumption that the price that a firm charges does not depend on how far it is beyond the bottom of its learning curve (p. 1119). This is tantamount to assuming, as we do, that the state space is finite.

[5]See the Online Appendix for a proof.

to counteract depreciation must increase with the stock of know-how. Our specification has this feature but, unlike the capital-stock model, is consistent with a discrete state space.[6]

The specific functional form we employ is

$$\Delta(e_n) = 1 - (1 - \delta)^{e_n},$$

where we refer to $\delta \in [0, 1]$ as the forgetting rate.[7] If $\delta > 0$, then $\Delta(e_n)$ is increasing and concave in e_n; $\delta = 0$ corresponds to the absence of organizational forgetting, the special case Cabral & Riordan (1994) analyzed. Other functional forms are plausible, and we explore some of them in Section 1.8.

Demand. The industry draws its customers from a large pool of potential buyers. In each period, one buyer enters the market and purchases the good from one of the two firms.[8] The utility that the buyer obtains by purchasing good n is $v - p_n + \varepsilon_n$, where p_n is the price of good n, v is a deterministic component of utility, and ε_n is a stochastic component that captures the idiosyncratic preference for good n of this period's buyer. ε_1 and ε_2 are unobservable to firms and are assumed to be independently and identically type 1 extreme value distributed with location parameter 0 and scale parameter $\sigma > 0$. The scale parameter governs the degree of horizontal product differentiation. As $\sigma \to 0$, goods become homogeneous.

[6]See Benkard (2004) for an alternative approximation to the capital-stock model.

[7]One way to motivate this functional form is to imagine that the stock of know-how is dispersed among a firm's workforce. In particular, assume that e_n is the number of skilled workers and that organizational forgetting is the result of labor turnover. Then, given a turnover rate of δ, $\Delta(e_n)$ is the probability that at least one of the e_n skilled workers leaves the firm.

[8]Since there is a different buyer in each period, buyers are non-strategic. Lewis & Yildirim (2002, 2005) consider models with a single buyer who optimally designs a multi-period procurement auction in order to influence the dynamics of the industry.

The buyer purchases the good that gives it the highest utility. Given our distributional assumptions the probability that firm n makes a sale is given by the logit specification

$$D_n(\mathbf{p}) = \Pr(q_n = 1) = \frac{\exp(\frac{v-p_n}{\sigma})}{\sum_{k=1}^{2} \exp(\frac{v-p_k}{\sigma})} = \frac{1}{1+\exp(\frac{p_n-p_{-n}}{\sigma})},$$

where $\mathbf{p} = (p_1, p_2)$ is the vector of prices and we adopt the convention of using p_{-n} to denote the price charged by the other firm. Demand effectively depends on differences in prices because we assume in line with Cabral & Riordan (1994) that the buyer always purchases from one of the two firms in the industry. In Section 1.8 we include an outside good in the specification.

State-to-state transitions. ¿From one period to the next, a firm's stock of know-how moves up or down or remains constant depending on realized demand $q_n \in \{0, 1\}$ and organizational forgetting $f_n \in \{0, 1\}$. The transition probabilities are

$$\Pr(e_n'|e_n, q_n) = \begin{cases} 1 - \Delta(e_n) & \text{if} \quad e_n' = e_n + q_n, \\ \Delta(e_n) & \text{if} \quad e_n' = e_n + q_n - 1, \end{cases}$$

where, at the upper and lower boundaries of the state space, we modify the transition probabilities to be $\Pr(M|M, 1) = 1$ and $\Pr(1|1, 0) = 1$, respectively. Note that the firm can increase its stock of know-how only if it makes a sale in the current period, an event that has probability $D_n(\mathbf{e})$; otherwise it runs the risk that its stock of know-how will decrease.

Bellman equation. Define $V_n(\mathbf{e})$ to be the expected net present value of firm n's cash flows if the industry is currently in state $\mathbf{e}$. The value function $\mathbf{V}_n : \{1, \ldots, M\}^2 \to [-\hat{V}, \hat{V}]$,

where $\hat{V}$ is a sufficiently large constant, is implicitly defined by the Bellman equation

$$V_n(\mathbf{e}) = \max_{p_n} D_n(p_n, p_{-n}(\mathbf{e}))(p_n - c(e_n)) + \beta \sum_{k=1}^{2} D_k(p_n, p_{-n}(\mathbf{e}))\overline{V}_{nk}(\mathbf{e}), \tag{1.1}$$

where $p_{-n}(\mathbf{e})$ is the price charged by the other firm in state $\mathbf{e}$, $\beta \in (0,1)$ is the discount factor, and $\overline{V}_{nk}(\mathbf{e})$ is the expectation of firm n's value function conditional on the buyer purchasing the good from firm $k \in \{1,2\}$ in state $\mathbf{e}$ as given by

$$\overline{V}_{n1}(\mathbf{e}) = \sum_{e_1'=e_1}^{e_1+1} \sum_{e_2'=e_2-1}^{e_2} V_n(\mathbf{e}') \Pr(e_1'|e_1,1) \Pr(e_2'|e_2,0), \tag{1.2}$$

$$\overline{V}_{n2}(\mathbf{e}) = \sum_{e_1'=e_1-1}^{e_1} \sum_{e_2'=e_2}^{e_2+1} V_n(\mathbf{e}') \Pr(e_1'|e_1,0) \Pr(e_2'|e_2,1). \tag{1.3}$$

The policy function $\mathbf{p}_n : \{1,\ldots,M\}^2 \to [-\hat{p}, \hat{p}]$, where $\hat{p}$ is a sufficiently large constant, specifies the price $p_n(\mathbf{e})$ that firm n sets in state $\mathbf{e}$.[9] Let $h_n(\mathbf{e}, p_n, p_{-n}(\mathbf{e}), \mathbf{V}_n)$ denote the maximand in the Bellman equation (1.1). Differentiating this so-called return function with respect to p_n and using the properties of logit demand we obtain the first-order condition (FOC):

$$0 = \frac{\partial h_n(\cdot)}{\partial p_n} = \frac{1}{\sigma} D_n(p_n, p_{-n}(\mathbf{e}))\Big(\sigma - (p_n - c(e_n)) - \beta\overline{V}_{nn}(\mathbf{e}) + h_n(\cdot)\Big).$$

Differentiating $h_n(\cdot)$ a second time yields

$$\frac{\partial^2 h_n(\cdot)}{\partial p_n^2} = \frac{1}{\sigma}\frac{\partial h_n(\cdot)}{\partial p_n}\Big(2D_n(p_n, p_{-n}(\mathbf{e})) - 1\Big) - \frac{1}{\sigma} D_n(p_n, p_{-n}(\mathbf{e})).$$

[9] In what follows we assume that $\hat{p}$ is chosen large enough to not constrain pricing behavior.

If the FOC is satisfied, then $\frac{\partial^2 h_n(\cdot)}{\partial p_n^2} = -\frac{1}{\sigma} D_n(p_n, p_{-n}(\mathbf{e})) < 0$. The return function $h_n(\cdot)$ is therefore strictly quasi-concave in p_n, so that the pricing decision $p_n(\mathbf{e})$ is uniquely determined by the solution to the FOC (given $p_{-n}(\mathbf{e})$).

Equilibrium. In our model, firms face identical demand and cost primitives. Asymmetries between firms arise endogenously as a consequence of their pricing decisions for realized demand and organizational forgetting. Hence, we focus attention on symmetric Markov perfect equilibria (MPE). In a symmetric equilibrium the pricing decision taken by firm 2 in state $\mathbf{e}$ is identical to the pricing decision taken by firm 1 in state $\mathbf{e}^{[2]}$, i.e., $p_2(\mathbf{e}) = p_1(\mathbf{e}^{[2]})$, and similarly for the value function. It therefore suffices to determine the value and policy functions of firm 1, and we define $V(\mathbf{e}) = V_1(\mathbf{e})$ and $p(\mathbf{e}) = p_1(\mathbf{e})$ for each state $\mathbf{e}$. Further, we let $\overline{V}_k(\mathbf{e}) = \overline{V}_{1k}(\mathbf{e})$ denote the conditional expectation of firm 1's value function and $D_k(\mathbf{e}) = D_k(p(\mathbf{e}), p(\mathbf{e}^{[2]}))$ the probability that the buyer purchases from firm $k \in \{1, 2\}$ in state $\mathbf{e}$.

Given this notation, the Bellman equation and FOC can be expressed as

$$F_{\mathbf{e}}^1(\mathbf{V}^*, \mathbf{p}^*) = -V^*(\mathbf{e}) + D_1^*(\mathbf{e})\left(p^*(\mathbf{e}) - c(e_1)\right) + \beta \sum_{k=1}^{2} D_k^*(\mathbf{e}) \overline{V}_k^*(\mathbf{e}) = 0, \tag{1.4}$$

$$F_{\mathbf{e}}^2(\mathbf{V}^*, \mathbf{p}^*) = \sigma - \left(1 - D_1^*(\mathbf{e})\right)\left(p^*(\mathbf{e}) - c(e_1)\right) - \beta \overline{V}_1^*(\mathbf{e}) + \beta \sum_{k=1}^{2} D_k^*(\mathbf{e}) \overline{V}_k^*(\mathbf{e}) = 0, \tag{1.5}$$

where we use asterisks to denote an equilibrium. The collection of equations (1.4) and (1.5) for all states $\mathbf{e} \in \{1, \ldots, M\}^2$ can be written more compactly as

$$\mathbf{F}(\mathbf{V}^*, \mathbf{p}^*) = \begin{bmatrix} F^1_{(1,1)}(\mathbf{V}^*, \mathbf{p}^*) \\ F^1_{(2,1)}(\mathbf{V}^*, \mathbf{p}^*) \\ \vdots \\ F^2_{(M,M)}(\mathbf{V}^*, \mathbf{p}^*) \end{bmatrix} = \mathbf{0}, \tag{1.6}$$

where $\mathbf{0}$ is a $(2M^2 \times 1)$ vector of zeros. Any solution to this system of $2M^2$ equations in $2M^2$ unknowns $\mathbf{V}^* = (V^*(1,1), V^*(2,1), \ldots, V^*(M,M))$ and $\mathbf{p}^* = (p^*(1,1), p^*(2,1), \ldots, p^*(M,M))$ is a symmetric equilibrium in pure strategies. A slightly modified version of Proposition 2 in Doraszelski & Satterthwaite (2007) establishes that such an equilibrium always exists for our model.

Parameterization. Our focus is on how learning-by-doing and organizational forgetting affect pricing behavior and the industry dynamics implied by that behavior. Accordingly, we explore the full range of values for the progress ratio ρ and the forgetting rate δ. To do so, we proceed as follows: First we specify a grid of 100 equidistant values of $\rho \in (0, 1]$. For each of them, we then use the homotopy algorithm described in Section 1.3 to trace the equilibrium as δ ranges from 0 to 1. Typically this entails solving the model for a few thousand intermediate values of δ.

Most empirical estimates of progress ratios are in the range of 0.7 to 0.95 (Dutton & Thomas 1984). However, a very steep learning curve with ρ much less than 0.7 may also capture a practically relevant situation. Suppose the first unit of a product is a hand-built prototype and the second unit is a guinea pig for organizing the production line. After this

point the gains from learning-by-doing are more or less exhausted and the marginal cost of production is close to zero.[10]

We note that empirical studies have found monthly rates of depreciation ranging from 4 to 25 percent of the stock of know-how (Benkard 2000, Argote et al. 1990). In the Online Appendix we show how to map these estimates that are based on a capital-stock model of learning-by-doing and organizational forgetting into in our specification. The implied values of the forgetting rate δ fall below 0.1.

We fix the values of the remaining parameters until Section 1.8 where we discuss their influence on the equilibrium and demonstrate the robustness of our conclusions. In our baseline parameterization, we set $M = 30$ and $m = 15$. The marginal cost at the top of the learning curve κ is equal to 10. For a progress ratio of $\rho = 0.85$, this implies that the marginal cost of production declines from a maximum value of $c(1) = 10$ to a minimum value of $c(15) = \ldots = c(30) = 5.30$. For $\rho = 0.15$, we have the case of a hand-built prototype where the marginal cost of production declines very quickly from $c(1) = 10$ over $c(2) = 1.50$ and $c(3) = 0.49$ to $c(15) = \ldots = c(30) = 0.01$.

Turning to demand, we set $\sigma = 1$ in our baseline parameterization. To illustrate, in the Nash equilibrium of a static price-setting game (obtained by setting $\beta = 0$) the own-price elasticity of demand ranges between -8.86 in state $(1, 15)$ and -2.13 in state $(15, 1)$ for a progress ratio of $\rho = 0.85$. The cross-price elasticity of firm 1's demand with respect to firm 2's price is 2.41 in state $(15, 1)$ and 7.84 in state $(1, 15)$. For $\rho = 0.15$ the own-price elasticity ranges between -9.89 and -1.00 and the cross-price elasticity between 1.00 and 8.05. These

[10]To avoid a marginal cost of close to zero, shift the cost function $c(e_n)$ by $\tau > 0$. While introducing a component of marginal cost that is unresponsive to learning-by-doing shifts the policy function by τ, the value function and the industry dynamics are left the same.

reasonable elasticities suggest that the results reported below are not artifacts of extreme parameterizations.

We set the discount factor to $\beta = \frac{1}{1.05}$. The discount factor can be thought of as $\beta = \frac{\zeta}{1+r}$, where $r > 0$ is the per-period discount rate and $\zeta \in (0, 1]$ is the exogenous probability that the industry survives from one period to the next. Consequently, our baseline parameterization corresponds to a variety of scenarios that differ in the length of a period. For example, it corresponds to a period length of one year, a yearly discount rate of 5 percent, and certain survival. Perhaps more interesting, it also corresponds to a period length of one month, a monthly discount rate of 1 percent (which translates into a yearly discount rate of 12.68 percent), and a monthly survival probability of 0.96. To put this—our focal scenario—in perspective, technology companies such as IBM and Microsoft had costs of capital in the range of 11 to 15 percent per annum in the late 1990s. Further, an industry with a monthly survival probability of 0.96 has an expected lifetime of 26.25 months. Thus this scenario is consistent with a pace of innovative activity that is expected to make the current generation of products obsolete within two to three years.

1.3. Computation

In this section we first describe a novel algorithm for computing equilibria that is based on homotopy methods. Then we turn to the Pakes & McGuire (1994) algorithm—the main tool in the literature initiated by Ericson & Pakes (1995)—and show that it is inadequate for characterizing the set of solutions to our model. A reader who is more interested in the economic implications of learning-by-doing and organizational forgetting may skip ahead to Section 1.4.

1.3.1. Homotopy Algorithm

The homotopy or path-following algorithm is described in detail in Chapter 3 and is designed to explore the set of equilibria in a systematic fashion. It is especially useful in models like ours that have multiple equilibria. Starting from a single equilibrium that has already been computed for a given parameterization of the model, the homotopy algorithm traces out an entire path of equilibria by varying a parameter of interest.[11]

As explained in Section 3.2, one of the requirements of homotopy method is that system of equations describing the equilibrium is "regular", i.e. it's Jacobian has full rank. While we have been unable to prove that our system is regular, we have been able to verify that the Jacobian always had full rank at all points along all paths taken by our homotopy algorithm.[12]

1.3.2. Pakes & McGuire (1994) Algorithm

The Pakes & McGuire (1994) algorithm or some other means for solving a system of non-linear equations (see, e.g., Judd 1998) is needed in order to compute a starting point for our homotopy algorithm. The Pakes & McGuire (1994) algorithm is the main tool in the literature initiated by Ericson & Pakes (1995). It is intuitively appealing because it combines value function iteration as familiar from dynamic programming with best reply dynamics (akin to Cournot adjustment) as familiar from static games.

Recall that $V_2(\mathbf{e}) = V_1(\mathbf{e}^{[2]})$ and $p_2(\mathbf{e}) = p_1(\mathbf{e}^{[2]})$ for each state $\mathbf{e}$ in a symmetric equilibrium and it therefore suffices to determine $\mathbf{V}$ and $\mathbf{p}$, the value and policy functions of

[11] See Zangwill & Garcia (1981) for an introduction to homotopy methods, Schmedders (1998, 1999) for an application to general equilibrium models with incomplete asset markets, and Berry & Pakes (2007) for an application to estimating demand systems.

[12] Our programs use Hompack (Watson, Billups & Morgan 1987, Watson, Sosonkina, Melville, Morgan & Walker 1997) written in Fortran 90 and are available from the authors upon request.

firm 1. The Pakes & McGuire (1994) algorithm is iterative. An iteration cycles through the states in some predetermined order, successively updating $\mathbf{V}$ and $\mathbf{p}$ as it progresses from one iteration to the next.

The strategic situation faced by firms in setting prices in state $\mathbf{e}$ is similar to a static game if the value of continued play is taken as given. The Pakes & McGuire (1994) algorithm computes the best reply of firm 1 against $p(\mathbf{e}^{[2]})$ in this game. The best reply serves to update the value and policy functions of firm 1 in state $\mathbf{e}$. More formally, let $h_1(\mathbf{e}, p_1, p(\mathbf{e}^{[2]}), \mathbf{V})$ be the maximand in the Bellman equation (1.1) after symmetry is imposed. The best reply of firm 1 against $p(\mathbf{e}^{[2]})$ in state $\mathbf{e}$ is given by

$$G^2_{\mathbf{e}}(\mathbf{V}, \mathbf{p}) = \arg\max_{p_1} h_1(\mathbf{e}, p_1, p(\mathbf{e}^{[2]}), \mathbf{V}) \tag{1.7}$$

and the value associated with it is

$$G^1_{\mathbf{e}}(\mathbf{V}, \mathbf{p}) = \max_{p_1} h_1(\mathbf{e}, p_1, p(\mathbf{e}^{[2]}), \mathbf{V}). \tag{1.8}$$

Write the collection of equations (1.7) and (1.8) for all states $\mathbf{e} \in \{1, \ldots, M\}^2$ as

$$\mathbf{G}(\mathbf{V}, \mathbf{p}) = \begin{pmatrix} G^1_{(1,1)}(\mathbf{V}, \mathbf{p}) \\ G^1_{(2,1)}(\mathbf{V}, \mathbf{p}) \\ \vdots \\ G^2_{(M,M)}(\mathbf{V}, \mathbf{p}) \end{pmatrix}. \tag{1.9}$$

Given an initial guess $\mathbf{x}^0 = (\mathbf{V}^0, \mathbf{p}^0)$, the Pakes & McGuire (1994) algorithm executes the iteration

$$\mathbf{x}^{k+1} = \mathbf{G}(\mathbf{x}^k), \quad k = 0, 1, 2, \ldots.$$

The algorithm aims to compute a fixed point $\mathbf{x} = \mathbf{G}(\mathbf{x})$ by continuing to iterate until the changes in the value and policy functions of firm 1 are deemed small (or a failure to converge is diagnosed). Any fixed point $\mathbf{x} = (\mathbf{V}^*, \mathbf{p}^*)$ of $\mathbf{G}$ is a symmetric equilibrium in pure strategies to our game.

Unlike our homotopy algorithm, the Pakes & McGuire (1994) algorithm does not lend itself to computing multiple equilibria. To identify more than one equilibrium (for a given parameterization of the model), the Pakes & McGuire (1994) algorithm must be restarted from different initial guesses. But different initial guesses may or may not lead to different equilibria. This, however, still understates the severity of the problem here: When there are multiple equilibria, the trial-and-error approach is sure to miss a substantial fraction of them even if an arbitrary number of initial guesses are tried. That is, our dynamic stochastic game has equilibria that cannot computed by the Pakes & McGuire (1994) algorithm.

Recall that the Pakes & McGuire (1994) algorithm continues to iterate until it reaches a fixed point $\mathbf{x} = \mathbf{G}(\mathbf{x})$. A necessary condition for it to converge is that the fixed point is locally stable. Specifically, consider the $(2M^2 \times 2M^2)$ Jacobian $\frac{\partial \mathbf{G}(\mathbf{x})}{\partial \mathbf{x}}$ at the fixed point and let $\rho\left(\frac{\partial \mathbf{G}(\mathbf{x})}{\partial \mathbf{x}}\right)$ be its spectral radius. The fixed point is locally stable under the Pakes & McGuire (1994) algorithm if $\rho\left(\frac{\partial \mathbf{G}(\mathbf{x})}{\partial \mathbf{x}}\right) < 1$, i.e., if all eigenvalues are within the complex unit circle. Given local stability, the algorithm converges provided that the initial guess is close (perhaps very close) to the fixed point. Conversely, the fixed point is unstable and cannot be computed by the Pakes & McGuire (1994) algorithm if $\rho\left(\frac{\partial \mathbf{G}(\mathbf{x})}{\partial \mathbf{x}}\right) \geq 1$.

In the remainder of this section we consider a parametric path $(\mathbf{x}(s), \delta(s)) \in \mathbf{F}^{-1}$ in the equilibrium correspondence, such as the path taken by our homotopy algorithm. Along this path we ask whether the equilibrium $\mathbf{x}(s)$ is locally stable or unstable under the Pakes &

McGuire (1994) algorithm when the forgetting rate is set to $\delta(s)$. Proposition 1 identifies a subset of equilibria that the Pakes & McGuire (1994) algorithm is sure to miss.

Proposition 1. *Let* $(\mathbf{x}(s), \delta(s)) \in \mathbf{F}^{-1}$. *(i) If* $\delta'(s) \leq 0$, *then* $\rho\left(\left.\frac{\partial \mathbf{G}(\mathbf{x}(s))}{\partial \mathbf{x}}\right|_{\delta=\delta(s)}\right) \geq 1$. *(ii) Moreover, the equilibrium* $\mathbf{x}(s)$ *remains unstable even if either dampening or extrapolation is applied to the Pakes & McGuire (1994) algorithm.*

Part (i) of Proposition 1 establishes that no equilibrium on the part of the equilibrium correspondence where $\delta'(s) \leq 0$ can be computed by the Pakes & McGuire (1994) algorithm. Whenever $\delta'(s)$ switches sign from positive to negative, the path that connects the unique equilibrium at $\delta = 0$ with the unique equilibrium at $\delta = 1$ bends backward and there are multiple equilibria. Conversely, whenever the sign of $\delta'(s)$ switches back from negative to positive, the path bends forward. Hence, holding fixed the forgetting rate, in between two equilibria with $\delta'(s) > 0$, there is one equilibrium with $\delta'(s) \leq 0$ that cannot be computed by the Pakes & McGuire (1994) algorithm. Similarly, a loop is necessarily composed of equilibria with $\delta'(s) > 0$ and equilibria with $\delta'(s) \leq 0$. The latter cannot be computed by the Pakes & McGuire (1994) algorithm.

Dampening and extrapolation are often applied to the Pakes & McGuire (1994) algorithm in the hope of improving its likelihood or speed of convergence. The iteration

$$\mathbf{x}^{k+1} = \omega \mathbf{G}(\mathbf{x}^k) + (1 - \omega)\mathbf{x}^k, \quad k = 0, 1, 2, \ldots,$$

is said to be dampened if $\omega \in (0, 1)$ and extrapolated if $\omega \in (1, \infty)$. Part (ii) of Proposition 1 establishes the futility of these attempts.[13]

[13]Dampening and extrapolation may, of course, still be helpful in computing equilibria with $\delta'(s) > 0$.

The ability of the Pakes & McGuire (1994) algorithm to provide a reasonably complete picture of the set of solutions to the model is limited beyond the scope of Proposition 1. As our computations indicate, some equilibria on the part of the equilibrium correspondence where $\delta'(s) > 0$ also cannot be computed by the Pakes & McGuire (1994) algorithm:

Result 1. *Let* $(\mathbf{x}(s), \delta(s)) \in \mathbf{F}^{-1}$. *If* $\delta'(s) > 0$, *then we may have* $\rho\left(\frac{\partial \mathbf{G}(\mathbf{x}(s))}{\partial \mathbf{x}}\Big|_{\delta=\delta(s)}\right) \geq 1$.

In the Online Appendix we provide a graphic illustration of Proposition 1 and Result 1.

As is well-known, not all Nash equilibria of static games are stable under best reply dynamics (see, e.g., Fudenberg & Tirole 1991).[14] Since the Pakes & McGuire (1994) algorithm incorporates best reply dynamics, it is reasonable to expect that this limits its usefulness. In the Online Appendix we argue that this is not the case. More precisely, we show that, holding fixed the value of continued play, the best reply dynamics are contractive and therefore converge to a unique fixed point irrespective of the initial guess. The value function iteration also is contractive holding fixed the policy function. Hence, each of the two building blocks of the Pakes & McGuire (1994) algorithm "works." What makes it impossible to obtain a large fraction of equilibria is the combination of value function iteration with best reply dynamics.

The Pakes & McGuire (1994) algorithm is also known as a pre-Gauss-Jacobi method. The subsequent literature has sometimes instead used a pre-Gauss-Seidel method (see, e.g., Benkard 2004, Doraszelski & Judd 2004). Whereas a Gauss-Jacobi method replaces the old guesses for the value and policy functions with the new guesses at the end of an iteration after all states have been visited, a Gauss-Seidel method updates after each state. This has

[14] More generally, in static games, Nash equilibria of degree -1 are unstable under any Nash dynamics, i.e., dynamics with rest points that coincide with Nash equilibria, including replicator and smooth fictitious play dynamics (Demichelis & Germano 2002).

the advantage that "information" is used as soon as it becomes available (see Chaps. 3 and 5 of Judd (1998) for an extensive discussion of Gaussian methods). While we have been unable to prove that Proposition 1 carries over to this alternative algorithm, we note that the Stein-Rosenberg theorem asserts, at least for certain systems of linear equations, that if the Gauss-Jacobi algorithm fails to converge then so does the Gauss-Seidel algorithm (see Proposition 6.9 in Section 2.6 of Bertsekas & Tsitsiklis 1997).

1.4. Equilibrium Correspondence

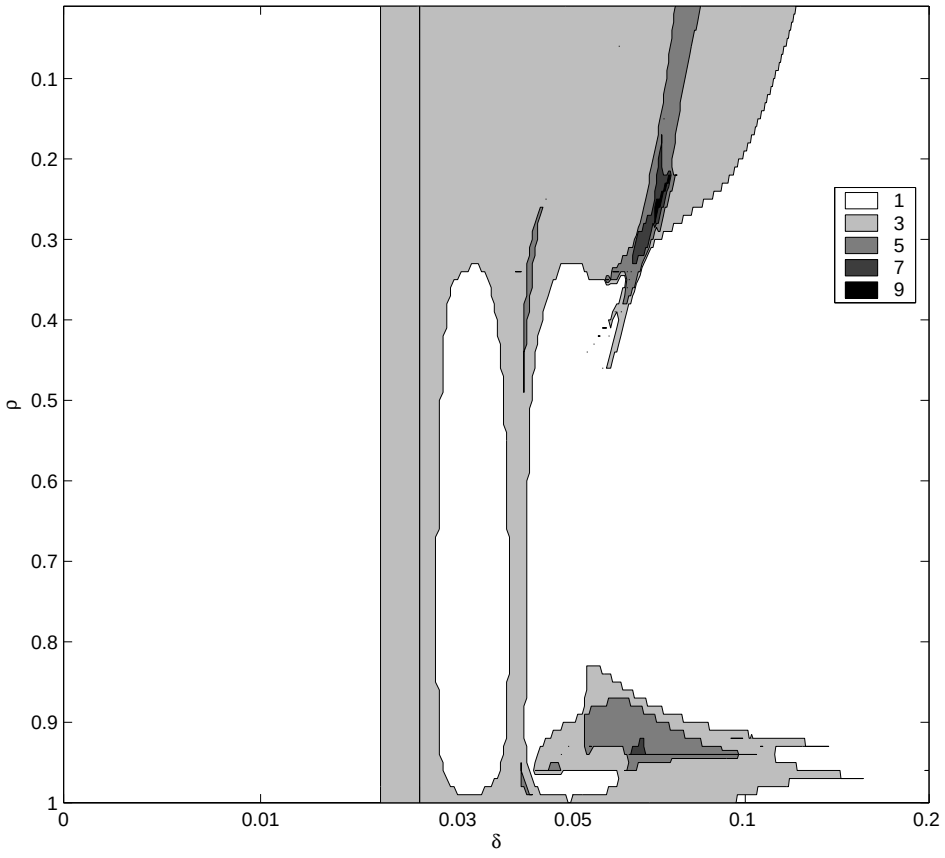

Figure 1.1. Number of equilibria.

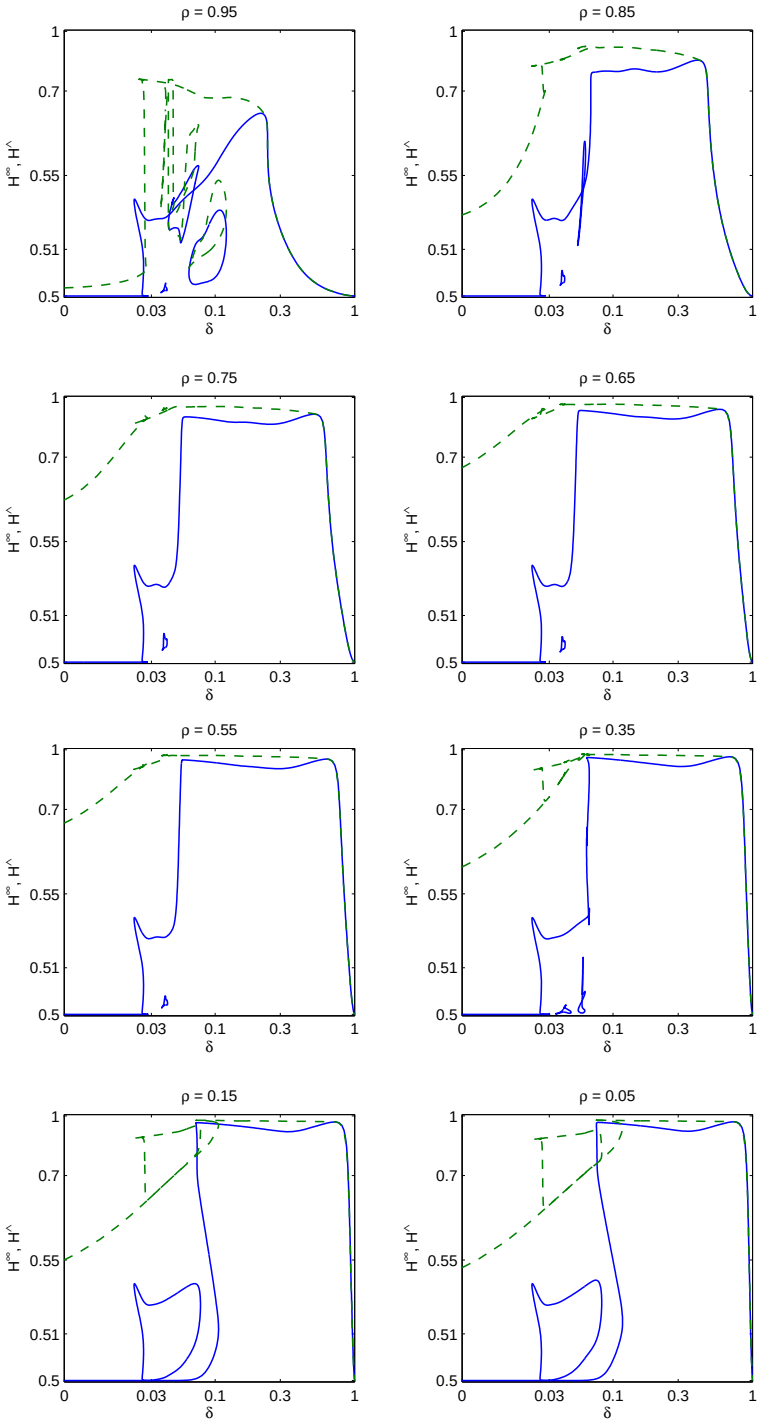

Figure 1.2. Limiting expected Herfindahl index H^{∞} (solid line) and maximum expected Herfindahl index $H^{\wedge}$ (dashed line).

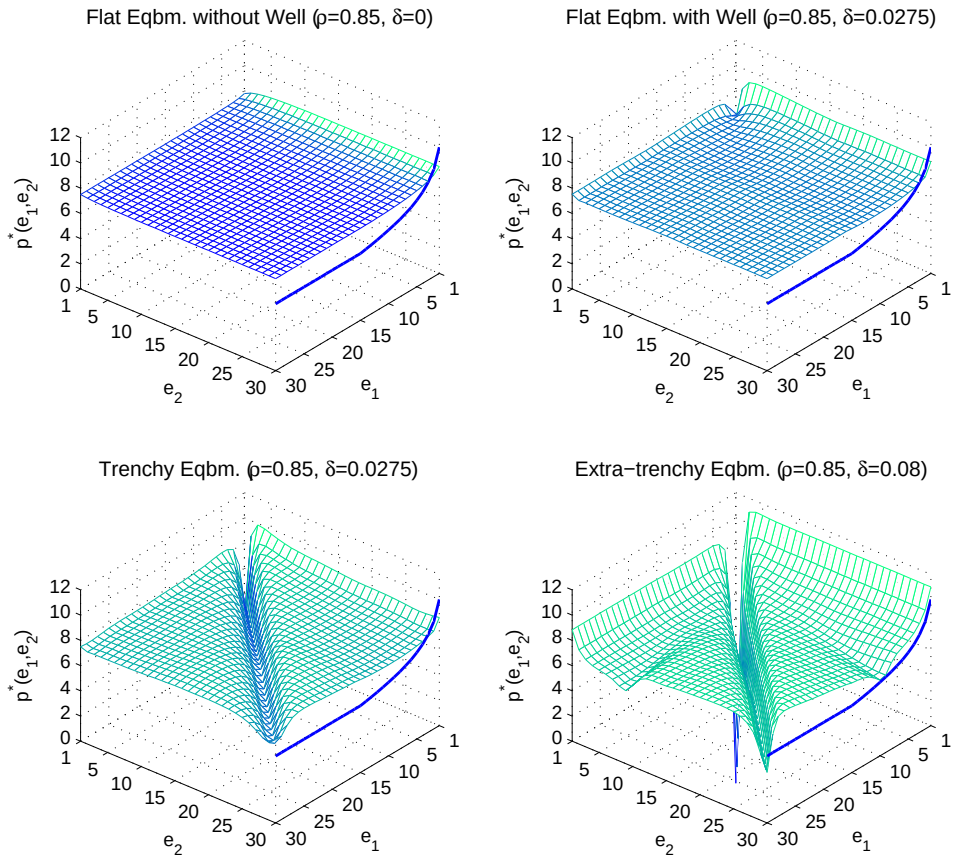

Figure 1.3. Policy function $p^*(e_1, e_2)$. Marginal cost $c(e_1)$ (solid line in $e_2 = 30$-plane).

This section provides an overview of the equilibrium correspondence. Section 1.5 analyzes industry dynamics and Section 1.6 characterizes the pricing behavior that drives it. Section 1.7 describes how organizational forgetting can lead to multiple equilibria.

In the absence of organizational forgetting, Cabral & Riordan (1994) show that the equilibrium is unique. The following proposition generalizes their result:

Proposition 2. *If organizational forgetting is either absent ($\delta = 0$) or certain ($\delta = 1$), then there is a unique equilibrium.*

Note that Proposition 2 pertains to both symmetric and asymmetric equilibria.[15] In what follows, we restrict attention to symmetric equilibria.

The cases of $\delta = 0$ and $\delta = 1$ are special in that they ensure that movements through the state space are unidirectional. Specifically, when $\delta = 0$, a firm can never move "backward" to a lower state, and when $\delta = 1$, it can never move "forward" to a higher state. In contrast, when $\delta \in (0, 1)$, a firm can move in either direction. Our computations show that this has a substantive impact on the set of equilibria:

Result 2. *If organizational forgetting is neither absent ($\delta = 0$) nor certain ($\delta = 1$), then there may be multiple equilibria.*

Figure 1.1 illustrates the extent of multiplicity. It shows the number of equilibria for each combination of forgetting rate δ and progress ratio ρ. Darker shades indicate more equilibria. As can be seen, we have found up to nine equilibria for some values of δ and ρ. Multiplicity is especially pervasive for forgetting rates δ in the empirically relevant range below 0.1; indeed, we always obtained a unique equilibrium for sufficiently large forgetting rates ($\delta \geq 0.15$).

In dynamic stochastic games with finite actions, Herings & Peeters (2004) have shown that generically the number of Markov perfect equilibria is odd. While they consider both symmetric and asymmetric equilibria, in a two-player game with symmetric primitives such as ours, asymmetric equilibria occur in pairs. Hence, their result immediately implies that generically the number of symmetric equilibria is odd in games with finite actions. Figure 1.1 suggests that this carries over to our setting with continuous actions.

[15] Recall that in a symmetric equilibrium the pricing decision taken by firm 2 in state $\mathbf{e}$ is identical to the pricing decision taken by firm 1 in state $\mathbf{e}^{[2]}$, i.e., $p_2(\mathbf{e}) = p_1(\mathbf{e}^{[2]})$, and similarly for the value function. In an asymmetric equilibrium this is not necessarily the case.

We next take a closer look at the set of equilibria. Let

$$\mathbf{F}^{-1} = \{(\mathbf{V}^*, \mathbf{p}^*, \delta) | \mathbf{F}(\mathbf{V}^*, \mathbf{p}^*, \delta) = \mathbf{0}\},$$

be the equilibrium correspondence, where $\mathbf{F}(\cdot)$ is the system of equations (1.6) that defines an equilibrium. Our homotopy algorithm traces out an entire path of equilibria by varying the forgetting rate (as explained in Section 1.3.1). We thus make explicit in our notation that the system of equations (1.6) depends on δ but, at the most basic level of the analysis, hold fixed all parameters other than δ. To extend the analysis we then vary the remaining parameters. For the progress ratio, in particular, we explore a grid of 100 equidistant values of $\rho \in (0, 1]$. We do not index $\mathbf{F}^{-1}$ by the remaining parameters, however, for notational simplicity.

We have the following result:

Result 3. *The equilibrium correspondence* $\mathbf{F}^{-1}$ *contains a unique path that connects the equilibrium at* $\delta = 0$ *with the equilibrium at* $\delta = 1$. *In addition,* $\mathbf{F}^{-1}$ *may contain (one or more) loops that are disjoint from the above path and from each other.*

Figure 1.2 illustrates Result 3. An equilibrium is defined in terms of a value and a policy function and is thus an element of a high-dimensional space. To succinctly describe it, we proceed in two steps.

First, we use the policy function to construct the probability distribution over next period's state $\mathbf{e}'$ given this period's state $\mathbf{e}$, i.e., the transition matrix that characterizes the Markov process of industry dynamics. This allows us to use stochastic process theory to analyze the Markov process of industry dynamics rather than rely on simulation. We compute the transient distribution over states in period t, $\mu^t(\cdot)$, starting from state $(1, 1)$.

This tells us how likely each possible industry structure is in period t, given that both firms began the game at the top of their learning curves. In addition, we compute the limiting (or ergodic) distribution over states, $\mu^{\infty}(\cdot)$.[16] The transient distribution captures short-run dynamics and the limiting distribution captures long-run (or steady-state) dynamics.

Second, we use the transient distribution over states in period t, $\mu^{t}(\cdot)$, to compute the expected Herfindahl index

$$H^{t} = \sum_{\mathbf{e}} \left(D_{1}^{*}(\mathbf{e})^{2} + D_{2}^{*}(\mathbf{e})^{2} \right) \mu^{t}(\mathbf{e}).$$

The time path of the expected Herfindahl index summarizes the implications of learning-by-doing and organizational forgetting for the dynamics of the industry. To the extent that the industry evolves asymmetrically, the expected Herfindahl index exceeds 0.5. The maximum expected Herfindahl index

$$H^{\wedge} = \max_{t \in \{1,\ldots,100\}} H^{t}$$

is therefore a summary measure of short-run industry concentration. In addition, we use the limiting distribution over states, $\mu^{\infty}(\cdot)$, to compute the limiting expected Herfindahl index H^{∞}, a summary measure of long-run industry concentration. If $H^{\infty} > 0.5$, then an asymmetric industry structure persists.

We visualize the equilibrium correspondence $\mathbf{F}^{-1}$ for a variety of progress ratios by plotting the maximum expected Herfindahl index $H^{\wedge}$ (dashed line) and the limiting expected Herfindahl index H^{∞} (solid line). As can be seen, there are multiple equilibria

[16] Let $\mathbf{P}$ be the $M^{2} \times M^{2}$ transition matrix. The transient distribution in period t is given by $\mu^{t} = \mu^{0}\mathbf{P}^{t}$, where μ^{0} is the $1 \times M^{2}$ initial distribution and $\mathbf{P}^{t}$ the t^{th} matrix power of $\mathbf{P}$. If $\delta \in (0,1)$, then the Markov process is irreducible because logit demand implies that the probability moving forward is always nonzero. That is, all its states belong to a single closed communicating class and the $1 \times M^{2}$ limiting distribution μ^{∞} solves the system of linear equations $\mu^{\infty} = \mu^{\infty}\mathbf{P}$. If $\delta = 0$ ($\delta = 1$), then there is also a single closed communicating class, but its sole member is state (M, M) $((1,1))$.

whenever the path that connects the equilibrium at $\delta = 0$ with the equilibrium at $\delta = 1$ folds back on itself. Moreover, the equilibrium correspondence contains one loop for $\rho \in \{0.75, 0.65, 0.55, 0.15, 0.05\}$ and two loops for $\rho \in \{0.95, 0.85, 0.35\}$, thus adding further equilibria.

Figure 1.2 is not necessarily a complete picture of the set of solutions to our model. As discussed in Section 1.3.1, no algorithm is guaranteed to find all equilibria, and our homotopy algorithm is no exception. We do find all equilibria along the path that connects the equilibrium at $\delta = 0$ with the equilibrium at $\delta = 1$, and we have been successful in finding a number of loops. But other loops may exist because, in order to trace out a loop, we must somehow compute at least one equilibrium on the loop, and doing so is problematic.

Types of equilibria. Despite the multiplicity, the equilibria of our game exhibit four typical patterns. One should recognize that these patterns, helpful as they are in understanding the range of behaviors that can occur, lie on a continuum and thus morph into each other as we change the parameter values.

Figure 1.3 exemplifies the policy functions of the typical equilibria.[17] The parameter values are $\rho = 0.85$ and $\delta \in \{0, 0.0275, 0.08\}$ and represent the median progress ratio across a wide array of empirical studies combined with the cases of no, low, and high organizational forgetting. The graph in the upper left panel of Figure 1.3 ($\rho = 0.85$ and $\delta = 0$) is typical for what we call a *flat equilibrium without well*. The policy function is very even over the entire state space. In particular, the price that a firm charges in equilibrium is fairly insensitive to its rival's stock of know-how. In a *flat equilibrium with well*, the policy function continues to be very even over most of the state space. However, price competition is intense in a neighborhood of state $(1, 1)$, which manifests itself as a "well" in the policy function (see the

[17]The corresponding value functions can be found in the Online Appendix.

upper right panel of Figure 1.3 for the case of $\rho = 0.85$ and $\delta = 0.0275$). The graph in the lower left panel of Figure 1.3 exemplifies a *trenchy equilibrium*. The parameter values are the same ($\rho = 0.85$ and $\delta = 0.0275$), thereby providing an instance of multiplicity. The policy function is more uneven and exhibits a "trench" along the diagonal of the state space. This trench extends from state $(1, 1)$ beyond the bottom of the learning curve in state (m, m) all the way to state (M, M). Hence, in a trenchy equilibrium, price competition between firms with similar stocks of know-how is extremely intense, but price competition abates once firms become asymmetric. Finally, in an *extra-trenchy equilibrium*, the policy function not only has a diagonal trench, but it also has a trench parallel to the edge of the state space. In an extra-trenchy equilibrium, price competition between symmetric firms is extremely intense. Furthermore, due to the sideways trench, there are also parts of the state space where the leader competes aggressively with the follower (see the lower right panel of Figure 1.3 for the case of $\rho = 0.85$ and $\delta = 0.08$).

Sunspots. For a progress ratio of $\rho = 1$ the marginal cost of production is constant at $c(1) = \ldots = c(M) = \kappa$, and there are no gains from learning-by-doing. It clearly is an equilibrium for firms to disregard their stocks of know-how and set the same prices as in the Nash equilibrium of a static price-setting game (obtained by setting $\beta = 0$). Since firms' marginal costs are constant, so are the static Nash equilibrium prices. Thus, we have an extreme example of a flat equilibrium with $p^*(\mathbf{e}) = \kappa + 2\sigma = 12$ and $V^*(\mathbf{e}) = \frac{\sigma}{1-\beta} = 21$ for all states $\mathbf{e} \in \{1, \ldots, M\}^2$. As Figure 1.1 shows, however, there are other equilibria for a range of forgetting rates δ below 0.1. Since the state of the industry has no bearing on primitives in case of $\rho = 1$, we refer to these equilibria as sunspots, but we note that they persist for $\rho \approx 1$.

In a sunspot, firms use the state to keep track of their sales. That is, the state serves merely as a coordination device. One of the sunspots is a trenchy equilibrium while the other one is, depending on the value of the forgetting rate, either a flat or a trenchy equilibrium. In the trenchy equilibrium the industry evolves towards an asymmetric structure where the leader charges a lower price than the follower and enjoys a higher probability of making a sale. Consequently, the net present value of cash flows to the leader exceeds that to the follower. The value in state $(1,1)$, however, is lower than in the static Nash equilibrium, i.e., $V^*(1,1) < 21$.[18] This indicates that value is destroyed as firms fight for dominance. More generally, the existence of sunspots suggests that the concept of Markov perfect equilibrium is richer than one may have thought.

In sum, accounting for organizational forgetting in a model of learning-by-doing leads to multiple equilibria and a rich array of pricing behaviors. In the following section, we explore what this entails for industry dynamics.

1.5. Industry Dynamics

Recall that the transient distribution over states in period t, $\mu^t(\cdot)$, starting from state $(1,1)$, captures short-run dynamics and the limiting distribution, $\mu^\infty(\cdot)$, captures long-run dynamics. Figures 1.4 and 1.5 display the transient distribution in period 8 and 32, respectively, and Figure 1.6 displays the limiting distribution for our four typical cases.[19] In the *flat equilibrium without well* ($\rho = 0.85$, $\delta = 0$, see upper left panels), the transient and limiting distributions are unimodal. The most likely industry structure is symmetric. For example,

[18] For a forgetting rate of $\delta = 0.0275$, for example, we have $V^*(28,21) = 25.43$ and $p^*(28,21) = 12.33$ for the leader, $V^*(21,28) = 22.39$ and $p^*(21,28) = 12.51$ for the follower, and $V^*(1,1) = 19.36$. For $\delta = 0.08$ we have $V^*(12,6) = 23.41$ and $p^*(12,6) = 11.96$ for the leader, $V^*(6,12) = 18.77$ and $p^*(6,12) = 12.45$ for the follower, and $V^*(1,1) = 15.94$.

[19] To avoid clutter, we do not graph states that have probability of less than 10^{-4}.

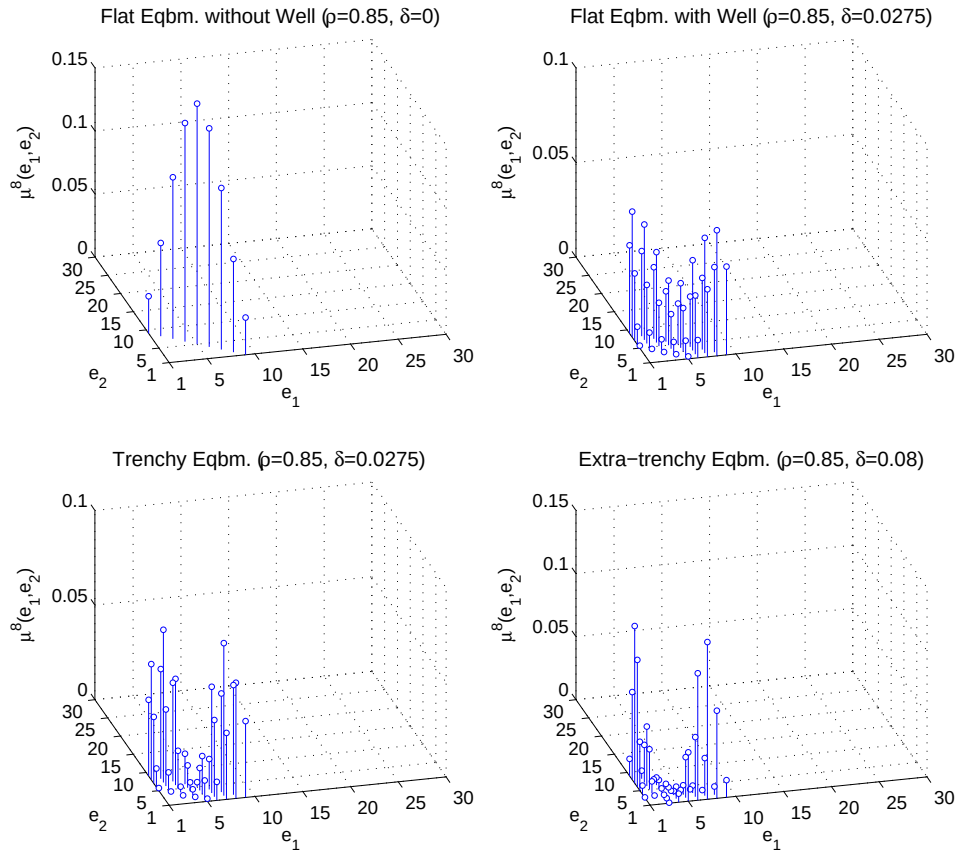

Figure 1.4. Transient distribution over states in period 8 given initial state $(1, 1)$.

the modal state is $(5, 5)$ in period 8, $(9, 9)$ in period 16, $(17, 17)$ in period 32, and $(30, 30)$ in period 64. Turning from the short run to the long run, the industry is most likely to remain in state $(30, 30)$ because, in the absence of organizational forgetting, both firms must eventually reach the bottom of their learning curves. In short, the industry starts symmetric and stays symmetric.

By contrast, in the *flat equilibrium with well* ($\rho = 0.85$, $\delta = 0.0275$, see upper right panels) the transient distributions are first bimodal and then unimodal as is the limiting distribution. The modal states are $(1, 8)$ and $(8, 1)$ in period 8, $(4, 11)$ and $(11, 4)$ in period 16, $(9, 14)$ and $(14, 9)$ in period 32, but the modal state is $(17, 17)$ in period 64 and the modal states of

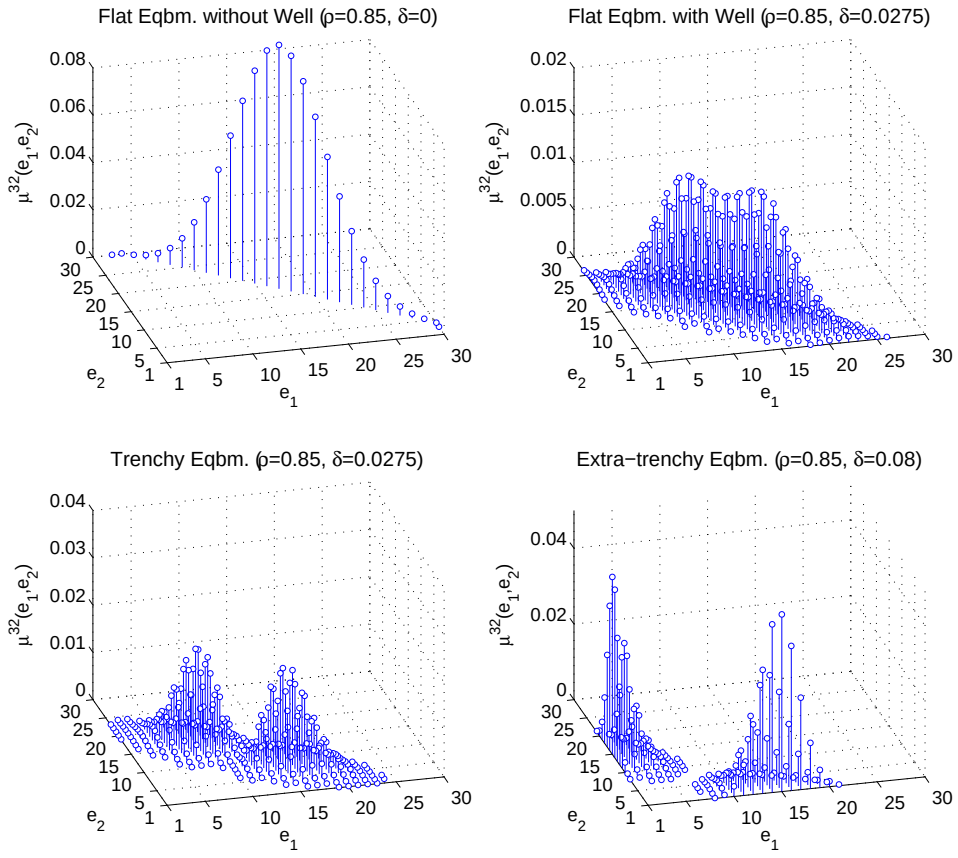

Figure 1.5. Transient distribution over states in period 32 given initial state $(1, 1)$.

the limiting distribution are $(24, 25)$ and $(25, 24)$. Thus, as times passes, firms compete on equal footing. In sum, the industry evolves first towards an asymmetric structure and then towards a symmetric structure. As we discuss in detail in the following section, the reason is that the well serves to build, but not to defend, a competitive advantage.

While the modes of the transient distributions are more separated and pronounced in the *trenchy equilibrium* ($\rho = 0.85$, $\delta = 0.0275$, see lower left panels) than in the flat equilibrium with well, the dynamics of the industry are similar at first. Unlike in the flat equilibrium with well, however, the industry continues to evolve towards an asymmetric structure. The modal states are $(14, 21)$ and $(21, 14)$ in period 64 and the modal states of the limiting distribution

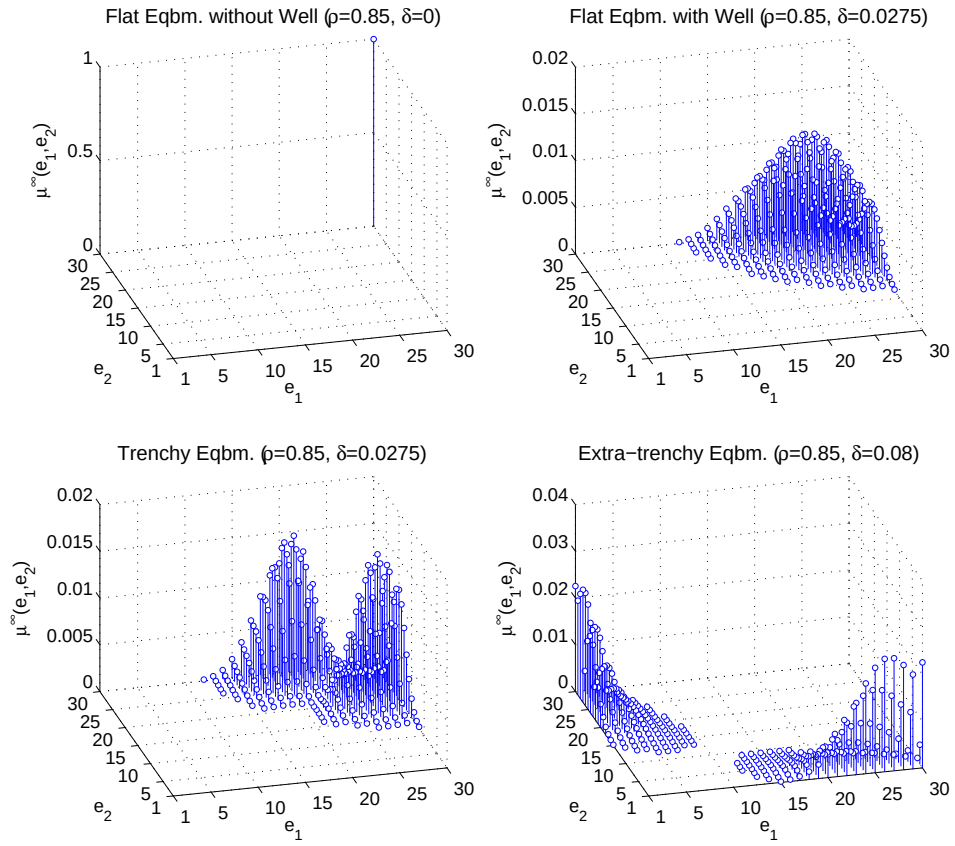

Figure 1.6. Limiting distribution over states.

are $(21, 28)$ and $(21, 28)$. Despite cost parity, however, the leader is more secure against future losses from organizational forgetting than the follower. Asymmetries persist as time passes because the diagonal trench serves to build and to defend a competitive advantage.

In the *extra-trenchy equilibrium* ($\rho = 0.85$, $\delta = 0.08$, see lower right panels), one firm never makes it down from the top of its learning curve due to the sideways trench. The transient and limiting distributions are bimodal, and the most likely industry structure is extremely asymmetric. The modal states are $(1, 7)$ and $(7, 1)$ in period 8, $(1, 10)$ and $(10, 1)$ in period 16, $(1, 15)$ and $(15, 1)$ in period 32, and $(1, 19)$ and $(19, 1)$ in period 64. The

modal states of the limiting distribution are $(1, 26)$ and $(26, 1)$. In short, one firm acquires a competitive advantage early on and maintains it with an iron hand.

Returning to Figure 1.2, our summary measures of industry concentration, the maximum expected Herfindahl index $H^{\wedge}$ (dashed line) and the limiting expected Herfindahl index H^{∞} (solid line), illustrate the fundamental economics of organizational forgetting. If organizational forgetting is sufficiently weak ($\delta \approx 0$), then asymmetries may arise but they cannot persist, i.e., $H^{\wedge} \geq 0.5$ and $H^{\infty} \approx 0.5$. Moreover, if asymmetries arise in the short run, they are modest. If organizational forgetting is sufficiently strong ($\delta \approx 1$), then asymmetries cannot arise in the first place, i.e., $H^{\wedge} \approx H^{\infty} \approx 0.5$. The reason is that organizational forgetting stifles investment in learning-by-doing altogether. By contrast, for intermediate degrees of organizational forgetting, asymmetries arise and persist. The asymmetry can be so pronounced that the leader is virtually a monopolist. This is because organizational forgetting predisposes the leader to defend its position aggressively. This more than offsets the increased vulnerability to organizational forgetting as the stock of know-how grows and therefore makes the leadership position more secure than it would have been in the absence of organizational forgetting.[20]

To summarize, contrary to what one might expect, organizational forgetting does not negate learning-by-doing. Rather, as can be seen in Figure 1.2, over a range of progress ratios ρ above 0.6 and forgetting rates δ below 0.1, learning-by-doing and organizational forgetting reinforce each other. Starting from the absence of both learning-by-doing ($\rho = 1$) and organizational forgetting ($\delta = 0$), a steeper learning curve, i.e., a lower progress ratio, tends to lead to a more asymmetric industry structure just as a higher forgetting rate does.

[20]Since the Markov process is irreducible if $\delta \in (0, 1)$, it is inevitable that the follower eventually overtakes the leader. However, as a practical matter, the expected time to a role reversal is so large that this possibility may be disregarded.

In the following section we analyze in more detail the pricing behavior that drives industry dynamics.

1.6. Pricing Behavior

Re-writing equation (1.5) shows that firm 1's price in state $\mathbf{e}$ satisfies

$$p^*(\mathbf{e}) = c^*(\mathbf{e}) + \frac{\sigma}{1 - D_1^*(\mathbf{e})}, \tag{1.10}$$

where the *virtual* marginal cost

$$c^*(\mathbf{e}) = c(e_1) - \beta\phi^*(\mathbf{e})$$

equals the *actual* marginal cost $c(e_1)$ minus the discounted *prize* $\beta\phi^*(\mathbf{e})$ from winning the current period's sale. The prize, to be determined in equilibrium, is given by

$$\phi^*(\mathbf{e}) = \overline{V}_1^*(\mathbf{e}) - \overline{V}_2^*(\mathbf{e})$$

and has two components. First, by winning a sale, firm 1 may move further down its a learning curve. We call this the *advantage-building* motive. Second, firm 1 may prevent firm 2 from moving further down its learning curve. We call this the *advantage-defending* motive. Winning the sale in expectation is worth $\overline{V}_1^*(\mathbf{e})$ to firm 1 and losing it is worth $\overline{V}_2^*(\mathbf{e})$. Pricing behavior thus hinges on the difference between these values of continued play.

The prize $\phi^*(\mathbf{e})$ is the wedge that causes dynamic pricing behavior to differ from static pricing behavior. To see this, recall that the FOC of a static price-setting game can be

written as

$$p^{\dagger}(\mathbf{e}) = c(e_1) + \frac{\sigma}{1 - D^{\dagger}(\mathbf{e})}, \tag{1.11}$$

where $D_k^{\dagger}(\mathbf{e}) = D_k(p^{\dagger}(\mathbf{e}), p^{\dagger}(\mathbf{e}^{[2]}))$ denotes the probability that, in the static Nash equilibrium, the buyer purchases from firm $k \in \{1, 2\}$ in state $\mathbf{e}$. Clearly, equation (1.10) reduces to equation (1.11) if either the firm is myopic ($\beta = 0$) or its prize is zero ($\phi^*(\mathbf{e}) = 0$). The difference in firms' pricing incentives depends on the difference in their virtual marginal costs. This difference, in turn, depends on the difference in their actual marginal costs and the difference in their prizes.

1.6.1. Price Bounds

Comparing equation (1.10) with equation (1.11) shows that equilibrium prices $p^*(\mathbf{e})$ and $p^*(\mathbf{e}^{[2]})$ coincide with the prices that obtain in a static Nash equilibrium with costs set to equal virtual marginal costs $c^*(\mathbf{e})$ and $c^*(\mathbf{e}^{[2]})$. Since in the static Nash equilibrium prices are increasing in either firm's cost (Vives 1999, p. 35) it follows that, as long as both firms' prizes are nonnegative, equilibrium prices are bounded above by static Nash equilibrium prices with costs set to equal actual marginal costs $c(e_1)$ and $c(e_2)$. More formally, if $\phi^*(\mathbf{e}) \geq 0$ and $\phi^*(\mathbf{e}^{[2]}) \geq 0$, then $p^*(\mathbf{e}) \leq p^{\dagger}(\mathbf{e})$ and $p^*(\mathbf{e}^{[2]}) \leq p^{\dagger}(\mathbf{e}^{[2]})$.

A sufficient condition for $\phi^*(\mathbf{e}) \geq 0$ for each state $\mathbf{e}$ is that the value function $V^*(e_1, e_2)$ is nondecreasing in e_1 and nonincreasing in e_2. Intuitively, it should not hurt firm 1 if it moves down its learning curve and it should not benefit firm 1 if firm 2 moves down its learning curve. This intuition is valid in the absence of organizational forgetting, and equilibrium prices are indeed bounded above by static Nash equilibrium prices:

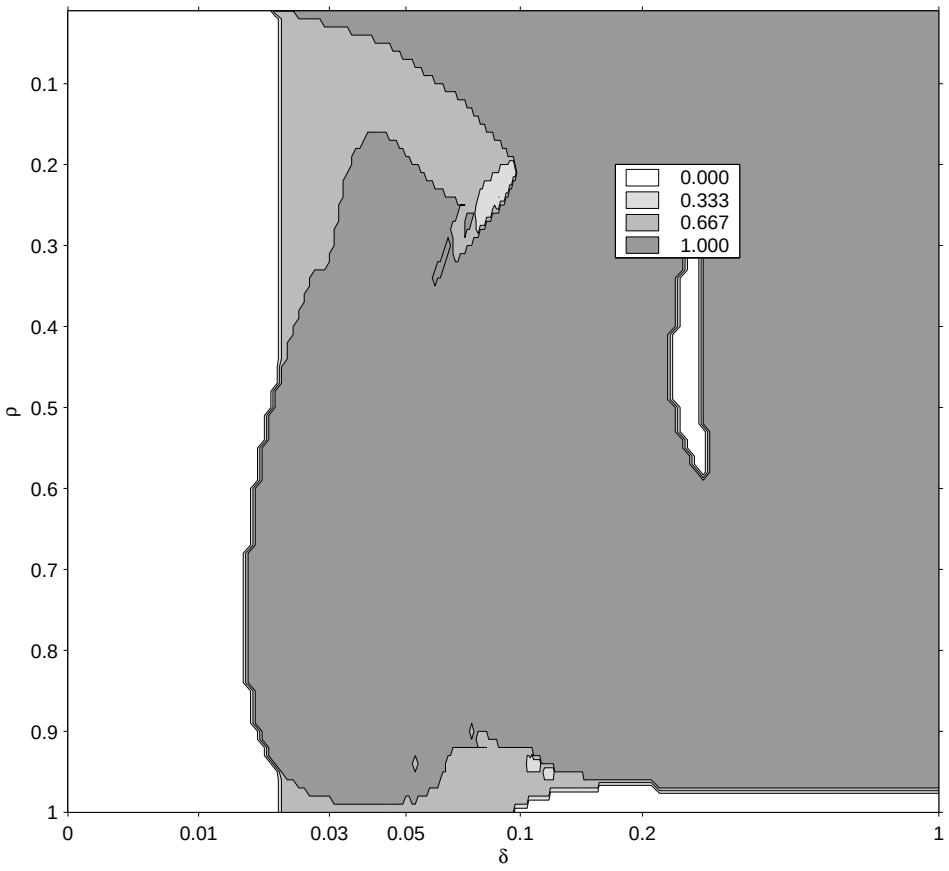

Figure 1.7. Share of equilibria violating upper bound on equilibrium prices (Result 4).

Result 4. *If organizational forgetting is absent ($\delta = 0$), then we have $p^*(\mathbf{e}) \leq p^\dagger(\mathbf{e})$ for all $\mathbf{e} \in \{1, \ldots, M\}^2$.*

Result 4 highlights the fundamental economics of learning-by-doing: as long as improvements in competitive position are valuable, firms use price cuts as investments to achieve them.

The following proposition complements Result 4 by providing a lower bound on equilibrium prices:

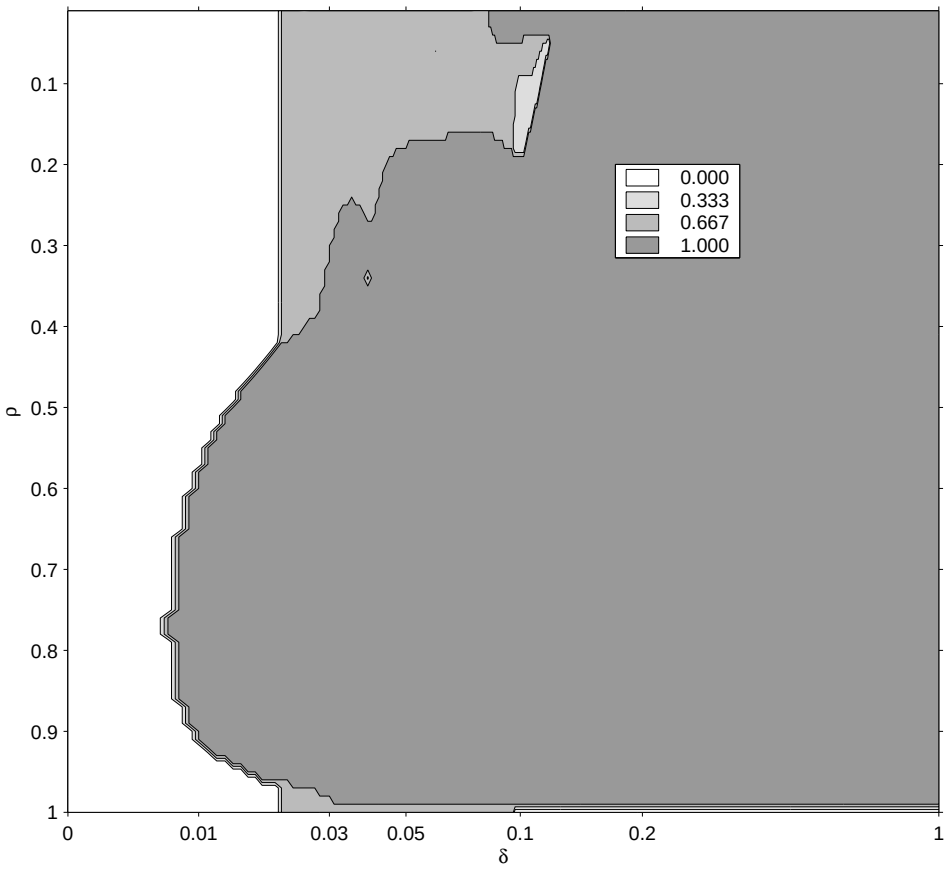

Figure 1.8. Share of equilibria violating lower bound on equilibrium prices (parts (i) and (ii) of Proposition 3).

Proposition 3. *If organizational forgetting is absent ($\delta = 0$), then we have (i) $p^*(\mathbf{e}) = p^\dagger(\mathbf{e}) = p^\dagger(m, m) > c(m)$ for all $\mathbf{e} \in \{m, \ldots, M\}^2$ and (ii) $p^*(\mathbf{e}) > c(m)$ for all $e_1 \in \{m, \ldots, M\}$ and $e_2 \in \{1, \ldots, m-1\}$.*

An immediate implication of part (i) of Proposition 3 is that diagonal trenches (and thus trenchy and extra-trenchy equilibria) cannot arise in the absence of organizational forgetting. In this case, prices are flat once both firms reach the bottom of their learning curves. To see this, note that given $\delta = 0$ the prize reduces to $\phi^*(\mathbf{e}) = V^*(e_1 + 1, e_2) - V^*(e_1, e_2 + 1)$. But once both firms reach the bottom of their learning curves, no further improvements in

competitive position are possible. Hence, as we show in the proof of Proposition 3, we have $V^*(\mathbf{e}) = V^*(\mathbf{e}')$ for all $\mathbf{e}, \mathbf{e}' \in \{m, \ldots, M\}^2$, so that the advantage-building and advantage-defending motives disappear. Consequently, equilibrium prices coincide with prices in the static Nash equilibrium which, in turn, are set above cost.

If the leader but not the follower has reached the bottom of its learning curve, then the leader no longer has an advantage-building motive but he continues to have an advantage-defending motive. This raises the possibility that the leader uses price cuts to delay the follower's progress in moving down its learning curve. However, part (ii) of Proposition 3 shows that there is a limit to how aggressively the leader will defend its advantage: below-cost-pricing is never optimal in the absence of organizational forgetting.

In the presence of organizational forgetting pricing behavior can become much more intricate. To begin with, the intuition that the value function $V^*(e_1, e_2)$ is nondecreasing in e_1 and nonincreasing in e_2 is not always valid:

Result 5. *If organizational forgetting is present ($\delta > 0$), then we may have $p^*(\mathbf{e}) > p^\dagger(\mathbf{e})$ for some $\mathbf{e} \in \{1, \ldots, M\}^2$.*

Figure 1.7 illustrates Result 5 by plotting the share of equilibria that violate the upper bound on equilibrium prices in Result 4.[21] Darker shades indicate higher shares. As can be seen, Result 4 continues to hold if organizational forgetting is very weak ($\delta \approx 0$) and possibly also if learning-by-doing is very weak ($\rho \approx 1$). Apart from these extremes (and a region around

[21] To take into account the limited precision of our computations, we take the upper bound in Result 4 to be violated if $p^*(\mathbf{e}) > p^\dagger(\mathbf{e}) + \epsilon$ for some $\mathbf{e} \in \{1, \ldots, M\}^2$, where ϵ is positive but small. Specifically, we set $\epsilon = 10^{-2}$, so that if prices are measured in dollars, then the upper bound must be violated by more than a cent. Given that the homotopy algorithm solves the system of equations up to a maximum absolute error of about 10^{-12}, Figure 1.7 therefore almost certainly understates the extent of violations.

$\delta = 0.25$ and $\rho = 0.45$), at least some, if not all, equilibria entail at least one state where equilibrium prices exceed static Nash equilibrium prices.

At first glance, Result 5 suggests that organizational forgetting makes firms less aggressive. This seems intuitive: After all, why invest in improvements in competitive position when they are transitory? Surprisingly, however, it turns out that organizational forgetting is a source of aggressive pricing behavior:

Result 6. *If organizational forgetting is present ($\delta > 0$), then we may have (i) $p^*(\mathbf{e}) < p^\dagger(\mathbf{e})$ for some $\mathbf{e} \in \{m, \ldots, M\}^2$ or (ii) $p^*(\mathbf{e}) \leq c(m)$ for some $e_1 \in \{m, \ldots, M\}$ and $e_2 \in \{1, \ldots, m-1\}$.*

Figure 1.8 depicts the share of equilibria that violate the lower bound on equilibrium prices in Proposition 3. As can be seen, unless organizational forgetting or learning-by-doing is very weak ($\delta \approx 0$), at least some, if not all, equilibria fail to obey Proposition 3. That is, the leader may be more aggressive in defending its advantage in the presence of organizational forgetting than in its absence. The most dramatic expression of this aggressive pricing behavior are the wells and trenches in the policy function.

1.6.2. Wells and Trenches

In equilibrium the price set by a firm is a best reply to the price set by its rival in each possible state of the world. One might wonder, though, whether actual firms placed in the environment we have modeled behave in such a manner. Benkard's (2004) analysis of the commercial-aircraft market provides a hint that they might. Lockheed sold the L-1011 aircraft at a price below its average variable cost for much of its 14-year lifespan. Given the nontrivial estimates of the forgetting rate in Benkard (2000), Lockheed's actions are

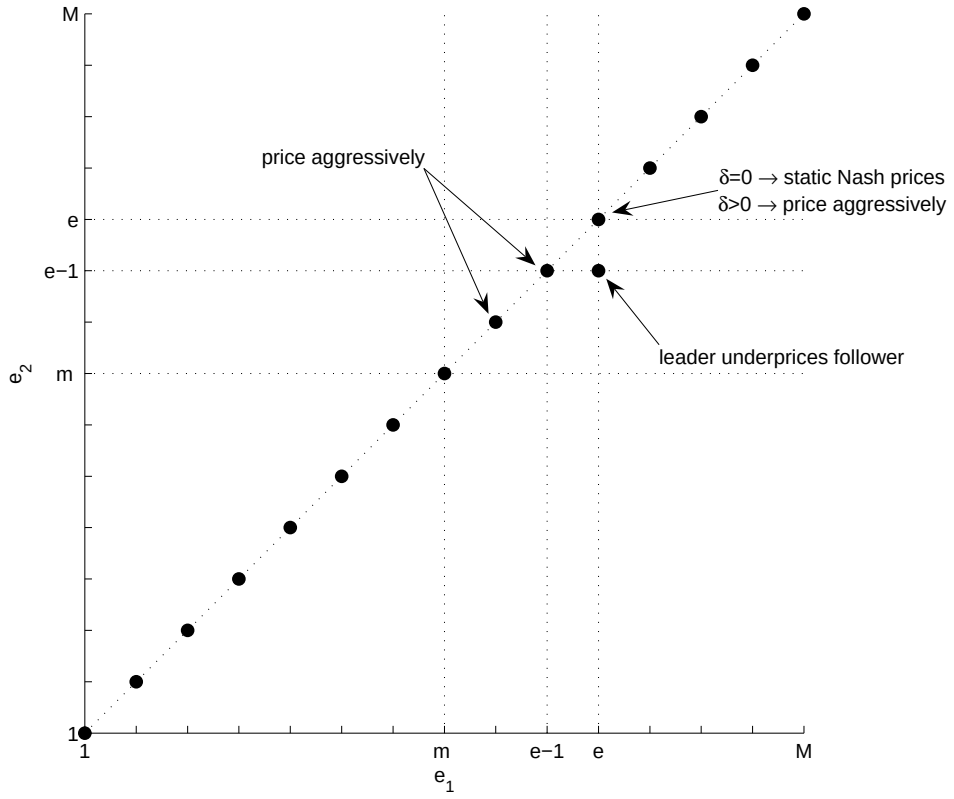

Figure 1.9. Diagonal trench.

consistent with the pricing behavior of a firm in the midst of a well or a trench. This section provides intuition for wells and trenches in order to explore whether the pricing behavior we have characterized is economically plausible and empirically relevant.

Wells. A *well*, as seen in the upper right panel of Figure 1.3, is a preemption battle that is fought by firms at the top of their learning curves. A well serves to build a competitive advantage as both firms use price cuts in the hope of being the first to move down the learning curve. Once one firm has moved ahead of the other, both the leader and the follower raise their price. The follower, in fact, surrenders by setting a much higher price than the leader. Yet, once the follower starts to move down its learning curve, the leader makes no attempt to defend its position. The competitive advantage is thus of a transitory nature.

	modal	leader				follower			
period	state	cost	prize	price	prob.	cost	prize	price	prob.
0	(1,1)	10.00	6.85	5.48	0.500	10.00	6.85	5.48	0.500
8	(8,1)	6.14	3.95	7.68	0.811	10.00	2.20	9.14	0.189
16	(11,4)	5.70	1.16	7.20	0.616	7.22	1.23	7.68	0.384
32	(14,9)	5.39	0.36	7.16	0.527	5.97	0.64	7.27	0.473
64	(17,17)	5.30	-0.01	7.31	0.500	5.30	-0.01	7.31	0.500
∞	(25,24)	5.30	-0.01	7.30	0.500	5.30	-0.00	7.30	0.500

Table 1.1. Cost, prize, price, and probability of making a sale. Flat equilibrium with well ($\rho = 0.85$, $\delta = 0.0275$).

A well arises when the first sale has profound consequences for the evolution of the industry. Table 1.1 provides details on firms' competitive positions at various points in time for our leading example of a flat equilibrium with well ($\rho = 0.85$, $\delta = 0.0275$).[22] Being the first to move down the learning curve, the leader has a lower cost and a higher prize and therefore charges a lower price and enjoys a higher probability of making a sale than the follower in the modal state $(8, 1)$ in period 8. As time passes and the follower moves down its learning curve, the competitive advantage of the leader begins to erode (see the modal state $(11, 4)$ in period 16) and eventually vanishes completely (see the modal state $(17, 17)$ in period 64). This erosion of the competitive advantage of the leader is reflected in the prize: While the leader's prize in state $(8, 1)$ is higher than the follower's (3.95 vs. 2.20), in state $(11, 4)$ the leader's prize is lower than the follower's (1.16 vs. 1.23). Although the competitive advantage is transitory, it is surely worth having: The prize in state $(1, 1)$ is 6.85 and justifies charging a price of 5.48 that is well below the marginal cost of 10. The well is therefore an investment in building a competitive advantage. It is deep to the extent that the competitive advantage can be sustained for at least some time.

[22] In the remainder of this section we assume, without loss of generality, that firm 1 is the leader and firm 2 the follower.

	leader				follower			
state	cost	prize	price	prob.	cost	prize	price	prob.
$(21, 21)$	5.30	2.14	5.26	0.50	5.30	2.14	5.26	0.50
$(21, 20)$	5.30	3.53	5.57	0.72	5.30	0.14	6.54	0.28
$(22, 20)$	5.30	3.22	6.44	0.76	5.30	-1.04	7.60	0.24
$(20, 20)$	5.30	2.16	5.24	0.50	5.30	2.16	5.24	0.50

Table 1.2. Cost, prize, price, and probability of making a sale. Trenchy equilibrium ($\rho = 0.85$, $\delta = 0.0275$).

Diagonal trenches. A *diagonal trench*, as seen in the lower panels of Figure 1.3, is a price war between symmetric or nearly symmetric firms. Like a well, a diagonal trench serves to build a competitive advantage. Unlike a well, however, a diagonal trench also serves to defend it. A diagonal trench is about acquiring and maintaining a permanent competitive advantage. Aggressive pricing is not confined to the top of the learning curve. On the contrary, it takes place all along the diagonal of the state space as each firm uses price cuts to push the state to "its" side of the diagonal and keep it there. A curious feature of a diagonal trench is that firms compete fiercely even though they have already exhausted the gains from learning-by-doing.

We can employ backward-induction-like logic, as illustrated in Figure 1.9, to gain intuition about the link between organizational forgetting and diagonal trenches. Consider state (e, e), where $e \geq m$, on the diagonal of the state space at or beyond the bottom of the learning curve. From part (i) of Proposition 3, without organizational forgetting, the advantage-building and advantage-defending motives disappear and equilibrium prices coincide with prices in the static Nash equilibrium. However, with organizational forgetting, the advantage-building and advantage-defending motives continue to operate. The advantage-building motive operates in state (e, e) because by winning a sale, the firm creates a "buffer

stock" of know-how against future losses from organizational forgetting. The advantage-defending motive operates because by winning a sale, the firm increases the likelihood that its rival slides back up its learning curve. Thus, organizational forgetting predisposes firms to compete fiercely even though they have already exhausted the gains from learning-by-doing. Table 1.2 illustrates this point by providing details on firms' competitive positions in various states for our leading example of a trenchy equilibrium ($\rho = 0.85$, $\delta = 0.0275$). As can be seen, the prize in state $(21, 21)$ is 2.14 and justifies charging a price of 5.26 that is a little below the marginal cost of 5.30 and a lot below the static Nash equilibrium price of 7.30.

Next consider state $(e, e-1)$ where firm 1 has a slight lead over firm 2 (see Figure 1.9). With organizational forgetting, the leader's prize from winning a sale is likely larger than the follower's. In our leading example, the leader's prize in state $(21, 20)$ is almost 25 times larger than the follower's. This indicates that winning a sale is considerably more valuable to the leader than the follower.

To see why, suppose the follower wins. In this case the follower may leapfrog the leader if the industry moves to state $(e-1, e)$, but the odds are against this possibility (by about $3:1$ in our leading example). The most likely possibility is that the industry moves back to state (e, e) where there is brutal price competition. In our leading example, such a move to state $(21, 21)$ happens with a probability of 0.32 and, if so, then the follower's expected cash flow in the next period decreases to $-0.02 = 0.50 \times (5.26 - 5.30)$ compared to $0.34 = 0.28 \times (6.54 - 5.30)$ if the industry were to remain in state $(21, 20)$.

By contrast, if the leader wins, then the industry is guaranteed to not move back to state (e, e), thereby avoiding the brutal price competition in this state. The leader also, of course, avoids the possibility of losing its competitive advantage by precluding movements to states such as $(e-1, e)$ and $(e-1, e-1)$. And the leader may even enhance its competitive advantage

by inducing movements to states such as $(e, e-2)$, $(e+1, e-1)$, and $(e+1, e-2)$. The most likely possibility is that the industry moves to state $(e+1, e-1)$. In our leading example, such a move to state $(22, 20)$ happens with a probability of 0.32 and, if so, the leader's expected cash flow in the next period increases to $0.87 = 0.76 \times (6.44 - 5.30)$ compared to $0.20 = 0.72 \times (5.57 - 5.30)$ if the industry were to remain in state $(21, 20)$.

To the extent that the leader's prize from winning a sale is larger than the follower's, the leader has a large advantage over the follower in terms of virtual marginal cost. Hence, the leader substantially underprices the follower. As a result the leadership position in state $(e, e-1)$ is more secure in the presence of organizational forgetting than in its absence. In our leading example, the probability that the leader defends its position is 0.79 (compared to 0.62 in the absence of organizational forgetting).

To complete the induction, consider finally state $(e-1, e-1)$ on the diagonal of the state space just below state (e, e) (see again Figure 1.9). Because the leadership position in state $(e, e-1)$ is so secure, both firms fight hard to attain it, thereby intensifying price competition in state $(e-1, e-1)$. In our leading example, the prize in state $(20, 20)$ is 2.16 (compared to 2.14 in state $(21, 21)$) and justifies charging a price of 5.24 (compared to 5.26). These effects cascade through the state space and give rise to diagonal trenches.

By fighting the price war in the diagonal trench the leader is able to acquire and maintain a permanent competitive advantage. In our leading example, the modal states of the limiting distribution are $(21, 28)$ and $(28, 21)$ where both firms are safely beyond the bottom of their learning curve and cost parity obtains. Yet, the leader underprices the follower (7.63 vs. 7.81) and thus enjoys higher probability of making a sale (0.55 vs. 0.45). The follower recognizes that to attain a competitive advantage it will have to "cross over" the diagonal trench. This discouraging prospect weakens the follower's motivation to improve its competitive position

and, in turn, strengthens the leader's motivation to preserve or strengthen its competitive position. What is critical about this price war is that it is a part of a Markov perfect equilibrium and, as such, is a credible threat the follower cannot ignore.

Interestingly enough, our leading example violates both the upper and the lower bound on equilibrium prices in Result 4 and Proposition 3, respectively, that obtain in the absence of organizational forgetting: Whereas both firms price higher in the most likely long-run industry structure than in the static Nash equilibrium (e.g., in states $(21, 28)$ and $(28, 21)$), "in the trench" they price lower (e.g., in states $(20, 20)$ and $(21, 21)$).[23] This illustrates Results 5 and 6 and, in turn, shows that in the presence of organizational forgetting the equilibrium may entail lackluster competition in some states as well as price wars in other states.

In sum, diagonal trenches are self-reinforcing mechanisms that lead to market dominance: if the leadership position is aggressively defended, symmetric firms fight a price war to attain it. This provide all the more reason to aggressively defend a competitive advantage because if it is lost and the industry moves back to the diagonal of the state space, then another price war ensues.

Sideways trenches. A sideways trench, as seen in the lower right panel of Figure 1.3, is a price war between very asymmetric firms. It is triggered when the follower starts to move down its learning curve. Similar to a diagonal trench, a sideways trench is about acquiring and maintaining a permanent competitive advantage. However, while a diagonal trench is about fighting an imminent threat, a sideways trench is about fighting a distant threat. Put informally, a sideways trench is the equilibrium manifestation of former Intel CEO Andy Grove's dictum "Only the paranoid survive." One can think of a sideways trench

[23]Recall that the static Nash equilibrium price is 7.30 is these states.

	leader				follower			
state	cost	prize	price	prob.	cost	prize	price	prob.
$(26, 1)$	5.30	6.43	8.84	0.90	10.00	0.12	11.00	0.10
$(26, 2)$	5.30	6.21	7.48	0.88	8.50	0.21	9.44	0.12
$(26, 3)$	5.30	5.16	6.94	0.85	7.73	0.27	8.65	0.15
$(26, 4)$	5.30	4.28	6.66	0.82	7.23	0.31	8.15	0.18
$(26, 5)$	5.30	3.70	6.49	0.79	6.86	0.34	7.81	0.21
$(26, 6)$	5.30	3.36	6.34	0.76	6.57	0.39	7.51	0.24
$(26, 7)$	5.30	3.04	6.14	0.73	6.34	0.58	7.15	0.27
$(26, 8)$	5.30	2.33	5.99	0.66	6.14	1.08	6.64	0.34
$(26, 9)$	5.30	1.17	6.24	0.51	5.97	1.71	6.29	0.49
$(26, 10)$	5.30	0.16	6.83	0.40	5.83	1.96	6.44	0.60

Table 1.3. Cost, prize, price, and probability of making a sale. Extra-trenchy equilibrium ($\rho = 0.85$, $\delta = 0.08$).

as an endogenously arising mobility barrier in the sense of Caves & Porter (1977). If the follower crashes through this mobility barrier—an unlikely event—he moves from being a docile competitor to being a viable threat. To prevent this from happening, the leader stalls the follower.

Table 1.3 provides details on firms' competitive positions in various states for our leading example of an extra-trenchy equilibrium ($\rho = 0.85$, $\delta = 0.08$). Recall that the modal state of the limiting distribution is $(26, 1)$. The sideways trench is evident in the decrease in the price charged by the leader between states $(26, 1)$ and $(26, 8)$ and the increase between states $(26, 8)$ and $(26, 10)$. To see how the leader can stall the follower, note that the follower has little chance of making it down its learning curve as long as the probability of winning a sale is less than the probability of losing a unit of know-how through organizational forgetting. Indeed, while $D_2^*(26, 1) = 0.10 > 0.08 = \Delta(1)$, we have $D_2^*(26, 2) = 0.12 < 0.15 = \Delta(2)$ and $D_2^*(26, 3) = 0.15 < 0.22 = \Delta(3)$. The ability of the leader to stall the follower at the top of its learning curve is reflected in the large prize in state $(26, 1)$. In state $(26, 2)$ the prize is almost as large because by winning a sale the leader may move the industry back

to state $(26, 1)$ in the next period. The prize falls in case the follower moves further down its learning curve because it takes progressively longer for the leader to force the follower back up its learning curve and because the lower cost of the follower makes it harder for the leader to do so. In the unlikely event that the follower crashes through the sideways trench in state $(26, 8)$, the prize of the leader falls sharply. At the same time the prize of the follower rises sharply because the follower turns from a docile competitor into a viable threat with an equal or larger share of the market.

Again, sideways trenches are self-reinforcing mechanisms that lead to market dominance: If organizational forgetting enables the leader to stall the follower in state (e_1, e_2), then this creates strong incentives for the leader to cut prices in states like $(e_1, e_2 + 1)$ and $(e_1, e_2 + 2)$ in order to deny the follower a sale and force it back up its learning curve. But because being in states like $(e_1, e_2 + 1)$ and $(e_1, e_2 + 2)$ is not especially attractive, the follower has little reason to cut price in state (e_1, e_2). This makes it even easier for the leader to stall the follower.

In sum, the four types of equilibria that we have identified in Section 1.4 give rise to distinct yet plausible pricing behaviors and, in turn, industry dynamics as discussed in Section 1.5. Rather than impeding it, organizational forgetting facilitates aggressive behavior. In the absence of organizational forgetting, the equilibria in our computations have always been flat either without or with well depending on the progress ratio. Moreover, the lower bound on equilibrium prices in part (i) of Proposition 3 rules out diagonal trenches (and thus trenchy and extra-trenchy equilibria). Indeed, in the absence of organizational forgetting, prices are flat once both firms reach the bottom of their learning curves. Similarly, if the leader but not the follower has reached the bottom of its learning curve, then part (ii) of Proposition 3 shows

ρ	0.95	0.85	0.75	0.65	0.55	0.35	0.15	0.05
$\bar{\delta}(\rho)$	0.55	0.60	0.62	0.71	0.78	0.81	0.88	0.90

Table 1.4. Critical value $\bar{\delta}(\rho)$ for investment stifling.

that the leader will not be overly aggressive in defending its advantage. This limits the scope for sideways trenches. In the presence of organizational forgetting, in contrast, firms may compete fiercely even though they have already exhausted the gains from learning-by-doing. Generally speaking, organizational forgetting is associated with "trenchier" equilibria, more aggressive behavior, and more concentrated industries both in the short run and in the long run.

There is clearly a limit to the enhancement of price-cutting incentives through organizational forgetting. If the forgetting rate δ is very large, then organizational forgetting stifles investment in learning-by-doing altogether. In particular, if δ exceeds the critical value $\bar{\delta}(\rho)$ listed in Table 1.4, then firms cannot expect to make it down their learning curves.[24] Hence, equilibrium prices at the top of the learning curve are close to prices in the static Nash equilibrium. Even then, however, price competition at the bottom of the learning curve is extremely intense as both firms seek to reduce the chance of being the first to slide back up the learning curve.

1.6.3. Dominance Properties

Traditional intuition suggests that learning-by-doing leads by itself to market dominance by giving a more experienced firm the ability to profitably underprice its less experienced rival. This enables the leader to widen its competitive advantage over time, thereby further enhancing its ability to profitably underprice the follower. Cabral & Riordan (1994) formalize

[24] We take $\bar{\delta}(\rho)$ to be the smallest forgetting rate such that state $(1,1)$ is the mode of the limiting distribution.

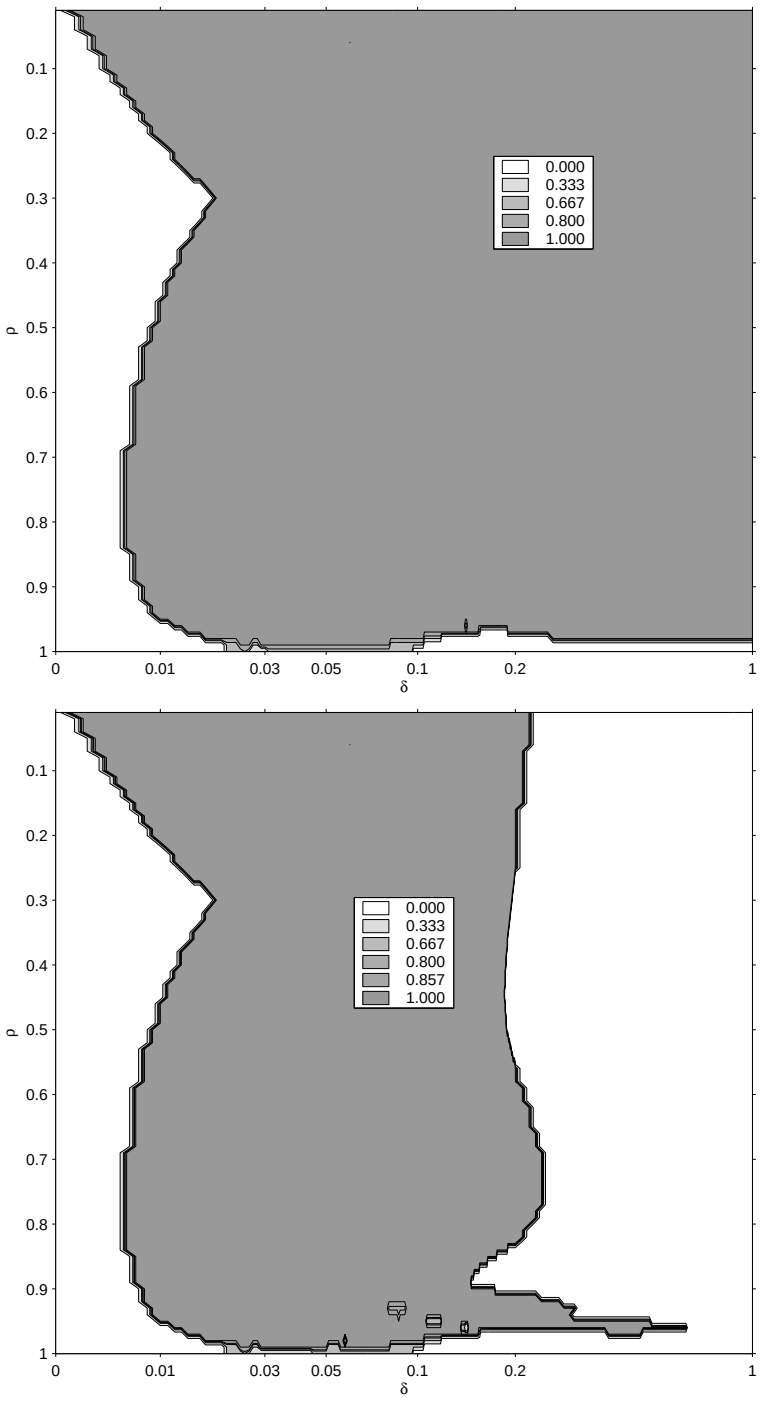

Figure 1.10. Share of equilibria violating IID (part (i) of Result 7, upper panel) and share of equilibria violating ID (part (ii) of Result 7, lower panel).

this idea with "two concepts of self-reinforcing market dominance" (p. 1115), increasing dominance (ID) and increasing increasing dominance (IID). An equilibrium exhibits ID if $p^*(\mathbf{e}) < p^*(\mathbf{e}^{[2]})$ whenever $e_1 > e_2$ and IID if $p^*(\mathbf{e}) - p^*(\mathbf{e}^{[2]})$ is decreasing in e_1. If ID holds, the leader charges a lower price than the follower and therefore enjoys a higher probability of making a sale. If IID holds, the gap between the leader's price and the follower's price widens with the length of the lead. Athey & Schmutzler (2001) extend this idea to dynamic games with deterministic state-to-state transitions. Their notion of weak increasing dominance describes the relationship between players' states and their actions and coincides with Cabral & Riordan's (1994) notion of ID.[25]

In the absence of organizational forgetting, Cabral & Riordan (1994) show that ID and IID hold provided that the discount factor β is sufficiently close to 1 (or, alternatively, sufficiently close to 0). Their main result carries over to our parameterization with $\beta = \frac{1}{1.05}$:

Result 7. *(i) If organizational forgetting is absent ($\delta = 0$), then IID holds. (ii) Thus, ID holds.*

Even though the equilibrium satisfies ID and IID, it is not clear that the industry is inevitably progressing towards monopolization. If the price gap is small, then the effect of ID and IID may be trivial.[26] In such a scenario, the leader charges a *slightly* lower price than the follower and this gap widens *a bit* over time. However, with even a modest degree of horizontal product differentiation, the firms still split sales more or less equally and thus move down the learning curve in tandem. Consequently, ID and IID may have no discernible

[25] Similar notions of increasing dominance have also been used by Vickers (1986) and Budd, Harris & Vickers (1993) in dynamic investment games.

[26] Indeed, Cabral & Riordan (1994) show that $p^*(\mathbf{e}) \to p^\dagger(m, m)$ for all $\mathbf{e} \in \{1, \ldots, M\}^2$ as $\beta \to 1$, i.e., both firms price *as if* at the bottom of their learning curves. This suggests that the price gap may be small for "reasonable" discount factors.

impact on industry structure and dynamics, either in the short run or in the long run. This is exactly what happens in the absence of organizational forgetting. For example, the flat equilibrium without well ($\rho = 0.85$, $\delta = 0$) satisfies IID and thus ID. Yet, the industry is likely to be a symmetric duopoly at all times. More generally, as Figure 1.2 shows, in the absence of organizational forgetting asymmetries are modest if they arise at all. In fact, although ID and IID hold, the maximum expected Herfindahl index is 0.67 (attained at $\rho = 0.65$). Hence, ID and IID are not sufficient for economically meaningful market dominance.

ID and IID are also not necessary for market dominance. To give an example, the extra-trenchy equilibrium ($\rho = 0.85$, $\delta = 0.08$) violates ID and thus IID because the leader coasts by charging a higher price if it is far ahead of the follower. Yet, the industry is likely to be a near-monopoly at all times. More generally, while the empirical studies of Argote et al. (1990), Darr et al. (1995), Benkard (2000), Shafer et al. (2001), and Thompson (2003) warrant the inclusion of organizational forgetting in a model of learning-by-doing, ID and IID may fail in its presence:

Result 8. *If organizational forgetting is present ($\delta > 0$), then (i) IID may fail and (ii) ID may fail.*

Figure 1.10 illustrates Result 8 by plotting the share of equilibria that violate IID (upper panel) and ID (lower panel). As can be seen, all equilibria fail to obey IID unless organizational forgetting or learning-by-doing is very weak. Even violations of ID are extremely common, especially for forgetting rates δ in the empirically relevant range below 0.1.

Of course, we do not mean to argue that the concepts of ID and IID have no place in the analysis of industry dynamics. It appears, however, that caution is warranted: Since ID and IID are neither necessary nor sufficient for economically meaningful market dominance,

	flat eqbm. without well	flat eqbm. with well	trenchy eqbm.	extra-trenchy eqbm.
leading example	$\rho = 0.85$, $\delta = 0$	$\rho = 0.85$, $\delta = 0.0275$	$\rho = 0.85$, $\delta = 0.0725$	$\rho = 0.85$, $\delta = 0.08$
preemption battle (well)	no	yes	no	no
price war triggered by imminent threat (diagonal trench)	no	no	yes	yes
price war triggered by distant threat (sideways trench)	no	no	no	yes
short-run market dominance	no	yes	yes	yes
long-run market dominance	no	no	yes, modest	yes, extreme
dominance properties	yes	no, mostly	no, mostly	no, mostly

Table 1.5. Pricing behavior and industry dynamics.

making inferences about the evolution of the industry on the basis of ID and IID alone may be misleading. We therefore suggest that these concepts be augmented by a search for telltale signs of market dominance such as wells and trenches and, perhaps even more important, by a direct examination of the industry dynamics implied by firms' pricing behavior.

1.6.4. Summary

Table 1.5 summarizes the broad patterns of pricing behavior and industry dynamics. Acknowledging that the know-how gained through learning-by-doing can be lost through organizational forgetting is evidently important. Generally speaking, organizational forgetting is associated with "trenchier" equilibria, more aggressive behavior, and more concentrated industries both in the short run and in the long run. Moreover, the dominance properties of firms' pricing behavior can break down in the presence of organizational forgetting.

The key difference between a model with learning-by-doing and organizational forgetting and a model with learning-by-doing alone is that in the former a firm can move both forward to a higher state and backward to a lower state whenever $\delta \in (0, 1)$. The possibility of bidirectional movements through the state space opens up new strategic possibilities for firms that work to enhance the advantage-building and advantage-defending motives. By winning a sale, a firm makes itself less vulnerable to future losses from organizational forgetting, thus enhancing the advantage-building motive. At the same time, it makes its rival more vulnerable to future losses from organizational forgetting, thus enhancing the advantage-defending motive. Because these additional benefits (like the benefits from learning-by-doing) are achieved by winning a sale, organizational forgetting can create strong incentives to cut prices. Thus, rather than impeding it, organizational forgetting can facilitate aggressive pricing behavior that manifests itself in wells and trenches.

1.7. Organizational Forgetting and Multiple Equilibria

While the equilibrium is unique if organizational forgetting is either absent ($\delta = 0$) or certain ($\delta = 1$), multiple equilibria are common for intermediate forgetting rates. Surprisingly, these equilibria range from "peaceful coexistence" to "trench warfare." Consequently, in addition to the degree of organizational forgetting, the equilibrium by itself is an important determinant of pricing behavior and industry dynamics.

Why do multiple equilibria arise in our model? To explore this question, think about the strategic situation faced by firms in setting prices in state $\mathbf{e}$. The value of continued play to firm n is given by the conditional expectation of its value function, $\overline{V}_{n1}(\mathbf{e})$ and $\overline{V}_{n2}(\mathbf{e})$. Holding the value of continued play fixed, the strategic situation in state $\mathbf{e}$ is thus akin to a static game. If the reaction functions in this game intersect more than once, then multiple

equilibria arise. On the other hand, we say that the model satisfies stagewise uniqueness if the reactions functions of the two firms intersect once irrespective of the value of continued play. This is indeed the case:

Proposition 4. *Statewise uniqueness holds.*

Note that the proof of Proposition 4 relies on the functional form of demand. This is reminiscent of the restrictions on demand (e.g., log-concavity) that Caplin & Nalebuff (1991) set forth to guarantee uniqueness of Nash equilibrium in their analysis of static price-setting games.

Given that the model satisfies statewise uniqueness, multiple equilibria must arise from firms' expectations regarding the value of continued play. To see this, consider again state $\mathbf{e}$. The intersection of the reaction functions constitutes a Nash equilibrium in prices in a subgame in which firm n believes that its value of continued play is given by $\overline{V}_{n1}(\mathbf{e})$ and $\overline{V}_{n2}(\mathbf{e})$. If firms have rational expectations, i.e., if the conjectured value of continued play is actually attained, then these prices constitute an equilibrium of our dynamic stochastic game. In our model, taking the value of continued play as given, the reaction functions intersect once because we have statewise uniqueness, but there may be more than one value of continued play that is consistent with rational expectations. In this sense multiplicity is rooted in the dynamics of the model.

The key driver of multiplicity is organizational forgetting. Dynamic competition with learning-by-doing and organizational forgetting is like racing down an upward-moving escalator. Unless a firm makes sales at a rate that exceeds the rate at which it loses know-how through organizational forgetting, its marginal cost is bound to increase. The inflow of know-how into the industry is one unit per period whereas in expectation the outflow in state $\mathbf{e}$

is $\Delta(e_1) + \Delta(e_2)$. Consider state (e, e), where $e \geq m$, on the diagonal of the state space at or beyond the bottom of the learning curve. If $1 \ll 2\Delta(e)$, then it is impossible that both firms reach the bottom of their learning curves and remain there. Knowing this, firms have no choice but to price aggressively. The result is trench warfare as each firm uses price cuts to push the state to its side of the diagonal and keep it there. If, however, $1 \gg 2\Delta(e)$, then it is virtually inevitable that both firms reach the bottom of their learning curves, and firms may as well price softly. In both cases, the primitives of the model tie down the equilibrium.

This is no longer the case if $1 \approx 2\Delta(e)$, setting the stage for multiple equilibria as diverse as peaceful coexistence and trench warfare. If firms believe that they cannot peacefully coexist at the bottom of their learning curves and that one firm will come to dominate the market, then both firms will cut their prices in the hope of acquiring a competitive advantage early on and maintaining it throughout. This naturally leads to trench warfare and market dominance. If, however, firms believe that they can peacefully coexist at the bottom of their learning curves, then neither firm cuts its price. Soft pricing, in turn, ensures that the anticipated symmetric industry structure actually emerges. A back-of-the-envelope calculation is reassuring here. Recall that $m = 15$ and $M = 30$ in our parameterization and observe that $1 = 2\Delta(15)$ implies $\delta \approx 0.05$, $1 = 2\Delta(20)$ implies $\delta \approx 0.03$, and $1 = 2\Delta(30)$ implies $\delta \approx 0.02$. This range of forgetting rates for which the inflow of know-how approximately equals the outflow is indeed where multiplicity prevails (see again Figure 1.1).

In general, a sufficient condition for uniqueness of equilibrium in a dynamic stochastic game with a finite state space is that the model satisfies statewise uniqueness and the movements through the state space are unidirectional. Statewise uniqueness precludes players' actions from giving rise to multiple equilibria and unidirectional movements preclude their expectations from doing so. To illustrate, recall that in the game at hand, a firm can never

move backward to a lower state if $\delta = 0$. Hence, once the industry reaches state (M, M), it remains there forever, so that the value of future play in state (M, M) coincides with the value of being in this state *ad infinitum*. In conjunction with statewise uniqueness, this uniquely determines the value of being in state (M, M). Next consider states $(M-1, M)$ and $(M, M-1)$. The value of future play in states $(M-1, M)$ and $(M, M-1)$ depends on the value of being in state (M, M). Statewise uniqueness ensures that firms' prices in states $(M-1, M)$ and $(M, M-1)$ as well as the value of being in these states are uniquely determined. Continuing to work backwards in this fashion establishes that the equilibrium is unique. If $\delta = 1$, then a firm can never move forward to a higher state and a similar argument anchored on state $(1, 1)$ establishes uniqueness of equilibrium.[27]

1.8. Robustness Checks

In this section, we discuss how the model specification and parameterization affect our results. In the interest of brevity, we confine ourselves to summarizing our robustness checks; the underlying figures and tables can be found in the Online Appendix.

1.8.1. Product Differentiation

Our baseline parameterization gives rise to a moderate degree of horizontal product differentiation. In the static Nash equilibrium, the own-price elasticity of demand ranges between -8.86 in state $(1, 15)$ and -2.13 in state $(15, 1)$ for a progress ratio of $\rho = 0.85$. The cross-price elasticity of firm 1's demand with respect to firm 2's price is 2.41 in state $(15, 1)$ and 7.84 in state $(1, 15)$. As σ is decreased from 1 to 0.2, the respective elasticities become -102.00, -0.00, 0.00, and 55.0. As σ increases from 1 over 2 to 10, the respective elasticities

[27] Proposition 2 provides an example of this type of proof.

become -4.38, -1.86, 2.10, and 3.88 in case of $\sigma = 2$ and -1.54, -1.24, 1.32, and 1.45 in case of $\sigma = 10$.

In case of weaker product differentiation with $\sigma = 0.2$, trenchier equilibria lead to more asymmetric industry structures. Conversely, in case of stronger product differentiation with $\sigma = 2$, we obtain more symmetric industry structures. With $\sigma = 10$, firms hardly compete any more and sales are split more or less equally between them. Multiple equilibria no longer arise because firms are essentially monopolists that do not interact strategically with each other. Finally, a higher degree of horizontal product differentiation makes the market that a firm has effective access to smaller, so that it becomes easier for organizational forgetting to stifle investment in learning-by-doing altogether.

1.8.2. Outside Good

We allow the buyer to choose an alternative made from a substitute technology (outside good 0) instead of purchasing from one of the two firms (inside goods 1 and 2). The probability that firm n makes a sale becomes

$$D_n(\mathbf{p}) = \frac{\exp(\frac{v-p_n}{\sigma})}{\exp(\frac{v_0-c_0}{\sigma}) + \sum_{k=1}^{2} \exp(\frac{v-p_k}{\sigma})},$$

where we assume that the outside good is supplied under conditions of perfect competition with price equal to marginal cost, $p_0 = c_0$. As $v_0 - c_0 \to -\infty$, $D_0(\mathbf{p}) = 1 - \sum_{n=1}^{2} D_n(\mathbf{p}) \to 0$ and we revert to the Cabral & Riordan (1994) setting in which the buyer always purchases from one of the two firms in the industry.

If we set $v = 10$ and $v_0 - c_0 = 0$, then $v - c(1) = v_0 - c_0$ and a firm at the top of its learning curve is on par with the outside good. The share of the outside good is quite small

in general. In the static Nash equilibrium, as the marginal cost of production declines, the share of the outside good declines from 0.63 in state $(1,1)$ over 0.33 in state $(2,2)$ and 0.15 in state $(4,4)$ to 0.03 in state $(15,15)$ for a progress ratio of $\rho = 0.85$. To further increase the attractiveness of the outside good we set $v_0 - c_0 \in \{3,5,10\}$. If $v_0 - c_0 = 5$, then the share of the outside good is quite large in general and declines from 1.00 in state $(1,1)$ over 0.99 in state $(2,2)$ and 0.93 in state $(4,4)$ to 0.69 in state $(15,15)$.

Multiple equilibria continue to arise in the presence of an outside good, although less frequently as the outside good becomes more attractive. In particular, we no longer have sunspots for a progress ratio of $\rho = 1$ because the outside good sufficiently constrains firms' pricing behavior even with $v_0 - c_0 = 0$. With $v_0 - c_0 = 10$ the equilibrium is unique because almost all consumers choose the outside good, so that the inflow of know-how into the industry is much smaller than the outflow. Finally, a more attractive outside good *de facto* makes the market smaller, so that it becomes easier for organizational forgetting to stifle investment in learning-by-doing altogether.

1.8.3. Choke Price

In the absence of organizational forgetting the equilibria in our computations have always been flat either without or with well depending on the progress ratio. As in Cabral & Riordan (1994) our logit specification for demand ensures that a firm always has a positive probability of making a sale and therefore must eventually reach the bottom of its learning curve. A natural concern is whether this stacks the deck against long-run market dominance to occur in the absence of organizational forgetting. To explore this issue, we assume that

the probability that firm n makes a sale is given by the linear specification

$$D_n(\mathbf{p}) = \min\left(\max\left(\frac{1}{2} - \frac{1}{4\sigma}(p_n - p_{-n}), 0\right), 1\right).^{28}$$

Note that due to the choke price in the linear specification, a firm is able to surely deny its rival a sale by pricing sufficiently aggressively.

For linear demand with $\sigma = 1$ our computations show that the industry evolves towards a symmetric structure in the absence of organizational forgetting ($\delta = 0$). With $\sigma = 0.2$, in contrast, firms at the top of their learning curves fight a preemption battle. The industry remains in an asymmetric structure as the winning firm takes advantage of the choke price to stall the losing firm at the top of its learning curve. Yet, we never found a trenchy or extra-trenchy equilibrium with linear demand in the absence of organizational forgetting. We are therefore confident that the flat equilibria that arise in the absence of organizational forgetting are not an artifact of the lack of a choke price with logit demand. At the same time, we continued to find trenchy and extra-trenchy equilibria with linear demand in the presence of organizational forgetting. Organizational forgetting has thus the same dramatic effect on firms' pricing behavior whether demand is logit or linear.

1.8.4. Frequency of Sales

Following Cabral & Riordan (1994) we take a period to be just long enough for a firm to make a sale. One might wonder whether the insights of the analysis are sensitive to this assumption. To explore this issue without fundamentally departing from our modeling framework, we divide a period into $K > 1$ subperiods. Assuming that one sale occurs in

[28] To allow for a fair comparison between linear and logit demand, we choose the slope parameter so that in the static Nash equilibrium the own-price elasticity of demand in state $(1, 1)$ is the same.

period	subperiod	flat eqbm. without well	flat eqbm. with well	trenchy eqbm.	extra-trenchy eqbm.
8	16	$(5,5)$	$(3,6.5)$, $(6.5,3)$	$(3,6.5)$, $(6.5,3)$	$(1,6.5)$, $(6.5,1)$
16	32	$(9,9)$	$(5.5,10)$, $(10,5.5)$	$(6,9.5)$, $(9.5,6)$	$(1,10)$, $(10,1)$
32	64	$(17,17)$	$(11.5,14)$, $(14,11.5)$	$(11,14.5)$, $(14.5,11)$	$(1,14)$, $(14,1)$
64	128	$(30,30)$	$(19.5,20)$, $(20,19.5)$	$(18,21.5)$, $(21.5,18)$	$(1,18)$, $(18,1)$
∞	∞	$(30,30)$	$(29,30)$, $(30,29)$	$(26,30)$, $(30,26)$	$(1,24)$, $(24,1)$

Table 1.6. Frequency of sales with $K = 2$. Modal states of transient and limiting distributions.

a subperiod, K sales occur in a period. If r is the discount rate per period, then $\frac{r}{K}$ is the discount rate per subperiod and $\beta = \frac{1}{1+\frac{r}{K}}$ the discount factor. We have to be careful not to inadvertently change the properties of learning-by-doing and organizational forgetting by changing the frequency of sales. For example, the reduction in marginal cost that is achievable by a period's worth of sales in the original specification has to be comparable to the reduction that is achievable by K subperiods' worth of sales in the alternative specification. To accomplish this, we take the state space to be $\{1,\ldots,K(M-1)+1\}^2$. The marginal cost and probability of forgetting of firm n in the alternative specification are given by $c\left(\frac{e_n-1}{K}+1\right)$ and $\Delta\left(\frac{e_n-1}{K}+1\right)$. Finally, we take $K(m-1)+1$ to be the stock of know-how at which a firm reaches the bottom of its learning curve.

We have computed equilibria for a progress ratio of $\rho = 0.85$ while doubling the frequency of sales by setting $K = 2$. We obtain a flat equilibrium without well ($\delta = 0$), a flat equilibrium with well ($\delta = 0.02$), a trenchy equilibrium ($\delta = 0.02$), and an extra-trenchy equilibrium ($\delta = 0.09$), similar to the four typical cases in Figure 1.3. Table 1.6 lists the modal states of the implied transient and limiting distributions. As can be seen, industry structure and

dynamics are comparable to those in Figures 1.4–1.6.[29] Overall, it appears that our results are not sensitive to the frequency of sales.

1.8.5. Learning-by-doing

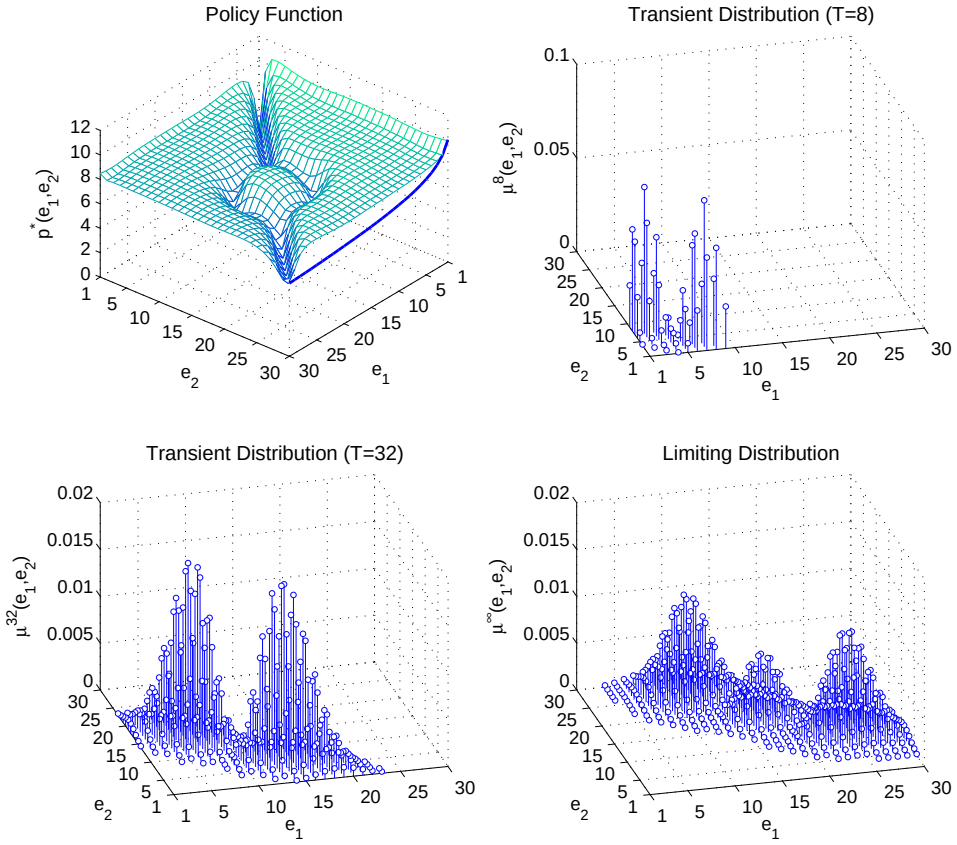

Figure 1.11. Bottomless learning. Policy function $p^*(e_1, e_2)$. Marginal cost $c(e_1)$ (solid line in $e_2 = 30$-plane) (upper left panel). Transient distribution over states in period 8 and 32 given initial state $(1, 1)$ (upper right and lower left panels). Limiting distribution over states (lower right panel). Plateau equilibrium ($\rho = 0.9$, $\delta = 0.04$).

[29]With the possible exception of the extra-trenchy equilibrium: While the modal states of the limiting distribution are $(1, 24)$ and $(24, 1)$ with a probability of 0.0086 each, the limiting distribution also has secondary peaks at states $(5.5, 9.5)$ and $(9.5, 5.5)$ with a probability of 0.0068 each. That is, there is some chance that the industry does not become extremely asymmetric as it does in our baseline parameterization with $K = 1$.

	asymmetric		symmetric	
period	state	prob.	state	prob.
8	$(2,7)$, $(7,2)$	0.0782	–	–
16	$(4,10)$, $(10,4)$	0.0357	–	–
32	$(6,14)$, $(14,6)$	0.0192	–	–
64	$(8,20)$, $(20,8)$	0.0143	$(15,15)$	0.0017
∞	$(11,25)$, $(25,11)$	0.0111	$(17,17)$	0.0072

Table 1.7. Bottomless learning. Most-likely asymmetric and symmetric states of transient and limiting distributions. Plateau equilibrium ($\rho = 0.9$, $\delta = 0.04$).

Following Cabral & Riordan (1994) we assume that $m < M$ represents the stock of know-how at which a firm reaches the bottom of its learning curve. To check the robustness of our results, we instead assume $m = M$. In this *bottomless learning* specification, we obtain another type of equilibrium in addition to the four typical cases in Figure 1.3. Figure 1.11 exemplifies the policy function of this *plateau equilibrium* (upper left panel), the transient distribution in period 8 and 32 (upper right and lower left panels), and the limiting distribution (lower right panel). The parameter values are $\rho = 0.9$ and $\delta = 0.04$.

As can be seen, the plateau equilibrium is similar to a trenchy equilibrium except that the diagonal trench is interrupted by a region (around state $(17,17)$) of very soft price competition. On this plateau both firms charge prices well above cost. This "cooperative" behavior contrasts markedly with the price war of the diagonal trench. While the most-likely industry structure is asymmetric in the long run in this example, there is also a substantial probability that the industry becomes symmetric: The modal states of the limiting distribution are $(11,25)$ and $(25,11)$ with a probability of 0.0111 each. Yet, the limiting distribution also has a secondary peak at state $(17,17)$ with a probability of 0.0072. Table 1.7 summarizes the dynamics of the industry by providing the most-likely asymmetric and symmetric states of

	asymmetric		symmetric	
period	state	prob.	state	prob.
8	–	–	$(4,5)$, $(5,4)$	0.0803
16	–	–	$(6,7)$, $(7,6)$	0.0434
32	$(4,17)$, $(17,4)$	0.0007	$(10,10)$	0.0312
64	$(8,21)$, $(21,8)$	0.0013	$(13,14)$, $(14,13)$	0.0247
∞	$(15,21)$, $(21,15)$	0.0075	–	–

Table 1.8. Bottomless learning. Most-likely asymmetric and symmetric states of transient and limiting distributions. Plateau equilibrium ($\rho = 0.9$, $\delta = 0.04$).

the transient and limiting distributions. As can be seen, the likelihood of cooperation goes up with time.

Table 1.8 provides another example of a plateau equilibrium. The parameter values are the same ($\rho = 0.9$ and $\delta = 0.04$), thereby providing another instance of multiplicity. In this case the most-likely industry structure is symmetric in the short run and asymmetric in the long run (see the Online Appendix for details). That is, the likelihood of cooperation goes down with time.

1.8.6. Organizational Forgetting

We take the probability $\Delta(e_n)$ that firm n loses a unit of know-how through organizational forgetting to be $1-(1-\delta)^{e_n}$, an increasing and concave function (as long as $\delta > 0$), to capture the idea that a firm with more know-how is more vulnerable to organizational forgetting. We alternatively take $\Delta(e_n)$ to be δ, a constant. This may be appropriate in situations in which there is a leading edge of know-how which, if not continually applied, is at risk of being lost.

Our results carry over to this *constant forgetting* specification. If organizational forgetting is sufficiently weak, then asymmetries may arise but they cannot persist. If organizational

forgetting is sufficiently strong, then asymmetries cannot arise in the first place because organizational forgetting stifles investment in learning-by-doing altogether. By contrast, for intermediate degrees of organizational forgetting, asymmetries arise and persist.

Multiple equilibria continue to arise in the constant forgetting specification. Multiplicity is especially pervasive for forgetting rates δ around or somewhat below 0.5. This reaffirms our notion that the primitives of the model tie down the equilibrium unless the inflow of know-how into the industry balances the outflow. The latter happens for forgetting rates around 0.5, and the nature of the equilibrium is therefore governed by firms' expectations regarding to value of continued play.

1.8.7. Entry and Exit

So far we have assumed that the industry is composed of a fixed number of firms. It is straightforward to extend the model to allow for entry and exit. The Online Appendix formally derives the general model; here, we briefly sketch it.

We assume that at any point in time there is a total of N firms, each of which can be either an incumbent firm or a potential entrant. Thus, if N^* is the number of incumbent firms, $N-N^*$ is the number of potential entrants. Once an incumbent firm exits the industry, it perishes and a potential entrant automatically takes its "slot" and has to decide whether or not to enter the industry. Potential entrants are drawn from a large pool. Hence, if a potential entrant chooses not to enter the industry in the current period, it disappears and its slot is given to another potential entrant in the subsequent period. In what follows we focus on the case of $N = 2$. Though alternatives are possible, we specify that an entrant comes into the industry at the top of the learning curve. Since the analysis of entry and exit requires a well-posed monopoly problem, we include an outside good.

To ensure the existence of an equilibrium, we use the approach in Doraszelski & Satterthwaite (2007). In each period, each potential entrant receives a privately observed draw S_n from a symmetric triangular distribution of possible set-up costs with support $[3, 6]$ and each incumbent firm receives a privately observed draw X_n from a symmetric triangular distribution of possible salvage values with support $[0, 3]$.[30] It is convenient to summarize the entry and exit decisions of firm n using an operating probability $\lambda_n(\mathbf{e})$, where $\mathbf{e} \in \{0, 1, \ldots, M\}^N$ is the state of the industry. If $e_n = 0$, firm n is a potential entrant and $\lambda_n(\mathbf{e})$ is the probability that it enters the industry in state $\mathbf{e}$; if $e_n \neq 0$, firm n is an incumbent firm and $\lambda_n(\mathbf{e})$ is the probability that it remains in the industry. A symmetric and anonymous Markov perfect equilibrium consists of a value function $V^*(\mathbf{e}) = V_1(\mathbf{e})$, a pricing function $p^*(\mathbf{e}) = p_1(\mathbf{e})$, and an operating probability $\lambda^*(\mathbf{e}) = \lambda_1(\mathbf{e})$ for firm 1.

Entry and exit do not alter the thrust of our results. Organizational forgetting remains a source of aggressive pricing behavior. Indeed, allowing for exit adds another component to the prize from winning a sale because by winning a sale, a firm may move the industry to a state in which its rival is likely to exit. But if the rival exits, then it may be replaced by an entrant that comes into the industry at the top of its learning curve or it may not be replaced at all. As a result, pricing behavior is more aggressive than in the basic model without entry nor exit. This leads to more pronounced asymmetries both in the short run and in the long run. It is even possible that the industry is monopolized.

Multiple equilibria continue to arise in the general model. Entry and exit exacerbate the multiplicity problem. Strikingly, in contrast to Proposition 2, there may be multiple equilibria even in the absence of organizational forgetting ($\delta = 0$). For a progress ratio

[30] This implies that some portion of set-up costs is sunk, thereby eliminating the possibility that a firm enters the industry merely because it hopes to draw a salvage value that exceeds its set-up cost.

of $\rho = 0.75$, for example, we found three equilibria. While these equilibria are flat either without or with well, the implied long-run industry structures range from symmetric (with the model state of the limiting distribution being $(30, 30)$) to monopolistic (with the modal states being $(0, 30)$ and $(30, 0)$). In the former equilibrium, once both firms have entered the industry, there may not be exit in the future (we have $\lambda^*(\mathbf{e}) = 1.00$ for all $\mathbf{e} \in \{1, \ldots, M\}^2$). Knowing this, firms may as well price softly, so that, in turn, the incentive to enter the industry is strong even if an incumbent must be faced (we have $\lambda^*(0, 1) = 0.84$). In the latter equilibrium, each firm uses price cuts to induce its rival to exit (we have $p^*(1, 1) = -36.95$ and $\lambda^*(2, 1) = 1.00$ but $\lambda^*(1, 2) = 0.76$). Given that post-entry pricing behavior is "predatory", the incentive to enter the industry is weak in the first place (we have $\lambda^*(0, 1) = 0.08$), thereby ensuring that the most-likely industry structure is monopolistic not only in the long run but also in the short run (the modal states of the transient distribution are $(0, 8)$ and $(8, 0)$ in period 8 and $(0, 30)$ and $(30, 0)$ in period 32).

1.9. Conclusions

Learning-by-doing and organizational forgetting have been shown to be important in a variety of industrial settings. Using the Markov-perfect equilibrium framework of Ericson & Pakes (1995) this paper provides a general model of dynamic competition that accounts for these economic fundamentals and shows how they shape industry structure and dynamics. We enhance the methodological foundations of this literature in two ways. First, we show that there are equilibria that cannot be computed by the Pakes & McGuire (1994) algorithm. Second, we propose a homotopy algorithm that allows us to describe in detail the structure of the set of equilibria of our dynamic stochastic game.

In contrast to the present paper, the theoretical literature on learning-by-doing has largely ignored organizational forgetting. Moreover, it has mainly focused on firms' pricing behavior. By directly examining industry dynamics, we are able to show that ID and IID may not be sufficient for economically meaningful market dominance. By generalizing the existing models of learning-by-doing through the addition of organizational forgetting, we are able to show that these dominance properties of firms' pricing behavior break down in the presence of even a small degree of organizational forgetting. Yet, it is precisely in the presence of organizational forgetting that market dominance ensues both in the short run and in the long run.

Our analysis of the role of organizational forgetting reveals that learning-by-doing and organizational forgetting are distinct economic forces. Organizational forgetting, in particular, does not simply negate learning-by-doing. The unique role played by organizational forgetting comes about because it makes bidirectional movements through the state space possible. Hence, dynamic competition with learning-by-doing and organizational forgetting is akin to racing down an upward moving escalator. As a consequence, a model with both learning-by-doing and organizational forgetting can give rise to aggressive pricing behavior, market dominance, and multiple equilibria, whereas a model with learning-by-doing alone cannot.

Diagonal and sideways trenches are part and parcel to the self-reinforcing mechanisms that lead to market dominance. Since the leadership position is aggressively defended, firms fight a price war to attain it. This provides all the more reason to aggressively defend the leadership position because if it is lost, then another price war ensues. This seems like a good story to tell. Our computations show that this is not just an intuitively sensible story but also a logically consistent one that—perhaps—plays out in real markets.

CHAPTER 2

Dynamic R&D and the Effectiveness of Policy Intervention in the Pharmaceutical Industry

2.1. Introduction

In 2005, The Bill and Melinda Gates Foundation have announced over $160 million in grants related to drugs and vaccines, covering every stage of their lifecycle, starting from initial discovery, following through clinical trials, and ending with sale subsidies. At the same time, the National Institutes of Health (NIH) have spent $2.9B on sponsoring clinical trials alone, which is more than 10% of spending by the firms in the industry. These numbers illustrate the effort that governments, NGOs and private charities exert to speed up the development of new treatments for various conditions, ranging from tropical diseases and AIDS to cancer.[1] In the vast majority of cases, these efforts aim to increase introduction of new drugs by profit-maximizing firms (as opposed to dedicated non-profit establishments).

This gives rise to the main question of this study – what is the best way to achieve the policy goal of increased output of new drugs? Is it best to finance the discovery of new substances with therapeutic potential? Or should a donor help offset the tremendous cost of the three stages of clinical trials that are required to determine whether the drug is sufficiently safe and effective? Or is it most efficient to expand markets for these drugs by subsidizing the sales? Yet another approach is to streamline and accelerate lengthy clinical trials and FDA review process.

In order to answer these questions, it is necessary to model the drug development process and the decisions made at each of several stages within it. Considering pre-clinical and clinical trials as well as FDA review, DiMasi, Hansen & Grabowski (2003) estimate the average time to bring a single drug to the market as 12 years; in addition, only 1 out of 5 drugs survives the development process, bringing the total accumulated cost to $800 million.

[1]This study will not deal with justification of such interventions, but accepts them as given. Common view is that targeted condition lacks the attention of Pharmaceutical companies due to lower profitability. Interested reader is pointed to Cockburn & Henderson (1996) for discussion of feasibility of public R&D funding.

Faced by such long and uncertain return to investment, the decision to continue with each consecutive stage of development involves not only the medical viability of the drug, but also a conjecture about situation in the market several years into the future. The basis for this conjecture is formed by the current number of competing drugs at various stages of development, as well as beliefs about the evolution of these over time. The latter, in turn, depend on the decision rules used by the firms.

To accurately describe the features of this decision-making process, this study formulates a dynamic stochastic model using the framework established by Ericson & Pakes (1995). Unlike most recent empirical papers using this framework[2], the model is formulated in continuous rather than discrete time, as suggested by Doraszelski & Judd (2004). Broadly speaking, while the discrete time model assumes periodic and simultaneous decisions by all of the agents, continuous time assumes sequential decisions with random order of moves. This matches the actual decision-making process, and offers important methodological advantages (the model would be intractable if formulated in discrete time). Additional feature of the model is the use of polynomial approximation to the value and policy functions (instead of more common discretization), which keeps the number of estimated parameters low, and further improves tractability.

The model is estimated structurally by means of Nested Pseudo-likelihood (NPL) method suggested by Aguirregabiria & Mira (2002) and Aguirregabiria & Mira (2007), which uses the equilibrium conditions of the model to estimate costs of each stage of the development process. In addition, a number of other parameters, such as the approval rate of FDA

[2]See, for example Bajari, Benkard & Levin (2007), Ryan (2005), Schmidt-Dengler (2005), Collard-Wexler (2005).

applications and average duration of each development stage, are estimated directly from data or calibrated prior structural estimation.

Once the model parameters and associated equilibrium are computed, it becomes possible to evaluate the impact of various policies through counterfactual experiments. Experiments consist of matching each policy intervention to a change in the appropriate model parameter, and computing a new equilibrium for this changed parametrization. These policy-affected equilibria, as well as the baseline estimate, are compared in terms of average flow of the new drugs through the development pipeline, with special emphasis on the new drugs entering the market.

The key finding is that interventions at later stages of the development process result in more new drugs entering the market than those at earlier ones. The reason for this is rooted in the uneven distribution of positive and negative effects of a policy across the development stages. A policy-induced increase in the number of drugs (both on the market and under development) necessarily reduces the value of each drug; to offset that, each policy offers a benefit of one sort or another. But the dynamic nature of the model means that negative effect is present at all stages, while the policy benefit occurs only at the single stage.

Naturally, the increased flow of drugs through the pipeline is the strongest at the stage affected by the policy. Moreover, earlier stages also see an increase in activity as firms expect to collect the policy benefit in the future. But in the stages following the policy intervention, only the negative effect remains, discouraging the firms from continuing the development and causing the effect of the policy to weaken with each subsequent stage, as industry reverts to its original state. Since policy interventions are set up to be comparable either in terms of average cost or direct impact on affected stage, earlier interventions seem to have lower impact on the number of drugs entering market – the final stage of a drug's life.

In order to make necessary assumptions, this study relies on a substantial body of existing work studying individual stages of drug development process. The Henderson-Cockburn research program (1994, 1994, 1996) develops a framework for describing initial ("discovery") research and finds such features as spillovers, economies of scale, and scope.

The literature on clinical trials includes such works as Danzon, Nicholson & Pereira (2003), Guedj & Scharfstein (2004), Dranove & Meltzer (1994) and Cockburn & Henderson (2001). These papers document a number of non-medical factors that affect the decision to continue the development into the next phase – size and type of firm, experience, and the perceived earning potential of the drug.

The market for drugs has been extensively studied by Berndt, Bui, Reily & Urban (1994, 1995) and Azoulay (2002) who document extensive advertising expense that plays at least as large a role as the drug quality, and show that first-mover advantage is present but but not always sustainable; these results suggest that differentiation in the drug markets is horizontal rather than vertical. Roberts (1999) finds correlation between market profitability and rate of new drug introduction, confirming presence of the profit motive in the R&D decisions. Grabowski & Vernon (1994) combine revenue data with development cost estimates of DiMasi et al. (2003) to show that average return on investment in pharmaceutical industry is on par with the economy average.

This paper contributes to the literature on pharmaceutical industry by aggregating the existing research on individual stages into a dynamic model of the entire development process. While such a model is analytically intractable, recent developments in computational economics have made it possible to estimate the model parameters and solve for the equilibrium numerically. The paper further contributed to literature on policy and dynamics by studying the timing of policy intervention.

The rest of the paper proceeds as follows: Section 2.2 starts by describing the pharmaceutical industry, Section 2.3 sets up the model of the drug development process, Section 2.4 discusses available data and estimation approach, and Section 2.5 presents the results.

2.2. R&D in the Pharmaceutical Industry

Pharmaceutical industry is characterized by a high intensity of R&D effort, meaning that new products (drugs and other treatment methods) constantly enter the market, driving the older ones out. In many modern industries, the focus of innovation is on research, which results in immediate changes to product; development is a relatively straightforward process of setting up the production. Pharmaceutical R&D is different in the sense that development (in the form of the several phases of trials) plays a much bigger role, taking considerable time and costing more than research. Moreover, US's Food and Drug Administration (FDA), or similar regulatory body in other countries requires firms to go through the complete set of trials for each new drug, so it makes sense to describe the R&D process as the sequence of stages in the "lifecycle" of a drug.

The research stage is referred to as *Initial discovery*; it results in the new chemical entities (NCE's), i.e. compounds believed to have therapeutic potential. These are typically patented and/or described in academic publications. The costs are relatively minor, as the research is limited to the laboratory experiments and wide adoption of the scientific discovery method has largely eliminated need for extensive trial-and-error search[3].

The next development stage for NCE is known as the *Pre-clinical trials* which are conducted on animals and test the safety of the drug, and last 2-3 years.

[3]See, for example Cockburn, Henderson & Stern (2001).

If pre-clinical trials are successful and the firm wants to proceed with the drug, it files *Investigational new drug* (IND) application to FDA. The application summarizes the results of pre-clinical trials and requests permission to start human (clinical) trials. The wait for this decision is relatively short as FDA has only 30 days to review the application and voice any objections.

Once the drug receives IND status, the firm can start the *Clinical trials* that are the central part of the development process, taking on average 9.7 years and consuming the bulk of the cost. They are conducted on human subjects. The FDA standards for these trials demand a control group that receives a placebo or an already approved treatment, random allocation of subjects to treatment and control groups, and a "double blind" requirement that this allocation is not known to either the subjects or the physicians evaluating their condition. The trials are grouped into three phases of increasing breadth and length:

- *Phase I* establishes basic safety of the drug and typically involves a small group of healthy subjects, since they are likely to suffer less harm than patients already weakened by the disease if the drug turns out to be unsafe.
- *Phase II* evaluates the efficiency of the drug in treating patients who have the target condition, which typically requires a larger group of subjects than Phase I.
- *Phase III* checks for any long-term side effects, which necessarily last a considerable amount of time and requires a broad range of subjects covering broad demographic range.

Within each phase, pharmaceutical firms conduct several trials with varying dosages of the drug, delivery methods, etc. Trials continue until the firm collects enough information

to determine whether the drug satisfies the objective of the respective phase, and whether the drug shows enough promise to continue development into the next phase.

Once the trials are complete, and if the firm is happy with their results, a *New drug application* (NDA) is submitted to FDA, which reviews the results of clinical trials and decides whether the drug is allowed to enter the market. The review process takes almost 3 years. Like with the trials, the decision to file the NDA application is not costless – in addition to application fee, firms have to compile and format the application materials, follow up with FDA and answer any questions that arise during the review process.

If FDA approves the drug, it is launched on the *Market* though a heavy advertising campaign. Over-the-counter drugs are advertised directly to consumer; prescription drugs are "detailed" to physicians by sales representatives. Advertising expenses remain a sizable fraction of sales even after the launch. Drugs are rarely withdrawn from the market (only in case of unexpected side-effects), but they are displaced over time by newer drugs or generics. First-mover advantage is considerable but not always permanent, as Azoulay (2002) demonstrates. Individual susceptibility of patients to drugs means that differentiation between drugs is horizontal rather than vertical.

The most common definition of market within the industry uses Anatomical-Therapeutic-Chemical classification, i.e. a market consists of the drugs that treat similar conditions using a specific chemical mechanism. The definition of markets along geographical boundaries is less common since the costs of entering additional countries are small compared to the expense of initial clinical trials; low transportation costs lead to active international trade. Division along consumer income and other demographic characteristics is even less common since various private and public health insurance schemes make same drugs available to widely varying groups of people.

Patent expiration & generics are a major factor contributing to gradual decline of profits and eventual elimination of the drug from the market. While the patent application is filed early in the development process, as of 1994 the patent clock starts when NDA is filed, with 20 years of protection granted to all drugs. After the patent expires, a generic (unbranded) version of the drug promptly enters the market[4]. Typically, brand loyalty and risk aversion of physicians ensure that branded drug retains some market share. Recent development of HMO's and implementation of drug-substitution mechanisms at the pharmacy level further weaken the profitability of original drug.

An important feature of the R&D process is the gradual elimination of drugs as they go through the development stages. Anecdotal evidence suggests that out of 5000 NCE's, 20 enter the pre-clinical trials, 5 receive IND status and only one is approved by FDA. Besides the two stages of FDA approval, a major source of such selection are the pharmaceutical companies themselves, that can (and do) discontinue drugs at every stage of the development process. Based on interviews during the data collection process, the reasons for such decisions are not limited to poor results of clinical trial, but can include "strategic considerations" or lack of funds[5]. This suggest building an economic model to describe these decisions.

2.3. Model

This section models the decisions of a pharmaceutical firm regarding the progress of drugs through development pipeline as the optimal strategy in a fully dynamic game. Subsection 2.3.1 sets up basic notation and introduces model parameters, and subsection 2.3.2 defines the equilibrium.

[4]While a generic also requires an FDA registration, Waxman-Hatch act of 1984 make it sufficient to demonstrate bio-equivalence with the original drug, essentially reducing the entry barrier to zero.

[5]Financing constraints can be ignored here, since it is common practice for small firms to sell drugs they cannot afford to develop to a larger company.

2.3.1. Assumptions and Notation

As previous section described, progress of drug through the development pipeline is a sequence of decisions to continue to the next stage or abandon the drug (here and below, word "stage" is used as a more general term than "phase of clinical trial", since stage can also include pre-clinical trial, FDA review and market.). This model will describe these decisions regarding an individual drug. Lacking data on clinical trial outcomes, the drugs are assumed to be identical (given same stage of development). This assumption is a "conservative" one, since it eliminates particularly good ("star") drugs from the model; it does not really affect underperforming drugs, since those are likely to be abandoned at previous stages of the development process.

The model assumes anonymous ownership of the drugs, essentially assuming that each drug is owned by a separate firm that has no other drugs, and is identical to any other firm on the market. This abstracts away from a possibility of a "large" firms that owns several drugs in each of development phases. While it is a realistic situation, and affects decision making in a meaningful way, but properly accounting for it would considerably complicate the model, and is not likely to generate any new insights since the decisions by both single- and multi-drug firms respond in similar way to outside factors.

Anonymous ownership also assumes away firms active on more than one market, but there is no evidence in industry literature to suggest use of multi-market strategies by Pharmaceutical firms, which is expected given sometimes widely different conditions and treatments that define each market.

The fundamental assumption of Ericson & Pakes (1995) modeling framework is that all the relevant information about an industry can be compressed into the few numbers of a state

vector. "Relevant information" here refers to a maximum information set that a firm within the industry can use to predict the actions of it's competitors and future development of the industry that it implies[6]. The solution concept is Markov-perfect equilibrium (MPE) that requires the firm's decisions to be optimal given current industry state, privately observed information and (rational) beliefs about the decision rules of competitors.

A consequence of such state-dependent strategies is that there is no history dependence, i.e. it does not matter how did the industry arrive at the present state. This prevents collusion (tacit or explicit) from being enforced by various punishment strategies. While allegations of price-fixing have been common during the 1990s, studies like Grabowski & Vernon (1994) demonstrated that proper computation of R&D expenses brings profitability of Pharmaceutical companies on par with other high-tech industries and it's own cost of capital.

In the model of this study, the state of a drug is given by it's current development stage, indexed by $s \in \{1:5\}$:

$\circ$ $s = 1:3$ correspond to Phases I-III of clinical trials,

$\circ$ $s = 4$ – drug submitted for FDA review, and

$\circ$ $s = 5$ – drug on the market.

State of the industry is characterized by a vector $x \in \mathbb{Z}_+^5$; its component is denoted as x_s represents the number of drugs at stage s. A combined state of a single drug and the industry around it is (s, x); note that $x_s \geq 1$ since the singled-out drug is included into x.

Given a drug entering stage s, the duration of the stage is assumed to be random and distributed as Exponential with average duration $1/\mu_s$ years. This means that regardless of

[6]This development need not be deterministic, as the model incorporates purely random events (moves by "nature") and decisions based on privately observed random draws.

how long the drug has been in the stage, the hazard rate[7] of stage ending in the subsequent year remains μ_s. This is generally consistent both with observed data, and the industry practice of doing a series of trials[8] within each phase until the firm has enough information to make the decision.

The decision in question is whether to abandon the drug or move on to next phase of clinical trials. The information set for this decision includes the results of the trial (privately observed), number of competing drugs at every stage of other firms (x) and lump-sum cost of next phase R_s.

The results of the clinical trial is consist of $\varepsilon = (\varepsilon_0, \varepsilon_1)$ – a pair of independent type I Extreme Value random variables with standard error parameter σ_s. If the the drug is abandoned, the firm owning it receives ε_0 ($\varepsilon_0 < 0$ means that the firm pays the amount); if it advances to next stage – firm receives ε_1. Realizations of these ε are assumed to incorporate several unobserved factors, including (a) the difference between R_s and actual cost of continuing to next stage, (b) chance of FDA approval, (c) future earning potential of the drug, (d) "salvage value" of a drug, either for treatment of other conditions[9] or as a source of spillovers. Vector ε is private information because most of (a)-(d) cannot be observed from outside the company; the academic publications that follow the completion of a trials phase reveal only partial information, and are delayed in time.

Without observing the private results of the trial (as is the case with a firm considering possible actions of its competitors), the decision can be characterized by a probability of

[7]Hazard rate refers to average number of events happening per a period of certain length. Here, the period is chosen to be year. Hazard rate can be interpreted as probability as long as it is no larger than one.
[8]The conditions of the trial – such as dosage, frequency, delivery methods, target demographics – are adjusted from one trial to the next in search of optimal performance.
[9]Pharmaceutical industry has numerous cases of drugs turning out to be effective in unintended roles, most famous of which is Viagra, which was originally developed as treatment for angina.

continuing to next phase. This probability will be endogenously determined in the model and is denoted as $p(s,x)$ – probability that drug will advance to phase s from $(s-1)$.

A drug submitted for FDA review ($s=4$) has a constant and exogenous probability of advancing given by p_5. Keeping probability of FDA as constant reflects the assumption that unlike firm's decision, FDA's approval process is not affected by situation on the market, and is driven entirely by therapeutic considerations. There is anecdotal evidence that large established firms have easier time getting their drugs through FDA (both in terms of time it takes to reach a decision, and decision itself), and it is reasonable to assume that FDA would take a different approach for drugs in what it views as "priority" areas. Accounting for identity of drug owner is not possible under anonymity assumption; effect of changing FDA priorities requires additional data.

A drug on the market ($s=5$) involves no decisions, but earns a steady stream of annual profits $R_5(x)$. Lacking meaningful information on quality of the drug, the functional form of $R_5(x)$ assumes the equal split of market by all drugs present on it.

$$R_5(x) = \frac{\pi M}{x_5},$$

where M is market size (in terms of total revenue), and $\pi < 1$ is the profit margin[10]. The assumption of equal market share does not negate first-mover advantage that is clearly a feature of drug markets, since first drug to reach the market starts earning profits earlier than others, these profits are larger, and increased number of drugs on the market discourages further entry.

[10] π is defined as share of revenues left after deducting production, marketing, operating and research costs. Development costs are excluded since model incorporates them explicitly (as R_s)

Just like with previous stages, hazard rate $\mu_5 > 0$ gives the likelihood of the stage ending. In the case of a drug on the market, however, the end of stage means permanent exit of a drug from the industry. This assumption reflects gradual erosion of drug's profitability by more advanced substitutes and generics.

Having discussed progress of drugs through the pipeline and their exit, it remains to describe the process that puts new drugs into the pipeline. The assumption here is that every year an average of λ_0 drugs enter the Phase I trials ($s = 1$). This assumption eliminates the transition from pre-clinical to clinical trials from the model; it is unavoidable since the transition involves two separate decisions – IND application by the firm, and FDA response to it. As described in Section 2.4 below, available data do not allow to separate these two decisions, preventing any meaningful use of them in the model. This assumption also reflects the fact that a lot of discovery research is conducted in academic rather than commercial labs, so unlike development decisions, the intensity of research is not necessarily affected by situation in the market. Instead, one of policy experiments simulates increasing output of discovery research as the result of increased funding of a particular field, increased public attention or scientific relevance, etc.

2.3.2. Equilibrium

The decision to continue with development will maximize the present expected value of a drug, conditional on private information, development stage of the drug and industry state. The result of this optimization is called value function and denoted as $V(s, x)$.

Computing present expected value of a drug requires an assumption about the progress of time in the model. Common assumption in Ericson & Pakes (1995) framework was discrete time, with a period of set length (e.g. year), which included the simultaneous decisions by

all firms and subsequent simultaneous transit of all players to new states according to these decisions and exogenous random processes.

In this case, discrete time is inappropriate for two major reasons. First, while there are industries where decision are made annually and simultaneously, decisions regarding drug development are made at random points in time throughout the year. Second, simultaneous transitions lead to insurmountable burden of computing the expectation over them.

Consider industry in state x; for any stage s, there are x_s drugs in that stage, and each of them can complete the trial in current period. Further, each of drugs completing the trial may (or may not) advance to the next phase. Overall, the number of possible combinations is $\left[x_5 \prod_{s=1}^{4} \frac{(x_s+1)(x_s+2)}{2}\right]$. Even for 5 drugs in each phase (entirely realistic situation, according to Table 2.1 below), that means almost a million combinations.

There is no opportunity to simplify the computation using independence; the changes in individual components x_s's are pair-wise dependent since a drug advancing to the next stage necessarily leaves the previous one. It turns out that with current computing technology, going through all of these realization takes prohibitively long time.

Instead, this study assumes the continuous timescale, meaning that individual trials end at random moments in the time continuum, and are instantly followed by a decision and transition to the next stage (or discontinuation). As a result, probability of two or more transitions occurring simultaneously is zero, so such events need not be considered. This leaves only the transitions of individual drugs, which permit only two possible outcomes and make the computation of expectation trivial.

Writing out expression for expected present value $V(s, x)$ requires some additional notation. Let $a_s \in \{0, 1\}$ denote decision to continue development to stage s, T_s – realized duration of this stage. Discounting of future profits and costs is parametrized by firm's cost

of capital ρ; given continuous time, discount factor applied to cash flow occurring at time t is $e^{-\rho t}$. Then the present value of a drug in stage $s = 1$ and facing industry state x_0 at $t = 0$ is given by:

$$V(1, x_0) = \max_{a_s} \mathbf{E}_{\varepsilon_s, T_s, x_t | x_0} \left\{ \begin{array}{l} e^{-\rho T_1} \{(1 - a_2) \varepsilon_{0,1} - a_2 (R_2 - \varepsilon_{1,1}) + a_2 \times \\ \times e^{-\rho T_2} \{(1 - a_3) \varepsilon_{0,2} - a_3 (R_3 - \varepsilon_{1,3}) + a_3 \times \\ \times e^{-\rho T_3} \{(1 - a_4) \varepsilon_{0,3} - a_4 (R_4 - \varepsilon_{1,3}) + a_4 \times \\ \times e^{-\rho T_4} p_5 \times \\ \times \int_0^{T_5} e^{-\rho q} R (x_{q+T_1+T_2+T_3+T_4}) \, dq \}\} \end{array} \right\}, \tag{2.1}$$

and expression for value of a drug in an arbitrary stage s can be obtained from (2.1) by dropping first $s - 1$ lines.

Even with simplification of transition process, (2.1) is not tractable. Instead, this study uses Bellman optimality principle that reduces the problem to solving for one decision at a time. Note that (2.1) can be restated as:

$$\begin{aligned} V(1, x) &= \max_{a_2} \mathbf{E}_{\varepsilon_1, T_1, x_{T_1}} \left\{ \begin{array}{l} e^{-\rho T_1} \{(1 - a_2) \varepsilon_{0,1} - a_2 (R_2 - \varepsilon_{1,1}) + a_2 \times \\ \times \max \mathbf{E}_{\varepsilon_s, T_s, x_t | x_{T_1}} [e^{-\rho T_2} \{(1 - a_3) \varepsilon_{0,2} - a_3 (R_3 - \varepsilon_{1,3}) + a_3 \times \\ \times e^{-\rho T_3} \{(1 - a_4) \varepsilon_{0,3} - a_4 (R_4 - \varepsilon_{1,3}) + a_4 \times \\ \times e^{-\rho T_4} p_5 \times \\ \times \int_0^{T_5} e^{-\rho q} R (x_{q+T_1+T_2+T_3+T_4}) \, dq \}\}] \end{array} \right\} \\ &= \max_{a_2} \mathbf{E}_{\varepsilon_1, T_1, x_{T_1}} \left\{ e^{-\rho T_1} \{(1 - a_2) \varepsilon_{0,1} - a_2 (R_2 - \varepsilon_{1,1}) + a_2 V(2, x_{T_1})\} \right. \end{aligned}$$

The next step is to consider an interpretation of continuous time as a limiting case of discrete-time model when the length of the period approaches zero. The approach is to take a short period of time Δ, write a discrete-time Bellman equation describing the change of

the value over that period, transform the equation and take a limit as $\Delta \longrightarrow 0$. Since the firm's information set does not include time, firm's decision and industry transitions are invariant of t, and the time subscript is dropped. Further details depend on the stage of the drug described by the equation ("our drug"), which is denoted as $\bar{s}$ (to distinguish it from arbitrary subscript s).

Consider a **drug in clinical trials** ($\bar{s} \in \{1, 2, 3\}$). It is easiest to write out the complete equation, and then explain individual terms:

$$
\begin{aligned}
V(\bar{s}, x) = (1 - \rho\Delta) \times \\
\times \left\{ \begin{array}{cc}
(\Delta\mu_{\bar{s}} + O(\Delta^2))\, \sigma \log\left[1 + \exp\left\{\dfrac{V\left(\bar{s}+1, x_{-s}^{+(s+1)}\right) - R_{\bar{s}+1}}{\sigma}\right\}\right] & (A) \\
+\sum_{s=1}^{5} (x_s - 1_{\bar{s}=s}) (\Delta\mu_s + O(\Delta^2)) \left[\begin{array}{c} p(s+1, x)\, V\left(\bar{s}, x_{-s}^{+(s+1)}\right) \\ +(1 - p(s+1, x))\, V(\bar{s}, x_{-s}) \end{array}\right] & (B) \\
+(\Delta\lambda_0 + O(\Delta^2))\, V(\bar{s}, x^{+1}) & (C) \\
+\left(1 - \left[\begin{array}{c} \sum_{s=1}^{5} [x_s \Delta\mu_s + O(\Delta^2)] \\ +\Delta p_0 + O(\Delta^2) + \Delta\lambda_0 + O(\Delta^2) \end{array}\right]\right) V(\bar{s}, x) & (D)
\end{array} \right\}
\end{aligned}
$$

Since the drug is in the trial, it does not generate any profit, so all there is the expected future value term. The discount factor is $e^{-\rho\Delta} \approx (1 - \Delta\rho)$. Now consider the sum in the curly brackets, which is described line-by-line:

(A). This term deals with our drug finishing clinical trial during the Δ period. The probability of this event is is $\Delta\mu_{\bar{s}}$. Probability that some other drug will finish stage s is $\Delta\mu_s$. Probability that these two events will happen simultaneously is $\Delta\mu_{\bar{s}}\Delta\mu_s = \Delta^2\mu_{\bar{s}}\mu_s = O(\Delta^2)$

as the limit is taken at $\Delta \longrightarrow 0$. So one can safely assume that any expected value associated with some other drug transiting along with ours is $O\left(\Delta^2\right)$.

The $\sigma \log\left[...\right]$ term is the expected value resulting from decision to abandon the drug $(a=0)$ or advance to next stage $(a=1)$:

$$\sigma \log\left[1+\exp\left\{\frac{V\left(\bar{s}+1, x_{-s}^{+(s+1)}\right)-R_{\bar{s}+1}}{\sigma}\right\}\right]=$$

$$= \mathbf{E}_{\varepsilon} \max_{a\in\{0,1\}}\left\{(1-a)\,\varepsilon_0 + a\left[\varepsilon_1 - R_{\bar{s}+1} + V\left(\bar{s}+1, x_{-s}^{+(s+1)}\right)\right]\right\} \tag{2.2}$$

where $V\left(\bar{s}+1, x_{-s}^{+(s+1)}\right)$ is the value realized if the drug advances, and $x_{-s}^{+(s+1)}$ means x with x_s decreased by one, and x_{s+1} increased by one to reflect transition of our drug to the next stage. It follows from properties of EV distribution that

$$\begin{aligned} p\left(\bar{s}+1, x\right) &= \mathbf{E}_{\varepsilon^i} a \\ &= \left[1+\exp\left\{-\frac{V\left(\bar{s}+1, x_{-s}^{+(s+1)}\right)-R_{\bar{s}+1}}{\sigma}\right\}\right]^{-1} \end{aligned} \tag{2.3}$$

(B). This term deals with other drugs completing the trials, receiving FDA approval or exiting the market. For a drug in stage s, probability of such event is $\Delta\mu_s$, probability of that for one in x_s drugs is $\Delta x_s \mu_s + O\left(\Delta^2\right)$, and expectation of more than one drug reaching the end of a stage is again $O\left(\Delta^2\right)$.

If a drug completes stage s, it advances to the next stage with probability $p\left(s+1, x\right)$, which causes the change in industry state and corresponding change in the expected value of our drug (x_{-s} means x with x_s reduced by one). To avoid separate notation for $s=5$ (drug on the market), let $p\left(6, x\right) \equiv 0$ (i.e. it cannot transit further).

(C). This term reflects the chance that a new drug enters Phase I trials. The probability of this $\Delta\lambda_0$, plus $O(\Delta^2)$ term that incorporates simultaneous events.

(D). This term represents the value realized if none of the above events occur during the Δ period. Appropriately, the probability is one minus sum of all other probabilities, and the value realized remains at $V(\bar{s}, x)$.

Now let's transform the equation. Split the (D) term and rearrange to obtain:

$$\rho\Delta V(\bar{s}, x) = (1 - \rho\Delta) \times$$
$$\times \left\{ \begin{array}{c} (\Delta\mu_{\bar{s}} + O(\Delta^2)) \left(\sigma \log \left[1 + \exp \left\{ \frac{V\left(\bar{s}+1, x_{-s}^{+(s+1)}\right) - R_{\bar{s}+1}}{\sigma} \right\} \right] - V(\bar{s}, x) \right) \\ + \sum_{s=1}^{5} (x_s - 1_{\bar{s}=s}) (\Delta\mu_s + O(\Delta^2)) \left[\begin{array}{c} p(s+1, x) V\left(\bar{s}, x_{-s}^{+(s+1)}\right) \\ + (1 - p(s+1, x)) V(\bar{s}, x_{-s}) \\ -V(\bar{s}, x) \end{array} \right] \\ + (\Delta\lambda_0 + O(\Delta^2)) \left[V(\bar{s}, x^{+1}) - V(\bar{s}, x) \right] \end{array} \right\}$$

Now note that for arbitrary constant π, $(1 - \rho\Delta)(\Delta\pi + O(\Delta^2)) = \Delta\pi + O(\Delta^2)$ and take the limit of the ratio of two sides of the equation as $\Delta \longrightarrow 0$, which gives:

$$\begin{aligned} \rho V(\bar{s}, x) = \mu_{\bar{s}} & \left(\sigma \log \left[1 + \exp \left\{ \frac{V\left(\bar{s}+1, x_{-s}^{+(s+1)}\right) - R_{\bar{s}+1}}{\sigma} \right\} \right] - V(\bar{s}, x) \right) \\ & + \sum_{s=1}^{5} (x_s - 1_{\bar{s}=s}) \mu_s \left[\begin{array}{c} p_s(s, x) V\left(\bar{s}, x_{-s}^{+(s+1)}\right) \\ + (1 - p_s(s, x)) V(\bar{s}, x_{-s}) \\ -V(\bar{s}, x) \end{array} \right] \\ & + \lambda_0 \left[V(\bar{s}, x^{+1}) - V(\bar{s}, x) \right] \end{aligned} \tag{2.4}$$

This is the standard form of Bellman equation for continuous time model, which sets the rate of change in value implied by discount rate ($\rho V(\bar{s}, x)$) equal to the expected rate of change provided by the model. The latter is a sum of products of hazard rates for each event and expected change in value as the result of that event.

The equation for a **drug under FDA review** ($\bar{s} = 4$) is constructed in a similar way, except that transition of our drug to the next stage is determined by FDA decision, which is an exogenous random event from the point of view of the firm who only know the probability of positive decision $p_{\bar{s}+1}$:

$$\begin{aligned} \rho V(\bar{s}, x) = {} & \mu_{\bar{s}} \left[p_{\bar{s}+1} V\left(\bar{s}+1, x_{-\bar{s}}^{+(\bar{s}+1)}\right) - V(\bar{s}, x) \right] \\ & + \sum_{s=1}^{5} (x_s - 1_{s=\bar{s}}) \mu_s \begin{bmatrix} p_s(s, x) V\left(\bar{s}, x_{-s}^{+(s+1)}\right) \\ + (1 - p_s(s, x)) V(\bar{s}, x_{-s}) \\ -V(\bar{s}, x) \end{bmatrix} \\ & + \lambda_0 \left[V(\bar{s}, x^{+1}) - V(\bar{s}, x) \right] \end{aligned} \tag{2.5}$$

Finally, **drug on the market** ($\bar{s} = 5$) does not involve any decisions, but it can vanish from the market, and it's value is still affected by new other drugs progressing through the pipeline, entering or exiting the market:

$$\begin{aligned} \rho V(\bar{s}, x) = {} & R_5(x) - \mu_{\bar{s}} V(\bar{s}, x) \\ & + \sum_{s=1}^{5} (x_s - 1_{s=\bar{s}}) \mu_s \begin{bmatrix} p_s(s, x) V\left(\bar{s}, x_{-s}^{+(s+1)}\right) \\ + (1 - p_s(s, x)) V(\bar{s}, x_{-s}) \\ -V(\bar{s}, x) \end{bmatrix} \\ & + \lambda_0 \left[V(\bar{s}, x^{+1}) - V(\bar{s}, x) \right] \end{aligned} \tag{2.6}$$

Taken together, equations (2.3), (2.4), (2.5), and (2.6) form the equilibrium conditions on value function $V(s,x)$ and policy function $p(s,x)$.

2.4. Data and Estimation Approach

The dataset[11] used in this study consists of year-drug observations of development stage $\bar{s}$ along with various additional information, most important of which is classification of the drug under Anatomical-Therapeutic-Chemical system that allows to group drugs into economically meaningful markets. The dataset covers 11 years (1995-2005) and is limited to markets with active research within the broader category of Infection drugs.

The dataset does not allow to properly model the transition from pre-clinical to clinical trials. This transition involves two separate decisions – firm choosing to file the IND application, and FDA reviewing it. Review takes at most 30 days, so dataset does not even define a stage for a drug under IND review. As the result, one cannot identify drivers of firm's decision to proceed separately from FDA approval probability.

While it would be useful to have results of clinical trials, those exist only in the form of abstracts, which describe the experiments conducted, various efficiency measures and side-effect rates. There seems to be no unified approach to measuring them and no formalized data are available, so trial results are viewed as unobserved.

Observing the stage of individual drugs across the years provides an estimate μ_s's – parameters driving the duration of each stage. Similar approach gives estimate of λ_0 (the arrival rate of drugs into Phase I), since dataset starts tracking drugs from pre-clinical stage.

Further, observed changes in stage indicate the decision to continue the development of the drug (indicated by advancement to next stage) or abandon it (indicated by transit

[11]Data are provided by PharmaProjects, a service of Informa UK Ltd.

to "Discontinued" stage). For drugs transiting out of FDA stage, this directly provides an estimate of p_5 – the probability of FDA approval, which is treated as exogenous in the model. The observed transitions out of clinical trial stages become observations on continuation decision of the firm (denoted a) and serve as the basis for structural estimation of the stage costs $R = \{R_s\}_{s=2}^4$. Structural estimation also requires knowledge of industry state x, which is constructed by aggregating stage data for all drugs in given market & year.

s	$\bar{x}_s$	s.e.	$\min x_s$	$\max x_s$
1	10.86	6.18	1	22
2	14.86	7.77	3	28
3	7.86	3.67	3	15
4	3.77	3.15	0	13
5	27.87	16.01	11	58

Table 2.1. Summary statistics of industry state vector x.

The identification of R is possible since the dataset records decisions taken by firms that face a wide range of industry states x. This variability results from observing both different markets and different moments in time. As a confirmation, Table 2.1 provides summary statistics for observed industry state x.

The estimation approach is Nested Pseudo-likelihood method of Aguirregabiria & Mira (2007), which essentially combines the value iteration of Pakes & McGuire (1994) with MLE estimation. Besides the stage costs stage, the estimation process simultaneously generates a model equilibrium, as defined by $\{V(\cdot), p(\cdot)\}$ pair.

The estimation proceeds iteratively, with each iteration k starting from (a) an equilibrium "candidate" $\left\{V^k(\cdot), p^k(\cdot)\right\}$ and (b) parameter estimate $\hat{R}^k$. The iteration updates them to $(k+1)$-th iterate in two steps:

(1) Pakes & McGuire (1994) update of the equilibrium candidate, which uses the equilibrium conditions (2.3), (2.4), (2.5), and (2.6). For each state $(\bar{s}, x)$, $\left\{V^{k+1}(\bar{s}, x), p^{k+1}(\bar{s}, x)\right\}$ are computed by plugging $\hat{R}^k$ and $V^k(\cdot), p^k(\cdot)$ into the equilibrium conditions and solving for $\{V(\bar{s}, x), p(\bar{s}, x)\}$.

(2) MLE estimation of parameter vector $\hat{R}^{k+1}$. The likelihood of a single observation j is constructed by computing the predicted continuation probability $p^{k+1}(s^j + 1, x^j)$ and matching it to the observed action a^j. After transformations, log-likelihood function reduces to:

$$L(R) = \sum_{j=1}^{N} \left[a^j \delta_j - \log(1 + \exp(\delta_j))\right] \tag{2.7}$$

$$\text{where } \delta_j = \frac{V^{k+1}(s^j + 1, x^j) - R_{s^j+1}}{\sigma_{s^j}} \tag{2.8}$$

The new parameter vector $\hat{R}^{k+1}$ is then selected to maximize this log-likelihood.

The iterations continue until the changes in $\left\{V^k(\cdot), p^k(\cdot)\right\}$ and $\hat{R}^k$ become smaller than pre-determined tolerance level.

There is no theoretical result that guarantees either existence or uniqueness of equilibrium in this kind of model. Doraszelski & Satterthwaite (2007) prove existence of Markov perfect equilibrium in certain kinds of dynamic games, but their results are not applicable here. Besanko, Doraszelski, Kryukov & Satterthwaite (2007) show that a dynamic game can have (economically meaningful) multiple equilibria.

Even if equilibrium exists, the algorithm presented above is not guaranteed to converge. There is a number of technical improvements (such as dampening) that help improve convergence, and indeed all cases considered in Section 2.5 did converge.

The potential multiplicity of equilibria is a much more serious issue; every estimation method for dynamic games assumes that if multiple equilibria exist, then same equilibrium is played out in every market. In pharmaceutical industry, however, this assumption might be at least partially justified since there are several large companies that are present in every market, so they are likely to use same strategies themselves which would force smaller firms to play same equilibrium as the best response.

A few additional notes are necessary to complete the description of model implementation and estimation.

Value function $V(\cdot)$ is represented as polynomial approximation, since discretization (computing and storing V for every possible value of $(\bar{s}, x)$ vector) is unfeasible due to the size of the state space. The polynomial approximation used in this exercise is a version of 2nd-order complete polynomial representation:

$$V(s,x) = v_0^s + \sum_{r=1}^{5} \left[v_r^s x_r + v_{r,r}^s x_r^2 \right] + \sum_{r=1}^{4} v_{r,r+1}^s x_r x_{r+1}. \tag{2.9}$$

Thus, the function $V(\cdot)$ is reduced to a set of coefficients $\mathcal{V} = \left\{ v_r^s, v_{r,q}^s \right\}_{r\in\{1..5\},q\in\{r,r+1\}}^{s\in\{1..6\}}$. The Pakes & McGuire (1994) update of equilibrium in this case consists of computing V^{k+1} for a pre-determined set of states, then using OLS formula to recover the coefficients. There is no need for approximating (or even tracking) $p(\cdot)$ since (2.3) provides an easy way of computing it from $V(\cdot)$.

As follows from (2.7), it is not possible to identify R_s separately from private information variance parameter σ_s. Instead, this study follows the traditional approach of holding σ_s constant at some pre-selected level. The specific values of σ_s's are based on level of $V(s,x)$ for each stage and reflect the wide distribution of possible trial outcomes.

Finally, the dataset is compiled of markets within the same size bracket, so there is no need to specially account for it in (2.9).

Once an estimate of R is obtained, it is used to evaluate impact of various policies as described in Section 2.5 below.

2.5. Results

2.5.1. Estimates

As mentioned in Section 2.4 above, a number of model parameters are estimated directly from data, and then estimate of the remaining parameters and model equilibrium are produced using the Nested Pseudo-likelihood method.

Phase s	$\hat{\mu}_s$	Implied phase duration (yrs)
1	0.3929	2.55
2	0.2663	3.76
3	0.2865	3.5
4	0.3455	2.89

Table 2.2. Estimates of stage duration parameters μ_s.

The estimates of μ_s – the probability that drug will complete the present stage within a year are presented in Table 2.2. Average annual arrival of new drugs is estimated at $\hat{\lambda}_0 = 4.05$.

Unfortunately, the PharmaProjects database does not track the sales of the drugs, so it is impossible to tell when any drug leaves the market. Instead, an assumed value of $\lambda_5 = 0.95$ is used, setting the average lifespan of a given drug equal to patent life of 20 years. In addition, a correction must be applied to the data on x_5, since in raw form they represent the number of all drugs released on the market since the records began. Since PharmaProjects started

tracking drugs in late 1980's, the average age of the drugs in the sample is about 10 years, so x_5 is multiplied by $(\lambda_5)^{10}$.

The probability of FDA granting a permission to enter the market is again estimated directly from data as $\hat{p}_5 = 0.7894$.

The profit margin is set as $\pi = 0.3$, according to an industry publication; this margin excludes R&D costs, since they are already accounted for in the model. All data come from markets placed into \$2-5 billion annual revenue bracket, so market size is set to average value: $M = 3500$ (in millions of dollars).

The discount rate ρ is set to 0.05, corresponding to long term real return on investment of 5%.

s	$\hat{R}_s$	$V(s, \mathbf{E}x)$	σ_s
1	–	22.18	–
2	13.18	30.27	10
3	100.15	112.33	25
4	383.36	458.27	100
5	–	635.37	–

Table 2.3. Estimated stage cost and stage value function (at average industry state) and assumed standard deviations of trials results.

The Nested Pseudo-likelihood estimate of the model successfully converged, providing both the estimates of stage costs R_s and equilibrium value function $V(\cdot)$. Table 2.3 reports the estimates of stage costs and value function for average industry state[12], along with values of $\{\sigma_s\}$ that were selected to match the scale of V and R.

Stage costs are increasing from one stage to another, which is consistent with each consecutive stage of development being longer and involving more subjects than the previous

[12]As described in 2.4, $V(\cdot)$ is represented by a set of polynomial coefficients, and author feels that reporting them would be less informative.

one; DiMasi et al. (2003) report similar relationship. Further, value of a drug in each subsequent stage is higher, which is reasonable as drugs more likely to reach the market. No R_s or σ_s are reported for $s = 1, 5$ since decision to enter Phase I ($s = 1$) is not included in the model due to lack of data, and the transition to the market ($s = 5$) is controlled by FDA.

A somewhat surprising finding is the relatively large cost of submitting a drug for FDA approval, since the application fee is substantially smaller than the estimate of $383 million. However, the estimate reflects an economic cost to the firm, which includes considerable effort involved in preparing the necessary application documents, and then following up with FDA officials, providing clarification on any unclear points, checking on the progress of the application, etc.

s	Observed $\bar{x}_s$	Predicted $\mathbf{E}x_s$
1	10.86	10.33
2	14.86	12.83
3	7.86	7.51
4	3.77	4.19
5	27.87	23.21

Table 2.4. Comparison of observed and predicted industry states.

To assess "goodness of fit", Table 2.4 compares the average observed industry state with the one predicted by the model. One cannot expect exact fit because model's prediction is a expectation over limiting (or ergodic) distribution[13], while observed data represent a relatively short period in life industry which cannot be reasonably expected to represent the limiting distribution.

[13]See 2.5.2 below for details of this computation.

2.5.2. Policy Experiments

Having estimated the model parameters, this study proceeds to conduct policy experiments by mapping each policy into a change in corresponding parameter, and computing equilibrium for that new parametrization.

To make the policy interventions comparable, the experiment setup is using "equal cost" principle. The quotation marks are there since the intervening agent is doing a simplified cost computation, assuming that the number of drugs flowing through the development pipeline will remain unchanged as the result of this policy. In reality, policy intervention will (or at least should) change number of drugs flowing through the system, but correcting for this requires multiple trial-and-error experiments with the model, and this study assumes that intervening agent is not sufficiently sophisticated to do that.

The specific experiments are:

- Subsidy offsetting part of stage cost R_s. The cost of this policy is naturally computed as the product of stage cost and average number of drugs that enter this stage each year. The cost of all experiments is set to match 10% decrease in cost of Phase III (R_3); corresponding decrease for Phase II (R_2) is 48%, for FDA filing (R_4) – 4%. As mentioned above, major part of R_4 is likely to be economic cost rather than actual payments by the firm, so subsidy might be harder to justify, and instead can be viewed as streamlining and facilitating the application process to reduce the economic costs of the firms.
- Subsidy towards the sales of the drug, making it accessible to those who previously could not afford it. This expands the market and hence increases profits of every

drug on it. The cost of this policy is computed as the product of subsidy percentage and total market revenue (M). The amount matching the other subsidies is 2%.

There is an additional issue related to sales subsidy. On average, Pharmaceutical firms spend 30% of their revenue on promotion. It is not reasonable to assume that this expense should apply to sales generated through subsidy, so profit margin on sales-related drugs is increased to reflect that.

In addition to subsidies, this study attempts to study effectiveness of several other policies, cost of which cannot be reliably predicted or matched to the cost of subsidy policies. Instead, this group of policy experiments aims to be internally consistent by using similar changes of model parameters:

- Shorter duration of development stages. This can be achieved through various institutional changes, such as greater sharing of information about clinical trials or streamlining the FDA review procedures. To achieve a visible effect, the assumption is that duration of each stage is reduced by 50%.
- Increased discovery research, which would result in additional drugs entering Phase I. While there are studies on productivity of research investment[14], the inability to model the transition into clinical trials prevents the proper cost computation for this policy. Instead, the experiment assumes 50% increase in number drugs entering Phase I.

The policy evaluation criteria is number of drugs entering the market per year; the flow of drugs through various stages of development is also reported to aid the understanding of mechanism by which policy works (or doesn't). This "flow" measure is preferable to "stock" of drugs on the market, since the latter is directly affected by an assumption on duration of

[14]See, for example, Henderson & Cockburn (1996).

profitable life of the drug (μ_5), which is somewhat arbitrary. Besides, the "flow" measure better matches the stated goal of increased introduction rate of new drugs.

Given a set of parameter values, an equilibrium is computed by iterating on Pakes-McGuire step described in Section 2.4. The estimated "baseline" equilibrium provides a convenient starting point for this computation, since best-reply dynamics of Pakes-McGuire update simulate the adjustment of an industry to a new environment. Convergence of this algorithm is again not guaranteed, but use of dampening helped ensure successful computation of all counterfactual equilibria that the experiments require.

The flow of drugs is computed as an expectation over a limiting distribution, i.e. the stable distribution of industry states that industry achieves after developing for a long period of time. While linear algebra offers a direct way to compute this distribution, it is not practical here due to large size of state space. Instead, this study simulates the development of the industry for a period of time, and takes an average over observed transitions (excluding first 20% of period, which eliminates effects of initial state). The duration of this simulation period is selected to be large enough to ensure that simulation error is well below the effect of the policy.

Table 2.5 lists the results of the experiments. Top row reports the average flow of drugs through stages of development pipeline for the baseline estimate, and the following rows report change to these numbers under policy experiments; the simulation standard errors are reported in brackets.

An expected observation on this table is that all policies have either positive (or insignificant) effect on the number of drugs entering the market.

More interesting, comparing the set of policies that affect costs or durations of several consecutive development stages, one finds that policies affecting later stages are more effective

Policy \ Stage	Ph. I	Ph. II	Ph. III	FDA	**Market**
Baseline estimate	4.090 (0.027)	3.397 (0.026)	2.124 (0.021)	1.454 (0.017)	1.157 (0.014)
Subsidy - FDA review	+.003 (.028)	+.121 (.027)	+.154 (.022)	+.**111** (.018)	+.093 (.015)
Subsidy - Phase III	−.008 (.028)	+.129 (.027)	+.**210** (.022)	+.087 (.018)	+.070 (.015)
Subsidy - Phase II	−.011 (.028)	−.**234** (.027)	+.070 (.021)	+.035 (.017)	+.021 (.015)
Subsidy - Sales	−.007 (.028)	+.154 (.026)	+.190 (.022)	+.136 (0.017)	+.**111** (.015)
Faster FDA review	−.013 (.028)	+.224 (.027)	+.295 (.022)	+.**214** (.019)	+.170 (.016)
Faster Phase III	+.002 (.028)	+.126 (.027)	+.**112** (.022)	+.060 (.017)	+.046 (.015)
Faster Phase II	−.017 (.028)	−.**047** (.025)	−.010 (.021)	−.002 (.018)	+.001 (.014)
Increased discovery	+**2.036** (0.040)	+1.195 (.035)	+.343 (.027)	+.129 (.021)	+.107 (.018)

Table 2.5. Results of policy experiments: Flow of drugs through development pipeline.

than those affecting earlier stages. There is just exception – Increased discovery. Despite affecting the earliest phase of trials, it shows higher effect than reduced duration of Phases II or III of clinical trials. However, as outlined above, the two sets of policies are not necessarily comparable.

Returning to the overall trend of interventions into later stages being more effective, it is worth investigating the reasons behind it. Note that the increase in the number of drugs flowing through the pipeline is usually the highest at the stage affected by the policy (indicated by bold font). The effect is smaller for earlier stages, and seems to be declining throughout the subsequent stages.

Under a limiting distribution, average number of drugs at a given stage remains constant, so average number of drugs entering and leaving the stage is the same[15], so any change in the flow of drugs from stage to stage is driven by the probability of advancement to the next

[15]Simulation confirms this theoretical conclusion.

stage $p(s,x)$. To look into these, consider the definition of this probability in (2.3). The only endogenous component of the expression is $V\left(\bar{s}+1, x_{-s}^{+(s+1)}\right)$ – the value that firm collects if it enters the stage.

Policy \ Stage	Ph. I	Ph. II	Ph. III	FDA	**Market**
Baseline	22.18	30.27	112.33	458.27	635.37
Subsidy - FDA review	+2.39	+2.60	+2.30	**−13.89**	−20.44
Subsidy - Phase III	+2.57	+3.01	**−5.68**	−11.15	−16.38
Subsidy - Phase II	+3.82	**−1.09**	−2.03	−3.95	−5.81
Subsidy - Sales	+3.77	+3.88	+3.02	+0.72	**−.51**
Faster FDA review	+5.03	+5.55	+5.23	**+2.08**	−25.54
Faster Phase III	+3.59	+3.07	**+1.02**	−4.81	−7.97
Faster Phase II	−0.28	**−0.73**	+0.41	+0.12	−0.19
Increased discovery	**−4.95**	−5.65	−9.66	−18.03	−26.24

Table 2.6. Changes to the Value function as the result of policy experiments.

Table 2.6 presents these values for each of the experiments, evaluated at average x under limiting distribution. As before, phase affected by the policy intervention is indicated by bold font. With the exception of Faster Phase II, it all experiments reveal a common pattern: Values for stages prior to the one affected by policy are increased (hence increasing chances of entering the phase; values for stages subsequent to intervention have the value reduced, hence reducing chance of any given drug continuing into that phase.

The reduction for stages past the innovation are easy to explain – Table 2.5 shows that the flow of drugs has increased, thus necessarily increasing the average number of drugs on the market and reducing the value of a drug at any given stage (since the market size remains unchanged in all but one experiment). Same effect applies to the stages prior to policy intervention, but it is offset by the policy benefit that the firms expect to collect at a future stage, leading to an overall positive change in value.

This explains why the flow of drugs through the pipeline increases up to the point of intervention, but shrinks with every stage past the intervention. So interventions into earlier stages of development process mean that shrinking occurs over a larger number of stages, resulting in a lower effect compared to policies affecting later stages.

It is important to note that the negative effect described above is a strategic phenomenon, since it represents response of the firm to policy-induced actions of the competing firms. An attempt to evaluate effectiveness of policy intervention outside of the framework of dynamic games, e.g. using a single-agent model would have missed this effect and overstated the impact of any policy intervention.

2.6. Conclusion

This study has constructed a dynamic model of the drug development process, structurally estimated it and evaluated the efficiency of various policy interventions by conducting counterfactual experiments. The major conclusion is that policies applied to earlier stages of development have a smaller effect on output of new drugs. The reason for this phenomenon is that a policy-driven increase in the number of drugs necessarily decreases the value of each drug, and while policy intervention ensures firms are more willing to invest into development when they expect to benefit from policy at a future stage, they are less likely to continue development into stages subsequent to policy intervention.

I am currently working on simulating the industry development over a short interval of time, studying how long it takes for policy intervention to take its effect. The simulations above concentrate on long-term average performance of the industry, and ignore the transition from the baseline industry structure. Short-term simulations will allow for a more

balanced evaluation of policies, based not just on their long-term effects, but how fast they will arrive.

Another on-going project is modifying the model to describe decision making by a multi-drug firm. Guedj & Scharfstein (2004) found that large firms with "portfolios" of drugs at every development stage behave differently from firms who have their business staked on a single drug. Proper treatment of this feature would change the model substantially, adding the firm's own drugs to the state space, altering the state transition process to keep track of individual firms with their portfolios, and possibly abandoning anonymity assumption, since firms might consider not just the number of competing drugs, but the identity of the firms that own them. On the positive side, this model will allow me to test whether the current coexistence of large and small firms is sustainable, or the industry leans towards concentration of the market in the hands of the few large firms.

A further expansion of this study would be incorporation of clinical trial outcomes into the estimate. This is a substantial undertaking, requiring a collection of clinical trial results and a method of comparing them across different markets (e.g. based on percentiles). In return, this approach will help to distinguish between medical and economic components of drug development decisions.

CHAPTER 3

A User's Guide to Solving Dynamic Stochastic Games Using the Homotopy Method (joint with Ulrich Doraszelski and Ron Borkovsky)

3.1. Introduction

There has been much interest in game-theoretic models of industry evolution and, in particular, in the framework introduced by Ericson & Pakes (1995) that is at the heart of a large and growing literature in industrial organization and other fields (see Doraszelski & Pakes (2007) and the references therein). Ericson & Pakes (1995) provide a model of dynamic competition in an oligopolistic industry with investment, entry, and exit. Their framework is designed to facilitate numerical analysis of a wide variety of phenomena that are too complex to be explored in analytically tractable models. Methods for computing Markov-perfect equilibria are therefore a key part of this stream of research. This paper contributes by providing a step-by-step guide to solving dynamic stochastic games using the homotopy method.

A particularly important concern in the literature following Ericson & Pakes (1995) is multiplicity of equilibria. The potential for multiplicity in their framework is widely recognized; see p. 570 of Pakes & McGuire (1994) and, more recently, the examples of multiple equilibria in Doraszelski & Satterthwaite (2007) and Besanko et al. (2007).

Multiple equilibria raise at least two issues. First, the existing structural estimation methods for dynamic oligopoly models such as Aguirregabiria & Mira (2007), Bajari et al. (2007), Pakes, Ostrovsky & Berry (2006), and Pesendorfer & Schmidt-Dengler (2003) rely on the assumption that the same equilibrium is being played in all geographic markets and/or time periods. Hence, multiple equilibria cast doubt on the estimation results unless one can convincingly argue that the same equilibrium is indeed being played in all geographic markets and/or time periods. Of course, this is trivially true if the equilibrium is unique. It would therefore be useful to be able to explore the set of equilibria of dynamic oligopoly

models to determine whether multiplicity exists in the empirically-relevant subset of the parameter space. The homotopy method offers a way of doing this.

Second, multiple equilibria limit what we can learn from conducting policy experiments. Strictly speaking, the most one can conclude from a policy experiment in the presence of multiple equilibria is that, were the change in policy to occur, one of several equilibria would be played, but which one is not known. However, if we are able to more fully characterize the set of equilibria, then it becomes possible to bound the range of outcomes that may be produced by a change in policy. The homotopy method is again a useful tool for this purpose.

Computing a Markov-perfect equilibrium of a dynamic stochastic game amounts to solving a large system of equations. To date the Pakes & McGuire (1994) algorithm has been used most often to compute equilibria of dynamic oligopoly models. This backward solution method falls into the broader class of Gaussian methods. The idea behind Gaussian methods is that it is harder to solve a large system of equations once than to solve smaller systems many times and that it may therefore be advantageous to break up a large system into small pieces. The drawback of Gaussian methods is that they offer no systematic approach to computing multiple equilibria. To identify more than one equilibrium (for a given parameterization of the model), the Pakes & McGuire (1994) algorithm must be restarted from different initial guesses. But different initial guesses may or may not lead to different equilibria. A similar remark applies to the stochastic approximation algorithm of Pakes & McGuire (2001), the other widely used method for computing equilibria.

This, however, still understates the severity of the problem. When there are multiple equilibria, the trial-and-error approach of restarting the Pakes & McGuire (1994) algorithm from different initial guesses is sure to miss a substantial fraction of them, regardless of how

many initial guesses are tried. That is, as shown in Chapter 1, if a dynamic stochastic game has multiple equilibria, then some of them cannot possibly be computed by the Pakes & McGuire (1994) algorithm. It is therefore important to consider alternative algorithms that can identify multiple equilibria and thus provide us with a more complete picture of the set of solutions to a dynamic stochastic game.

The homotopy method allows us to explore the equilibrium correspondence in a systematic fashion. The homotopy method is a type of path-following method. Starting from a single equilibrium that has already been computed for a given parameterization of the model, the homotopy algorithm traces out an entire path of equilibria by varying one or more selected parameters of the model. Whenever we can find such a path and multiple equilibria are the result of the path folding back on itself, then the homotopy method is guaranteed to identify them. We note at the outset that it is not assured that any given path computes all possible equilibria at a given value of the parameter vector.

In this paper we discuss the theory of the homotopy method as well as HOMPACK90, a suite of Fortran90 routines developed by Watson et al. (1997) that implements this method. We also discuss potential problems that one may encounter in using HOMPACK90 to solve dynamic stochastic games and offer some guidance as to how to resolve them.

We then present two examples of dynamic stochastic games and show, step by step, how to solve them using the homotopy method. In order to use the homotopy method, one must formulate a problem as a system of equations. Our first example, the learning-by-doing model of Chapter 1, is particularly well-suited for the homotopy method because it is straightforward to express the equilibrium conditions as a system of equations. We discuss in detail how this is done. Moreover, we illustrate the computational demands of the homotopy method using the learning-by-doing model as an example.

Our second example, the quality ladder model of Pakes & McGuire (1994), presents a complication. As investment cannot be negative, the problem that a firm has to solve is formulated using a complementary slackness condition, a combination of equalities and inequalities. We show how to reformulate this complementary slackness condition as a system of equations that is amenable to the homotopy method. In fact, we offer several such reformulations that may be useful if complications arise.

In sum, this paper provides a step-by-step guide to solving dynamic stochastic games using the homotopy method. Our goal here is not to provide a comprehensive treatment of the theory of the homotopy method (see Zangwill & Garcia (1981) for an excellent introduction to the homotopy method) or the possibilities for implementing this method on a computer (see Allgower & Georg (1992) among others); rather, it is to enable to reader to start using HOMPACK90 as quickly as possible. To this end, we also make the codes for the learning-by-doing and quality ladder models available on our homepages. We include additional detailed instructions on how to set up and use these codes with the codes themselves.

Most of this paper is devoted to explaining how to use the homotopy method to explore the equilibrium correspondence of a dynamic stochastic game in a systematic fashion. To this end, we use the homotopy algorithm to trace out an entire path of solutions to a system of equations by varying a parameter of interest. This type of application is refereed to as a *natural-parameter homotopy*. The homotopy method has other applications. A so-called *artificial homotopy* can be used to obtain a solution for a particular parameterization of a system of equations; it aims to compute just one equilibrium. An *all-solutions homotopy* can

sometimes be used to obtain all solutions to systems of equations with certain properties; it aims to compute all equilibria. We briefly discuss these applications at the end of the paper.[1]

3.2. The Theory of the Homotopy Method

A Markov-perfect equilibrium of a dynamic stochastic game consists of values, i.e., expected net present values of per-period payoffs, and policies, i.e., strategies, for each player in each state. Values are typically characterized by Bellman equations and policies by optimality conditions (e.g., first-order conditions). Collecting Bellman equations and optimality conditions for each player in each state, the equilibrium conditions amount to a system of equations of the form

$$\mathbf{H}(\mathbf{x}) = \mathbf{0},$$

where $\mathbf{x}$ is the vector of the unknown values and policies and $\mathbf{0}$ is a vector of zeros, and we use boldface to distinguish between vector and scalars. Hence, computing an equilibrium of a dynamic stochastic game amounts to solving a system of typically highly nonlinear equations.

Various methods are available for solving a system of nonlinear equations (see, e.g., Chapter 5 of Judd 1998). Gaussian methods such as the Pakes & McGuire (1994) algorithm most often used to solve for equilibria of dynamic stochastic games are not guaranteed to converge. Moreover, they offer no systematic approach to computing multiple equilibria and, when multiple equilibria exist, they are unable to compute some of them (Besanko et al. 2007). Unlike the Pakes & McGuire (1994) algorithm, some nonlinear solvers – notably

[1]The homotopy method has also been applied in other contexts, including general equilibrium models with incomplete asset markets (Schmedders 1998, Schmedders 1999). See also Eaves & Schmedders (1999) for a summary of other applications to general equilibrium models, Berry & Pakes (2007) for an application to estimating demand systems, and Bajari, Hong, Krainer & Nekipelov (2006) for an application to estimating static games of incomplete information.

Newton's method – are guaranteed to converge provided that the system of equations satisfies certain conditions and the initial guess that the user provides to the algorithm as a starting point is close to the final solution. However, like the Pakes & McGuire (1994) algorithm, these algorithms are limited in their ability to compute multiple equilibria because, to find a particular equilibrium, an initial guess must be supplied that is close (perhaps very close) to it.

The homotopy method allows us to explore the equilibrium correspondence in a systematic fashion. It is therefore especially useful in models that have multiple equilibria. Starting from a single equilibrium that has already been computed for a given parameterization of the model, the homotopy method traces out an entire path of equilibria by varying a parameter of interest. Recall that the equilibrium conditions depend on the parameterization of the model. Making this dependence explicit, the above system of equations becomes

$$\mathbf{H}(\mathbf{x}, \lambda) = \mathbf{0}, \tag{3.1}$$

where $\mathbf{H} : \mathbb{R}^{N+1} \to \mathbb{R}^N$, $\mathbf{x} \in \mathbb{R}^N$ is the vector of unknown values and policies, and $\mathbf{0} \in \mathbb{R}^N$ is a vector of zeros. $\lambda \in [0, 1]$ is the so-called homotopy parameter. Depending on the application at hand, the homotopy parameter maps into one or more of the parameters of the model. The object of interest is the equilibrium correspondence

$$\mathbf{H}^{-1} = \{(\mathbf{x}, \lambda) | \mathbf{H}(\mathbf{x}, \lambda) = \mathbf{0}\}.$$

The homotopy method aims to trace out entire paths of equilibria in $\mathbf{H}^{-1}$.

Example. An example is helpful to explain how the homotopy method works. Let

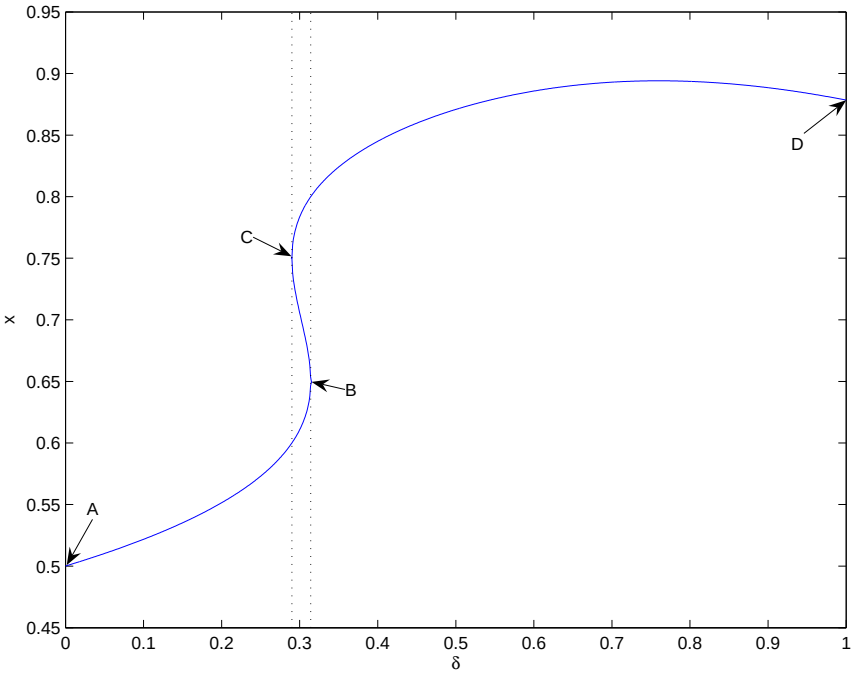

Figure 3.1. Example.

$N = 1$ and consider the equation $H(x, \lambda) = 0$, where

$$H(x, \lambda) = -15.289 - \frac{\lambda}{1 + \lambda^4} + 67.500x - 96.923x^2 + 46.154x^3.$$

Here we do not use boldface for x and 0 since they are scalars. This equation implicitly relates a variable x with a parameter λ. The set of solutions $H^{-1} = \{(x, \lambda)|H(x, \lambda) = 0\}$ is graphed in Figure 3.1. There evidently are multiple solutions to $H(x, \lambda) = 0$, e.g., $x = 0.610$, $x = 0.707$, and $x = 0.783$ at $\lambda = 0.3$. Finding these solutions is trivial with the graph in hand, but the graph is less than straightforward to draw even in this very simple case. Whether one solves $H(x, \lambda) = 0$ for x taking λ as given or for λ taking x as given, the result is a multi-valued correspondence, not a single-valued function.

To apply the homotopy method, we introduce an auxiliary variable s that indexes each point on the graph starting at point A for $s = 0$ and ending at point D for $s = \bar{s}$. The graph is then just the parametric path given by a pair of functions $(x(s), \lambda(s))$ satisfying $H(x(s), \lambda(s)) = 0$ or, equivalently, $(x(s), \lambda(s)) \in H^{-1}$. While there are infinitely many such pairs, there is a simple way to characterize a member of this family. Differentiate $H(x(s), \lambda(s)) = 0$ with respect to s to obtain

$$\frac{\partial H(x(s), \lambda(s))}{\partial x} x'(s) + \frac{\partial H(x(s), \lambda(s))}{\partial \lambda} \lambda'(s) = 0. \tag{3.2}$$

This differential equation in two unknowns $x'(s)$ and $\lambda'(s)$ captures the condition that is required to remain "on path." One possible approach for tracing out a path in H^{-1} is thus to solve equation (3.2) for the ratio $\frac{x'(s)}{\lambda'(s)} = -\frac{\partial H(x(s),\lambda(s))/\partial \lambda}{\partial H(x(s),\lambda(s))/\partial x}$ that indicates the direction of the next step along the path from s to $s + ds$. This approach, however, creates difficulties because the ratio may switch from $+\infty$ to $-\infty$, e.g., at point B in Figure 3.1. So instead of solving for the ratio, we simply solve for each term of the ratio. This insight implies that the graph of H^{-1} in Figure 3.1 is the solution to the system of differential equations

$$x'(s) = \frac{\partial H(x(s), \lambda(s))}{\partial \lambda}, \tag{3.3}$$

$$\lambda'(s) = -\frac{\partial H(x(s), \lambda(s))}{\partial x}. \tag{3.4}$$

Equations (3.3) and (3.4) are the so-called basic differential equations for our example. In our example, note that if $\lambda = 0$, then $H(x, \lambda) = 0$ is easily solved for $x = 0.5$. This provides the initial condition (point A in Figure 3.1). From there the homotopy method uses the basic differential equations to determine the next step along the path. It continues to follow the path – step-by-step – until it reaches $\lambda = 1$ (point D). Whenever $\lambda'(s)$ switches sign

from negative to positive (point B), the path is bending backward and there are multiple solutions. Conversely, whenever the sign of $\lambda'(s)$ switches back from positive to negative (point C), the path is bending forward.[2]

Returning to the general case with $N > 1$, our goal is to explore the equilibrium correspondence $\mathbf{H}^{-1} = \{(\mathbf{x}, \lambda) | \mathbf{H}(\mathbf{x}, \lambda) = \mathbf{0}\}$ that depends on the homotopy parameter λ. Proceeding as in our example, a parametric path is a set of functions $(\mathbf{x}(s), \lambda(s)) \in \mathbf{H}^{-1}$. Differentiating $\mathbf{H}(\mathbf{x}(s), \lambda(s)) = \mathbf{0}$ with respect to s yields the conditions that are required to remain on path

$$\frac{\partial \mathbf{H}(\mathbf{x}(s), \lambda(s))}{\partial \mathbf{x}} \mathbf{x}'(s) + \frac{\partial \mathbf{H}(\mathbf{x}(s), \lambda(s))}{\partial \lambda} \lambda'(s) = \mathbf{0}, \tag{3.5}$$

where $\frac{\partial \mathbf{H}(\mathbf{x}(s), \lambda(s))}{\partial \mathbf{x}}$ is the $(N \times N)$ Jacobian of $\mathbf{H}$ with respect to $\mathbf{x}$, $\mathbf{x}'(s)$ and $\frac{\partial \mathbf{H}(\mathbf{x}(s), \lambda(s))}{\partial \lambda}$ are $(N \times 1)$ vectors, and $\lambda'(s)$ is a scalar. This system of N differential equations in $N + 1$ unknowns, $x_i'(s)$, $i = 1, \ldots, N$, and $\lambda'(s)$, has a solution that obeys the basic differential equations

$$y_i'(s) = (-1)^{i+1} \det \left(\left[\frac{\partial \mathbf{H}(\mathbf{y}(s))}{\partial \mathbf{y}} \right]_{-i} \right), \quad i = 1, \ldots, N + 1, \tag{3.6}$$

where $\mathbf{y}(s) = (\mathbf{x}(s), \lambda(s))$, and the notation $[\cdot]_{-i}$ is used to indicate that the ith column is removed from the $(N \times (N+1))$ Jacobian $\frac{\partial \mathbf{H}(\mathbf{y}(s))}{\partial \mathbf{y}}$ of $\mathbf{H}$ with respect to $\mathbf{y}$. Note that equation (3.6) reduces to equations (3.3) and (3.4) if $\mathbf{x}$ is a scalar instead of a vector. For the general case, a proof that the basic differential equations (3.6) satisfy the conditions in equation

[2]The orientation of the path taken by the homotopy method is arbitrary. Reversing the signs of the basic differential equations implies, perhaps more intuitively, that $\lambda'(s)$ switches sign from positive to negative at point B.

(3.5) that are required to remain on path can be found in Garcia & Zangwill (1979) and on pp. 27–28 of Zangwill & Garcia (1981).

Regularity and smoothness requirements. A closer inspection of the basic dif-

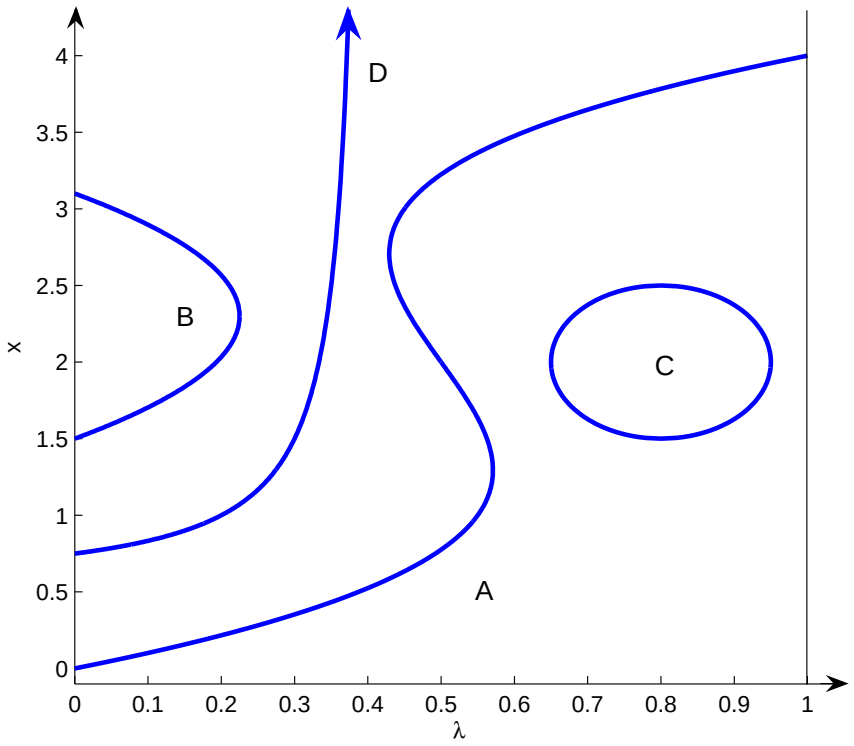

Figure 3.2. Examples of solution paths if $\mathbf{H}$ is regular.

ferential equations (3.6) reveals a potential difficulty. If the Jacobian $\frac{\partial \mathbf{H}(\mathbf{y}(s))}{\partial \mathbf{y}}$ is not of full rank at some point $\mathbf{y}(s)$ on the solution path, then the determinant of each of its square submatrices is zero. Thus, according to the basic differential equations (3.6), $y_i'(s) = 0$, $i = 1, \ldots, N+1$, and the homotopy method is stuck at point $\mathbf{y}(s)$. A central condition in the mathematical literature on the homotopy method is thus that the Jacobian must have full rank at all points on the solution path. If so, the homotopy is called regular. More formally, $\mathbf{H}$ is regular if rank $\left(\frac{\partial \mathbf{H}(\mathbf{y})}{\partial \mathbf{y}}\right) = N$ for all $\mathbf{y} \in \mathbf{H}^{-1}$. The regularity requirement – and a certain smoothness requirement to be discussed below – ensures that the set of solutions

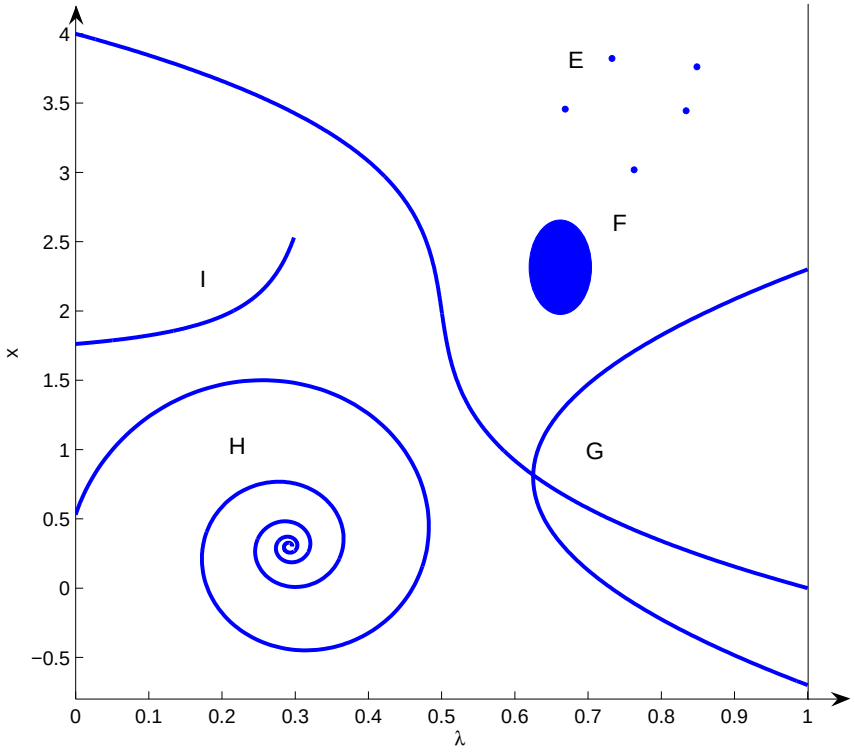

Figure 3.3. Examples of solution paths if **H** is not regular.

$\mathbf{H}^{-1}$ consists only of continuous paths. Figure 3.2 shows examples of possible solution paths if **H** is regular: (A) paths that start at $\lambda = 0$ and end at $\lambda = 1$; (B) paths that start and end at $\lambda = 0$ or $\lambda = 1$; (C) loops; and (D) paths that start at $\lambda = 0$ or $\lambda = 1$ but never end because x (or a component of $\mathbf{x}$ in the case of a vector) tends to $+\infty$ or $-\infty$.[3] Figure 3.3 shows examples of solution paths that are ruled out by the regularity requirement: (E) isolated equilibria; (F) continua of equilibria; (G) branching point;[4] (H) paths of infinite length that start at $\lambda = 0$ or $\lambda = 1$ and converge to single points (spirals); and (I) paths that start at $\lambda = 0$ or $\lambda = 1$ but suddenly terminate.

[3]The mathematical literature on the homotopy method rules out paths like (D) by imposing a boundary freeness requirement (see, e.g., Chapter 3 of Zangwill & Garcia 1981).

[4]More formally, (G) is a so-called pitchfork bifurcation. The regularity requirement also rules out transcritical (X-shaped) bifurcations but is consistent with other types of bifurcations (saddle-node and double saddle-node). See Golubitsky & Schaeffer (1985) for an introduction to bifurcation theory.

In practice, it is often hard to establish regularity because the Jacobian of a system of equations that characterizes the equilibria of a dynamic stochastic game formulated in the Ericson & Pakes (1995) framework tends to be intractable. This stems partly from the fact that the Jacobian for such a system is typically quite large because the system includes at least two equations (Bellman equation and optimality condition) for each state of the industry, and even "small" models with few firms and few states per firm tend to have hundreds of industry states.

The other major requirement of the homotopy method is smoothness in the sense of differentiability. This yields solution paths that are smooth and free of sudden turns or kinks. Formally, if $\mathbf{H}$ is continuously differentiable in addition to regular, then the set of solutions $\mathbf{H}^{-1}$ consists only of continuously differentiable paths. This result is known as the path theorem and essentially follows from the implicit function theorem (see, e.g., p. 20 of Zangwill & Garcia 1981). Moreover, for a path to be described by the basic differential equations (3.6) it must be the case that $\mathbf{H}$ is twice continuously differentiable in addition to regular. This result is known as the BDE theorem (see, e.g., pp. 27–28 of Zangwill & Garcia 1981).

The smoothness requirement is non-trivial and easily violated, for example, by non-negativity constraints on components of $\mathbf{x}$, say because investment cannot become negative, or by distributions with non-differentiable cumulative distribution functions such as the uniform distribution that is often used to model random scrap values and setup costs. Section 3.5 explains how to deal with such complications.

To understand why smoothness is necessary, consider an analogy: Imagine that the solution path is lined with railroad tracks and that the homotopy algorithm follows these

tracks just as a train would. Like a train the homotopy algorithm can follow a curve in the tracks, perhaps at a reduced speed, but the train derails if the tracks take a sudden turn.

There is a subtle difference between the homotopy method, a mathematical theory, and the homotopy algorithm, a computational method. In theory, the homotopy method is used to describe solution paths. In practice, a homotopy algorithm takes discrete steps along such a path. This can be beneficial because the homotopy algorithm may succeed in tracing out a solution path even if the regularity and/or smoothness requirements are violated; as the homotopy algorithm proceeds along the solution path in discrete steps, it may skip over points at which one or both of these requirements are violated. However, this also can lead to a complication. As the homotopy algorithm proceeds in discrete steps, it may jump from one solution path to another, thus failing to trace out either path in its entirety. These issues are discussed further in Section 3.5.5.

3.3. The HOMPACK90 Software Package

HOMPACK90 is as a suite of Fortran90 routines that traces out a path in

$$\mathbf{H}^{-1} = \{\mathbf{y} | \mathbf{H}(\mathbf{y}) = 0\}\,.$$

The notation $\mathbf{y} = (\mathbf{x}, \lambda) \in \mathbb{R}^{N+1}$ underlines that the homotopy method does not make a distinction between the unknown variables $\mathbf{x} \in \mathbb{R}^N$ and the homotopy parameter $\lambda \in [0, 1]$. The detailed description of HOMPACK90 is given in Watson et al. (1987) and Watson et al. (1997). Here we just give a brief overview that is meant to enable the reader to start using HOMPACK90 as quickly as possible.

In order to use HOMPACK90, the user must provide Fortran90 code for the system of equations and its Jacobian. In addition, the user must supply HOMPACK90 with an initial

	dense Jacobian	sparse Jacobian
ODE based	FIXPDF	FIXPDS
normal flow	FIXPNF	FIXPNS
augmented Jacobian	FIXPQF	FIXPQS

Table 3.1. Path-following algorithms and dense vs. sparse Jacobian in HOMPACK90.

condition in the form of a solution to the system of equations for a particular parameterization. HOMPACK90 then traces out a solution path. HOMPACK90 offers several different path-following algorithms as well as storage formats for the Jacobian of the system of equations. Table 3.1 gives an overview. Below we proceed to discuss the differences between the various path-following algorithms and storage formats as well as ways to generate initial conditions. In Section 3.4.3 we then compare the implications of the various path-following algorithms and storage formats for the performance of HOMPACK90.

The output of HOMPACK90 includes a sequence of solutions to the system of equations, saved to binary files[5], and an exit flag that indicates a normal ending or several kinds of failure. We discuss some of the potential problems in Section 3.5.5.

3.3.1. Path-Following Algorithms

HOMPACK90 traces out a parametric path $\mathbf{y}(s) \in \mathbf{H}^{-1}$ as a sequence of points, indexed by k. The kth point in the sequence is $\left\{s^k, \mathbf{y}^k\right\}$, where $\mathbf{y}^k$ is understood to represent $\mathbf{y}(s^k)$. The step along the path from one point to the next starts by choosing $\Delta s = s^{k+1} - s^k$. HOMPACK90 adjusts this step length based on the curvature of the path. Then HOMPACK90 computes the next point $\mathbf{y}^{k+1}$ using a two-phase method. The predictor phase generates a guess at the solution $\mathbf{y}^{k+1}$; the corrector phase then improves that guess using a version

[5]This functionality was added by us and is not a part of the original HOMPACK90.

of Newton's method. The difference between algorithms lies in the implementation of the predictor and corrector phases.

ODE based. The predictor phase of the ordinary differential equation (ODE) based path-following algorithm is a direct application of the system of differential equations (3.5). It first solves the system of linear equations

$$\frac{\partial \mathbf{H}\left(\mathbf{y}^k\right)}{\partial \mathbf{y}} \Delta \mathbf{y} = 0 \tag{3.7}$$

to obtain $\Delta \mathbf{y}$ and then computes the guess as $\mathbf{y}^{k+1} = \mathbf{y}^k + \Delta \mathbf{y} \Delta s$. As the predictor step tends to be very precise, the algorithm typically goes through several such steps before a corrector step becomes necessary.

Normal flow. The predictor phase uses a Hermite cubic extrapolation from the previous two points. While the Hermite cubic extrapolation is much easier to compute than solving the system of linear equations (3.7), it is also much cruder. The corrector phase is thus necessary at every step.

Augmented Jacobian. This path-following algorithm is similar to the normal flow algorithm except that it takes a more sophisticated approach to the corrector phase.

3.3.2. Jacobian

HOMPACK90 requires the user to provide a routine that returns the Jacobian $\frac{\partial \mathbf{H}(\mathbf{y})}{\partial \mathbf{y}}$ at a given point $\mathbf{y}$. In many applications, including dynamic stochastic games, the number of equations and unknowns is large but any given equation involves only a small number of unknowns (because the transitions from one state to the next are typically restricted to a small set of "nearby" states), leading to a Jacobian with most elements being zero. Such

a Jacobian is called sparse and can be more efficiently represented using a sparse matrix storage format that consists of a list of the non-zero elements with corresponding row and column indices rather than as a dense matrix that consists of the entire set of $N(N+1)$ elements. Due to the key role of the Jacobian in the homotopy algorithm, we next discuss some details on ways to compute and represent it.

Numerical vs. analytical Jacobian. The easiest way to compute the Jacobian is to do so numerically using a one- or two-sided finite-difference scheme (see, e.g., Chapter 7 of Judd 1998). However, we found that, due to the limited precision of numerical differentiation, the ODE-based algorithm takes small steps and this increases the time needed to complete the entire path significantly; Normal flow and Augmented Jacobian algorithms are more robust to imprecise Jacobians.

The obvious solution is to use analytical instead of numerical differentiation, but this carries a high fixed cost of deriving, coding, and debugging the Jacobian. Instead, we use ADIFOR, a program that analytically differentiates Fortran code. ADIFOR is described in Bischof, Khademi, Mauer & Carle (1996); here we just give a brief overview.[6]

The input to ADIFOR is the Fortran90 code that returns $\mathbf{H}(\mathbf{y})$ at a given point $\mathbf{y}$. ADIFOR analyses this code and from it generates the new code. This code receives a pair of $((N+1)\times 1)$ vectors $(\mathbf{y}, \Delta\mathbf{y})$ and returns the $(N\times 1)$ vector

$$\Delta\mathbf{H} = \frac{\partial\mathbf{H}(\mathbf{y})}{\partial\mathbf{y}}\Delta\mathbf{y}.$$

Thus, we obtain the jth column of the Jacobian via a single call to the ADIFOR generated code with $\Delta\mathbf{y}$ set to the jth basis vector. Repeating this for $j = 1, \ldots, N+1$ we assemble the entire Jacobian.

[6]ADIFOR can be obtained at http://www-unix.mcs.anl.gov/autodiff/ADIFOR/.

Dense vs. sparse Jacobian. Taking advantage of the sparse nature of the Jacobians in dynamic stochastic games offers a decrease in computation time, and in fact we show in Section 3.4.3 that this decrease is substantial. The additional efficiency comes from lower memory requirements and faster linear algebra operations. In addition, the Jacobian of a very large system of equations may exceed the available memory unless it is stored as a sparse matrix.

The use of sparse Jacobians is complicated by two issues. First, there is additional coding because the user must specify the "sparsity structure," i.e., the row and column indices of potentially non-zero elements. In practice, this means going through the system of equations and identifying the elements of $\mathbf{y}$ that are involved in a given equation.

Second, the sparse and dense versions of the various path-following algorithms take different approaches to solving systems of linear equations. In all cases, the linear algebra routines in HOMPACK90 were selected for speed not reliability, which means that they can and do fail for certain problems. Our experience has been that the sparse linear solver is more likely to fail than the dense linear solver; Section 3.5.6 gives more details and some solutions.

3.3.3. Initial Condition

The final input to HOMPACK90 is an initial condition in the form of a solution to the system of equations for the particular parameterization associated with $\lambda = 0$. In some cases, if the parameterization associated with $\lambda = 0$ is trivial, the solution can be derived analytically. A good example is the case of a zero discount rate that turns the dynamic stochastic game into a set of disjoint static games played out in every state. Another example is a particular

parameterization that makes movements through state space unidirectional and thus allows the game to be solved by backwards induction (see, e.g., Judd, Schmedders & Yeltekin 2002).

More generally, a solution for a particular parameterization can be computed numerically using a number of approaches such as Gaussian methods including (but not limited to) the Pakes & McGuire (1994) algorithm, other nonlinear solvers (see Ferris, Judd & Schmedders 2007), and artificial homotopies (see Section 3.6.1). Finally, one can use a solution obtained by tracing out a path along a different parameter as an initial condition (see Section 3.5.1 for an example of path-following along several parameters).

3.4. Example 1: The Learning-by-Doing Model

We begin with the learning-by-doing model of Chapter 1 because it is particularly well-suited for the homotopy method; as explained above, the dynamic programming problem that a firm has to solve leads to a system of equations that satisfies the smoothness requirement.

3.4.1. Model

The description of the model is presented in Section 1.2. There we prove that the model has a unique equilibrium if $\delta = 0$ or $\delta = 1$. It is therefore natural to use the homotopy method to trace out the equilibrium correspondence by varying δ from 0 to 1. We thus make the forgetting rate δ a function of the homotopy parameter λ and set

$$\delta(\lambda) = \delta^{start} + \lambda \left(\delta^{end} - \delta^{start}\right).$$

In particular, if $\delta^{start} = 0$ and $\delta^{end} = 1$, then the homotopy method traces out the equilibrium correspondence from $\delta(0) = 0$ to $\delta(1) = 1$. To explore the role of learning-by-doing, we repeat this procedure for 100 evenly spaced values of $\rho \in [0.01, 1]$.

It is worth re-stating some material from 1.2 to show how we describe the equilibrium as a system of equations in the form given by (3.1). Define the vector of unknown values and policies in equilibrium as

$$\mathbf{x} = [V(1,1), V(2,1), \ldots, V(M,1), V(1,2), \ldots, V(M,M), p(1,1), \ldots, p(M,M)]'.$$

The Bellman equation and first-order condition in state $\mathbf{e}$ are

$$H^1_{\mathbf{e}}(\mathbf{x},\lambda) = -V(\mathbf{e}) + D_1(\mathbf{e})(p(\mathbf{e}) - c(e_1)) + \beta \sum_{k=1}^{2} D_k(\mathbf{e})\overline{V}_k(\mathbf{e}) = 0, \tag{3.8}$$

$$H^2_{\mathbf{e}}(\mathbf{x},\lambda) = 1 - (1 - D_1(\mathbf{e}))(p(\mathbf{e}) - c(e_1)) - \beta\overline{V}_1(\mathbf{e}) + \beta \sum_{k=1}^{2} D_k(\mathbf{e})\overline{V}_k(\mathbf{e}) = 0. \tag{3.9}$$

The collection of equations (3.8) and (3.9) for all states $\mathbf{e} \in \{1, \ldots, M\}^2$ can be written more compactly as

$$\mathbf{H}(\mathbf{x},\lambda) = \begin{bmatrix} H^1_{(1,1)}(\mathbf{x},\lambda) \\ H^1_{(2,1)}(\mathbf{x},\lambda) \\ \vdots \\ H^2_{(M,M)}(\mathbf{x},\lambda) \end{bmatrix} = \mathbf{0}, \tag{3.10}$$

where $\mathbf{0} \in \mathbb{R}^{2M^2}$ is a vector of zeros. Any solution to this system of $2M^2$ equations in $2M^2$ unknowns $\mathbf{x} \in \mathbb{R}^{2M^2}$ is a symmetric equilibrium in pure strategies (for a given value of $\lambda \in [0,1]$).[7]

Code. A set of code that allows the user to compute equilibria of the learning-by-doing model using the homotopy method is available on the authors' homepages. It includes (i)

[7] A slightly modified version of Proposition 2 in Doraszelski & Satterthwaite (2007) establishes that such an equilibrium always exists.

Matlab code that implements the Pakes & McGuire (1994) algorithm that we use to compute a starting point for the homotopy algorithm (see Section 1.3 for a detailed description); (ii) Fortran90 code that includes HOMPACK90 and the implementation of the learning-by-doing model; and (iii) additional Matlab code that analyzes the output of the homotopy algorithm. More detailed information is included within the code itself.

3.4.2. Equilibrium Correspondence

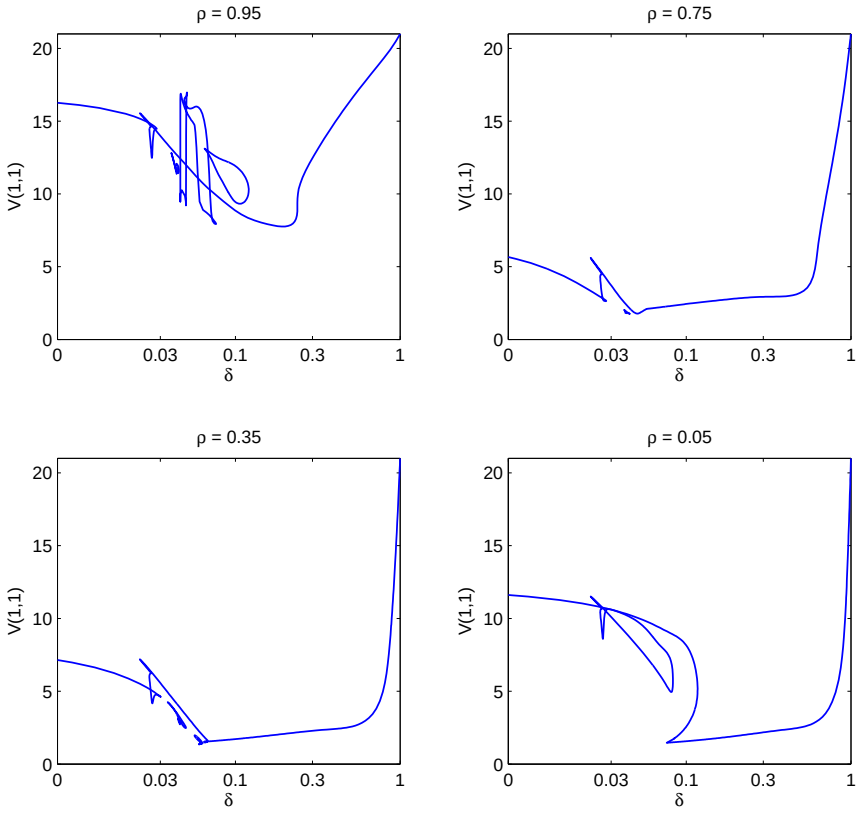

Figure 3.4. Initial firm value $V(1,1)$. Learning-by-doing model.

Equilibrium correspondence is discusses in detail in Section 1.4. Figure 1.1 shows the number of equilibria as a function of the progress ratio ρ and the forgetting rate δ. The subset of parameterizations that yield three equilibria is fairly large, and we have found up

to nine equilibria for some values of ρ and δ. It is not a coincidence that the number of equilibria at each parameterization is odd; see the discussion in Section 1.4.

Multiplicity is especially pervasive for progress ratios and forgetting rates that are broadly consistent with empirical studies of learning-by-doing and organizational forgetting ($\rho \geq 0.7$ and $\delta \leq 0.1$). Moreover, Sections 1.5 and 1.6 show that these multiple equilibria describe a rich array of pricing behaviors that are economically meaningful and that are quite different in terms of implied industry structure and dynamics. Consequently, in addition to the parameterization, the equilibrium itself is an important determinant of pricing behavior and industry dynamics. This reinforces our earlier point that it is important to explore the equilibrium correspondence of structurally estimated models in order to determine whether multiplicity exists in the empirically relevant subset of the parameter space.

Recall that we made the forgetting rate δ a function of the homotopy parameter λ and applied the homotopy method repeatedly for a series of values for the progress ratio ρ. To visualize the set of equilibria as a correspondence of δ for a specific value of ρ, we need a way to summarize each equilibrium as a single number. The value function in the initial state $(1, 1)$ is the value of a firm at the onset of the industry; $V(1, 1)$ is thus an economically meaningful summary of an equilibrium. As we are also interested in long-run industry concentration, we further compute the expected Herfindahl index, computed as described in Section 1.4.

Figures 3.4 and 1.2 visualize the equilibrium correspondence either in terms of $V(1, 1)$ or in terms of Herfindhals for a variety of different progress ratios ρ. In Chapter 1 we prove that the model has unique equilibria at $\delta = 0$ and $\delta = 1$. Hence, if the system of equations that characterizes the equilibria is regular, then there must be a path connecting them. We

observe multiple equilibria whenever this path bends back on itself. Moreover, we have been able to identify one or more loops that are disjoint from this path.

In some places the various solution paths appear to intersect each other. A case in point is the loop in the upper left panel of Figure 1.2 that appears to twice intersect the path from $\delta = 0$ and $\delta = 1$. While such an intersection seemingly resembles the branching point (G) in Figure 3.3, the fact that two equilibria give rise to the same expected Herfindahl index does not mean that the equilibria themselves are the same. We have indeed verified that the various solution paths do not intersect. Thus, the intersections in Figures 3.4 and 1.2 do not violate the regularity requirement.

While we have been able to identify some loops, we note that other loops may exist because, in order to trace out a loop, we must somehow compute at least one equilibrium on the loop. Unfortunately, there is no sure fire of way of doing so. Figures 1.1,3.4,1.2 are therefore not necessarily a complete mapping of the equilibria.

3.4.3. Performance

HOMPACK90 offers several different path-following algorithms and storage formats for the Jacobian of the system of equations. Moreover, the user can compute the Jacobian either numerically or analytically. Below we present the results of a series of experiments that are designed to highlight the implications of these choices for the performance of HOMPACK90. We have traced out the main path of the equilibrium correspondence from $\delta = 0$ to $\delta = 1$ for a progress ratio of $\rho = 0.75$ (as shown the upper right panel of Figure 1.2). We set the precision in HOMPACK90 to 10^{-10} (see Section 3.5.5 for a discussion). We use ADIFOR to analytically compute the Jacobian. All experiments are conducted on a Linux machine with a 64-bit 1GHz AMD Athlon CPU and 4GB of RAM.

algorithm / Jacobian	time (h:m)	#steps	time/step (s)
ODE based / dense, ana.	22:50	1596	51.5
normal flow / dense, ana.	28:59	2197	47.5
aug. Jacobian / dense, ana.	25:25	2250	40.7
ODE based / sparse, ana.	1:28	1579	03.4
normal flow / sparse, ana.	1:44	2197	02.9
aug. Jacobian / sparse, ana.	2:43	2195	04.5

Table 3.2. Performance. Path-following algorithms and dense vs. sparse Jacobian. Learning-by-doing model.

Path-following algorithms. A major issue is the trade-off between robustness and computation time. Computation time is the product of the number of steps it takes to trace out the entire path and the average time per step. This involves yet another trade off because these two determinants of computation time are affected in opposite ways by the size of the step that the homotopy algorithm takes from one point to the next. Optimally adjusting the step size is a highly non-trivial problem and the algorithm that does this is a major part of HOMPACK90.

Turning to the choice of a specific path-following algorithm, Watson et al. (1997) describe the normal flow algorithm as the baseline offering a reasonable compromise between robustness and computation time. The ODE based algorithm is described as the most robust but slowest and the augmented Jacobian algorithm as the least robust but fastest.

Table 3.2 shows that the ODE based algorithm is not always slower than the other path-following algorithms. On the contrary, in our experiments the ODE based algorithm turns out to be fastest: While it takes more time to complete each step, it takes fewer steps to complete the path.

To further investigate this somewhat unexpected finding, in Table 3.3 we contrast the performance of the different path-following algorithms on separate portions of the solution

algorithm / Jacobian	"complicated" ($\delta \in [0, 0.03]$) time (h:m)	#steps	time/step (s)	"simple" ($\delta \in (0.03, 1]$) time (h:m)	#steps	time/step (s)
ODE based / sparse, ana.	0:31	507	3.7	0:57	1072	3.2
normal flow / sparse, ana.	0:18	292	3.9	1:26	1905	2.8
aug. Jacobian / sparse, ana.	0:26	290	5.4	2:17	1905	4.3

Table 3.3. Performance. "Complicated" vs. "simple" segment of path. Learning-by-doing model

path. The ODE based algorithm is faster on the "simple" segment of the path ($\delta \in (0.03, 1]$) without multiplicity and much curvature (so that the unknowns change gradually with the homotopy parameter) but slower on the "complicated" segment of the path ($\delta \in [0, 0.03]$). The reason may lie in the different step size adjustment procedures of the different path-following algorithms. Indeed, as a closer analysis of the output of HOMPACK90 reveals, the ODE based algorithm takes much larger (and thus fewer) steps than the other path-following algorithms on the "simple " segment of the path. In contrast, on the "complicated" segment of the path, the ODE based algorithm takes much smaller (and thus more) steps than the other path-following algorithms.[8]

Returning to Table 3.2, the comparison between the normal flow and augmented Jacobian algorithms is not clear-cut either. The dense augmented Jacobian algorithm takes less time for each step but requires more steps, thereby leading to an overall decrease of computation time. In contrast, the sparse augmented Jacobian algorithm takes more time for each step but requires fewer steps, thereby leading to an overall increase in computation time. While the sparse augmented Jacobian algorithm takes only two fewer steps than the sparse normal flow algorithm in Table 3.2, Borkovsky, Doraszelski & Satterthwaite (2007) have found that

[8]Further investigation revealed that the normal flow and augmented Jacobian algorithms indeed limit the maximum step size (as set in the input variable SSPAR(5)). We kept it at the default value to make for a more fair comparison between the different path-following algorithms. Increasing the maximum step size also appears to increase the likelihood that the homotopy algorithm strays from the solution path.

algorithm / Jacobian	time (h:m)	#steps	time/step (s)
ODE based / sparse, ana.	1:28	1,579	3.4
ODE based / sparse, num.	>6:27	>10,000	2.3
normal flow / sparse, ana.	1:44	2,197	2.9
normal flow / sparse, num.	1:22	2,197	2.3

Table 3.4. Performance. Path-following algorithms and numerical vs. analytical Jacobian. Learning-by-doing model.

in some applications both the dense and the sparse versions of the augmented Jacobian algorithm sometimes take up to 20 percent fewer steps than their normal flow counterparts.

Jacobian. As is obvious from Table 3.2, the dense-Jacobian algorithms require considerably more computation time. A closer analysis reveals that the additional computation time required by the dense-Jacobian algorithms is spent performing linear algebra operations on the Jacobians. Overall, this makes an overwhelmingly strong case for using sparse Jacobians.

With regard to the dense-Jacobian algorithms, we have found that the choice between a numerical, hand-coded analytical, or ADIFOR-generated analytical Jacobian has a negligible effect on the time per step. This is because the time required to compute the Jacobian is dwarfed by the time required to solve the system of linear equations that the algorithm must solve to compute the next step along the path.

Turning to the sparse-Jacobian algorithms, precision is a key advantage of analytically computed Jacobians. Table 3.4 shows that, while the ODE based algorithm succeeds in completing the solution path with an analytical Jacobian, it fails to do so with a numerical Jacobian; in particular, it spends much time tracing out a short segment of the path and stops at $\delta = 0.096$ where it reaches the maximum number of steps.

On the other hand, the normal flow algorithm requires the same number of steps to complete the solution path regardless of whether an analytical or numerical Jacobian is

used. Interestingly, the path is computed more quickly when a numerical Jacobian is used (presumably because computing the numerical Jacobian requires less time than computing the analytical Jacobian due to the column-by-column approach that ADIFOR requires to assemble to Jacobian). Thus, it appears that the lower precision of the numerical Jacobian is problematic for the ODE based algorithm but not for the other path-following algorithms. The likely reason is that, in contrast to the other two path-following algorithms, the ODE based algorithm uses the Jacobian not just in corrector but also in the predictor phase.

Overall, our results make an overwhelmingly strong case for using sparse Jacobians. There are also good reasons to prefer analytical over numerical Jacobians, especially because ADIFOR makes the process of computing analytical Jacobians very easy. Finally, we conclude that performance is at least partly problem-specific. We therefore recommend conducting experiments on the particular application at hand. The gains from experimentation can be substantial, and experimentation is virtually costless once the system of equations and the Jacobian have been coded.

3.5. Example 2: The Quality Ladder Model

We next consider the quality ladder model of Pakes & McGuire (1994). The quality ladder model presents a complication that stems from the non-negativity constraint on investment. The problem that a firm has to solve is formulated using a complementary slackness condition, a combination of equalities and inequalities, rather than a first-order condition, an equation, as in the learning-by-doing model in Section 3.4. However, the homotopy method operates on a system of equations. We show how to resolve this problem by reformulating the complementary slackness condition as a system of equations.

3.5.1. Model

The description of the model is abridged; please see Pakes & McGuire (1994) for details. To simplify the exposition, we restrict attention to a duopoly without entry and exit in what follows.[9]

Firms and states. The state of firm $n \in \{1, 2\}$ is $\omega_n \in \{1, \ldots, M\}$ and reflects its product quality. The vector of firms' states is $\omega = (\omega_1, \omega_2) \in \{1, \ldots, M\}^2$ and we use $\omega^{[2]}$ to denote the vector (ω_2, ω_1). Each period firms first compete in the product market and then make investment decisions. The state in the next period is determined by the stochastic outcomes of these investment decisions and an industry-wide depreciation shock which stems from an increase in the quality of an outside alternative. In particular, firm n's state evolves according to the law of motion

$$\omega_n' = \omega_n + \tau_n - \eta,$$

where $\tau_n \in \{0, 1\}$ is a random variable governed by firm n's investment $x_n \geq 0$ and $\eta \in \{0, 1\}$ is an industry-wide depreciation shock. If $\tau_n = 1$, the investment is successful and the quality of firm n increases by one level. The probability of success is $\frac{\alpha x_n}{1+\alpha x_n}$, where $\alpha > 0$ is a measure of the effectiveness of investment. If $\eta = 1$, the industry is hit by a depreciation shock and the qualities of all firms decrease by one level; this happens with probability $\delta \in [0, 1]$.

Below we first describe the static model of product market competition and then turn to investment dynamics.

[9]It is straightforward to extend the quality ladder model to allow for entry and exit. The key is to do this in a way that guarantees the existence of an equilibrium; see Doraszelski & Satterthwaite (2007) for details. The Online Appendix of Besanko et al. (2007) contains a formal derivation of the learning-by-doing model with entry and exit. Setup costs and scrap values are drawn from triangular distributions that yield cumulative distribution functions that are once but not twice continuously differentiable, yet Besanko et al. (2007) did not encounter a problem. If a problem is encountered in another application, we suggest using a Beta(k, k) distribution with $k \geq 3$ to ensure that the system of equations is at least twice continuously differentiable.

Product market competition. The product market is characterized by price competition with vertically differentiated products. There is a continuum of consumers. Each consumer purchases at most one unit of one product. The utility a consumer derives from purchasing product n is $g(\omega_n) - p_n + \epsilon_n$, where

$$g(\omega_n) = \begin{cases} \omega_n & \text{if} \quad 1 \leq \omega_n \leq \omega^*, \\ \omega^* + \ln\left(2 - \exp\left(\omega^* - \omega_n\right)\right) & \text{if} \quad \omega^* < \omega_n \leq M \end{cases}$$

maps the quality of the product into the consumer's valuation for it, p_n is the price, and ϵ_n represents the consumer's idiosyncratic preference for product n. There is an outside alternative, product 0, which has utility ϵ_0. Assuming that the idiosyncratic preferences $(\epsilon_0, \epsilon_1, \epsilon_2)$ are independently and identically type 1 extreme value distributed, the demand for firm n's product is

$$D_n(\mathbf{p}; \omega) = m \frac{\exp\left(g(\omega_n) - p_n\right)}{1 + \sum_{j=1}^{2} \exp\left(g(\omega_j) - p_j\right)},$$

where $\mathbf{p} = (p_1, p_2)$ is the vector of prices and $m > 0$ is the size of the market (the measure of consumers).

Firm n chooses the price p_n of product n to maximize profits. Hence, firm n's profits in state ω are

$$\pi_n(\omega) = \max_{p_n} D_n(p_n, p_{-n}(\omega); \omega)\left(p_n - c\right),$$

where $p_{-n}(\omega)$ is the price charged by the other firm and $c \geq 0$ is the marginal cost of production. Given a state ω, there exists a unique Nash equilibrium of the product market game (Caplin & Nalebuff 1991). It is found easily by numerically solving the system of first-order conditions corresponding to firms' profit-maximization problem. Note that the quality

ladder model differs from the learning-by-doing model in that product market competition does not directly affect state-to-state transitions and, hence, $\pi_n(\omega)$ can be computed before the Markov-perfect equilibria of the dynamic stochastic game are computed via the homotopy method. This allows us to treat $\pi_n(\omega)$ as a primitive in what follows.

Bellman equation and complementary slackness condition. Define $V_n(\omega)$ to be the expected net present value of firm n's cash flows if the industry is currently in state ω. The value function $\mathbf{V}_n : \{1, \ldots, M\}^2 \to \mathbb{R}$ is implicitly defined by the Bellman equation

$$V_n(\omega) = \max_{x_n \geq 0} \pi_n(\omega) - x_n + \beta \left(\frac{\alpha x_n}{1 + \alpha x_n} W_n^1(\omega) + \frac{1}{1 + \alpha x_n} W_n^0(\omega) \right), \tag{3.11}$$

where $\beta \in (0, 1)$ is the discount factor and $W_n^{\tau_n}(\omega)$ is the expectation of firm n's value function conditional on an investment success ($\tau_n = 1$) and failure ($\tau_n = 0$), respectively, as given by

$$\begin{aligned} W_n^{\tau_n}(\omega) \quad = \quad & \sum_{\eta \in \{0,1\}, \tau_{-n} \in \{0,1\}} \delta^{\eta} (1 - \delta)^{1 - \eta} \left(\frac{\alpha x_{-n}(\omega)}{1 + \alpha x_{-n}(\omega)} \right)^{\tau_{-n}} \left(\frac{1}{1 + \alpha x_{-n}(\omega)} \right)^{1 - \tau_{-n}} \\ & \times V_n \Big(\max \left\{ \min \left\{ \omega_n + \tau_n - \eta, M \right\}, 1 \right\}, \max \left\{ \min \left\{ \omega_{-n} + \tau_{-n} - \eta, M \right\}, 1 \right\} \Big), \end{aligned}$$

where $x_{-n}(\omega)$ is the investment of the other firm in state ω. Note that the min and max operators merely enforce the bounds of the state space.

The policy function $\mathbf{x}_n : \{1, \ldots, M\}^2 \to [0, \infty)$ specifies the investment of firm n in state ω. Solving the maximization problem on the right-hand side of the Bellman equation (3.11),

we obtain the complementary slackness condition

$$\begin{aligned} -1+\beta\frac{\alpha}{(1+\alpha x_n)^2}\left(W_n^1(\omega)-W_n^0(\omega)\right) &\leq 0, \\ x_n\left(-1+\beta\frac{\alpha}{(1+\alpha x_n)^2}\left(W_n^1(\omega)-W_n^0(\omega)\right)\right) &= 0, \\ x_n &\geq 0. \end{aligned} \tag{3.12}$$

The investment decision $x_n(\omega)$ is uniquely determined by the solution to complementary slackness condition. It follows that

$$x_n(\omega)=\max\left\{0,\frac{-1+\sqrt{\beta\alpha\left(W_n^1(\omega)-W_n^0(\omega)\right)}}{\alpha}\right\} \tag{3.13}$$

if $W_n^1(\omega)-W_n^0(\omega)\geq 0$ and $x_n(\omega)=0$ otherwise.

Equilibrium. We restrict attention to symmetric Markov-perfect equilibria. In a symmetric equilibrium, the investment decision taken by firm 2 in state ω is identical to the investment decision taken by firm 1 in state $\omega^{[2]}$, i.e., $x_2(\omega)=x_1(\omega^{[2]})$, and similarly for the value functions. It therefore suffices to determine the value and policy functions of firm 1, and we define $V(\omega)=V_1(\omega)$ and $x(\omega)=x_1(\omega)$ for each state ω. Similarly, we define $W^{\tau_1}(\omega)=W_1^{\tau_1}(\omega)$ for each state ω.

Parameterization. As explained above, the homotopy algorithm traces out an entire path of equilibria by varying one or more parameters of interest. We allow α and δ to vary, while holding the remaining parameters fixed at the values shown in Table 3.5. The effectiveness of investment α is a natural parameter to vary because the equilibrium trivially involves no investment if $\alpha=0$. Moreover, this equilibrium is unique. In addition to α, we allow the rate of depreciation δ to vary because experience suggests that the rate

parameter	M	m	c	ω^*	β
value	18	5	5	12	0.925

Table 3.5. Parameter values. Quality ladder model.

of depreciation is often a key determinant of industry structure and dynamics (see, e.g., Besanko & Doraszelski 2004, Besanko, Doraszelski, Lu & Satterthwaite 2006). Note that in the quality ladder model the equilibrium may not be unique at either $\delta = 0$ or $\delta = 1$.

Taken together, we make the vector comprising α and δ a function of the homotopy parameter λ:

$$\begin{bmatrix} \alpha(\lambda) \\ \delta(\lambda) \end{bmatrix} = \begin{bmatrix} \alpha^{start} \\ \delta^{start} \end{bmatrix} + \lambda \begin{bmatrix} \alpha^{end} - \alpha^{start} \\ \delta^{end} - \delta^{start} \end{bmatrix}.$$

For example, if $\delta^{start} = 0$ and $\delta^{end} = 1$ while $\alpha^{start} = \alpha^{end}$, then the homotopy algorithm traces out the equilibrium correspondence from $\delta(0) = 0$ to $\delta(1) = 1$, holding all other parameter values fixed. Setting different starting and ending values for one or more of these parameters allows us to explore the set of equilibria by moving through the parameter space in various directions. In general, given any starting and ending values for the parameter vector, the homotopy algorithm can trace out an entire path of equilibria by moving along the line in parameter space that connects the starting and ending values.

System of equations. Due to the non-negativity constraint on investment, we obtained a complementary slackness condition instead of a first-order condition as in the learning-by-doing model in Section 3.4. To apply the homotopy method, we must reformulate the combination of equalities and inequalities in (3.12) as equalities. In the next section we describe one way to do this.

Code. A set of code that allows the user to compute equilibria of the quality ladder model using the homotopy method is available on the authors' homepages.

3.5.2. The Zangwill & Garcia (1981) Reformulation of the Complementary Slackness Condition

The homotopy method operates on equations. Therefore, a model that includes a complementary slackness condition, a combination of equalities and inequalities, must be reformulated as a system of equations.

Consider a general complementary slackness condition on a scalar variable x:

$$\begin{aligned} A(x) &\leq 0, \\ B(x) &\leq 0, \\ A(x)\,B(x) &= 0, \end{aligned} \tag{3.14}$$

where $A(x)$ and $B(x)$ are functions of x. A complementary slackness condition may arise from an optimization problem with a non-negativity constraint as in the quality ladder model in Section 3.5.1. It may also arise if a model contains min or max operators. For example, the equation $x = \min\{a(x), b(x)\}$ is equivalent to

$$\begin{aligned} x - a(x) &\leq 0, \\ x - b(x) &\leq 0, \\ (x - a(x))\,(x - b(x)) &= 0. \end{aligned}$$

Note that the equation $x = \min\{a(x), b(x)\}$ has a kink when $a(x) = b(x)$ and hence does not satisfy the smoothness requirement of the homotopy method.

Zangwill & Garcia (1981) offer a reformulation of the complementary slackness condition that consists entirely of equations that are continuously differentiable to an arbitrary degree

(see pp. 65–68).[10] The idea is to introduce another scalar variable ζ and consider the system of equations

$$A(x) + [\max\{0, \zeta\}]^k = 0, \tag{3.15}$$

$$B(x) + [\max\{0, -\zeta\}]^k = 0, \tag{3.16}$$

where $k \in \mathbb{N}$. From equations (3.15) and (3.16), it follows that

$$\zeta = \begin{cases} [-A(x)]^{1/k} & \text{if} \quad A(x) < 0, \\ -[-B(x)]^{1/k} & \text{if} \quad B(x) < 0, \\ 0 & \text{if} \quad A(x) = B(x) = 0. \end{cases} \tag{3.17}$$

Using the fact that $\max\{0, -\zeta\} \max\{0, \zeta\} = 0$ and the solution for ζ in equation (3.17), it is easy to see that the system of equations (3.15) and (3.16) is equivalent to the complementary slackness condition in (3.14). Moreover, this system is $(k-1)$ times continuously differentiable with respect to ζ. Hence, by choosing k large enough, we can satisfy the smoothness requirement of the homotopy method.

Example: The quality ladder model. An example is helpful in understanding how the Zangwill & Garcia (1981) reformulation works. Consider the complementary slackness condition (3.12) in the quality ladder model in Section 3.5. Using the fact that we focus on symmetric equilibria in order to eliminate firm indices and multiplying through by $(1 + \alpha x(\omega))^2$ to simplify the expressions that arise in what follows, the complementary slackness

[10] As Zangwill & Garcia (1981) is out of print, a more easily accessible source may be Charnes, Garcia & Lemke (1977).

condition (3.12) can be restated as

$$\begin{aligned} -(1+\alpha x(\omega))^2 + \beta\alpha\left(W^1(\omega) - W^0(\omega)\right) &\leq 0, \\ x(\omega)\left(-(1+\alpha x(\omega)^2 + \beta\alpha\left(W^1(\omega) - W^0(\omega)\right)\right) &= 0, \\ x(\omega) &\geq 0. \end{aligned} \tag{3.18}$$

Applying the Zangwill & Garcia (1981) reformulation to the complementary slackness condition (3.18) yields the equations

$$-(1+\alpha x(\omega))^2 + \beta\alpha\left(W^1(\omega) - W^0(\omega)\right) + \left[\max\left\{0, \zeta(\omega)\right\}\right]^k = 0, \tag{3.19}$$

$$-x(\omega) + \left[\max\left\{0, -\zeta(\omega)\right\}\right]^k = 0. \tag{3.20}$$

The terms $\left[\max\left\{0, \zeta(\omega)\right\}\right]^k$ and $\left[\max\left\{0, -\zeta(\omega)\right\}\right]^k$ serve as slack variables that ensure that the inequalities in (3.12) are satisfied and the fact that $\left[\max\left\{0, \zeta(\omega)\right\}\right]^k \left[\max\left\{0, -\zeta(\omega)\right\}\right]^k = 0$ ensures that the equality in (3.12) holds.

We can now proceed to define the system of homotopy equations using equations (3.19) and (3.20) along with the Bellman equation

$$-V(\omega) + \pi_1(\omega) - x(\omega) + \beta\left(\frac{\alpha x(\omega)}{1+\alpha x(\omega)}W^1(\omega) + \frac{1}{1+\alpha x(\omega)}W^0(\omega)\right) = 0, \tag{3.21}$$

where we substitute for $W^{\tau_1}(\omega)$ using the definition

$$\begin{aligned} W^{\tau_1}(\omega) = \sum_{\eta\in\{0,1\},\tau_2\in\{0,1\}} & \delta^\eta(1-\delta)^{1-\eta}\left(\frac{\alpha x(\omega^{[2]})}{1+\alpha x(\omega^{[2]})}\right)^{\tau_2}\left(\frac{1}{1+\alpha x(\omega^{[2]})}\right)^{1-\tau_2} \\ & \times V\Big(\max\left\{\min\left\{\omega_1+\tau_1-\eta, M\right\}, 1\right\}, \max\left\{\min\left\{\omega_2+\tau_2-\eta, M\right\}, 1\right\}\Big). \end{aligned}$$

This yields a system of $3M^2$ equations in the $3M^2$ unknowns $V(1,1),\ldots,V(M,M)$, $x(1,1),\ldots,x(M,M)$, and $\zeta(1,1),\ldots,\zeta(M,M)$.

Two problems arise: First, because we have added the slack variables, this system of equations is relatively large with $3M^2$ equations and unknowns. This leads to increased memory requirements and computation time. Second, this system of equations yields an extremely sparse Jacobian. Note that the rows of the Jacobian corresponding to equation (3.20) each have only one or two non-zero elements. Also note that each column of the Jacobian corresponding to a slack variable has only one non-zero element. We have found that such a Jacobian tends to cause HOMPACK90's sparse linear equation solver to fail; this is discussed further in Section 3.5.6.

We address these problems by solving equation (3.20) for $x(\omega)$ and then substituting for $x(\omega)$ in equations (3.19) and (3.21).[11] This reduces the system of $3M^2$ equations in $3M^2$ unknowns to a system of $2M^2$ equations in $2M^2$ unknowns. Moreover, it eliminates the rows and columns of the Jacobian that included only one or two non-zero elements; thus, we have eliminated the excessive sparsity that tends to cause HOMPACK90's sparse linear equation solver to fail.

To this end, define the vector of unknowns in equilibrium as

$$\mathbf{x} = [V(1,1), V(2,1), \ldots, V(M,1), V(1,2), \ldots, V(M,M), \zeta(1,1), \ldots, \zeta(M,M)]'.$$

[11] We thank Karl Schmedders for suggesting this approach.

The equations in state ω are

(3.22)
$$H^1_\omega(\mathbf{x},\lambda) = -V(\omega) + \pi_1(\omega) - x(\omega) + \beta\left(\frac{\alpha x(\omega)}{1+\alpha x(\omega)}W^1(\omega) + \frac{1}{1+\alpha x(\omega)}W^0(\omega)\right) = 0,$$

(3.23)
$$H^2_\omega(\mathbf{x},\lambda) = -(1+\alpha x(\omega))^2 + \beta\alpha\left(W^1(\omega) - W^0(\omega)\right) + \left[\max\{0,\zeta(\omega)\}\right]^k = 0,$$

where we substitute for $W^{\tau_1}(\omega)$ using the definition

(3.24)
$$\begin{aligned} W^{\tau_1}(\omega) &= \sum_{\eta\in\{0,1\},\tau_2\in\{0,1\}} \delta^\eta(1-\delta)^{1-\eta}\left(\frac{\alpha x(\omega^{[2]})}{1+\alpha x(\omega^{[2]})}\right)^{\tau_2}\left(\frac{1}{1+\alpha x(\omega^{[2]})}\right)^{1-\tau_2} \\ &\quad \times V\Big(\max\{\min\{\omega_1+\tau_1-\eta, M\},1\}, \max\{\min\{\omega_2+\tau_2-\eta, M\},1\}\Big), \end{aligned}$$

and for $x(\omega)$ using

(3.25)
$$x(\omega) = \left[\max\{0,-\zeta(\omega)\}\right]^k,$$

obtained from equation (3.20). Note that (3.22) and (3.23) are equations that are used to construct the system of homotopy equations, while (3.24) and (3.25) are simply definitional shorthands for terms that appear in equations (3.22) and (3.23). The collection of equations (3.22) and (3.23) for all states $\omega \in \{1,\ldots,M\}^2$ can be written more compactly as

(3.26)
$$\mathbf{H}(\mathbf{x},\lambda) = \begin{bmatrix} H^1_{(1,1)}(\mathbf{x},\lambda) \\ H^1_{(2,1)}(\mathbf{x},\lambda) \\ \vdots \\ H^2_{(M,M)}(\mathbf{x},\lambda) \end{bmatrix} = \mathbf{0},$$

where $\mathbf{0} \in \mathbb{R}^{2M^2}$ is a vector of zeros. Any solution to this system of $2M^2$ equations in $2M^2$ unknowns, $\mathbf{x} \in \mathbb{R}^{2M^2}$, is a symmetric equilibrium in pure strategies (for a given value of $\lambda \in [0, 1]$).[12] The equilibrium investment decision $x(\omega)$ in state ω is recovered by substituting the equilibrium slack variable $\zeta(\omega)$ into definition (3.25).

In general, our approach of replacing a model variable with a slack variable can be taken only if one of the equations in the Zangwill & Garcia (1981) formulation admits a closed-form solution for a model variable (in case of the quality ladder model, we solved equation (3.20) for the investment decision $x(\omega)$). This is always the case if a model variable is constrained to be above/below a constant, as with the non-negativity constraint in the quality ladder model. However, it is possible that none of the equations in the Zangwill & Garcia (1981) formulation admits a closed-form solution for a model variable. Suppose, for example, we impose an upper bound on the sum of firms' investments in each state, i.e., $x_n(\omega) + x_{-n}(\omega) \leq L(\omega)$, in the quality ladder model (say because firms are competing for a scare resource).[13] Then in solving an equation corresponding to (3.15) or (3.16) for $x_n(\omega)$, one finds that $x_n(\omega) = f(\zeta_n(\omega), x_{-n}(\omega)) = f(\zeta_n(\omega), x_n(\omega^{[2]}))$. That is, the closed-form solution for firm n's policy in state ω, $x_n(\omega)$, is a function of its rival's policy in state ω, $x_{-n}(\omega)$, and thus its own policy in state $\omega^{[2]}$, $x_n(\omega^{[2]})$. In this case, it is impossible to find a closed-form solution for $x_n(\omega)$ as a function of only $\zeta_n(\omega)$ and thus it is impossible to eliminate the model variable $x_n(\omega)$. On the other hand, in this case, the Jacobian of the system formulated using the "pure" version of the Zangwill & Garcia (1981) formulation is no longer as sparse, thereby reducing our motivation for replacing a model variable with a slack variable in the first place.

[12] A simplified version of Proposition 3 in Doraszelski & Satterthwaite (2007) establishes that such an equilibrium always exists.

[13] See Besanko et al. (2006) for a more concrete example.

3.5.3. Equilibrium Correspondence

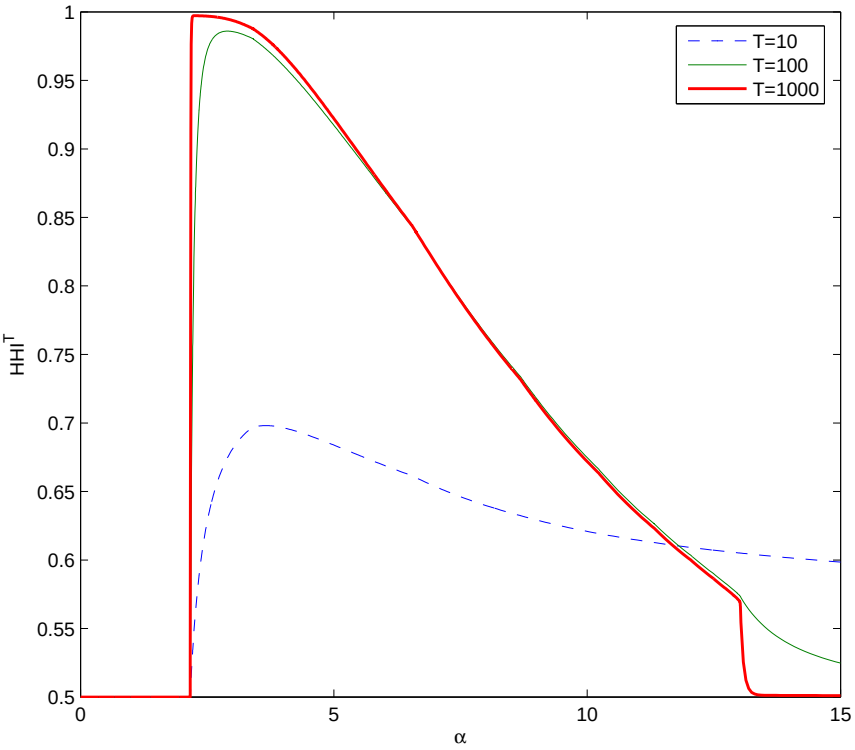

Figure 3.5. Transient expected Herfindahl index HHI^T at $T \in \{10, 100, 1000\}$ along α with $\delta = 0.7$. Quality ladder model.

We set $k = 2$ in the Zangwill & Garcia (1981) formulation of the quality ladder model. It follows that the system of homotopy equations is continuously differentiable.[14] We explore the equilibrium correspondence by allowing two parameters to vary: the effectiveness of investment α and the rate of depreciation δ. To visualize the equilibrium correspondence we graph the expected Herfindahl index

$$HHI^T = \sum_{\omega \in \{1,\ldots,M\}^2} \left[\left(D_1(\mathbf{p}(\omega);\omega)\right)^2 + \left(D_2(\mathbf{p}(\omega);\omega)\right)^2 \right] \mu^T(\omega),$$

[14] In general, if HOMPACK90 encounters problems when $k = 2$, we recommend setting $k > 2$; this ensures that the system of equations is at least twice continuously differentiable.

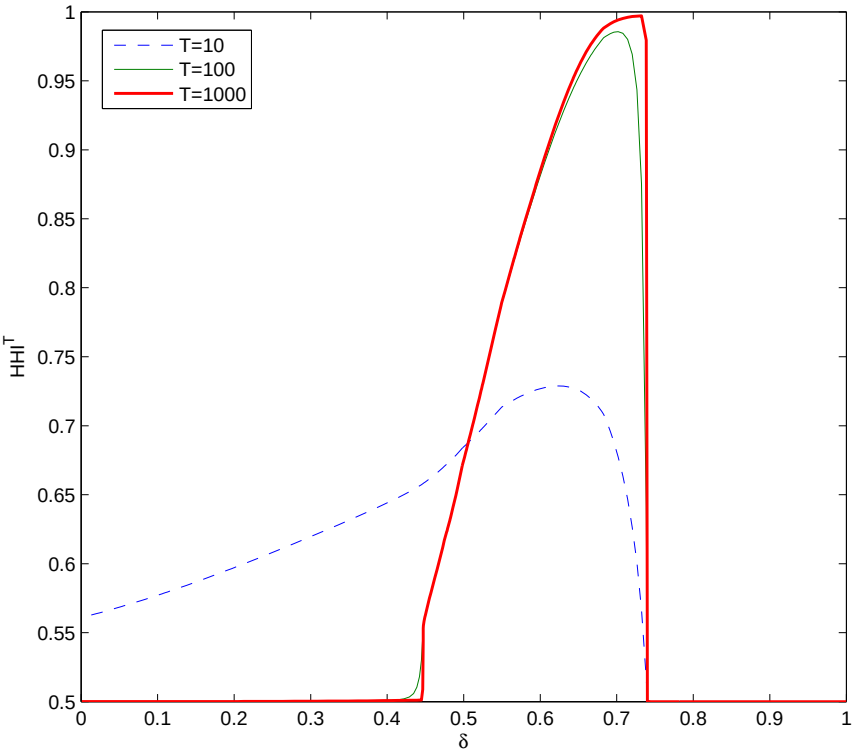

Figure 3.6. Transient expected Herfindahl index HHI^T at $T \in \{10, 100, 1000\}$ along δ with $\alpha = 3$. Quality ladder model.

where μ^T is the transient distribution over states in period T, starting from state $(1, 1)$ in period 0. We use a transient distribution rather than the limiting distribution as in the learning-by-doing model because there may be several closed communicating classes.[15] When there are multiple closed communicating classes, one cannot compute a single limiting distribution; rather, one must compute a separate limiting distribution for each closed communicating class. So, instead, we compute the transient distribution at various points in time $T \in \{10, 100, 1000\}$. The transient distribution accounts for the probability of reaching any one of the closed communicating classes. In addition, given a discount factor of $\beta = 0.925$

[15] A closed communicating class is a subset of states that the industry never leaves once it has entered it.

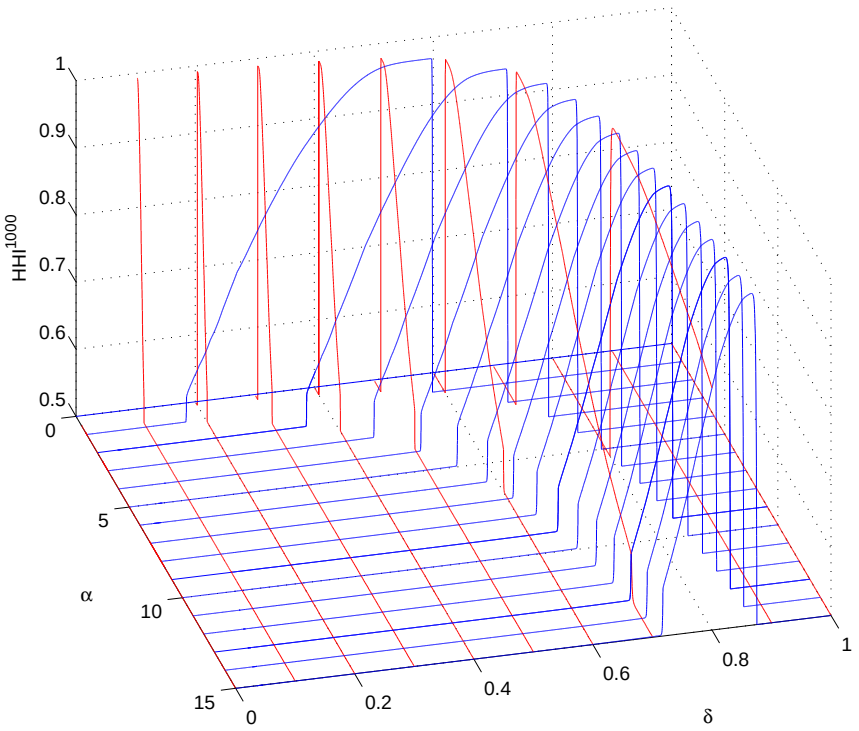

Figure 3.7. Transient expected Herfindahl index HHI^{1000} along α and δ. Quality ladder model.

we take a period to be one year; therefore, anything that happens beyond a certain point in time may be considered economically irrelevant.

We present the results in Figures 3.5 and 3.6. The industry concentration, both in the short- and in the long-run, is affected by α and δ in non-trivial ways. While the homotopy algorithm computes continuous solution paths, the expected Herfindahl indexes in Figures 3.5 and 3.6 appears to change almost discontinuously in some places. This happens because the shape of the transient distribution, and with it the value of the expected Herfindahl index, changes abruptly as investment in certain states goes to zero. In particular, if investment in state $(1, 1)$ is zero, then both firms are stuck at the lowest possible quality level. As soon as $x(1, 1) > 0$, however, the industry takes off, thereby giving rise to a nontrivial transient distribution that assigns positive probability to asymmetric industry structures.

M	ω^*	#equations	time (h:m:s)	#steps	time/step (s)
9	6	162	0:00:15	931	0.02
18	12	648	0:24:27	7608	0.19
27	18	1458	5:08:27	21457	0.86

Table 3.6. Scalability. Normal flow algorithm with sparse analytic Jacobian. Quality ladder model.

For example, with $\delta = 0.7$ investment rises from zero to positive around $\alpha = 2.17$ to cause the abrupt change in the expected Herfindahl index in Figure 3.5; with $\alpha = 3$ investment drops from positive to zero around $\delta = 0.74$.

Figure 3.7 illustrates the ability of the homotopy algorithm to criss-cross the parameter space. It combines several slices through the equilibrium correspondence to show how the expected Herfindahl index HHI^{1000} depends jointly on the effectiveness of investment α and the rate of depreciation δ.

Despite our best efforts, we have not uncovered any multiple equilibria in the quality ladder model; this does not necessarily mean that they do not exist.

3.5.4. Scalability

We use the quality ladder model to assess the scalability of HOMPACK90. We change the number of quality levels M, and thus the number of equations $R = 2M^2$, and adjust the quality cutoff ω^* accordingly. We trace out the equilibrium correspondence along $\alpha \in [0, 15]$ with $\delta = 0.7$ held fixed.

The results are presented in Table 3.6. It appears that the total computation time increases approximately as a third-order polynomial in the number of equations R. It is to be expected that the time per step increases in the number of equations since solving the system of linear equations becomes more burdensome. The rate of increase is approximately

proportional to $R^{\frac{3}{2}}$. More surprisingly, the number of steps increases in the number of equations. Again the rate of increase is approximately proportional to $R^{\frac{3}{2}}$.

The reason for this latter result is the following. Recall from Section 3.5.2 that the quality ladder model exhibits a kink as the investment in a state switches from zero to positive or vice versa in response to a change in the parameter values (see equation (3.13)). While the Zangwill & Garcia (1981) reformulation of the complementary slackness condition smoothes out this kink, it inevitably does so by introducing additional curvature into the solution path. This forces the homotopy algorithm to take small steps. Moreover, the larger the state space, the more kinks there potentially are in the quality ladder model and the more additional curvature is introduced by the Zangwill & Garcia (1981) reformulation. This argument implies that the homotopy algorithm should take large steps and proceed quickly as long as the solution path does not exhibit kinks. Indeed, irrespective of the size of the state space, the homotopy algorithm takes less than a hundred steps to traverse the segment along which investment is positive for all states; the rest of the steps are needed to trace out the segment along which investment in some state switches from zero to positive or vice versa.

3.5.5. Troubleshooting

If HOMPACK90 successfully follows a path to its end, it indicates a normal ending (exit flag 1). The end of the path may be associated with either $\lambda = 1$ or $\lambda = 0$. The latter case, in turn, may indicate genuine multiplicity of equilibria (see case B in Figure 3.2) or that the homotopy algorithm "turned around" and backtracked along the path until it returned to the starting point. HOMPACK90 may also fail to follow a path to its end for other reasons.

In the remainder of this section, we detail several types of failures that may occur and give tips for troubleshooting these problems.

With any type of failure, it is good practice to first verify that the regularity and smoothness requirements are satisfied. To check for regularity, we compute the condition numbers of Jacobians along the path. If the condition numbers increase as the homotopy algorithm approaches the point of failure, it is very likely due to a violation of the regularity requirement.[16] It may be possible to avoid this type of failure by making a small change in the parameter values of the model or by relaxing the precision setting so that HOMPACK90 takes larger steps and is thus more likely to "skip over" the singularity.

The homotopy algorithm does not check for smoothness and it is entirely possible that it would successfully follow a path to its end even if the smoothness requirement were violated. In general, however, it is advisable to formulate the problem such that the smoothness requirement is satisfied (see Sections 3.5.2 and 3.5.7).

HOMPACK90 may abort if the precision setting is too stringent (exit flags 2 and 6) or too lax (exit flag 5). In the latter case, the homotopy algorithm takes a step and ends up too far from the path to be able to return to it; this often happens on segments with high curvature. The solution is to adjust the precision setting.

HOMPACK90 may reach the maximum number of steps (exit flag 3). While the obvious solution is to increase the maximum number of steps, it is worth investigating if the homotopy algorithm proceeds slowly because the precision setting is too stringent. The tighter the precision setting, the narrower the "band" around the solution path in which the homotopy algorithm aims to stay and thus the smaller the steps that it takes. Also recall from Section

[16] A matrix is singular if its condition number is infinite. A large condition number signifies that a matrix is nearly singular, see pp. 67–70 of Judd (1998).

3.4.3 that the numerical Jacobian often lacks the precision that allows the ODE based algorithm to take long steps and proceed quickly. Finally, in the normal flow and augmented Jacobian algorithms, the maximum step size (as set in the input variable SSPAR(5)) can be increased.

If the homotopy algorithm progresses very slowly in the vicinity of the initial condition, then a useful trick is to allow the homotopy algorithm to instead begin at the parameterization originally designated as the end point and proceed "backwards" toward the parameterization originally designated as the starting point. This may alleviate the problem in cases where it allows the homotopy algorithm to approach the segment of high curvature from a segment of low curvature. We suspect that this occurs because some of the path-following algorithms – namely, the normal flow and augmented Jacobian algorithms – predict the next step on the solution path using several previous steps. A segment of low curvature on the solution path may therefore provide the homotopy algorithm with "data" on the path that serves as a good indication of the direction in which to proceed.

If a solution path gets sufficiently close to another, then the homotopy algorithm may jump from one path to another and, in doing so, may fail to traverse the path in its entirety. Similarly, the homotopy algorithm may also jump between one or more segments of the same path. If path jumping is suspected to occur, then it is advisable to tighten the precision setting and/or decrease the maximum step size in order to force the homotopy algorithm to remain close to the current solution path.

3.5.6. The Linear Solver

All the path-following algorithms must solve a system of linear equations of the form $\mathbf{Az} = \mathbf{b}$ in each step, where the matrix $\mathbf{A}$ is constructed from the Jacobian $\frac{\partial \mathbf{H}(\mathbf{y})}{\partial \mathbf{y}}$ (see, e.g., equation

(3.7) in case of the ODE based algorithm).[17] The final and perhaps most troubling reason that HOMPACK90 may fail to follow a path to its end is a failure of the linear solver (exit flag 4). This occurs if the Jacobian is (nearly) singular; again it is good practice to verify that the regularity requirement is satisfied. If this is the case, it is likely that the linear solver cannot handle the problem at hand.

The dense and sparse algorithms in HOMPACK90 differ not only in the storage format of the Jacobian but also in the low-level numerical linear algebra routines. In our experience, the dense linear solver has been relatively robust, while the sparse linear solver has sometimes failed. The dense algorithms in HOMPACK90 use QR decomposition – a direct method – to solve linear systems. The sparse algorithms use the iterative generalized minimal residual (GMRES) method (Saad & Schultz 1986) coupled with incomplete LU (ILU) preconditioning. Thus, HOMPACK90 solves $(\mathbf{Q}^{-1}\mathbf{A})\,\mathbf{z} = (\mathbf{Q}^{-1}\mathbf{b})$, where $\mathbf{Q}$ is the ILU preconditioner of $\mathbf{A}$. $\mathbf{Q}$ is chosen to make $(\mathbf{Q}^{-1}\mathbf{A})$ close to diagonal and easy to evaluate.

Both Layne Watson (the principal author of HOMPACK90) and Ken Judd (an authority on numerical methods in economics) acknowledge that the GMRES method can and does fail for some problems. There is no guidance as to which problems are susceptible but we strongly suspect problems with extremely sparse Jacobians. As explained in Section 3.5.2, constructing a system of equations for the quality ladder model using the "pure" version of the Zangwill & Garcia (1981) formulation yields such a sparse Jacobian. Our proposal is to reduce the size and sparsity of the Jacobian by eliminating variables. Since this proposal may

[17] The Jacobian $\frac{\partial \mathbf{H}(\mathbf{y})}{\partial \mathbf{y}}$ is a $(N \times (N+1))$ matrix whereas the linear solver requires a square matrix. All three path-following algorithms therefore add a row to the Jacobian. This extra row is simply a basis vector in case the ODE-based and normal flow algorithms and the augmented Jacobian algorithm uses a vector that is tangent to the solution path.

not be applicable or successful in other applications, we discuss in Section 3.5.7 a number of additional reformulations of the complementary slackness conditions.

In addition, we also offer the following suggestions: (i) Reorder the unknowns and/or equations to change the order of the columns and rows of the Jacobian. (ii) Use the dense algorithms if the dimension of the problem is less than several hundred equations. (iii) Increase the limit on the number of GMRES iterations and/or increase the "k" in GMRES(k) (GMRES(k) is restarted every k iterations until the residual norm is small enough (Watson et al. 1997)). (iv) Remove the ILU preconditioning and use GMRES by itself to solve the linear system. (v) Replace the sparse linear solver in HOMPACK90.[18]

3.5.7. Other Reformulations of the Complementary Slackness Condition

The user has considerable freedom in formulating the system of equations that characterizes the equilibria of a dynamic stochastic game and some formulations work better than others in some applications. In what follows we present additional reformulations of the complementary slackness conditions. The additional reformulations of the complementary slackness conditions are applicable to problems with simple inequality constraints that force a variable to be above or below a constant.[19]

First reformulation. Recall that in the quality ladder model the investment decision is[20]

$$x(\omega) = \max\left\{0, \frac{-1 + \sqrt{\beta\alpha\left(W^1(\omega) - W^0(\omega)\right)}}{\alpha}\right\}$$

[18]We thank Layne Watson for some of these suggestions. We warn the reader that implementing some of them requires in-depth knowledge of HOMPACK90.

[19]More general formulations for simple inequality constraints other than the non-negativity constraint in the quality ladder model are available from the authors upon request.

[20]We have eliminated the firm indices because we restrict attention to symmetric equilibria.

if $(W^1(\omega) - W^0(\omega)) \geq 0$ and $x(\omega) = 0$ otherwise. Taken together, the above can be restated as

$$x(\omega) = \frac{-1 + \sqrt{\max\left\{1, \beta\alpha\left(W^1(\omega) - W^0(\omega)\right)\right\}}}{\alpha}. \tag{3.27}$$

It follows that the complementary slackness condition (3.18) is equivalent to equation (3.27). However, we cannot simply replace the complementary slackness condition (3.18) with equation (3.27) because the max operator in the argument of the square root introduces a kink, thereby violating the smoothness requirement of the homotopy method. But we can eliminate this kink through a change of variables. To see this, let

$$\xi^k(\omega) = \beta\alpha\left(W^1(\omega) - W^0(\omega)\right) - 1, \tag{3.28}$$

where $k \in \mathbb{N}$ is odd.[21] This allows us to restate the investment decision as

$$x(\omega) = \frac{-1 + \sqrt{\left[\max\left\{0, \xi(\omega)\right\}\right]^k + 1}}{\alpha}. \tag{3.29}$$

Thus we can replace the complementary slackness condition (3.18) with equations (3.28) and (3.29). By setting $k \geq 3$ we ensure that the system of equations is twice continuously differentiable.

We can further eliminate the model variable $x(\omega)$ by using equation (3.29) to substitute for it in equations (3.28) and (3.21) to construct a system of $2M^2$ equations in the $2M^2$ unknowns $V(1,1), \ldots, V(M,M)$ and $\xi(1,1), \ldots, \xi(M,M)$.

[21] Note if k is odd, then $\xi(\omega) = (\beta\alpha(W^1(\omega) - W^0(\omega)) - 1)^{1/k}$, which follows from equation (3.28), is always well-defined.

Besides the quality ladder model, this reformulation of the complementary slackness condition can also be applied in other models of investment wherein investment is constrained to be above or below a constant and the first-order condition is quadratic.

Second reformulation. Consider an unconstrained version of the quality ladder model. The investment decision is

$$\theta(\omega) = \frac{-1 + \sqrt{\beta\alpha \left(W^1(\omega) - W^0(\omega)\right)}}{\alpha} \tag{3.30}$$

if $W^1(\omega) - W^0(\omega) \geq 0$ whereas otherwise the problem has no solution. We can recover the investment decision in the constrained version of the model as follows.

$$x(\omega) = \max\left\{0, \theta(\omega)\right\} \tag{3.31}$$

Hence, the complementary slackness condition (3.18) is equivalent to equations (3.30) and (3.31). Unfortunately, the max operator in equation (3.31) introduces a kink, thereby violating the smoothness requirement. We address this problem by letting $\phi(\omega)^k$ instead of $\theta(\omega)$ be the investment decision in the unconstrained version of the model, where $k \in \mathbb{N}$ is odd. This yields the following system of equations:

$$[\phi(\omega)]^k - \frac{-1 + \sqrt{\beta\alpha \left(W^1(\omega) - W^0(\omega)\right)}}{\alpha} = 0 \tag{3.32}$$

$$x(\omega) - \max\{0, [\phi(\omega)]^k\} = 0 \tag{3.33}$$

We can also eliminate the model variable $x(\omega)$ by using equation (3.33) to substitute for it in equations (3.32) and (3.21) to construct a system of $2M^2$ equations in the $2M^2$ unknowns $V\left(1,1\right), \ldots, V(M,M)$ and $\phi(1,1), \ldots, \phi(M,M)$.

In general, this formulation can be used when the first-order condition of the unconstrained problem has a solution. Hence, to present this formulation within the context of the quality ladder model, we have to assume that the investment decision in the unconstrained version of the model is always well defined. Although we have not proven this, a sufficient condition is that the value function is nondecreasing in a firm's state, and this does seem to be the case for the quality ladder model (and many other applications). Moreover, the first-order condition of the unconstrained problem must be expressed in a manner that ensures that its unique solution is the desired optimum, i.e., the second-order condition must hold at the unique solution to the chosen formulation of the first-order condition.

3.6. Artificial and All-Solutions Homotopies

Below we discuss some other uses of the homotopy method. We first introduce the distinction between natural-parameter homotopies, which trace out an entire path of solutions by varying a parameter of interest, and artificial homotopies, which obtain a solution for a particular parameterization. Then we present all-solutions homotopies that aim to obtain all solutions to systems of equations with certain properties.

3.6.1. Artificial Homotopies

While the homotopy method can be used to trace out an entire path of solutions by varying a parameter of interest, it has another important application, namely obtaining a solution to a system of equations for a particular parameterization. Consider the system of equations

$$\mathbf{F}(\mathbf{x}) = \mathbf{0} \tag{3.34}$$

where $\mathbf{F} : \mathbb{R}^N \to \mathbb{R}^N$ and $\mathbf{0} \in \mathbb{R}^N$ is a vector of zeros and define

$$\mathbf{H}(\mathbf{x}, \lambda) = \lambda \mathbf{F}(\mathbf{x}) + (1 - \lambda)(\mathbf{x} - \mathbf{a}), \tag{3.35}$$

where $\mathbf{a} \in \mathbb{R}^N$ is a vector. An artificial homotopy traces out a path from $\lambda = 0$, where the solution to $\mathbf{H}(\mathbf{x}, 0) = \mathbf{0}$ is $\mathbf{x} = \mathbf{a}$, to $\lambda = 1$. As $\mathbf{H}(\mathbf{x}, 1) = \mathbf{F}(\mathbf{x})$, the choice of $\mathbf{a}$ does not matter; as long as it reaches $\lambda = 1$, the homotopy algorithm finds a solution to the system of equations (3.34). So, if the goal is to solve a dynamic stochastic game for a single parameterization (e.g., to obtain an initial condition as in Section 3.3.3), then HOMPACK90 offers the functionality to do so.

An artificial homotopy tends to be more robust than a homotopy that follows a "natural" parameter of the model. Watson et al. (1997) prove that if $\mathbf{F}$ is twice continuously differentiable and the Jacobian of $\mathbf{F}$ has full rank at any solution to the system of equations (3.34), then almost all starting points $\mathbf{a}$ will result in a path that has finite length and satisfies regularity at every point. Thus, the artificial homotopy will succeed in tracing out the entire path and, therefore, in solving the system of equations (3.34) with probability one. In practice, if the homotopy algorithm strays off the solution path to some point $\left(\bar{\mathbf{x}}, \bar{\lambda}\right)$ where $\mathbf{H}(\bar{\lambda}, \bar{\mathbf{x}}) \neq 0$, then it can change the value of the starting point from $\mathbf{a}$ to $\bar{\mathbf{a}}$ such that

$$\bar{\lambda}\mathbf{F}(\bar{\mathbf{x}}) + \left(1 - \bar{\lambda}\right)(\bar{\mathbf{x}} - \bar{\mathbf{a}}) = 0.$$

The homotopy algorithm then returns to the task of tracing out a solution path, starting from point $(\bar{\mathbf{a}}, 0)$, until it finds a solution to the system of equations (3.34). Note that in changing $\mathbf{a}$ to $\bar{\mathbf{a}}$, the homotopy algorithm simply elects to proceed along a different path, in particular, one that passes through the point $(\bar{\lambda}, \bar{\mathbf{x}})$ to which it has strayed.

Devising a globally convergent algorithm for solving dynamic stochastic games is complicated by the fact that, while we can generally guarantee that the system of equations is twice continuously differentiable, we cannot establish regularity (although, of course, successful usage of natural-parameter homotopies suggests that regularity holds for most – if not all – parameterizations). It remains to be seen how reliable artificial homotopies for dynamic stochastic games are, especially in comparison to the Pakes & McGuire (1994) algorithm and other nonlinear solvers (see Ferris et al. 2007). Moreover, while an artificial homotopy is extremely robust, it may prove to be less efficient than nonlinear solvers that are based on Newton's method.

3.6.2. All-Solutions Homotopies

The problem of finding all solutions to a system of equations is largely unresolved in the mathematics literature. Indeed, as already noted, there is no guarantee that the homotopy method finds all the equilibria of a dynamic stochastic game.

In some cases, it is possible to exploit the structure of the system of equations. For example, the system of equations that characterizes the Nash equilibria of a finite game is polynomial (see, e.g., McKelvey & McLennan 1996). For polynomial systems, in turn, there are methods that are sure to find all solutions. These so-called all-solutions homotopies have been implemented in the freely-available software package Gambit (McKelvey, McLennan & Turocy 2006) and used by Bajari, Hong & Ryan (2004) in the context of static games. Judd & Schmedders (2004) use all-solutions homotopies to construct a computational uniqueness proof for a class of dynamic stochastic games in which movements through the state space are unidirectional and the primitives are given by polynomials. We refer the reader to Chapter 18 of Zangwill & Garcia (1981) for further details on all-solutions homotopies.

While there is little reason to believe that all-solutions homotopies can be extended to general classes of dynamic stochastic games, we emphasize that both natural-parameter and artificial homotopies can and have been used to identify multiple solutions. Our experience suggests that following different parameters (in the case of natural-parameter homotopies) is often a successful strategy. Wolf & Sanders (1996) provide a number of additional suggestions, including using different starting points $\mathbf{a}$ (in the case of artificial homotopies); using a complex homotopy parameter or multiple real homotopy parameters; and allowing the homotopy algorithm to proceed beyond $\lambda = 1$ in the hope that it will bend back and find another solution at $\lambda = 1$. While none of these suggestions is foolproof, in our view, striving to find some solutions is at least a first step toward finding all solutions.

References

Ackerberg, D., Benkard, L., Berry, S. & Pakes, A. (2005), Econometric tools for analyzing market outcomes, *in* J. Heckman, ed., 'Handbook of Econometrics', Vol. 6, North-Holland, Amsterdam, p. forthcoming.

Aguirregabiria, V. & Mira, P. (2002), 'Swapping the nested fixed point algorithm: A class of estimators for discrete markov decision models', *Econometrica* **70**(5), 1519–1543.

Aguirregabiria, V. & Mira, P. (2007), 'Sequential estimation of dynamic discrete games', *Econometrica* **75**(1), 1–53.

Alchian, A. (1963), 'Reliability of progress curves in airframe production', *Econometrica* **31**(4), 679–693.

Allgower, E. & Georg, K. (1992), 'Continuation and path following', *Acta Numerica* pp. 1–64.

Argote, L., Beckman, S. & Epple, D. (1990), 'The persistence and transfer of learning in an industrial setting', *Management Science* **36**(2), 140–154.

Athey, S. & Schmutzler, A. (2001), 'Investment and market dominance', *Rand Journal of Economics* **32**(1), 1–26.

Azoulay, P. (2002), 'Do pharmaceutical sales respond to scientific evidence?', *Journal of Economics and Management Strategy* **11**(4), 551–594.

Bailey, C. (1989), 'Forgetting and the learning curve: A laboratory study', *Management Science* **35**(3), 340–352.

Bajari, P., Benkard, L. & Levin, J. (2007), 'Estimating dynamic models of imperfect competition', *Econometrica* **75**(5), 1331–1370.

Bajari, P., Hong, H., Krainer, J. & Nekipelov, D. (2006), Estimating static models of strategic interactions, Working paper, University of Minnesota, Minneapolis.

Bajari, P., Hong, H. & Ryan, S. (2004), Identification and estimation of discrete games of complete information, Working paper, Duke University, Durham.

Baloff, N. (1971), 'Extension of the learning curve: Some empirical results', *Operational Reserach Quarterly* **22**(4), 329–340.

Benkard, L. (2000), 'Learning and forgetting: The dynamics of aircraft production', *American Economic Review* **90**(4), 1034–1054.

Benkard, L. (2004), 'A dynamic analysis of the market for wide-bodied commercial aircraft', *Review of Economic Studies* **71**(3), 581–611.

Berndt, E., Bui, L., Reily, D. & Urban, G. (1994), The roles of marketing, product quality and price commpetition in the growth and composition of the U.S. anti-ulcer drug industry, Working paper no. 4904, NBER, Cambridge.

Berndt, E., Bui, L., Reily, D. & Urban, G. (1995), 'Information, marketing, and pricing in the U.S. antiulcer drug market', *American Economic Review* **85**(2), 100–105.

Berry, S. & Pakes, A. (2007), 'The pure characteristics demand model', *International Economic Review* **48**(4), 1193–1225.

Bertsekas, D. & Tsitsiklis, J. (1997), *Parallel and distributed computation: Numerical methods*, Athena Scientific, Belmont.

Besanko, D. & Doraszelski, U. (2004), 'Capacity dynamics and endogenous asymmetries in firm size', *Rand Journal of Economics* **35**(1), 23–49.

Besanko, D., Doraszelski, U., Kryukov, Y. & Satterthwaite, M. (2007), Learning-by-doing, organizational forgetting, and industry dynamics, Working paper, Harvard University, Cambridge.

Besanko, D., Doraszelski, U., Lu, L. & Satterthwaite, M. (2006), Capacity investment dynamics in oligopolistic industries, Working paper, Northwestern University, Evanston.

Bischof, C., Khademi, P., Mauer, A. & Carle, A. (1996), 'ADIFOR 2.0: Automatic differentiation of Fortran 77 programs', *IEEE Computational Science and Engineering* **3**(3), 18–32.

Bohn, R. (1995), 'Noise and learning in semiconductor manufacturing', *Management Science* **41**(1), 31–42.

Borkovsky, R., Doraszelski, U. & Satterthwaite, M. (2007), A dynamic quality ladder oligopoly with spillovers, Working paper, Northwestern University, Evanston.

Budd, C., Harris, C. & Vickers, J. (1993), 'A model of the evolution of duopoly: Does the asymmetry between firms tend to increase or decrease?', *Review of Economic Studies* **60**(3), 543–573.

Cabral, L. & Riordan, M. (1994), 'The learning curve, market dominance, and predatory pricing', *Econometrica* **62**(5), 1115–1140.

Cabral, L. & Riordan, M. (1997), 'The learning curve, predation, antitrust, and welfare', *Journal of Industrial Economics* **45**(2), 155–169.

Caplin, A. & Nalebuff, B. (1991), 'Aggregation and imperfect competition: On the existence of equilibrium', *Econometrica* **59**(1), 26–59.

Caves, R. & Porter, M. (1977), 'From entry barriers to mobility barriers: Conjectural decisions and contrived deterrence to new competition', *Quarterly Journal of Economics* **91**(2), 241–261.

Charnes, A., Garcia, C. & Lemke, C. (1977), 'Constructive proofs of theorems relating to f(x)=y, with applications', *Mathematical Programming* **12**(1), 328–343.

Cockburn, I. & Henderson, R. (1994), 'Racing to invest? The dynamics of competition in ethical drug discovery', *Journal of Economics and Management Strategy* **3**(3), 481–519.

Cockburn, I. & Henderson, R. (1996), 'Public-private interaction in pharmaceutical research', *Proceedings of the National Academy of Sciences* **93**, 12725–12730.

Cockburn, I. & Henderson, R. (2001), 'Scale and scope in drug development: unpacking the advantages of size in pharmaceutical research', *Journal of Health Economics* **20**, 1033–1057.

Cockburn, I., Henderson, R. & Stern, S. (2001), The diffusion of science-driven drug discovery: Organizational change in pharmaceutical research, Working paper no. 7359, NBER.

Collard-Wexler, A. (2005), Plant turnover and demand fluctuations in the ready-mix concrete industry, Working paper, New York University, New York.

Danzon, M., Nicholson, S. & Pereira, N. (2003), Productivity in pharmaceutical-biotechnology R&D: the role of experience and alliances, Working paper no. 9615, NBER, Cambridge.

Darr, E., Argote, L. & Epple, D. (1995), 'The acquisition, transfer, and depreciation of knowledge in service organizations: Productivity in franchises', *Management Science* **41**(11), 1750–1762.

Dasgupta, P. & Stiglitz, J. (1988), 'Learning-by-doing, market structure and industrial and trade policies', *Oxford Economics Papers* **40**(2), 246–268.

DeJong, J. (1957), 'The effects of increasing skills on cycle time and its consequences for time standards', *Ergonomics* **1**(1), 51–60.

Demichelis, S. & Germano, F. (2002), 'On (un)knots and dynamics in games', *Games and Economic Behavior* **41**, 46–60.

DiMasi, J., Hansen, R. & Grabowski, H. (2003), 'The price of innovation: new estimates of drug development costs', *Journal of Health Economics* **22**(2), 151–185.

Doraszelski, U. & Judd, K. (2004), Avoiding the curse of dimensionality in dynamic stochastic games, Technical working paper no. 304, NBER, Cambridge.

Doraszelski, U. & Pakes, A. (2007), A framework for applied dynamic analysis in IO, *in* M. Armstrong & R. Porter, eds, 'Handbook of Industrial Organization', Vol. 3, North-Holland, Amsterdam, pp. 1887–1966.

Doraszelski, U. & Satterthwaite, M. (2007), Computable Markov-perfect industry dynamics: Existence, purification, and multiplicity, Working paper, Harvard University, Cambridge.

Dranove, D. & Meltzer, D. (1994), 'Do important drugs reach the market sooner?', *Rand Journal of Economics* **25**(3), 402–423.

Dudley, L. (1972), 'Learning and productivity changes in metal products', *American Economic Review* **62**(4), 662–669.

Dutton, J. & Thomas, A. (1984), 'Treating progress functions as a managerial opportunity', *Academy of Management Review* **9**(2), 235–247.

Eaves, C. & Schmedders, K. (1999), 'General equilibrium models and homotopy methods', *Journal of Economic Dynamics and Control* **23**(9–10), 1249–1279.

Ericson, R. & Pakes, A. (1995), 'Markov-perfect industry dynamics: A framework for empirical work', *Review of Economic Studies* **62**, 53–82.

Ferris, M., Judd, K. & Schmedders, K. (2007), Solving dynamic games with Newton's method, Working paper, University of Wisconsin, Madison.

Fudenberg, D. & Tirole, J. (1983), 'Learning-by-doing and market performance', *Bell Journal of Economics* **14**(2), 522–530.

Fudenberg, D. & Tirole, J. (1991), *Game theory*, MIT Press, Cambridge.

Garcia, C. & Zangwill, W. (1979), 'An approach to homotopy and degree theory', *Mathematics of Operations Reserach* **4**(4), 390–405.

Ghemawat, P. & Spence, M. (1985), 'Learning curve spillovers and market performance', *Quarterly Journal of Economics* **100**(Supplement), 839–852.

Golubitsky, M. & Schaeffer, D. (1985), *Singularities and groups in bifurcation theory*, Vol. 1, Springer, New York.

Grabowski, H. & Vernon, J. (1994), 'Returns to R&D on new drug introductions in the 1980s', *Journal of Health Economics* **13**, 383–406.

Gruber, H. (1992), 'The learning curve in the production of semiconductor memory chips', *Applied Economics* **24**, 885–895.

Guedj, I. & Scharfstein, D. (2004), 'Organizational scope and investment: Evidence from the drug development strategies of biopharmaceutical firms', *mimeo* .

Hatch, N. & Mowery, D. (1998), 'Process innovation and learning by doing in semiconductor manufacturing', *Management Science* **44**(1), 1461–1477.

Henderson, R. & Cockburn, I. (1994), 'Measuring competence? Exploring firm effects in pharmaceutical research', *Strategic Management Journal* **15**, 63–74.

Henderson, R. & Cockburn, I. (1996), 'Scale, scope, and spillovers: The determinants of research productivity in drug discovery', *Rand Journal of Economics* **7**(1), 32–59.

Herings, J. & Peeters, R. (2004), 'Stationary equilibria in stochastic games: Structure, selection, and computation', *Journal of Economic Theory* **118**, 32–60.

Hirsch, W. (1952), 'Progress functions of machine tool manufacturing', *Econometrica* **20**(1), 81–82.

Hirschmann, W. (1964), 'Profit from the learning curve', *Harvard Business Review* **42**(1), 125–139.

Irwin, D. & Klenow, P. (1994), 'Learning-by-doing spillovers in the seminconductor industry', *Journal of Political Economy* **102**(6), 1200–1227.

Jarmin, R. (1994), 'Learning by doing and competition in the early rayon industry', *Rand Journal of Economics* **25**(3), 441–454.

Judd, K. (1998), *Numerical methods in economics*, MIT Press, Cambridge.

Judd, K. & Schmedders, K. (2004), A computational approach to proving uniqueness in dynamic games, Working paper, Hoover Institution, Stanford.

Judd, K., Schmedders, K. & Yeltekin, S. (2002), Optimal rules for patent races, Working paper, Hoover Institution, Stanford.

Kilbridge, M. (1962), 'A model of industrial learning', *Management Science* **8**(4), 516–527.

Levy, F. (1965), 'Adaptation in the production process', *Management Science* **11**(6), 136–154.

Lewis, T. & Yildirim, H. (2002), 'Managing dynamic competition', *American Economic Review* **92**(4), 779–797.

Lewis, T. & Yildirim, H. (2005), 'Managing switching costs in multiperiod procurements with strategic buyers', *International Economic Review* **46**(4), 1233–1269.

Lieberman, M. (1984), 'The learning curve and pricing in the chemical processing industries', *Rand Journal of Economics* **15**(2), 213–228.

McKelvey, R. & McLennan, A. (1996), Computation of equilibria in finite games, *in* H. Amman, D. Kendrick & J. Rust, eds, 'Handbook of Computational Economics', North-Holland, Amsterdam, pp. 87–142.

McKelvey, R., McLennan, A. & Turocy, T. (2006), Gambit: Software tools for game theory, Technical report, California Institute of Technology, Pasadena.

Pakes, A. (2008), Theorey and empirical work on imperfectly competitive markets, Fisher-Schultz lecture, Harvard University, Cambridge.

Pakes, A. & McGuire, P. (1994), 'Computing Markov-perfect Nash equilibria: Numerical implications of a dynamic differentiated product model', *Rand Journal of Economics* **25**(4), 555–589.

Pakes, A. & McGuire, P. (2001), 'Stochastic algorithms, symmetric Markov perfect equilibrium, and the "curse" of dimensionality', *Econometrica* **69**(5), 1261–1281.

Pakes, A., Ostrovsky, M. & Berry, S. (2006), 'Simple estimators for the parameters of discrete dynamic games (with entry/exit examples)', *Rand Journal of Economics* **forthcoming**.

Pesendorfer, M. & Schmidt-Dengler, P. (2003), Identification and estimation of dynamic games, Working paper no. 9726, NBER, Cambridge.

Pisano, G. (1994), 'Knowledge, integration, and the locus of learning: An empirical analysis of process development', *Strategic Management Journal* **15**, 85–100.

Preston, L. & Keachie, E. (1964), 'Cost functions and progress functions: An integration', *American Economic Review* **54**(2), 100–107.

Roberts, P. (1999), 'Product innovation, product-market competition and persistent profitability in the U.S. pharmaceutical industry', *Strategic Management Journal* **20**(7), 655–670.

Ross, D. (1986), 'Learning to dominate', *Journal of Industrial Economics* **34**(4), 337–353.

Ryan, S. (2005), The costs of environmental regulation in a concentrated industry, Working paper, Duke University, Durham.

Saad, Y. & Schultz, M. (1986), 'GMRES: A generalized minimal residual algorithm for solving nonsymmetric linear systems', *SIAM Journal on Scientific and Statistical Computing* **7**(3), 856–869.

Schmedders, K. (1998), 'Computing equilibria in the general equilibrium model with incomplete asset markets', *Journal of Economic Dynamics and Control* **22**, 1375–1401.

Schmedders, K. (1999), 'A homotopy algorithm and an index theorem for the general equilibrium model with incomplete asset markets', *Journal of Mathematical Economics* **32**(2), 225–241.

Schmidt-Dengler, P. (2005), The timing of new technology adoption: The case of MRI, Working paper, LSE, London.

Shafer, S., Nembhard, D. & Uzumeri, M. (2001), 'The effects of worker learning, forgetting, and heterogeneity on assembly line productivity', *Management Science* **47**(12), 1639–1653.

Spence, M. (1981), 'The learning curve and competition', *Bell Journal of Economics* **12**(1), 49–70.

Thompson, P. (2001), 'How much did the Liberty shipbuilders learn? New evidence for an old case study', *Journal of Political Economy* **109**(1), 103–137.

Thompson, P. (2003), How much did the Liberty shipbuilders forget?, Working paper, Florida International University, Miami.

Thornton, R. & Thompson, P. (2001), 'Learning from experience and learning from others: An exploration of learning and spillovers in wartime shipbuilding', *American Economic Review* **91**(5), 1350–1368.

Vickers, J. (1986), 'The evolution of market structure when there is a sequence of innovations', *Journal of Industrial Economics* **35**(1), 1–12.

Vives, X. (1999), *Oligopoly pricing: Old ideas and new tools*, MIT Press, Cambridge.

Watson, L., Billups, S. & Morgan, A. (1987), 'HOMPACK: A suite of codes for globally convergent homotopy algorithms', *ACM Transcations on Mathematical Software* **13**(3), 281–310.

Watson, L., Sosonkina, M., Melville, R., Morgan, A. & Walker, H. (1997), 'Algorithm 777: HOMPACK90: A suite of Fortran 90 codes for globally convergent homotopy algorithms', *ACM Transcations on Mathematical Software* **23**(4), 514–549.

Wixted, M. & Ebbesen, E. (1991), 'On the form of forgetting', *Psychological Science* **2**(6), 409–415.

Wolf, D. & Sanders, S. (1996), 'Multiparameter homotopy methods for finding DC operating points of nonlinear circuits', *IEEE Transactions on Circuits and Systems – I: Fundamental Theory and Applications* **43**(10), 824–838.

Wright, T. (1936), 'Factors affecting the cost of airplanes', *Journal of Aeronautical Sciences* **3**(4), 122–128.

Zangwill, W. & Garcia, C. (1981), *Pathways to solutions, fixed points, and equilibria*, Prentice Hall, Englewood Cliffs.

Zimmerman, M. (1982), 'Learning effects and the commercialization of new energy technologies: The case of nuclear power', *Bell Journal of Economics* **13**(2), 297–310.

APPENDIX

Proofs from Chapter 1

PROOF OF PROPOSITION 1. *Part (i):* The basic differential equations (3.6) set

$$\delta'(s) = \det\left(\frac{\partial \mathbf{F}(\mathbf{x}(s), \delta(s))}{\partial \mathbf{x}}\right).$$

The Jacobian $\frac{\partial \mathbf{F}(\mathbf{x}(s),\delta(s))}{\partial \mathbf{x}}$ is a $(2M^2 \times 2M^2)$ matrix and therefore has an even number of eigenvalues. Its determinant is the product of its eigenvalues. Hence, if $\delta'(s) \leq 0$, then there exists at least one real nonnegative eigenvalue. (Suppose to the contrary that all eigenvalues are either complex or real and negative. Since the number of complex eigenvalues is even, so is the number of real eigenvalues. Moreover, the product of a conjugate pair of complex eigenvalues is positive, as is the product of an even number of real negative eigenvalues.)

To relate the Pakes & McGuire (1994) algorithm to our homotopy algorithm, consider a parametric path $(\mathbf{x}(s), \delta(s)) \in \mathbf{F}^{-1}$ in the equilibrium correspondence. We show in the Online Appendix that

$$\left.\frac{\partial \mathbf{G}(\mathbf{x}(s))}{\partial \mathbf{x}}\right|_{\delta=\delta(s)} = \frac{\partial \mathbf{F}(\mathbf{x}(s), \delta(s))}{\partial \mathbf{x}} + \mathbf{I}, \tag{.1}$$

where $\mathbf{I}$ denotes the $(2M^2 \times 2M^2)$ identity matrix.

The proof is completed by recalling a basic result from linear algebra: Let $\mathbf{A}$ be an arbitrary matrix and $\sigma(\mathbf{A})$ its spectrum. Then $\sigma(\mathbf{A} + \mathbf{I}) = \sigma(\mathbf{A}) + 1$ (see Proposition A.17 in Appendix A of Bertsekas & Tsitsiklis 1997). Hence, because $\frac{\partial \mathbf{F}(\mathbf{x}(s),\delta(s))}{\partial \mathbf{x}}$ has at least one real nonnegative eigenvalue, it follows from equation (.1) that $\left.\frac{\partial \mathbf{G}(\mathbf{x}(s))}{\partial \mathbf{x}}\right|_{\delta=\delta(s)}$ has at least one real eigenvalue equal to or bigger than unity. Hence, $\rho\left(\left.\frac{\partial \mathbf{G}(\mathbf{x}(s))}{\partial \mathbf{x}}\right|_{\delta=\delta(s)}\right) \geq 1$.

Part (ii): Consider the iteration $\mathbf{x}^{k+1} = \tilde{\mathbf{G}}(\mathbf{x}^k) = \omega\mathbf{G}(\mathbf{x}^k) + (1-\omega)\mathbf{x}^k$, where $\omega > 0$. Using equation (.1) its Jacobian at $(\mathbf{x}(s), \delta(s)) \in \mathbf{F}^{-1}$ is

$$\left.\frac{\partial\tilde{\mathbf{G}}(\mathbf{x}(s))}{\partial\mathbf{x}}\right|_{\delta=\delta(s)} = \omega \left.\frac{\partial\mathbf{G}(\mathbf{x}(s))}{\partial\mathbf{x}}\right|_{\delta=\delta(s)} + (1-\omega)\mathbf{I} = \omega\frac{\partial\mathbf{F}(\mathbf{x}(s),\delta(s))}{\partial\mathbf{x}} + \mathbf{I}.$$

As before it follows that $\rho\left(\left.\frac{\partial\tilde{\mathbf{G}}(\mathbf{x}(s))}{\partial\mathbf{x}}\right|_{\delta=\delta(s)}\right) \geq 1$. □

Proof of Proposition 2. We rewrite the Bellman equations and FOCs in state $\mathbf{e}$ as

$$V_1 = D_1(p_1,p_2)\left(p_1 - c(e_1) + \beta\left(\overline{V}_{11} - \overline{V}_{12}\right)\right) + \beta\overline{V}_{12}, \tag{.2}$$

$$V_2 = D_2(p_1,p_2)\left(p_2 - c(e_2) + \beta\left(\overline{V}_{22} - \overline{V}_{21}\right)\right) + \beta\overline{V}_{21}, \tag{.3}$$

$$0 = \frac{\sigma}{D_2(p_1,p_2)} - \left(p_1 - c(e_1) + \beta\left(\overline{V}_{11} - \overline{V}_{12}\right)\right), \tag{.4}$$

$$0 = \frac{\sigma}{D_1(p_1,p_2)} - \left(p_2 - c(e_2) + \beta\left(\overline{V}_{22} - \overline{V}_{21}\right)\right), \tag{.5}$$

where, to simplify the notation, V_n is shorthand for $V_n(\mathbf{e})$, $\overline{V}_{nk}$ for $\overline{V}_{nk}(\mathbf{e})$, p_n for $p_n(\mathbf{e})$, etc. and we use the fact that $D_1(p_1,p_2) + D_2(p_1,p_2) = 1$.

Case (i): First suppose $\delta = 0$. The proof proceeds in a number of steps. In step 1, we establish that the equilibrium in state (M,M) is unique. In step 2a, we assume that there is a unique equilibrium in state (e_1+1, M), where $e_1 \in \{1,\ldots,M-1\}$, and show that this implies that the equilibrium in state (e_1, M) is unique. In step 2b, we assume that there is a unique equilibrium in state (M, e_2+1), where $e_2 \in \{1,\ldots,M-1\}$, and show that this implies that the equilibrium in state (M, e_2) is unique. By induction, steps 1, 2a, and 2b establish uniqueness along the upper edge of the state space. In step 3, we assume that there is a unique equilibrium in states (e_1+1, e_2) and (e_1, e_2+1), where

$e_1 \in \{1, \ldots, M-1\}$ and $e_2 \in \{1, \ldots, M-1\}$, and show that this implies that the equilibrium in state (e_1, e_2) is unique. Hence, uniqueness in state $(M-1, M-1)$ follows from uniqueness in states $(M, M-1)$ and $(M-1, M)$, uniqueness in state $(M-2, M-1)$ from uniqueness in states $(M-1, M-1)$ and $(M-2, M)$, etc. Working backwards gives uniqueness in states $(e_1, M-1)$, where $e_1 \in \{1, \ldots, M-1\}$. This, in turn, gives uniqueness in states $(e_1, M-2)$, where $e_1 \in \{1, \ldots, M-1\}$, etc.

Step 1: Consider state $\mathbf{e} = (M, M)$. From the definition of the state-to-state transitions in Section 1.2, we have

$$\overline{V}_{11} = \overline{V}_{12} = V_1, \quad \overline{V}_{21} = \overline{V}_{22} = V_2.$$

Imposing these restrictions and solving equations (.2) and (.3) for V_1 and V_2, respectively, yields

$$V_1 = \frac{D_1(p_1, p_2)(p_1 - c(e_1))}{1 - \beta}, \tag{.6}$$

$$V_2 = \frac{D_2(p_1, p_2)(p_2 - c(e_2))}{1 - \beta}. \tag{.7}$$

Simplifying equations (.4) and (.5) yields

$$0 = \frac{\sigma}{D_2(p_1, p_2)} - (p_1 - c(e_1)) = F_1(p_1, p_2), \tag{.8}$$

$$0 = \frac{\sigma}{D_1(p_1, p_2)} - (p_2 - c(e_2)) = F_2(p_1, p_2). \tag{.9}$$

The system of equations (.8) and (.9) determines equilibrium prices. Once we have established that there is a unique solution for p_1 and p_2, equations (.6) and (.7) immediately ascertain that V_1 and V_2 are unique.

Let $p_1^\natural(p_2)$ and $p_2^\natural(p_1)$ be defined by

$$F_1(p_1^\natural(p_2), p_2) = 0, \quad F_2(p_1, p_2^\natural(p_1)) = 0$$

and set $F(p_1) = p_1 - p_1^\natural(p_2^\natural(p_1))$. The p_1 that solves the system of equations (.8) and (.9) is the solution to $F(p_1) = 0$, and this solution is unique provided that $F(p_1)$ is strictly monotone. The implicit function theorem yields

$$F'(p_1) = 1 - \frac{\left(-\frac{\partial F_1}{\partial p_2}\right)}{\frac{\partial F_1}{\partial p_1}} \frac{\left(-\frac{\partial F_2}{\partial p_1}\right)}{\frac{\partial F_2}{\partial p_2}}.$$

Straightforward differentiation shows that

$$\frac{\left(-\frac{\partial F_1}{\partial p_2}\right)}{\frac{\partial F_1}{\partial p_1}} = \frac{-\frac{D_1(p_1,p_2)}{D_2(p_1,p_2)}}{-\frac{1}{D_2(p_1,p_2)}} = D_1(p_1, p_2) \in (0, 1),$$
$$\frac{\left(-\frac{\partial F_2}{\partial p_1}\right)}{\frac{\partial F_2}{\partial p_2}} = \frac{-\frac{D_2(p_1,p_2)}{D_1(p_1,p_2)}}{-\frac{1}{D_1(p_1,p_2)}} = D_2(p_1, p_2) \in (0, 1).$$

It follows that $F'(p_1) > 0$.

Step 2a: Consider state $\mathbf{e} = (e_1, M)$, where $e_1 \in \{1, \ldots, M-1\}$. We have

$$\overline{V}_{12} = V_1, \quad \overline{V}_{22} = V_2.$$

Imposing these restrictions and solving equations (.2) and (.3) for V_1 and V_2, respectively, yields

$$V_1 = \frac{D_1(p_1, p_2)(p_1 - c(e_1) + \beta\overline{V}_{11})}{1 - \beta D_2(p_1, p_2)}, \tag{.10}$$

$$V_2 = \frac{D_2(p_1, p_2)(p_2 - c(e_2) - \beta\overline{V}_{21}) + \beta\overline{V}_{21}}{1 - \beta D_2(p_1, p_2)}. \tag{.11}$$

Substituting equations (.10) and (.11) into equations (.4) and (.5) and dividing through by $\frac{1-\beta}{1-\beta D_2(p_1,p_2)}$ and $\frac{1}{1-\beta D_2(p_1,p_2)}$, respectively, yields

$$0 = \frac{(1-\beta D_2(p_1,p_2))\sigma}{(1-\beta)D_2(p_1,p_2)} - \big(p_1 - c(e_1) + \beta\overline{V}_{11}\big) = G_1(p_1,p_2), \tag{.12}$$

$$0 = \frac{(1-\beta D_2(p_1,p_2))\sigma}{D_1(p_1,p_2)} - \big(p_2 - c(e_2) - \beta(1-\beta)\overline{V}_{21}\big) = G_2(p_1,p_2). \tag{.13}$$

The system of equations (.12) and (.13) determines equilibrium prices as a function of $\overline{V}_{11}$ and $\overline{V}_{21}$. These are given by $V_1(e_1+1,M)$ and $V_2(e_1+1,M)$, respectively, and are unique by hypothesis. As in step 1, once we have established that there is a unique solution for p_1 and p_2, equations (.10) and (.11) immediately ascertain that, in state $\mathbf{e} = (e_1, M)$, V_1 and V_2 are unique.

Proceeding as in step 1, set $G(p_1) = p_1 - p_1^\natural(p_2^\natural(p_1))$, where $p_1^\natural(p_2)$ and $p_2^\natural(p_1)$ are defined by $G_1(p_1^\natural(p_2), p_2) = 0$ and $G_2(p_1, p_2^\natural(p_1)) = 0$, respectively. We have to show that $G(\cdot)$ is strictly monotone. Straightforward differentiation shows that

$$\frac{\left(-\frac{\partial G_1}{\partial p_2}\right)}{\frac{\partial G_1}{\partial p_1}} = \frac{-\frac{D_1(p_1,p_2)}{(1-\beta)D_2(p_1,p_2)}}{-\frac{1-\beta D_2(p_1,p_2)}{(1-\beta)D_2(p_1,p_2)}} = \frac{D_1(p_1,p_2)}{1-\beta D_2(p_1,p_2)} \in (0,1),$$

$$\frac{\left(-\frac{\partial G_2}{\partial p_1}\right)}{\frac{\partial G_2}{\partial p_2}} = \frac{-\frac{(1-\beta)D_2(p_1,p_2)}{D_1(p_1,p_2)}}{-\frac{1-\beta D_2(p_1,p_2)}{D_1(p_1,p_2)}} = \frac{(1-\beta)D_2(p_1,p_2)}{1-\beta D_2(p_1,p_2)} \in (0,1).$$

It follows that $G'(p_1) > 0$.

Step 2b: Consider state $\mathbf{e} = (M, e_2)$, where $e_2 \in \{1, \ldots, M-1\}$. We have

$$\overline{V}_{11} = V_1, \quad \overline{V}_{21} = V_2.$$

The argument is completely symmetric to the argument in step 2a and therefore omitted.

Step 3: Consider state $\mathbf{e} = (e_1, e_2)$, where $e_1 \in \{1, \ldots, M-1\}$ and $e_2 \in \{1, \ldots, M-1\}$. The system of equations (.4) and (.5) determines equilibrium prices as a function of $\overline{V}_{11}$, $\overline{V}_{12}$, $\overline{V}_{21}$, and $\overline{V}_{22}$. These are given by $V_1(e_1+1, e_2)$, $V_1(e_1, e_2+1)$, $V_2(e_1+1, e_2)$, and $V_2(e_1, e_2+1)$, respectively, and are unique by hypothesis. As in step 1, once we have established that there is a unique solution for p_1 and p_2, equations (.2) and (.3) immediately ascertain that, in state $\mathbf{e} = (e_1, e_2)$, V_1 and V_2 are unique.

Let $H_1(p_1, p_2)$ and $H_2(p_1, p_2)$ denote the RHS of equation (.4) and (.5), respectively. Proceeding as in step 1, set $H(p_1) = p_1 - p_1^\natural(p_2^\natural(p_1))$, where $p_1^\natural(p_2)$ and $p_2^\natural(p_1)$ are defined by $H_1(p_1^\natural(p_2), p_2) = 0$ and $H_2(p_1, p_2^\natural(p_1)) = 0$, respectively. We have to show that $H(\cdot)$ is strictly monotone. Straightforward differentiation shows that

$$\frac{\left(-\frac{\partial H_1}{\partial p_2}\right)}{\frac{\partial H_1}{\partial p_1}} = \frac{-\frac{D_1(p_1,p_2)}{D_2(p_1,p_2)}}{-\frac{1}{D_2(p_1,p_2)}} = D_1(p_1, p_2) \in (0,1),$$

$$\frac{\left(-\frac{\partial H_2}{\partial p_1}\right)}{\frac{\partial H_2}{\partial p_2}} = \frac{-\frac{D_2(p_1,p_2)}{D_1(p_1,p_2)}}{-\frac{1}{D_1(p_1,p_2)}} = D_2(p_1, p_2) \in (0,1).$$

It follows that $H'(p_1) > 0$.

Case (ii): Next suppose $\delta = 1$. A similar induction argument as in the case of $\delta = 0$ can be used to establish the claim except that in the case of $\delta = 1$ we anchor the argument in state $(1, 1)$ rather than state (M, M). □

PROOF OF PROPOSITION 3. *Part (i):* Consider the static Nash equilibrium. The FOCs in state $\mathbf{e}$ are

$$p_1^\dagger(\mathbf{e}) = c(e_1) + \frac{\sigma}{1 - D_1(p_1^\dagger(\mathbf{e}), p_2^\dagger(\mathbf{e}))}, \tag{.14}$$

$$p_2^\dagger(\mathbf{e}) = c(e_2) + \frac{\sigma}{1 - D_2(p_1^\dagger(\mathbf{e}), p_2^\dagger(\mathbf{e}))}. \tag{.15}$$

Equations (.14) and (.15) imply $p_1^\dagger(\mathbf{e}) > c(e_1)$ and $p_2^\dagger(\mathbf{e}) > c(e_2)$ and thus in particular $p^\dagger(m,m) > c(m)$. In addition, $p^\dagger(\mathbf{e}) = p^\dagger(m,m)$ because $c(e_1) = c(e_2) = c(m)$ for all $\mathbf{e} \in \{m, \ldots, M\}^2$.

Turning to our dynamic stochastic game, suppose that $\delta = 0$. The proof of part (i) proceeds in a number of steps, similar to the proof of Proposition 2. In step 1, we establish that equilibrium prices in state (M, M) coincide with the static Nash equilibrium. In step 2a, we assume that the equilibrium in state $(e_1 + 1, M)$, where $e_1 \in \{m, \ldots, M-1\}$, coincides with the equilibrium in state (M, M) and show that this implies that the equilibrium in state (e_1, M) does the same. In step 2b, we assume that the equilibrium in state $(M, e_2 + 1)$, where $e_2 \in \{m, \ldots, M-1\}$, coincides with the equilibrium in state (M, M) and show that this implies that the equilibrium in state (M, e_2) does the same. In step 3, we assume that the equilibrium in states $(e_1 + 1, e_2)$ and $(e_1, e_2 + 1)$, where $e_1 \in \{m, \ldots, M-1\}$ and $e_2 \in \{m, \ldots, M-1\}$, coincides with the equilibrium in state (M, M) and show that this implies that the equilibrium in state (e_1, e_2) does the same. Also similar to the proof of Proposition 2, we continue to use V_n as shorthand for $V_n(\mathbf{e})$, $\overline{V}_{nk}$ for $\overline{V}_{nk}(\mathbf{e})$, p_n for $p_n(\mathbf{e})$, etc.

Step 1: Consider state $\mathbf{e} = (M, M)$. From the proof of Proposition 2, equilibrium prices are determined by the system of equations (.8) and (.9). Since equations (.8) and (.9)

are equivalent to equations (.14) and (.15), equilibrium prices are $p_1 = p_1^\dagger$ and $p_2 = p_2^\dagger$. Substituting equation (.8) into (.6) and equation (.9) into (.7) yields equilibrium values

$$V_1 = \frac{\sigma D_1(p_1, p_2)}{(1-\beta) D_2(p_1, p_2)}, \tag{.16}$$

$$V_2 = \frac{\sigma D_2(p_1, p_2)}{(1-\beta) D_1(p_1, p_2)}. \tag{.17}$$

Step 2a: Consider state $\mathbf{e} = (e_1, M)$, where $e_1 \in \{m, \ldots, M-1\}$. Equilibrium prices are determined by the system of equations (.12) and (.13). Given $\overline{V}_{11} = V_1(e_1+1, M) = V_1(M, M)$ and $\overline{V}_{21} = V_2(e_1+1, M) = V_2(M, M)$, it is easy to see that, in state $\mathbf{e} = (e_1, M)$, $p_1 = p_1(M, M)$ and $p_2 = p_2(M, M)$ are a solution. Substituting equation (.12) into (.10) and equation (.13) into (.11) yields equilibrium values $V_1 = V_1(M, M)$ and $V_2 = V_2(M, M)$ as given by equations (.16) and (.17).

Step 2b: Consider state $\mathbf{e} = (M, e_2)$, where $e_2 \in \{m, \ldots, M-1\}$. The argument is completely symmetric to the argument in step 2a and therefore omitted.

Step 3: Consider state $\mathbf{e} = (e_1, e_2)$, where $e_1 \in \{m, \ldots, M-1\}$ and $e_2 \in \{m, \ldots, M-1\}$. Equilibrium prices are determined by the system of equations (.4) and (.5). Given $\overline{V}_{11} = V_1(e_1+1, e_2) = V_1(M, M)$, $\overline{V}_{12} = V_1(e_1, e_2+1) = V_1(M, M)$, $\overline{V}_{21} = V_2(e_1+1, e_2) = V_2(M, M)$, and $\overline{V}_{22} = V_2(e_1, e_2+1) = V_2(M, M)$, it is easy to see that, in state $\mathbf{e} = (e_1, e_2)$, $p_1 = p_1(M, M)$ and $p_2 = p_2(M, M)$ are a solution. Substituting equation (.4) into (.2) and equation (.5) into (.3) yields equilibrium values $V_1 = V_1(M, M)$ and $V_2 = V_2(M, M)$ as given by equations (.16) and (.17).

Part (ii): We show that $p_2(\mathbf{e}) > c(m)$ for all $e_1 \in \{1, \ldots, m-1\}$ and $e_2 \in \{m, \ldots, M\}$. The claim follows because $p^*(\mathbf{e}) = p_2(\mathbf{e}^{[2]})$.

The proof of part (ii) proceeds in two steps. In step 1, we establish that the equilibrium price of firm 2 in state (e_1, M), where $e_1 \in \{1, \ldots, m-1\}$, exceeds $c(m)$. In step 2, we assume that the equilibrium in state (e_1, e_2+1), where $e_1 \in \{1, \ldots, m-1\}$ and $e_2 \in \{m, \ldots, M-1\}$, coincides with the equilibrium in state (e_1, M) and show that this implies that the equilibrium in state (e_1, e_2) does the same.

Step 1: Consider state $\mathbf{e} = (e_1, M)$, where $e_1 \in \{1, \ldots, m-1\}$. From the proof of Proposition 2, equilibrium prices are determined by the system of equations (.12) and (.13). Using the fact that in equilibrium $\overline{V}_{21} \geq 0$, equation (.13) implies $p_2 > c(m)$. Substituting equation (.12) into (.10) and equation (.13) into (.11) yields equilibrium values

$$V_1 = \frac{\sigma D_1(p_1, p_2)}{(1-\beta) D_2(p_1, p_2)}, \tag{.18}$$

$$V_2 = \frac{\sigma D_2(p_1, p_2) + \beta D_1(p_1, p_2) \overline{V}_{21}}{D_1(p_1, p_2)}. \tag{.19}$$

Step 2: Consider state $\mathbf{e} = (e_1, e_2)$, where $e_1 \in \{1, \ldots, m-1\}$ and $e_2 \in \{m, \ldots, M-1\}$. Equilibrium prices are determined by the system of equations (.4) and (.5). Given $\overline{V}_{12} = V_1(e_1, e_2+1) = V_1(e_1, M)$ and $\overline{V}_{22} = V_2(e_1, e_2+1) = V_2(e_1, M)$, it is easy to see that, in state $\mathbf{e} = (e_1, e_2)$, $p_1 = p_1(e_1, M)$ and $p_2 = p_2(e_1, M)$ are a solution. Substituting equation (.4) into (.2) and equation (.5) into (.3) yields equilibrium values $V_1 = V_1(e_1, M)$ and $V_2 = V_2(e_1, M)$ as given by equations (.18) and (.19). □

PROOF OF PROPOSITION 4. We rewrite the FOCs in state $\mathbf{e}$ as

$$0 = \frac{\sigma}{D_2(p_1,p_2)} - \left(p_1 - c(e_1) + \beta\left(\overline{V}_{11} - \overline{V}_{12}\right)\right), \tag{.20}$$

$$0 = \frac{\sigma}{D_1(p_1,p_2)} - \left(p_2 - c(e_2) + \beta\left(\overline{V}_{22} - \overline{V}_{21}\right)\right), \tag{.21}$$

where, to simplify the notation, $\overline{V}_{nk}$ is shorthand for $\overline{V}_{nk}(\mathbf{e})$, p_n for $p_n(\mathbf{e})$, etc. and we use the fact that $D_1(p_1,p_2) + D_2(p_1,p_2) = 1$. The system of equations (.20) and (.21) determines equilibrium prices. We have to establish that there is a unique solution for p_1 and p_2 irrespective of $\overline{V}_{11}$, $\overline{V}_{12}$, $\overline{V}_{21}$, and $\overline{V}_{22}$.

Let $H_1(p_1,p_2)$ and $H_2(p_1,p_2)$ denote the RHS of equation (.20) and (.21), respectively. Proceeding as in step 3 of the proof of Proposition 2, set $H(p_1) = p_1 - p_1^\natural(p_2^\natural(p_1))$, where $p_1^\natural(p_2)$ and $p_2^\natural(p_1)$ are defined by $H_1(p_1^\natural(p_2),p_2) = 0$ and $H_2(p_1,p_2^\natural(p_1)) = 0$, respectively. We have to show that $H(\cdot)$ is strictly monotone. Straightforward differentiation shows that

$$\frac{\left(-\frac{\partial H_1}{\partial p_2}\right)}{\frac{\partial H_1}{\partial p_1}} = \frac{-\frac{D_1(p_1,p_2)}{D_2(p_1,p_2)}}{-\frac{1}{D_2(p_1,p_2)}} = D_1(p_1,p_2) \in (0,1),$$

$$\frac{\left(-\frac{\partial H_2}{\partial p_1}\right)}{\frac{\partial H_2}{\partial p_2}} = \frac{-\frac{D_2(p_1,p_2)}{D_1(p_1,p_2)}}{-\frac{1}{D_1(p_1,p_2)}} = D_2(p_1,p_2) \in (0,1).$$

It follows that $H'(p_1) > 0$. □