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ABSTRACT

This dissertation contains two topics. The first topic focuses on how to use information design to minimize costs of implementing a policy that guarantees 100% passing rate of all participants by providing enough compensation for the their effort. The second topic explores an auction setting which involves financially distressed business insider and deep-pocketed investor.

Chapter 1 studies the design of a feedback system that encourages a group of agents with different ability levels to complete a task, given that they are sufficiently compensated for their effort. The goal is to design the feedback system in such a way that guarantees all agents finish the task regardless of their ability and at the same time minimizes the compensation required. In the model, a group of agents faces a task that requires costly effort to complete. An agent's progress toward task-completion is determined by both his ability and effort. The agent does not know his ability nor his progress toward task-completion, but the effort is his private information. A principal can monitor each agent's progress, but she has no information about an agent's ability nor effort. The principal's job is to provide enough compensation for the agents so that they all complete the task in the end. Meanwhile, the principal wants to keep the total compensation to a minimum. An optimal feedback system balances the trade-off between revealing too much information and letting the agents enjoy the information-rent from their privately known effort, versus revealing too little information and resulting in some agents over-exerting effort. The first-best outcome is one in which the principal's expenditure exactly equals the aggregate

amount of effort that is required so that all agents successfully finish the task. I first show under what conditions a feedback system can be designed to achieve the first-best outcome, and when the first-best outcome is not attainable, what the optimal feedback system should be. I then demonstrate that once more flexibility is allowed in designing such system, the first-best outcome is always attainable.

Chapter 2 studies an auction model with potentially budget-constrained business insiders and a deep-pocketed investor who is only constrained by his lack of insider knowledge but not his budget. I show that the outsider status sometimes puts the deep-pocketed investor in danger of the winner's curse. As a result, when the probability of business insiders being budget-constrained is low, the outsider bids more conservatively. Consequently, the auction's allocation efficiency is higher when insiders are less likely to be constrained by their budget. However, the seller does not necessarily benefit from this increase in allocation efficiency, as the expected revenue can decrease as a result of the deep-pocketed investor shading his bid. The presence of more business insiders helps alleviate the winner's curse and increase the seller's expected revenue in some situations. But in certain cases, for example, when the outsider's valuation depends on the average of the insiders' valuations, having many insiders in the auction only makes the outsider more cautious in his bidding. Therefore, aside from insiders' budget, the relationship between the outsider's valuation and those of the insiders' is equally if not more important from the seller's perspective. In addition, I also demonstrate that a reservation price has different implications for the allocation efficiency and the seller's expected revenue. In particular, a reservation price always hurt the former – the higher the reservation price, the lower the allocation efficiency. However, a moderate reservation price can improve the seller's revenue.

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CHAPTER 1

Task Completion with Minimum Compensation

1.1. Introduction

Self-assessment plays a crucial role in people's decision making when they try to achieve certain goals. A person sometimes takes on a task without knowing how well-suited his ability is in accomplishing such tasks. Only by directly observing and/or by receiving feedback from a relevant authority on the progress toward completing the task does the person learn about his ability. The newly gained knowledge on his ability, in turn, influences the person's decision regarding whether he should quit or keep going. For example, although many youths dream of the wealthy and glorious life of becoming professional athletes, only the few who are certain about their extraordinary talent set out to pursue a career in sports. A prompt and accurate feedback system is valuable to individuals – gifted or not – so that they can allocate their time and effort efficiently in deciding if they want to play sports for a living. Similarly, many college students choose their major not only based on their interests but also on how good they are in a particular field, with the latter usually comes from the grades they get from relevant courses. Again, an accurate feedback system can help the students make such decisions. In contrast, the lack of an accurate feedback system can result in misallocation of human capital. The civil service examination system in Imperial China is a good example. Candidates only got informed of whether they had passed the exam. Those who failed might end up with false hope that success was just around the corner. As a result, the civil service examination system absurdly produced a handful of "lifetime pupils" who kept dreaming of passing the exam even in their late nineties. However, there are circumstances that the person who is in charge of providing feedback can benefit from such vagueness in information disclosure. For example, a mother who teaches her son how to ride a bike may choose not to reveal his true progress, for otherwise the child may get discouraged and give up if his progress remains poor after putting in considerable amount of effort.

It seems that in the examples mentioned above, the authority (referred to as the principal in the remaining text, with the pronoun "she") has a monopoly in information, in the sense that she alone can observe the progress an individual (referred to as the agent in the remaining text, with the pronoun "he") makes toward task completion. This is true, but it is not the whole story. In most real world situations, the agent also possesses private information that is not observable by the principal – his effort. For example, if two students get the same grade for a class with one studied day and night diligently for a whole semester while the other partied all the time only until one day before the exam and began to study, it would be reasonable to assume that these two students have different ability. That is why I used the term "self-assessment" at the beginning of the introduction. The progress feedback by the principal is certainly useful, but only when combined with an agent's private information of his effort. In some settings, the latter aspect of the information asymmetry is not important, as the whole point of the feedback is to help the agent make rational decisions. Thus, the agent has no incentive to cheat. However, in some other settings, the latter aspect of the information asymmetry cannot be overlooked. For example, if the principal, aside from providing feedback on the agent's progress, also has to compensate the agent for his effort, then she cannot ignore the agent's incentive to exaggerate his effort. More often than not, there is nothing the principal can do to directly prevent the agent from lying about his effort. This type of situations can arise when the completion of the task is not merely for the agent's own benefit. For example, many countries want their citizens to receive universal education to a certain level. If the government cannot force its people to study until the desired education level is achieved, it has to provide enough reward so that even the least talented individual would willingly put in as much effort as it takes. Given that the government cannot distinguish who the least talented individuals are, those who do not need to spend as much effort to obtain the required level of education thus can enjoy information rent on their private knowledge of their ability. But note that an agent's private knowledge of his ability is known by him before hand. Rather, he acquires this knowledge through the principal's feedback Therefore, in order to economize on her expenditure, the principal has to monetize her information on the agent's progress. How to design such a compensation/feedback system optimally is the focus of this paper.

In particular, I consider the setting in which there is a task to be completed by a group of agents. Each agent completes the task individually, that is, the task is not a group project. An agent's progress toward task completion is determined by two factors: effort and ability. The agents cannot observe their progress directly. Rather, they have to learn about their progress through the feedback provided by a principal, who can observe each agent's progress. However, only the agent knows how much effort he has put into the task. Effort is costly. It is the principal's obligation to provide enough compensation for the agents' effort so that all agents complete the task in the end. Neither the principal nor an agent himself knows the agent's ability beforehand, but the distribution of the agents' ability is common knowledge. Using the feedback provided by the principal, an agent will rationally update his belief regarding his ability, given the effort he has spent on the task. It is the agents who decide when to solicit feedback from the principal.

However, the principal can charge the agents for the feedback that she provides. The principal also decides how vague the feedback should be. I will explain the meaning of vagueness when I present the model in more detail in coming sections. Upon completing the task, an agent get a cash reward from the principal, the amount of which is set at the outset.

At the very beginning, the principal announces the cash reward for completing the task, the feedback policy (the vagueness), and the fee that has to be paid in exchange for her feedback. Upon seeing the principal's announcement, an agent decides whether he wants to participate, and if so, how much effort he will spend before he pays the fee and gets feedback on his progress. In case that the agent has not completed the task already[1], he decides whether he wants to continue or to quit, and in case of continuing, how much more effort to spend before asking for feedback once again. Aside from the initial announcement, the principal's role is completely passive – she is fully committed to the preset feedback policy and only provides feedback when asked by an agent. In this paper, I assume that the feedback policy can be vague, but it is always truthful. It will become clear later that this assumption is not essential, for as long as the principal is fully committed to a feedback policy, any biased feedback policy has a truthful counterpart. For ease of understanding, in the remaining text, I refer to the task as a *test*, the cash reward as the *passing reward*, the fee paid for feedback as the *test fee*. The principal's net expenditure is the difference between the passing reward and test fee. Her goal is to minimize her net expenditure under the premise that all agents pass the test in the end. It is worth pointing out that although an agent only claims the passing reward once, he may choose to pay the test fee multiple times.

To get a brief idea of how the feedback policy, the passing reward, and the test fee relate to each other, it is helpful to first understand how the principal and an agent's interests both

[1]By assuming that an agent's progress is only observable by the principal, I also assume that the agent can complete the task without knowing it, unless the principal officially tells him so.

contradict and align with each other. The contradiction in their interests should be relatively easy to see. Suppose, in contrast to the assumption that an agent's ability is neither known by himself nor by the principal beforehand, an agent is fully aware of his own ability. Then every agent would claim that he has the lowest ability and thus needs to spend the highest amount of effort possible, which further translates into the highest compensation. There is nothing the principal can do to extract the truth from the agents. Therefore, the principal has to set the passing reward high enough to compensate for the effort spent by the least talented agent. When an agent's ability is not known by himself at the outset, however, he can only learn about his ability (maybe gradually) from the feedback he gets from the principal, which gives the principal a chance to lower her expenditure. Since the net expenditure is the passing reward, which the principal has to hand out to everyone eventually, minus the test fee, which may be collected multiple times, there are two means that the principal can explore. First, she can try to lower the passing reward that she has to offer. This can be achieved by letting the agents spend some effort before they acquire any information regarding their progress. Since the effort that has already been spent is sunk, the passing reward only needs to be high enough to compensate for any additional effort that is still required. Alternatively, the principal can try to collect more test fee from the agents. She can do so either by letting them pay the test fee many times over, or by charging a large amount directly. Intuitively, she cannot do both, because a large test fee will make the agents more conservative in test-taking so that they can avoid paying the test fee many times. In contrast, a moderate test fee will make the agents seek feedback more often. Up until this point, the discussion has focused mainly on the contradiction between the principal's interest and that of an agent. To see the alignment in their interest, imagine that the principal sets a test fee so large that an agent only takes the test when he is absolutely sure that he can pass, i.e.

by assuming that he has the lowest possible ability outright. Although this scheme completely deprives the agents of their information rent, the principal herself does not benefit from it. In fact, her net expenditure is exactly the same as in a world in which the agents know their ability at the outset. In essence, the principal does not want to withhold too much information from the agents, as the more effort is wasted, the more baseline compensation has to be. By baseline compensation, I mean the compensation needed were the principal able to extract full surplus from the agents through the test fee. Based on the discussion above, the connection between the feedback policy, the passing reward, and the test fee can be summarized as follows. The feedback policy, in terms of its vagueness, determines how valuable feedback is to the agents. The value of this information, together with the price to obtain such information, i.e. the test fee, determines when an agent would seek such information. An accurate and early revelation on the principal's part means giving up some information rent to the agents. A vague or late revelation may be helpful to reduce the agent's information rent, as they can either reduce the passing reward or increase the total test fee, but they also increase the baseline compensation. Therefore, the key to the optimal design on the feedback policy, the passing reward, and the test fee lies in balancing these two contradicting forces.

In Section 1.2, I present the formal model and use the details contained in the model to further explain the contradicting and aligning aspects of the interests of the agents and that of the principal. Special attention is paid to the ideal outcome from the principal's perspective when the first-best-minimum-expenditure (FBME) can be achieved. FBME refers to the situation in which each agent spends just the right amount of effort to pass the test and the principal extracts all surplus from the agents. It requires the same net expenditure as if the principal knew each agent's ability and thus could pay just enough to get them to pass the test. This is the absolute minimum net expenditure. In Section 1.3 to Section 1.5, I assume that the passing reward and test fee are both independent of the number of tests an agent has taken. Under this assumption, Section 1.3 explores the full-disclosing feedback policy that enables an agent to know his ability right after a test is taken. I show that the feasibility of FBME depends on how large the test fee can be set without making the agents become too conservative in test taking that leads to the most talented agent over exerting effort, and at the same time, how small the passing reward can be set so that it is enough to prevent the least talented agent from quitting once he learns about his low ability. I show under what conditions is FBME can be achieved, and when it is not achievable, how to find the optimal passing reward-test fee combination that minimizes the principal's net expenditure. I show that sometimes is may be optimal to let the top talent over spend some effort, so that by the time the least talented finds out about his misfortune, not too much additional effort is required from him to keep going and pass the test. I also find the absolute upper bound on the principal's expenditure and show that she is always better off than she would have been were the agents know their ability beforehand. Section 1.4 explores the opposite end of disclosing power: the minimal-disclosing feedback policy that reveals as little information as possible to the agents after each test. Here, the principal only discloses whether an agent has passed the test. I show that to achieve FBME, it is not always optimal for the principal to try to set the test fee as large as possible, given that it is not too large to make the agents over conservative in test taking. On the contrary, sometimes it may be best to waive the test fee all together. Similar to its full-disclosing counterpart, a crucial aspect of the minimal-feedback policy is to make sure that an agent does not quit after failing a test at any point. The difference between the two types of feedback policies is that with full-disclosing, an agent does not need to pay the test fee again and again to get a better estimate of his ability. Therefore, the test fee is no longer a factor that influences an agent's decision once he has taken a test. In contrast, a failing agent may have to retake the test for unknown number of times were he choose to continue. As a result, a high test fee can become a limiting factor at any stage under the minimal-disclosing feedback policy. I further show that comparing to the full-disclosing feedback policy, FBME is easier to achieve with minimal-disclosing feedback policy when the agents' ability is widely distributed. When FBME is not feasible, I demonstrate the procedure that the principal should follow to find the optimal passing reward-test fee combination that minimizes her net expenditure. Section 1.5 explores feedback policies that have disclosing power sitting between the two extremes discussed in the previous two sections. I first show that under certain circumstances, this type of feedback policy can indeed produce FBME outcome when the previous two feedback policies fail to do so. However, the optimal policy is too case-depend and no meaningful general conclusion can be drawn. Therefore, my approach is to start from a full-disclosing feedback policy and examine how to modify it into a partial-disclosing feedback policy that results in less net expenditure. Finally, in Section 1.6, I relax the assumption that the passing reward and test fee must both be independent of the number of tests an agent has taken. I show that without this assumption, FBME can always be achieve – both with the full-disclosing feedback policy and with the minimal-disclosing feedback policy. Since both are special cases of the partial-disclosing feedback policy, the conclusion that FBME is feasible with a partial-disclosing feedback policy is automatically true.

1.1.1. Related Work

This paper is closely related to Ely and Szydlowski (2020 henceforth ES) [13]. In ES, an agent works on a task of uncertain difficulty and the principal maximizes the agent's effort by committing to a policy of disclosure of information about the task difficulty. The uncertain of the task's difficulty in ES and the uncertainty of an agent's ability in this research share the same nature, as both boil down to the uncertainty of required effort to complete the task. However, a major difference is that in ES, the principal's objective is to induce the agent to work as much as possible, whereas in this paper, the principal does not benefit from any extra effort from the agents beyond task-completion. Another key difference is that in ES, the principal only controls the information disclosure but not the agent's reward for completing the task, which is exogenous. In this paper, the agent's reward for completing the task is set by the principal to provide proper incentive to prevent the agent from quitting. In addition, the agent decides when to acquire information regarding his progress from the principal. Therefore, the agent trades off not only between the reward from completing the task and the cost of effort as in ES, but also the cost of acquiring information and the risk of over spending effort.

This paper is also related to research on career concerns. In his seminal paper, Holmström (1998) [20] studies the incentive problem arising from a person taking unobserved actions to affect performance and thus influences the wage process. Kovrijnykh (2007) [22] and Martinez (2009) [26] both analyze career concerns models with history-dependent effort and point out that, in such environments,current effort influences the market's beliefs about future effort. However, neither analyzes how disclosing or withholding information from the worker about his performance affects the worker's behavior. In contrast, Dewatripont, Jewitt and Tirole (1999) [11], Mukherjee (2008) [28], Koch and Peyrache (2001) [21], and Hansen (2012) [19] examine how varying the amount of information available to the labor market about worker performance affects the incentives that underlie effort provision in career concerns models. A common theme is that limiting the amount of performance information available to the labor market can be optimal.

Aside from the analysis of performance appraisal and career concerns,there is a growing literature that studies performance feedback within other contracting frameworks. Aoyagi (2011) [2] and Ederer (2010) [12] analyze information disclosure in dynamic tournaments. Aoyagi shows that when the principal controls the agents' effort incentive through the use of a feedback policy, the optimal feedback policy that maximizes the agents' expected effort is either the no-feedback policy that reveals no information, or the full-feedback policy that reveals all his information. There is no heterogeneity in worker ability in Aoyagi's paper. In contrast, Ederer shows that, when workers have unknown ability, and effort and ability are complements in production, there is a trade-off of performance feedback between evaluation effect and motivation effect.

I first consider a fixed pair of fee and reward, which is supposed to be independent of how many times feedback has been provided to a particular agent. I discuss how to choose such pairs under different feedback policies and what outcomes can be achieved. Then I look into more flexible designs that allow the fee-reward schedules to be contingent on the number of times feedback has been provided and show that with this type of flexibility, the first-best outcome is always achievable.

1.2. Model

A unit mass of agents (he) each faces a common task. The amount of effort required to complete the task depends on the agent's ability, which is distributed according to $\{a_1, p_1; ...; a_N, p_N\}$, with $a_1 > a_2 > ... > a_N > 0$ and $\sum_{i=1}^{N} p_i = 1$. An agent's progress toward task completion is characterized by a *score function* $S(a, e)$ that has ability as the first argument and effort as the second. The task is completed if and only if $S(a, e) \geq x$, with x exogenously given. Call this x the *passing score*.

Assumption 1 (Score function). *The score function is assumed to have the following properties:*

(1) $S(a, 0) = 0$ *and* $\frac{\partial S(a,e)}{\partial e} > 0$ *for all* $a \in \{a_1, ..., a_N\}$*;*

(2) For any $e > 0$*,* $S(a, e) > S(a', e)$ *if and only if* $a > a'$*;*

(3) There exists $e > 0$ *such that* $S(a_N, e) \geq x$*.*

Property 1 says that an agent can always increase his score by exerting more effort; Property 2 implies that an agent with higher ability scores higher than his less talented peer who spends the same amount of effort; Property 3 guarantees that even the least talented person can complete the task as long as he spends enough effort. Given the assumptions on $S(a, e)$, it is clear that inverse of the score function in e exists for any $a \in \{a_1, ..., a_N\}$. Define the *effort function* $\mathcal{E}(a, s)$ to be this inverse. That is, $S(a, \mathcal{E}(a, s)) = s$ for all $a \in \{a_1, ..., a_N\}$ and $s > 0$. To simplify notation, for each $i \in \{1, ..., N\}$, let $\varepsilon_i = \mathcal{E}(a_i, x)$, i.e., the effort required for an agent with ability a_i to complete the task.[2]

[2]When the distribution of ability has finite support, as in our model, one can define the type space using ε_i instead. However, without introducing ability, the connection between effort and progress becomes obscure.

Effort is costly. All agents share the same cost function $C(e)$ with e being the effort. The cost function is such that $C(0) = 0$ and $C'(e) > 0$ for all $e > 0$. That is, cost only occurs when effort is spent, and it strictly increases with the amount of effort. For simplicity, I assume that the the cost of effort coincides with effort, i.e. $C(e) = e$. This assumption simplifies the notation without compromising the generality of the analysis. Participation is voluntary, and the agents are free to quit anytime.

There is a single principal (she) who has the *obligation* of letting everyone complete the task. *Ex-ante*, neither the principal nor the agent knows about the agent's ability – only its distribution is common knowledge. The score function $S(\cdot, \cdot)$ and the passing grade x are also common knowledge to both parties. An agent's effort is his private information that cannot be observed by the principal. Since participation is voluntary, the principal has to compensate the agents for their efforts. I assume that the principal does not personally benefit from the agent's efforts in any way. Her sole purpose is to let everyone complete the task. The principal wants to keep her overall expenditure as low as possible. One can think of the situation as a training requirement being delegated to the principal, by her superior or by a third-party, for a fixed amount of up-front payment. In this case, the principal wants to minimize her expenditure, given that the training requirement is met. Alternatively, one can also think of the principal as a policy maker who cares about some social outcome, say, a government that wants to eradicate illiteracy. As a benevolent government, it hopes to achieve this goal not by force, but by providing rewards to encourage people to get basic education. However, as all governments do, it has to keep its budget in mind.

The principal administers *tests* to keep track of the agents' progress and provide feedback to the agents. Administrating tests is costless to the principal. The design of the tests involves two dimensions: monetary and non-monetary. The monetary dimension consists of *passing reward* r and *test fee* b. The passing reward r is paid by the principal to the agent when the latter passes the test. The test fee b, which is paid to the principal by the agent, only comes into effect when an agents fails a test.[3] I first discuss the optimal design of the monetary dimension by choosing a fixed pair of (r, b) that does not depend on how many times an agent has taken the test. I then explore whether taking a more flexible approach by choosing a sequence of $\{(r_i, b_i)\}_{i=1}^{M}$ pairs that are contingent on the number of tests that an agent has taken can improve the financial outcome for the principal. For example, by setting $\{(r_1, b_1), (r_2, b_2)\} = \{(4, 3), (2, 1)\}$, the agent gets a passing reward of 4 if he passes the test on the first attempt and pays a test fee of 3 if he fails. In the latter case, if the agent chooses to re-take the test at some point in the future, he gets a passing reward of 2 if he passes and pays a test fee of 1 if he fails again.

The non-monetary dimension of the test-design is the feedback policy that the principal is committed to. First, I consider a full-disclosing feedback policy that, once combined with an agent's private information of the effort he has spent, enables him to learn about his ability immediately after he receives the feedback. This can be achieved by revealing the test score directly to the agent. The agent can then learn about his ability $\tilde{a}$ by searching for $\tilde{a}$ such that $S(\tilde{a}, e) = s$, where e and s are his effort and score, respectively. At the opposite extreme in terms of disclosing power, I look into a minimal-disclosing feedback policy that only informs an agent of whether he has passed the test or not. Under most circumstances, an agent would not be able to learn precisely what his ability is only by knowing that he has failed the previous test. But this information would still help him rule out certain possibilities, because if his ability is sufficiently

[3]Perhaps a more natural way to model the payment scheme is to make the test fee applicable for each test taken, regardless of passing or not. However, by setting the passing reward equal to $r + b$, the two payment schemes become equivalent. Yet as will become clear below, making the test fee only applicable in case of failure simplifies the notation.

high, he should have passed the test with the amount of effort he has already spent. At last, I move on to consider other feedback policies with disclosing power sitting in between the two extremes mentioned above. An example of this type of feedback policy is the letter grades: after failing a test, the agent may not be able to know his ability exactly, but he can nevertheless narrow down its possibilities more than he otherwise would with the minimal-disclosing feedback policy.

In terms of notation, I use t_i to denote the effort level at which the i^{th} test is taken. For example, if an agent spends effort $\tilde{e}$ and takes the test for the first time, then $t_1 = \tilde{e}$. Conversely, if another agent failed a test once and later re-take the test after spending a total effort of $\tilde{e}$, then $t_2 = \tilde{e}$ for this agent. The timeline is as follows:

- The principal designs the tests, including both the monetary dimension (passing reward r and test fee b) and the non-monetary dimension (feedback policy), and makes them public to the agents;
- An agent decides whether to participate, and in case of participation, how much effort to spend before taking the test (t_1);
- The agent takes the test after the amount of effort reaches t_1;
- The principal observes the agent's score. If the agent has passed the test, the passing reward r is paid to the agent by the principal, and there is no further interaction between the agent and the principal; If the agent fails, the principal receives the test fee b from the agent and provides him with feedback according to the feedback policy announced at the beginning;
- Upon receiving the feedback, the agent updates his belief about his ability and decides whether to continue. If the agent decides to continue, he further makes a decision

on how much total effort (including the effort he has already spent before failing the previous test) to spend before re-taking the test (t_2);

- The agent re-takes the test after the total amount of effort reaches t_2;

  ⋮

- This process goes on until the agent passes the test or decides not to continue.

Assuming that the principal's obligation of letting everyone complete the task is fulfilled, then *ex-ante*, an agent's expected payoff is

$$r - \big(\mathbb{E}[\text{number of tests taken}] - 1\big) \cdot b - \mathbb{E}[\text{total cost of effort}],$$

whereas the principal's net expenditure is

$$r - \big(\mathbb{E}[\text{number of tests taken}] - 1\big) \cdot b.$$

Seemingly, the principal and the agents have a strong conflict of interest, as a lower expenditure for the principal means a lower expected payoff to the agents. However, when the test system is designed properly, the principal may be able to decrease her net expenditure without sacrificing the agents' payoff – by limiting the amount of "unnecessary effort" wasted by the agents. Since participation is voluntary, the agents' *ex-ante* participation constraint must be respected, which effectively sets the aggregate cost of effort as the lower bound for the principal's expenditure. Therefore, if much effort is expected to be "wasted", in the sense that the agents know, before hand, that some of them are going to spend more effort than needed, then the principal's expenditure may still be large, even if she is able to extract all the surplus from the agents. One immediate observation is that the principal's expenditure is always bounded below by $\sum_{j=1}^{N} p_j \varepsilon_j$, which is the least amount of aggregate effort required so that everyone completes the task. For this reason, I call it the *first-best minimum expenditure (FBME)*.

A natural question that may arise at this point is when, if ever, "wasted" effort can be desirable for the principal. Intuitively, to avoid over-exerting effort, an agent needs to find out about his ability as soon as possible, so that he can spend just the right amount of effort to pass the test. From the principal's perspective, however, letting an agent learn about his ability at an early stage has the drawback that in case the agent finds out that his ability is so limited that it requires a large amount of effort for him to pass the test, he will be reluctant to do so unless the passing reward is set to be very large. This is a problem for the principal as she has to offer this large passing reward to everyone. This point can be easily understood in a fictitious setting in which all agents know their ability from the very beginning. Since the principal cannot observe the agent's effort, she has to offer $r \geq \varepsilon_N$ so that the least talented agents are willing to complete the task. But this leaves considerable information rent for the more talented agents. For example, those with ability a_1 end up with a surplus of $\varepsilon_N - \varepsilon_1$. Thus, in order to minimize her expenditure, the principal has to balance the trade-off between:

(1) withholding information and letting effort go to waste;

(2) revealing information and giving up information rent.

In the subsequent sections, I start my analysis by assuming a fixed (r, b) pair. I discuss how to choose (r, b) optimally under different feedback policies, and what outcome can be achieved. Later, I allow for flexible $\{(r_i, b_i)\}_{i=1}^{M}$ pairs that are contingent on the number of tests that an agent has already taken. I show how to choose $\{(r_i, b_i)\}_{i=1}^{M}$ optimally, and then demonstrate that FBME is always feasible in this case.

1.3. Full-Disclosing

When a full-disclosing feedback policy is adapted, an agent receives his test score after a test is taken. Using the score function and how much effort he has spent, the agent is able to calculate his ability. This further allows him to calculate the additional effort needed before passing the test. If the passing reward is not enough to cover the extra cost of effort, the agent quits immediately. Otherwise, he continues to spend just enough effort to pass the test. In the discussion that follows, I make the additional assumption that "speculative" test taking is not allowed, by requiring $t_1 \geq \varepsilon_1$. This can be done by imposing a huge penalty on those whose test score is less than $S(a_N, \varepsilon_1)$. As will become clear by the end of this section, this assumption is not essential, since such speculative behavior will not happen if (r, b) is chosen properly. The details can be found in the Appendix.

Under the full-disclosing feedback policy, the agent's problem boils down to choosing how much effort to spend before taking the first test, i.e. choosing t_1. Once the first test is taken, the agent's ability becomes clear, and his subsequent decision is straightforward. The following proposition allows me to further simplify the agent's problem:

Proposition 1. *When the feedback policy is full-disclosing, it can only be optimal for an agent to choose $t_1 = \varepsilon_i$ for some $i \in \{1, ..., N\}$.*

Proof. Suppose, on the contrary, that $t_1 \neq \varepsilon_i$ for all $i \in \{1, ..., N\}$. Clearly, $t_1 > \varepsilon_N$ is not optimal, since setting $t_1 = \varepsilon_N$ saves effort, and the agent can still pass the test for sure. The case with $t_1 < \varepsilon_1$, as discussed earlier, is ruled out by assumption. Assume that $\varepsilon_k < t_1 < \varepsilon_{k+1}$ $(k = 1, ..., N-1)$, and compare it to the alternative of $\tilde{t}_1 = \varepsilon_k$. Let a_n $(n = 1, ..., N)$ be the agent's ability. If $n \leq k$, then the agent passes the test at either t_1 or $\tilde{t}_1$, only with less effort in the latter case; If $n > k$, and $r \geq \varepsilon_n - \tilde{t}_1 (> \varepsilon_n - t_1)$, then the agent's final payoff is $r - b - \varepsilon_n$ whether he has chosen t_1 or $\tilde{t}_1$ in the first place, since he will proceed to pass the test anyway; If $n > k$, and $r < \varepsilon_n - t_1 (< \varepsilon_n - \tilde{t}_1)$, then the agent quits in either case, but with $\tilde{t}_1$, the sunk cost of effort is less than that with t_1. The only remaining case to consider is $n > k$ and $\varepsilon_n - t_1 \leq r < \varepsilon_n - \tilde{t}_1$. That is, if the agent finds it worthwhile to continue to pass the test were he to take the test at t_1 but not so if the test is taken at $\tilde{t}_1$. However, $r < \varepsilon_n - \tilde{t}_1$ implies that $r - b - \varepsilon_n < -b - \varepsilon_k$, the left-hand-side of which is the agent's payoff with t_1 while the right-hand-side is the agent's payoff with $\tilde{t}_1$. Therefore, in all cases, $t_1 > \varepsilon_k$ is (weakly) dominated by $\tilde{t}_1 = \varepsilon_k$. □

Proposition 1 implies that the agent's problem can be simplified into choosing $i \in \{1, 2, ..., N\}$ so that the *ex-ante* expected utility $U^{fd}(i; (r, b))$ is maximized by setting $t_1 = \varepsilon_i$:

$$\max_{i \in \{1,\ldots,N\}} \quad U^{fd}(i; (r, b)) := \sum_{j=1}^{i} p_j r + \sum_{j=i+1}^{N} p_j \Big(\max\{0, r - (\varepsilon_j - \varepsilon_i)\} - b \Big) - \varepsilon_i.$$

When an agent is indifferent between two effort levels to take the first test, I assume that the lower one will be chosen. Clearly, achieving FBME requires $t_1 = \varepsilon_1$, for otherwise those with ability a_1 would have already spent more effort than needed by the time they take the first test. Furthermore, implementing $t_1 = \varepsilon_1$ requires $r \geq \varepsilon_N - \varepsilon_1$ so that those with ability a_N still go on to complete the task. Having such a high lower bound on r is apparently bad news for the principal. However, she may nevertheless be able to reduce her net expenditure by choosing a large test fee b. One disadvantage of choosing a large b is that the daunting test fee would make the agents more avert of failure. As a result, they may choose $t_1 > \varepsilon_1$ to increase the chance of passing the test in one sitting, which makes FBME unattainable. In contrast, if the test fee is negligible, the agents would want to learn about their ability as early as possible to avoid over exerting effort. Lemma 1 shows that when r is set high enough to guarantee 100% final passing rate (for example, $r \geq \varepsilon_N - \varepsilon_1$), the choice of t_1 depends solely on b.

Lemma 1. *Let $k, l \in \{1, ..., N\}$ and $k < l$. If $r \geq \varepsilon_N - \varepsilon_k$, an agent's preference between $t_1 = \varepsilon_k$ and $t_1 = \varepsilon_l$ is determined by b only. In particular, there exist a threshold test fee*

$$\beta^{fd}(k,l) = \frac{\sum_{j=1}^{k} p_j(\varepsilon_l - \varepsilon_k) + \sum_{j=k+1}^{l} p_j(\varepsilon_l - \varepsilon_j)}{\sum_{j=k+1}^{l} p_j} > 0, \tag{1.1}$$

such that the agent prefers $t_1 = \varepsilon_k$ to $t_1 = \varepsilon_l$ if and only if $b \leq \beta^{fd}(k,l)$.

Proof. With $r \geq \varepsilon_N - \varepsilon_k$,

$$U^{fd}(k;(r,b)) - U^{fd}(l;(r,b)) = \sum_{j=1}^{k} p_j(\varepsilon_l - \varepsilon_k) + \sum_{j=k+1}^{l} p_j(\varepsilon_l - \varepsilon_j) - \left(\sum_{j=k+1}^{l} p_j\right) \cdot b.$$

Therefore, $U^{fd}(k;(h,b)) \geq U^{fd}(l;(h,b))$ if and only if

$$b \leq \frac{\sum_{j=1}^{k} p_j(\varepsilon_l - \varepsilon_k) + \sum_{j=k+1}^{l} p_j(\varepsilon_l - \varepsilon_j)}{\sum_{j=k+1}^{l} p_j} = \beta^{fd}(k,l).$$

The fact that $\beta^{fd}(k,l) > 0$ is guaranteed by the assumption $k < l$. □

Lemma 1 is consistent with the intuition that an agent would prefer to take the test early when test fee is small. Therefore, to implement $t_1 = \varepsilon_1$, the principal faces an upper bound on b. Beyond this upper bound, the agents will take the first test after spending more effort than ε_1. Another restriction that the principal faces were she to implement $t_1 = \varepsilon_1$ is that the passing reward r has the lower bound of $\varepsilon_N - \varepsilon_1$, below which an agent whose ability is at the lower end loses incentive to continue. In addition, the (r, b) combination has to respect the agent's participation constraint. Note that the restriction imposed by the participation constraint does not create extra difficulty for the principal to achieve FBME, as it can be relaxed simply by reducing the test fee b and/or increasing the passing reward r, neither of which compromises the implementability of $t_1 = \varepsilon_1$. In fact, it is the opposite that is problematic for the principal. That is, if the participation constraint is slack when b is raised to its upper bound and r is reduced to its lower bound. In that case, FBME is not achievable. According to this argument, the following conclusion can be drawn:

Proposition 2 (Conditions for achieving FBME)**.** *When the feedback policy is full-disclosing, FBME is achievable if and only if*

$$U^{fd}(1;(\underline{R}^{fd}(1),\bar{B}^{fd}(1))) = \underline{R}^{fd}(1) - \sum_{j=2}^{N} p_j \bar{B}^{fd}(1) - \sum_{j=1}^{N} p_j \varepsilon_j \leq 0,$$

where $\underline{R}^{fd}(1) = \varepsilon_N - \varepsilon_1$ *and* $\bar{B}^{fd}(1) = \min_{i\in\{2,\dots,N\}} \beta^{fd}(1,i)$.

Proof. To implement $t_1 = \varepsilon_1$, (r, b) has to meet the following conditions:

(1) $r \geq \varepsilon_N - \varepsilon_1$;

(2) $b \leq \min_{i\in\{2,\dots,N\}} \beta^{fd}(1,i)$;

(3) $U^{fd}(1;(r,b)) = r - \sum_{j=2}^{N} p_j b - \sum_{j=1}^{N} p_j \varepsilon_j \geq 0$.

The first condition is needed so that everyone completes the task; The second condition builds on Lemma1 and ensures that the agents choose $t_1 = \varepsilon_1$; The third condition is the participation constraint. As mentioned earlier, neither increasing r nor decreasing b changes an agent's decision to choose $t_1 = \varepsilon_1$ and subsequently keep working until he can pass the test. Therefore, if $U^{fd}(1;(\underline{R}^{fd}(1),\bar{B}^{fd}(1))) < 0$, the principal can increase r and/or decrease b until $U^{fd}(1;(r,b)) = 0$. In contrast, if $U^{fd}(1;(\underline{R}^{fd}(1),\bar{B}^{fd}(1))) > 0$, there is nothing the principal can do to reduce her expenditure without changing the agent's choice of t_1 or whether to keep working until the task is completed, as reducing expenditure necessarily requires decreasing r and/or increasing b. □

Based on Proposition 2, the following corollary is immediate:

Corollary 2.1 (Minimum Expenditure to Implement $t_1 = \varepsilon_1$)**.** *When the conditions in Proposition 2 is not satisfied, the minimum expenditure required to implement $t_1 = \varepsilon_1$ is*

$$EX^{fd}(1; (\underline{R}^{fd}(1), \bar{B}^{fd}(1))) = \underline{R}^{fd}(1) - \sum_{j=2}^{N} p_j \bar{B}^{fd}(1)$$

with $\underline{R}^{fd}(1)$ and $\bar{B}^{fd}(1)$ as defined in Proposition 2.

When the condition in Proposition 2 is not satisfied, FBME is not achievable. In this case, the principal can either implement $t_1 = \varepsilon_1$ and leave some surplus to the agents, or implement $t_1 = \varepsilon_i$ for some $i > 1$ instead. However, unlike $t_1 = \varepsilon_1$, some $t_1 = \varepsilon_i$'s ($i > 1$) are not implementable. Intuitively, this is because to implement any $t_1 = \varepsilon_i > \varepsilon_1$, a test fee that is too low will make the agents take the first test earlier than ε_i, while a test fee that is too high will make the agents take the first test later than ε_i. Therefore, implementing any $t_1 = \varepsilon_i > \varepsilon_1$ is restricted by both a lower bound and an upper bound on b. When there does not exist any b that satisfies both the upper bound and the lower bound at the same time, such t_1 is not implementable.

Proposition 3 (Implementability under Full-Disclosing Feedback Policy)**.** *Under the full-disclosing feedback policy, $t_1 = \varepsilon_1$ and $t_1 = \varepsilon_N$ are always implementable. For $i \in \{2, ..., N-1\}$, $t_1 = \varepsilon_i$ is implementable if and only if*

$$\max_{k \in \{1,...,i-1\}} \beta^{fd}(k, i) < \min_{l \in \{i+1,...,N\}} \beta^{fd}(i, l).$$

Proof. The implementability of $t_1 = \varepsilon_1$ is discussed earlier. To implement $t_1 = \varepsilon_N$, we can choose any $b > \max_{j\in\{1,...,N-1\}} \beta^{fd}(j, N)$ and $r \geq \varepsilon_N$. We now discuss the case of implementing $t_1 = \varepsilon_i$ with $i \in \{2, ..., N-1\}$.

($\Rightarrow$) If $t_1 = \varepsilon_i$ is implementable, then there exist (r, b), $r \geq \varepsilon_N - \varepsilon_i$, such that $U^{fd}(i; (r, b)) \geq U^{fd}(l; (r, b))$ for all $l \in \{i+1, ..., N\}$ and $U^{fd}(i; (r, b)) > U^{fd}(k; (r, b))$ for all $k \in \{1, ..., i-1\}$. Since $r \geq \varepsilon_N - \varepsilon_i$ implies $r > \varepsilon_N - \varepsilon_l$, it follows from the definition of $\beta^{fd}(i, l)$ that $b \leq \min_{l\in\{i+1,...,N\}} \beta^{fd}(i, l)$.

The case with $\beta^{fd}(k, i)$ is slightly more complicated, as $r \geq \varepsilon_N - \varepsilon_i$ does not necessarily imply $r \geq \varepsilon_N - \varepsilon_k$. Let $M = \max\{j : \varepsilon_j - \varepsilon_k \leq r\}$. To put it plainly, M is defined in such a way that an agent whose ability is lower than a_M chooses to drop out after he fails the first test. Clearly, $M \geq i$, as otherwise $t_1 = \varepsilon_i$ cannot be optimal. Then

$$
\begin{aligned}
&U^{fd}(i; (r, h)) - U^{fd}(k; (r, h)) \\
=&\left(r - \sum_{j=1}^{i} p_j \varepsilon_i - \sum_{j=i+1}^{N} p_j (b + \varepsilon_j)\right) - \left(\sum_{j=1}^{M} p_j r - \sum_{j=1}^{k} p_j \varepsilon_k - \sum_{j=k+1}^{M} p_j (b + \varepsilon_j) - \sum_{j=M+1}^{N} p_j (b + \varepsilon_k)\right).
\end{aligned}
$$

$U^{fd}(i; (r, b)) > U^{fd}(k; (r, b))$ if and only if

$$
\begin{aligned}
b >& \frac{\sum_{j=1}^{k} p_j(\varepsilon_i - \varepsilon_k) + \sum_{j=k+1}^{i} p_j(\varepsilon_i - \varepsilon_j) + \sum_{j=M+1}^{N} p_j[(\varepsilon_j - \varepsilon_k) - r]}{\sum_{j=k+1}^{i} p_j} \\
=&\beta^{fd}(k, i) + \frac{\sum_{j=M+1}^{N} p_j[(\varepsilon_j - \varepsilon_k) - r]}{\sum_{j=k+1}^{i} p_j} \geq \beta^{fd}(k, i),
\end{aligned}
$$

with the last inequality following from $\varepsilon_j - \varepsilon_k > r$ for all $j \in \{M+1, ..., N\}$. The equality holds if $M = N$.

When $t_1 = \varepsilon_i$ is implemented by (r, b), $b > \beta^{fd}(k, i) + \frac{\sum_{j=M+1}^{N} p_j[(\varepsilon_j - \varepsilon_k) - r]}{\sum_{j=k+1}^{i} p_j}$ must hold for all $k \in \{1, ..., i-1\}$, which certainly implies $b > \beta^{fd}(k, i)$ for all $k \in \{1, ..., i-1\}$. Putting things together, one concludes that b must satisfy

$$\max_{k \in \{1,...,i-1\}} \ \beta^{fd}(k, i) < b \leq \min_{l \in \{i+1,...,N\}} \ \beta^{fd}(i, l),$$

and $\max_{k \in \{1,...,i-1\}} \ \beta^{fd}(k, i) < \min_{l \in \{i+1,...,N\}} \ \beta^{fd}(i, l)$ is thus required for such b to exist.

($\Leftarrow$) Choose any (r, b) such that $\max_{k \in \{1,...,i-1\}} \ \beta^{fd}(k, i) < b \leq \min_{l \in \{i+1,...,N\}} \ \beta^{fd}(i, l)$ and $r \geq \varepsilon_N - \varepsilon_1$[4]. Then, by definition of $\beta^{fd}(k, i)$ and $\beta^{fd}(i, l)$, $t_1 = \varepsilon_i$ is preferred to all $t_1 = \varepsilon_l$ $(l > i)$ and $t_1 = \varepsilon_k$ $(k < i)$, and $r \geq \varepsilon_N - \varepsilon_1$ guarantees that all agents continue to spend enough effort to complete the task. □

It is worth pointing out that Proposition 3 only says which $t_1 = \varepsilon_i$'s are implementable using certain (r, b) pairs, but in terms of expenditure minimization, this is only one side of the story. As discussed earlier, the principal's expenditure is determined by two factors: the aggregate effort, which is determined by which t_1 to implement, and how much surplus has to be left to the agents when this t_1 is implemented. Assume that $t_1 = \varepsilon_i$ satisfies the condition in Proposition 3 and thus can be implemented. In the proof of Proposition 3, the passing reward is set to be $r \geq \varepsilon_N - \varepsilon_1$. In theory, when $t_1 = \varepsilon_i$ is chosen by the agents, a passing reward of $r \geq \varepsilon_N - \varepsilon_i < \varepsilon_N - \varepsilon_1$ should be enough to encourage the least talented agents to keep going. However, if the principal tries to decrease her expenditure by lowering r from $\varepsilon_N - \varepsilon_1$ to $\varepsilon_N - \varepsilon_i$, the lower bound on b

[4]Note that $r \geq \varepsilon_N - \varepsilon_i$ cannot guarantee $t_1 = \varepsilon_i$ is preferred to all $t_1 = \varepsilon_k$ $(k < i)$, as the lower bound on b can increase above $\beta^{fd}(k, i)$ if $M < N$. With $r \geq \varepsilon_N - \varepsilon_1$, one can be certain that $M = N$.

starts to increase as a result.[5] If, at some point, the lower bound on b ceases to be smaller than the upper bound, $t_1 = \varepsilon_i$ cannot be implemented anymore.

Proposition 2 explores when full-surplus-extraction is possible were $t_1 = \varepsilon_1$ to be implemented. Clearly, full-surplus-extraction is always possible when $t_1 = \varepsilon_N$, as this can be done simply by choosing any $b > \max_{j\in\{1,\dots,N-1\}} \beta^{fd}(j,N)$ and $r = \varepsilon_N$. In this case, the test fee is so high that the agents will not take the test unless they are a hundred percent sure that they can pass, and they are willing to do so as long as the passing reward is large enough. Proposition 4, in turn, deals with the remaining cases of $t_1 = \varepsilon_i$, $i \in \{2,\dots,N-1\}$.

Proposition 4 (Condition for Full-Surplus-Extraction)**.** *Assume that $t_1 = \varepsilon_i$ ($i \in \{2,\dots,N-1\}$) is implementable. Then full-surplus-extraction is possible with $t_1 = \varepsilon_i$ if and only if*

$$U^{fd}(i;(\underline{R}^{fd}(i),\bar{B}^{fd}(i))) = \underline{R}^{fd}(i) - \sum_{j=i+1}^{N} p_j \bar{B}^{fd}(i) - \sum_{j=1}^{i} p_j\varepsilon_i - \sum_{j=i+1}^{N} p_j\varepsilon_j \le 0,$$

where $\bar{B}^{fd}(i) = \min_{l\in\{i+1,\dots,N\}} \beta^{fd}(i,l)$ and

$$\underline{R}^{fd}(i) = \max\left\{\varepsilon_N - \varepsilon_i, \inf\left\{r : U^{fd}(k,(r,\bar{B}^{fd}(i))) < U^{fd}(i,(r,\bar{B}^{fd}(i)))\ \forall\ k = 1,\dots,i-1\right\}\right\}.$$

PROOF. As before, the principal's ability of surplus-extraction depends on how much she can raise b and/or lower r without impacting the agents' choice of t_1 nor resulting in some agents quit before the task is completed. To what extent can r be lowered is bounded by two factors.[6]

[5] Recall that

$$b > \beta^{fd}(k,i) + \frac{\sum_{j=M+1}^{N} p_j[(\varepsilon_j - \varepsilon_k) - r]}{\sum_{j=k+1}^{i} p_j} \tag{1.2}$$

has to be satisfied in order to have $t_1 = \varepsilon_i$ keep being optimal. The right-hand-side increases when r decreases.

[6] In fact, the extent to which r can be lowered is bounded by three factors, the first of which being the participation constraint. However, in the discussion of surplus-extraction, this factor is not really relevant, since when this constraint becomes binding, full-surplus-extraction is achieved.

The first factor is with regard to the least-talented agent's *interim* incentive to continue to pass the test, which is given by $r \geq \varepsilon_N - \varepsilon_i$; The second factor deals with the agent's *ex-ante* incentive to choose $t_1 = \varepsilon_i$ in the first place. This last factor is irrelevant in the precious discussion of implementing $t_1 = \varepsilon_1$ for the reason that will become clear in the analysis that follows.

Take any $k \in \{1, ..., i-1\}$, and let M again be such that $M = \max\{j : \varepsilon_j - \varepsilon_k \leq r\}$. Then

$$U^{fd}(i; (r, b)) - U^{fd}(k; (r, b)) = \sum_{j=M+1}^{N} p_j[r - (\varepsilon_j - \varepsilon_k)] + \sum_{j=k+1}^{i} p_j[b - (\varepsilon_i - \varepsilon_j)] - \sum_{j=1}^{k} p_j(\varepsilon_i - \varepsilon_k).$$

To implement $t_1 = \varepsilon_i$, we need $U^{fd}(i; (r, b)) > U^{fd}(k; (r, b))$ for all $k < i$. Note that $U^{fd}(i; (r, b)) - U^{fd}(k; (r, b))$ is immune to changes in r for $r \geq \varepsilon_N - \varepsilon_k$. It strictly decreases if we further lower r from $\varepsilon_N - \varepsilon_k$ to $\varepsilon_N - \varepsilon_i$. For $t_1 = \varepsilon_i$ to remain to be implemented, we need

$$r \geq \inf\left\{r : U^{fd}(k, (r, b)) < U^{fd}(i, (r, b)) \; \forall \; k = 1, ..., i-1\right\}, \tag{1.3}$$

taking b as given.[7] Hence, although reducing r helps with surplus-extraction, it may also compromise the implementability of $t_1 = \varepsilon_i$. In contrast, increasing b enlarges $U^{fd}(i; (r, b)) - U^{fd}(k; (r, b))$, which, in turn, relaxes the restriction on r given by Equation 1.3. Given that doing so is also beneficial for surplus-extraction, there is no downside in setting b as large as possible, as long as it is not so large that agents find any $t_1 = \varepsilon_l > \varepsilon_i$ more attractive. Therefore, the smallest r that the principal can choose without compromising the implementability of

[7] Alternatively, this lower bound on r can be explicitly expressed as

$$r \geq \max_{k=1,...,i-1} \frac{-\sum_{j=k+1}^{i} p_j b + \sum_{j=1}^{k} p_j(\varepsilon_i - \varepsilon_k) + \sum_{j=k+1}^{i} p_j(\varepsilon_i - \varepsilon_j) + \sum_{j=M+1}^{N} p_j(\varepsilon_j - \varepsilon_k)}{\sum_{j=M+1}^{N} p_j}.$$

$t_1 = \varepsilon_i$ is thus given by $\underline{R}^{fd}(i)$.[8] At this point, it should be clear that when $t_1 = \varepsilon_1$ is implemented, there is no other choice of t_1 that comes before ε_1, and as a result, the principal can safely reduce r without worrying about doing so may make some $t_1 < \varepsilon_1$ more attractive.

We have shown that $\left(\bar{B}^{fd}(i), \underline{R}^{fd}(i)\right)$ is the combination of the largest test fee and smallest passing reward that can be chosen to implement $t_1 = \varepsilon_i$. Therefore, the principal is not able to fully extract surplus if $U^{fd}(i; (\underline{R}^{fd}(i), \bar{B}^{fd}(i))) > 0$. When $U^{fd}(i; (\underline{R}^{fd}(i), \bar{B}^{fd}(i))) < 0$, however, she can simply raise r so that $U^{fd}(i; r, \bar{B}^{fd}(i))) = 0$[9] $\square$

In case that a certain t_1 can be implemented and in addition, full-surplus-extraction is feasible with this choice of t_1, the principal's expenditure would just be the aggregate effort from all the agents. For example, suppose that $t_1 = \varepsilon_i$ is being implemented with full-surplus-extraction. Then the corresponding net expenditure is $\sum_{j=1}^{i} p_j \varepsilon_i + \sum_{j=i+1}^{N} p_j \varepsilon_j$. The specific combination of the passing reward and the test fee does not influence the expenditure, as long as it satisfies all the constraints.

As mentioned earlier, sometimes it is in the principal's interest to let the agents take the test at an early stage, as they would then be able to spend just the right amount of effort according to their abilities. In this way, even if some information rent has to be given to the agents, the principal might still be able to pay less than she otherwise would were she to let the agents take the test at a later stage so that she can extract all the surplus from the agents. For this reason, while seeking for the optimal passing reward and test fee combination, one cannot focus solely on those t_1's under which full-surplus-extraction is possible. Instead, it is necessary to look at the

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[8]See Appendix 3.1 for a related discussion.

[9]Note that lowering b until $U^{fd}(i; (\underline{R}^{fd}(i), b)) = 0$ is not always a valid method, as any decrease in b from $\bar{B}^{fd}(i)$ may result in an increase in the lower bound for r to sustain the implementability of $t_1 = \varepsilon_i$. This happens when $\max\left\{r : U^{fd}(k, (r, \bar{B}^{fd}(i)) = U^{fd}(i, (r, \bar{B}^{fd}(i))),\ k = 1, ..., i-1\right\} > \varepsilon_N - \varepsilon_i$. However, lowering b and increasing r collaboratively is also a viable approach to meet the participation constraint.

minimum expenditure associated with each potential t_1 that can be implemented and compare all such expenditures to find the truly optimal one.

Corollary 4.1 (Minimum Expenditure to Implement $t_1 = \varepsilon_i$)**.** *When the condition in Proposition 4 is not met, the minimum expenditure required to implement $t_1 = \varepsilon_i$ is*

$$EX^{fd}(i; (\underline{R}^{fd}(i), \bar{B}^{fd}(i))) = \underline{R}^{fd}(i) - \sum_{j=i+1}^{N} p_j \bar{B}^{fd}(i),$$

with $\underline{R}^{fd}(i)$ and $\bar{B}^{fd}(i)$ as defined in Proposition 4.

With the results above, the principal's expenditure minimization problem can be fully solved. The procedure is summarized in the Theorem below:

Theorem 1 (Expenditure Minimization under Full-Disclosing Feedback Policy)**.** *When the feedback policy is full-disclosing, the principal can minimize her expenditure in the following steps:*

(1) Starting from $i = 1$, list all the implementable $t_1 = \varepsilon_i$'s in ascending order and index them by $\{\mathcal{I}_1, \mathcal{I}_2, ..., \mathcal{I}_K\}$. Proposition 3 indicates $\mathcal{I}_1 = 1$ and $\mathcal{I}_K = N$.

(2) Starting from $k = 1$, check whether full-surplus-extraction is possible under $t_1 = \varepsilon_{\mathcal{I}_k}$. If not, calculate the corresponding minimum expenditure using Corollary 2.1 in case $\mathcal{I}_k = 1$ or 4.1 in case $\mathcal{I}_k > 1$ and move on to $t_1 = \varepsilon_{\mathcal{I}_{k+1}}$. Stop checking as soon as full-surplus-extraction becomes possible for the first time[10].

[10]Recall that $t_1 = \varepsilon_{\mathcal{I}_{i+1}}$ requires more aggregate effort than $t_1 = \varepsilon_{\mathcal{I}_i}$, and thus requires more expenditure if full-surplus-extraction is possible when $t_1 = \varepsilon_{\mathcal{I}_i}$ is implemented.

(3) *Suppose the previous step stops at* k^*. *Compare the minimum expenditures associated with each* $t_1 = \mathcal{I}_1, ..., \mathcal{I}_{k^*}$ *and choose the smallest. This is the minimum expenditure that can be achieved with a full-disclosing feedback policy.*

Remark. *In Step 2 of Theorem 1, there is no need to check the rest of the list as soon as full-surplus-extraction becomes feasible for the first time. This is because as one moves down the list of implementable* t_1*'s, the aggregate effort increases. Since aggregate effort acts as a lower bound for the principal's expenditure were the corresponding* t_1 *to be implemented, the principal cannot do any better by going further down the list if full-surplus-extraction has already become possible. She does need to, however, compare all the previous* t_1*'s, since they all involve smaller aggregate effort. Sometimes it is cheaper to implement* $t_1 = \varepsilon_k$ *than to implement* $\tilde{t}_1 = \varepsilon_l > \varepsilon_k$ *even if full-surplus-extraction is not feasible with* t_1 *but feasible with* $\tilde{t}_1$. *An example of this situation is provided in Appendix 3.2. Appendix 3.3 gives an example in which* $\tilde{t}_1$ *is indeed cheaper to implement than* t_1, *given that full-surplus-extraction is possible with the former but not with the latter.*

An interesting question to ask is whether the agents' initial ignorance of their own ability always benefits the principal. In other words, is the principal's expenditure always strictly less than ε_N, the minimum amount required if the agents have complete information about their ability while the principal does not? The answer is yes. This result is formalized in Proposition 5.

Proposition 5 (Expenditure Upper Bound under Full-Disclosing Feedback Policy)**.** *Under the full-disclosing feedback policy, the principal's expenditure is bounded above by*

$$\sum_{j=1}^{\mathcal{I}_{K-1}} p_j \varepsilon_{\mathcal{I}_{K-1}} + \sum_{j=\mathcal{I}_{K-1}+1}^{N} p_j \varepsilon_j,$$

where $\mathcal{I}_{K-1}$ is the second-largest element of the list in Step 1 of Theorem 1. Note that this upper limit of the principal's expenditure is strictly less than ε_N, the expenditure required when the agents know their ability beforehand.

Proof. With $t_1 = \varepsilon_{\mathcal{I}_{K-1}}$, the aggregate effort is $\sum_{j=1}^{\mathcal{I}_{K-1}} p_j \varepsilon_{\mathcal{I}_{K-1}} + \sum_{j=\mathcal{I}_{K-1}+1}^{N} p_j \varepsilon_j$. Therefore, proving Proposition 5 is equivalent to proving that full-surplus-extraction is always possible when $t_1 = \varepsilon_{\mathcal{I}_{K-1}}$ is implemented. Set $\tilde{b} = \beta^{fd}(\mathcal{I}_{K-1}, N)$ and $\tilde{r} = \varepsilon_N$. Apparently, $t_1 = \varepsilon_{\mathcal{I}_{K-1}}$ can be implemented by this pair of $(\tilde{r}, \tilde{b})$. By definition of $\beta^{fd}(\mathcal{I}_{K-1}, N)$, the agent is indifferent between $t_1 = \varepsilon_{\mathcal{I}_{K-1}}$ and $t_1 = \varepsilon_N$, with the passing reward being $\tilde{r}$ and the test fee being $\tilde{b}$. As $U^{fd}(N; (\tilde{r}, \tilde{b})) = \tilde{r} - \sum_{j=1}^{N} p_j \varepsilon_N = \tilde{r} - \varepsilon_N = 0$, it follows that $U^{fd}(\mathcal{I}_{K-1}; (\tilde{r}, \tilde{b})) = U^{fd}(N; (\tilde{r}, \tilde{b})) = 0$. Therefore, full-surplus-extraction is possible when $t_1 = \varepsilon_{\mathcal{I}_{K-1}}$. □

In this section, I have analysed the principal's expenditure minimization problem with a fixed pair of (r, b) and the full-disclosing feedback policy. I find hat FBME is achievable under certain circumstances, but not always. The major limitations come from the lower bound on r to guarantee 100% passing rate and the upper bound on b to guarantee that $t_1 = \varepsilon_1$ is chosen by the agents. When FBME is not feasible, I have explored the implementability of other choices of t_1 and the corresponding expenditure needed. It could be the case that letting some effort go waste by inducing the agents to postpone their test taking can be beneficial to the principal, if doing so enables the principal to extract more surplus from the agents. However, it could also be the case that the principal finds it more desirable to leave some surplus to the agents, if extracting surplus results in too much effort being wasted. The trade-off between wasted effort and information rent is at the heart of the issue.

Loosely speaking, the full-disclosing feedback policy can be very undesirable if there are a small number of agents whose abilities are extremely low, as the passing reward is always bounded below by the lowest ability, regardless of how many people actually have the lowest ability. In these occasions, the minimal-disclosing feedback policy discussed in next section can be more desirable, as it helps postpone an agent from realizing that his ability is among the lowest.

1.4. Minimal-Disclosing

When the feedback policy is minimal-disclosing, the agent only learns whether he has passed the test. In case of failure, the agent knows that his ability is not among those who would already have passed the test with the effort he has exerted. However, he does not know exactly how much more effort is required of him to pass the test. Consequently, in case he decides to continue and retake the test at some point, he may fail again and be called upon to make a decision of a similar nature. Therefore, unlike the full-disclosing case in which the agent's decision can be simplified into choosing how much effort to spend before taking the first test (t_1 in the previous section), the agent's decision under the minimal-disclosing feedback policy must include a complete test-taking-plan consisting of all the effort levels at which a test is to be taken, in case that he has failed at all the previous attempts. I temporarily denote a test taking plan by $\{e_1, e_2, ..., e_M\}$, meaning that an agent takes his first test after spending effort of e_1, and if he fails, he re-take the test when his total effort reaches e_2, and so on. The final element e_M is the last test that the agent is willing to take. If $e_M < \varepsilon_N$, it means that the agent quits without completing the test, in case that he has spent an effort of e_M and still fails. If $e_M \geq \varepsilon_N$, the agent is sure to pass at e_M, and possibly earlier.

Given any test-taking-plan $\{e_1, e_2, ..., e_M\}$, there is a continuation payoff, which I temporarily denote by $V\left(e_i; (r, b)\right)$, associated with each e_i ($i = 1, ..., M - 1$), such that

$$
\begin{aligned}
V\left(e_i; (r, b)\right) =& P\{e_{i+1} \geq \mathcal{E}(a, x) | e_i < \mathcal{E}(a, x)\} \cdot r \\
& + P\{e_{i+1} < \mathcal{E}(a, x) | e_i < \mathcal{E}(a, x)\} \cdot \left[V\left(e_{i+1}; (r, b)\right) - b\right] - \left(e_{i+1} - e_i\right).
\end{aligned} \tag{1.4}
$$

That is, if an agent fails the test after spending effort e_i, his expected payoff moving forward is $V\left(e_i; (r, b)\right)$. The first component of $V\left(e_i; (r, b)\right)$ builds on the possibility that the agent may pass the test the next time he takes it. The second component comes from the possibility that the agent may fail again at his next attempt. In that case, the agent pays the test fee and gets the continuation payoff associated with the newly-failed test, $V\left(e_{i+1}; (r, b)\right)$. Regardless of the outcome of the next test, the agent has to spend an additional effort of $e_{i+1} - e_i$ so that he can take the next test as planned, and thus the third component.

Similar to the case with full-disclosing, in which it is never optimal for an agent to take a test with an effort level that does not equal to any of “just enough” amount of effort that makes the agents with a specific ability pass the test, a test-taking-plan under the minimal-disclosing policy should not include any test-taking that does not directly correspond to any ability levels. In other words, an agent should never take a test with some effort that is “too much for a particular ability but too little for any lower abilities”. If this is true, the analysis would become much easier, as the choice set of potential test-taking-plans becomes finite. This intuition is formally addressed and proved in the following Proposition:

Proposition 6. *Under the minimal-disclosing feedback policy, a test-taking-plan* $\{e_1, ..., e_M\}$ *cannot be optimal if there exist any* $i \in \{1, ..., M\}$ *such that* $e_i \neq \varepsilon_j$ *for some* $j \in \{1, ..., N\}$.

PROOF. Suppose that the test-taking-plan $\{e_1, e_2, ..., e_M\}$ is optimal, but there exist $e_i \in \{e_1, e_2, ..., e_M\}$ such that $e_i \neq \varepsilon_j$ for all $j \in \{1, ..., N\}$. Apparently, $e_i > \varepsilon_N$ is not optimal, since setting $e_i = \varepsilon_N$ instead does not hurt the probability of passing the test while saves effort. Similarly, $e_i < \varepsilon_1$ is not optimal, since all agents are sure to fail at this point – they pay the test fee in exchange for nothing.[11] All that remains to be shown is that $\varepsilon_k < e_i < \varepsilon_{k+1}$ $(k = 1, ..., N-1)$ cannot be optimal, either.

If $i = M$, that is, if there is no test scheduled after an agent fails at e_i, then by taking this test at $\tilde{e}_i = \varepsilon_k$ instead, the agent is able to save effort in the amount of $e_i - \varepsilon_k$ without sacrificing his chance of passing. Thus, $e_M \notin \{\varepsilon_1, ..., \varepsilon_N\}$ is not optimal. Similarly, if $i < M$, then by taking this test at $\tilde{e}_i = \varepsilon_k$ instead, the agent's chance of passing the test on the i^{th} attempt does not change, while he is able to save effort in the amount of $e_i - \varepsilon_k$. If the agent fails the i^{th} test, he is scheduled to re-take the test once the total amount of effort he has spent reaches e_{i+1}. His continuation payoff at that time, $V\left(e_{i+1}; (r, b)\right)$, is the same whether $e_i > \varepsilon_k$ or $\tilde{e}_i = \varepsilon_k$, as the continuation payoff is history-independent. What happened before e_{i+1} under the two scenarios are also the same, in terms of the amount of total effort and the number of test fee payment. Therefore, by moving the i^{th} test from $e_i > \varepsilon_k$ to $\tilde{e}_i = \varepsilon_k$, an agent is strictly better off in case his ability is greater than a_k and thus is able to pass the test on the i^{th} attempt, while has the same payoff otherwise. Thus, the original test schedule is not optimal. This concludes the proof. □

[11] Note that with full-disclosing, I make an additional assumption that $t_1 < \varepsilon_1$ is not allowed, even though that assumption is not necessary, as is shown in the Appendix. Here, with minimal disclosing, a similar assumption is not needed for the apparent reason that nobody gains anything from taking a test with less effort than ε_1. They would be better off if they eliminate this test from their test-taking-plan or not participate at all.

Proposition 6 implies that the agent's problem can be simplified into choosing a subset of $\{\varepsilon_1, ..., \varepsilon_N\}$ as a test-taking-plan that maximizes the *ex-ante* expected utility. For ease of discussion, I provide the following definitions:

Definition. *A* ***test schedule*** $\mathcal{T} = \{t_1, ..., t_{|\mathcal{T}|}\}$ $(t_1 < t_2 < ... < t_{|\mathcal{T}|})$ *is the set of index corresponding to the test-taking-plan* $\{e_{t_1}, ..., e_{t_{|\mathcal{T}|}}\} \subseteq \{\varepsilon_1, ..., \varepsilon_N\}$*. For example, if an agent plans to take tests at* $\{\varepsilon_1, \varepsilon_2, \varepsilon_5\}$*, then the corresponding test schedule is* $\{1, 2, 5\}$*, with* $t_1 = 1$*,* $t_2 = 2$ *and* $t_3 = 5$*.*

A test schedule $\mathcal{T}$ *is* ***acceptable*** *if* $t_{|\mathcal{T}|} = N$*. Apparently, only an acceptable test schedule guarantees one hundred percent passing rate.*

A test schedule $\hat{\mathcal{T}}$ *is a* ***sub-schedule of*** $\mathcal{T}$ *if* $\hat{\mathcal{T}} \subsetneqq \mathcal{T}$*. Furthermore,* $\hat{\mathcal{T}}$ *is a* ***direct-sub-schedule of*** $\mathcal{T}$ *if* $\hat{\mathcal{T}} = \mathcal{T} \setminus \{t_i\}$ $(t_i \in \{t_1, ..., t_{|\mathcal{T}|-1}\})$*. For example, given the test schedule* $\mathcal{T} = \{1, 2, 3, 5\}$*,* $\{1, 3, 5\}$ *is both a sub-schedule and a direct-sub-schedule of* $\mathcal{T}$*, whereas* $\{3, 5\}$ *is a sub-schedule but not a direct-sub-schedule.*

An agent's problem can thus be expressed as choosing a test schedule $\mathcal{T} = \{t_1, ..., t_{|\mathcal{T}|}\}$ with which his *ex-ante* expected utility $U^{md}(\mathcal{T}; (r, b))$ is maximized:[12][13]

$$(1.5) \qquad \max_{\mathcal{T} \subseteq \{1,...,N\}} \quad U^{md}(\mathcal{T}; (r, b)) := \sum_{m=0}^{|\mathcal{T}|-1} \sum_{j=t_m+1}^{t_{m+1}} p_j(r - mb - \varepsilon_{t_{m+1}}) - \sum_{j=t_{|\mathcal{T}|}+1}^{N} p_j(|\mathcal{T}| \cdot b + \varepsilon_{t_{|\mathcal{T}|}}).$$

The principal's expenditure associated with test schedule $\mathcal{T} = \{t_1, ..., t_{|\mathcal{T}|}\}$ is simply

$$(1.6) \qquad EX^{md}(\mathcal{T}; (r, b)) = \sum_{m=0}^{|\mathcal{T}|-1} \sum_{j=t_m+1}^{t_{m+1}} p_j(r - mb) - \sum_{j=t_{|\mathcal{T}|}+1}^{N} p_j|\mathcal{T}|b.$$

[12]For ease of notation, let $t_0 = 0$.

[13]The tie-breaking rule in case two test schedules deliver the same expected utility will be stated later.

Recall that under the full-disclosing feedback policy, there are $N+1$ possibilities of t_1 that the agent may choose from (including not participating). In contrast, under the minimal-disclosing feedback policy, there are 2^N possible test schedules that can arise (including not participating). Again, the first question needs to be answered is whether a specific test schedule is implementable. And if so, how much surplus can be extracted from the agents. Just as before, the relation between the agent's expected utility (Equation 1.5) and the principal's expenditure (Equation 1.6) makes it clear that both the test schedule being implemented and the principal's surplus-extraction capacity under that test schedule influence the expenditure. The only test schedule under which FBME can possibly be achieved is $\{1, ..., N\}$, as this is the only way agents with all ability levels are able to spend just the right amount of effort to pass the test. Any other test schedule will inevitably result in some agents spend more effort than necessary. For example, assume $N = 5$ and the test schedule $\{1, 2, 3, 5\}$ is to be implemented. Then those with ability a_4 and a_5 alike will pass the test after spending ε_5 in total effort. For its uniqueness, I use the special notation $\mathcal{T}^{FB}$ for the test schedule $\{1, 2, ..., N\}$.

My approach is to first assume that all agents will eventually pass the test (which can certainly be achieved by setting r large enough) and focus solely on how agents choose among different test schedules. Thereafter, I investigate when r is reduced to extract more surplus, what restrictions should be taken into account to guarantee that all agents indeed pass the test in the end. As can be expected, the choice of b should not be neglected during this process. In all that follows, I focus solely on acceptable test schedules and thus omit the word "acceptable" in most cases for succinctness.

To begin with, notice that if $\mathcal{T}$ is an acceptable test schedule, then an agent's *ex-ante* expected utility associated with it simplifies into

$$U^{md}(\mathcal{T};(r,b)) = r - \sum_{m=0}^{|\mathcal{T}|-1} \sum_{j=t_m+1}^{t_{m+1}} p_j(mb + \varepsilon_{t_{m+1}}), \tag{1.7}$$

which immediately implies that an agent's preference between any two acceptable test schedules is determined by the test fee alone. If an agent is indifferent between two test schedules, I assume that the tie is broken in the principal's favor. That is, the agent chooses the test schedule that has the greater expected test fee. To study the implementability of any test schedule, it is necessary to first prove that an agent's preference among test schedules remains consistent after each test that he takes. That is, the test schedule that delivers the highest *ex-ante* expected payoff also yields the highest *interim* expected payoff should the agent fails a test. Given a test schedule $\mathcal{T} = \{t_1, ..., t_{|\mathcal{T}|}\}$ and $i \in \{1, ..., |\mathcal{T}| - 1\}$, denote the remaining test schedule after an agent fails at ε_{t_i} by $\mathcal{T}_{t_i}$, i.e., $\mathcal{T}_{t_i} := \{j \in \mathcal{T} : j > t_i\}$. For example, if $\mathcal{T} = \{t_1, t_2, t_3, t_4\} = \{1, 3, 4, 7\}$, then $\mathcal{T}_{t_2} = \mathcal{T}_3 = \{4, 7\}$. Note that $\mathcal{T}_{t_i}$, by definition, is necessarily a sub-schedule of $\mathcal{T}$. Keeping the assumption that all agents eventually pass the test, the continuation payoff at ε_{t_i} is:[14]

$$V^{md}(\mathcal{T}_{t_i};(r,b)) = r - \sum_{m=i}^{|\mathcal{T}|-1} \sum_{j=t_m+1}^{t_{m+1}} \tilde{p}_j \left[(m-i)b + (\varepsilon_{t_{m+1}} - \varepsilon_{t_i})\right], \tag{1.8}$$

where $\tilde{p}_j = \dfrac{p_j}{\sum_{n=t_i+1}^{N} p_n}$ is the updated probability that the agent's ability is a_j ($j > i$), given that the agent fails at ε_i. Using the continuation payoff, the agent's *ex-ante* expected utility can be

[14]Although the expression of the continuation payoff here is slightly different from the one given by Equation 1.4, they carry the same meaning.

re-written as

$$(1.9)\quad U^{md}(\mathcal{T};(r,b)) = \sum_{m=0}^{i-1}\sum_{j=t_m+1}^{t_{m+1}} p_j(r-mb-\varepsilon_{t_{m+1}}) + \sum_{t_i+1}^{N} p_j\left[V^{md}(\mathcal{T}_{t_i};(r,b)) - ib - \varepsilon_{t_i}\right], \quad \forall\, t_i \in \mathcal{T}.$$

I next demonstrate that an optimal test schedule stays optimal at each *interim* stage, so that once an agent chooses a test schedule, he has no incentive to deviate from it.

Proposition 7 (Preference Consistency at *Interim* Stage)**.** *Suppose that the test schedule* $\mathcal{T} = \{t_1, ..., t_{|\mathcal{T}|}\}$ *is ex-ante optimal, in the sense that* $U^{md}(\mathcal{T};(r,b)) \geq U^{md}(\mathcal{T}';(r,b))$ *for any acceptable test schedule* $\mathcal{T}'$. *Then for all* $i \in \{1, ..., |\mathcal{T}| - 1\}$, $V^{md}(\mathcal{T}_{t_i};(r,b)) \geq V^{md}\big((\mathcal{T}' \cup \{t_i\})_{t_i};(r,b)\big)$.

Proof. Suppose, on the contrary, that there exists $\mathcal{T}' = \{t'_1, ..., t'_{|\mathcal{T}'|}\}$ and $i \in \{1, ..., |\mathcal{T}|-1\}$, such that $V^{md}(\mathcal{T}_{t_i};(r,b)) < V^{md}\big((\mathcal{T}' \cup \{t_i\})_{t_i};(r,b)\big)$. Consider the alternative test schedule $\tilde{\mathcal{T}} = \big(\mathcal{T} \setminus \mathcal{T}_{t_i}\big) \cup (\mathcal{T}' \cup \{t_i\})_{t_i}$ which is made up of the "first half" of $\mathcal{T}$ prior to and including t_i and the "second half" of $\mathcal{T}'$ that is no less than t_i. The agent's *ex-ante* expected payoff under $\tilde{\mathcal{T}}$ is then

$$\begin{aligned} U^{md}(\tilde{\mathcal{T}};(r,b)) &= \sum_{m=0}^{i-1}\sum_{j=t_m+1}^{t_{m+1}} p_j(r-mb-\varepsilon_{t_{m+1}}) + \sum_{t_i+1}^{N} p_j\left[V^{md}\big((\mathcal{T}' \cup \{t_i\})_{t_i};(r,b)\big) - ib - \varepsilon_{t_i}\right] \\ &> \sum_{m=0}^{i-1}\sum_{j=t_m+1}^{t_{m+1}} p_j(r-mb-\varepsilon_{t_{m+1}}) + \sum_{t_i+1}^{N} p_j\left[V^{md}(\mathcal{T}_{t_i};(r,b)) - ib - \varepsilon_{t_i}\right] = U^{md}(\mathcal{T};(r,b)), \end{aligned}$$

contradicting with $U^{md}(\mathcal{T};(r,b))$ being the largest among all acceptable test schedules. Therefore, such $\mathcal{T}'$ does not exist. □

Proposition 7 implies that an *ex-ante* optimal test schedule is indeed *interim* optimal. Combining with the previous observation that the agent's preference between any two acceptable test schedules is determined by the test fee alone, it follows that a test schedule is **implementable** if and only if it delivers the highest *ex-ante* expected payoff under some b. As in the full-disclosing case, I use $\beta^{md}(\mathcal{T}, \tilde{\mathcal{T}})$ to denote the threshold test fee that determines an agent's preference between $\mathcal{T}$ and $\tilde{\mathcal{T}}$, with the latter being preferred when $b > \beta^{md}(\mathcal{T}, \tilde{\mathcal{T}})$.

Clearly, $\mathcal{T}^{FB}$ is implementable, as for any other test schedule $\mathcal{T} \neq \mathcal{T}^{FB}$,

$$U^{md}(\mathcal{T}^{FB}; (r, b)) \geq U^{md}(\mathcal{T}; (r, b)) \quad \Leftrightarrow \quad b \leq \underbrace{\frac{\sum_{m=0}^{|\mathcal{T}|-1} \sum_{j=t_m+1}^{t_{m+1}} p_j(\varepsilon_{t_{m+1}} - \varepsilon_j)}{\sum_{m=0}^{|\mathcal{T}|-1} \sum_{j=t_m+1}^{t_{m+1}} p_j(j - m - 1)}}_{>0}.$$

That is, $\beta^{md}(\mathcal{T}^{FB}, \mathcal{T}) > 0$ for all $\mathcal{T} \neq \mathcal{T}^{FB}$. As long as b is set to be small enough, say, $b = 0$, agents choose $\mathcal{T}^{FB}$ for sure. This is intuitive because if tests are free to take, there is no reason for an agent to bare the risk of over-exerting efforts. However, the expression above, by itself, is too complicated to provide any insight to the problem at hand. And it tells little about the implementability of other test schedules, which may be cheaper to implement under certain circumstances, as we have seen in the full-disclosing case. Intuitively, as the test fee increases, agents may "skip" a test at some point. If the test fee were to be increased even more, the agents would skip more tests. One implication of this intuition is that as b increases, the alternative test schedules that are immediately competing with a given test schedule are those with only one test in the original schedule being skipped, but not those with multiple tests being skipped. This intuition is correct, but with one caveat . It is true that if a given test schedule is preferred to all its direct-sub-schedules, then it is also preferred to all other sub-schedules. In other words, if an agent considers whether he would be better off by skipping some tests from the test schedule that he currently has, he only needs to consider if any of the direct-sub-schedules are more attractive. If no direct-sub-schedule offers a higher *ex-ante* expected utility to the agent than the original test schedule does, then there is no need to look at any other sub-schedules. The caveat is that the argument above only compares a test schedule to its sub-schedules, it may well be the case that even though a certain test schedule is preferred to all its sub-schedules, there exists another test schedule that is not a sub-schedule of the original test schedule, but nevertheless offers a higher payoff to the agent. It is the existence of this caveat that considerably complicates the analysis of the minimal-disclosing feedback policy. Before diving into this complexity, I first formalize the intuition regarding the dominance of a test schedule to its sub-schedules.

Lemma 2. *Let $\mathcal{T} = \{t_1, ..., t_{|\mathcal{T}|}\}$ be an acceptable test schedule. If $\mathcal{T}$ is preferred to all its direct-sub-schedules, then it is also preferred to all other sub-schedules.*

Proof. It is straightforward to verify that $\forall\ i \in \{1, ..., |\mathcal{T}| - 1\}$,

$$U^{md}(\mathcal{T}; (r, b)) \geq U^{md}(\mathcal{T} \setminus \{t_i\}; (r, b)) \quad \Leftrightarrow \quad b \leq \beta^{md}(\mathcal{T}, \mathcal{T} \setminus \{t_i\}) = \frac{\sum_{j=t_{i-1}+1}^{t_i} p_j(\varepsilon_{t_{i+1}} - \varepsilon_{t_i})}{\sum_{j=t_i+1}^{N} p_j}.$$

Note that $\forall\ k \in \{1, ..., |\mathcal{T}| - 1\}$, $k \neq i$,

(1.10)

$$\beta^{md}(\mathcal{T} \setminus \{t_i\}, \mathcal{T} \setminus \{t_i, t_k\}) = \begin{cases} \dfrac{\sum_{j=t_{k-1}+1}^{t_k} p_j(\varepsilon_{t_{k+2}} - \varepsilon_{t_k})}{\sum_{j=t_k+1}^{N} p_j} > \beta^{md}(\mathcal{T}, \mathcal{T} \setminus \{t_k\}), & \text{if } k = i - 1 \\ \dfrac{\sum_{j=t_{k-2}+1}^{t_k} p_j(\varepsilon_{t_{k+1}} - \varepsilon_{t_k})}{\sum_{j=t_k+1}^{N} p_j} > \beta^{md}(\mathcal{T}, \mathcal{T} \setminus \{t_k\}), & \text{if } k = i + 1 \\ \dfrac{\sum_{j=t_{k-1}+1}^{t_k} p_j(\varepsilon_{t_k+1} - \varepsilon_{t_k})}{\sum_{j=t_k+1}^{N} p_j} = \beta^{md}(\mathcal{T}, \mathcal{T} \setminus \{t_k\}), & \text{if } k \neq i \pm 1 \end{cases}.$$

Equation 1.10 shows that $\beta^{md}(\mathcal{T} \setminus \{t_i\}, \mathcal{T} \setminus \{t_i, t_k\}) \geq \beta^{md}(\mathcal{T}, \mathcal{T} \setminus \{t_k\})$ under all circumstances. Therefore, $b \leq \beta^{md}(\mathcal{T}, \mathcal{T} \setminus \{t_k\})$ implies that $\mathcal{T} \setminus \{t_i\}$ is preferred to $\mathcal{T} \setminus \{t_i, t_k\}$, which further implies $\mathcal{T}$ is preferred to $\mathcal{T} \setminus \{t_i, t_k\}$. Using the argument above inductively leads to the conclusion that $\mathcal{T}$ is preferred to all its acceptable sub-schedules. □

Since all acceptable test schedules other than $\mathcal{T}^{FB}$ are sub-schedules of $\mathcal{T}^{FB}$, the following conclusion is immediate:

Proposition 8 (Implementing $\mathcal{T}^{FB}$)**.** *Given that r is large enough so that* 100% *passing rate is guaranteed,* $\mathcal{T}^{FB}$ *is the most preferred test schedule if and only if*

$$b \leq \bar{B}^{md}(\mathcal{T}^{FB}) := \min_{i \in \{1,\dots,N-1\}} \frac{p_i(\varepsilon_{i+1} - \varepsilon_i)}{\sum_{j=i+1}^{N} p_j}.$$

PROOF. Simply notice that

$$\beta^{md}(\mathcal{T}^{FB}, \mathcal{T}^{FB} \setminus \{t_i\}) = \beta^{md}(\mathcal{T}^{FB}, \mathcal{T}^{FB} \setminus \{i\}) = \frac{p_i(\varepsilon_{i+1} - \varepsilon_i)}{\sum_{j=i+1}^{N} p_j},$$

and the rest follows directly from Lemma 2.

□

As mentioned before, Lemma 2 provides the condition under which a given test schedule is preferred to all its sub-schedules, but it says nothing about whether it is also preferred to those that are not its sub-schedules. For example, suppose $N = 4$. Lemma 2 tells us when the test schedule $\{1, 3, 4\}$ is preferred to $\{3, 4\}$, $\{1, 4\}$, and $\{4\}$, but it offers no help in deciding whether it is preferred to $\{2, 3, 4\}$ or $\{2, 4\}$. Therefore, except for $\mathcal{T}^{FB}$, Lemma 2 alone does not suffice to determine if a test schedule is implementable.

Before addressing this issue, I want to point out another major difference between the minimal-disclosing feedback policy and the full-disclosing feedback policy. As I have shown in the previous section, when the feedback policy is full-disclosing, the feasibility of FBME is limited by the lower bound on r and the upper bound on b. And $\bar{B}^{fd}(1)$ weakly dominates any $b < \bar{B}^{fd}(1)$. I now demonstrate that the upper bound on b, as given by Proposition 8, may not be a limiting force on the principal's surplus-extraction ability under the minimal-disclosing rule. In fact, it may not even be optimal to set b at the upper bound given by Proposition 8.

To see this, note that the continuation payoff at ε_n under $\mathcal{T}^{FB}$ is

$$V^{md}(\mathcal{T}_n^{FB};(r,b)) = r - \frac{\sum_{j=n+1}^{N} p_j \left\{[j-(n+1)]b + (\varepsilon_j - \varepsilon_n)\right\}}{\sum_{j=n+1}^{N} p_j}, \quad n = 1, ..., N-1.$$

When $b = 0$, it simplifies into

$$V^{md}(\mathcal{T}_n^{FB};(r,0)) = r - \frac{\sum_{j=n+1}^{N} p_j(\varepsilon_j - \varepsilon_n)}{\sum_{j=n+1}^{N} p_j}.$$

The *ex-ante* participation constraint requires $r \geq \sum_{j=1}^{N} p_j \varepsilon_j$. In addition, to satisfy all the *interim* participation constraints, r has to satisfy

$$r \geq \max_{n\in\{1,...,N-1\}} \left\{ \frac{\sum_{j=n+1}^{N} p_j(\varepsilon_j - \varepsilon_n)}{\sum_{j=n+1}^{N} p_j} \right\}.$$

If $\sum_{j=1}^{N} p_j \varepsilon_j \geq \max_{n\in\{1,...,N-1\}} \left\{ \frac{\sum_{j=n+1}^{N} p_j(\varepsilon_j - \varepsilon_n)}{\sum_{j=n+1}^{N} p_j} \right\}$, i.e., if the *ex-ante* participation constraint is the binding constraint, full-surplus-extraction can be easily achieved by setting $b = 0$ and $r = \sum_{j=1}^{N} p_j \varepsilon_j$. In contrast, if $\sum_{j=1}^{N} p_j \varepsilon_j < \max_{n\in\{1,...,N-1\}} \left\{ \frac{\sum_{j=n+1}^{N} p_j(\varepsilon_j - \varepsilon_n)}{\sum_{j=n+1}^{N} p_j} \right\}$, then by setting $b = 0$, the principal has to leave at least a surplus of

$$\max_{n\in\{1,...,N-1\}} \left\{ \frac{\sum_{j=n+1}^{N} p_j(\varepsilon_j - \varepsilon_n)}{\sum_{j=n+1}^{N} p_j} \right\} - \sum_{j=1}^{N} p_j \varepsilon_j$$

to the agents. Unlike in the full-disclosing case, this surplus cannot be reduced merely by increasing b, because as b increases, the continuation payoffs, except for $V^{md}(\mathcal{T}^{FB}_{N-1};(r,b))$, all decrease correspondingly. It thus requires raising r to keep the *interim* participation constraints satisfied. Whether it is profitable for the principal to do so depends on how r and b are related in the *ex-ante* participation constraint and in the *interim* ones. More specifically, when using $n = \operatorname{argmax}_{n\in\{1,\dots,N-1\}} \left\{ \frac{\sum_{j=n+1}^{N} p_j(\varepsilon_j - \varepsilon_n)}{\sum_{j=n+1}^{N} p_j} \right\}$ to denote the *interim* stage that has the binding participation constraint when $b = 0$, it follows that raising b helps reduce the net expenditure if and only if

$$-\frac{\partial U^{md}/\partial b}{\partial U^{md}/\partial r}\big(\mathcal{T}^{FB};(r,0)\big) > -\frac{\partial V^{md}/\partial b}{\partial V^{md}/\partial r}\big(\mathcal{T}^{FB}_{l};(r,0)\big)$$

$$\Leftrightarrow \quad \sum_{j=1}^{N} p_j(j-1) > \frac{\sum_{j=l+1}^{N} p_j[j-(l+1)]}{\sum_{j=l+1}^{N} p_j}.$$

In this case, the effect on surplus-extraction by increasing b is more than enough to compensate for the necessary increase in r required to keep the *interim* participation constraint at ε_l respected, and therefore the net effect is a reduction in the principal's expenditure.

Another complication that may arise is that as b increases, the binding *interim* participation constraint may change. In general, let

$$R^{md}(\mathcal{T}^{FB}_{n}, b) = \frac{\sum_{j=n+1}^{N} p_j \left\{[j-(n+1)]b + (\varepsilon_j - \varepsilon_n)\right\}}{\sum_{j=n+1}^{N} p_j} \quad (n \in \{1,\dots,N-1\}), \tag{1.11}$$

be the minimum passing reward required to satisfy the the *interim* participation constraint at ε_n for a given test fee b under $\mathcal{T}^{FB}$, and

$$R^{md}(\mathcal{T}^{FB}_{0}, b) = \sum_{j=1}^{N} p_j[(j-1)b + \varepsilon_j]$$

be the minimum passing reward required to satisfy the *ex-ante* participation constraint with the same test fee b, then full surplus-extraction is possible under $\mathcal{T}^{FB}$ if and only if $R^{md}(\mathcal{T}_0^{FB}, b) \geq \max_{n\in\{1,...,N-1\}} R^{md}(\mathcal{T}_n^{FB}, b)$ for some $b \in [0, \bar{B}^{md}(\mathcal{T}^{FB})]$. In other words, for FBME to be feasible, there must exist some test fee that implements $\mathcal{T}^{FB}$ and at the same time requires the highest passing reward from the *ex-ante* point of view. Proposition 9 formalizes this argument.

Proposition 9 (Conditions for Achieving FBME)**.** *Divide the set* $\{1, ..., N-1\}$ *into* $\mathcal{L}$ *and* $\mathcal{G}$*, with*

$$\mathcal{L} = \left\{ n \in \{1, ..., N-1\} : \frac{dR^{md}(\mathcal{T}_n^{FB}, b)}{db} < \frac{dR^{md}(\mathcal{T}_0^{FB}, b)}{db} \right\}$$

and

$$\mathcal{G} = \left\{ n \in \{1, ..., N-1\} : \frac{dR^{md}(\mathcal{T}_n^{FB}, b)}{db} \geq \frac{dR^{md}(\mathcal{T}_0^{FB}, b)}{db} \right\}.$$

For each $n \in \{1, ..., N-1\}$*, let* b_n *be such that* $R^{md}(\mathcal{T}_n^{FB}, b_n) = R^{md}(\mathcal{T}_0^{FB}, b_n)$*.*[15]

When $\mathcal{L}, \mathcal{G} \neq \emptyset$*, FBME is feasible if and only if*

$$\left[\max_{n\in\mathcal{L}}\{b_n\}, \min_{n\in\mathcal{G}}\{b_n\}\right] \bigcap \left[0, \bar{B}^{md}(\mathcal{T}^{FB})\right] \neq \emptyset;$$

When $\mathcal{L} = \emptyset$*, FBME is feasible if and only if* $\min_{n\in\mathcal{G}}\{b_n\} \geq 0$*;*

When $\mathcal{G} = \emptyset$*, FBME is feasible if and only if* $\max_{n\in\mathcal{L}}\{b_n\} \leq \bar{B}^{md}(\mathcal{T}^{FB})$*.*

Proof. It is clear from Equation 1.11 that for $n \in \mathcal{L}$, $R^{md}(\mathcal{T}_0^{FB}, b) \geq R^{md}(\mathcal{T}_n^{FB}, b)$ if $b \geq b_n$, whereas for $n \in \mathcal{G}$, $R^{md}(\mathcal{T}_0^{FB}, b) \geq R^{md}(\mathcal{T}_n^{FB}, b)$ if $b \leq b_n$. For FBME to be feasible, it is necessary and sufficient that there exists $b^* \in [0, \bar{B}^{md}(\mathcal{T}^{FB})]$ such that $R^{md}(\mathcal{T}_0^{FB}, b^*) \geq$

[15]It is straightforward to calculate

$$b_n = \left(\sum_{j=1}^{N} p_j(j-1) - \frac{\sum_{j=n+1}^{N} p_j[j-(n+1)]}{\sum_{j=n+1}^{N} p_j} \right)^{-1} \left(\frac{\sum_{j=n+1}^{N} p_j(\varepsilon_j - \varepsilon_n)}{\sum_{j=n+1}^{N} p_j} - \sum_{j=1}^{N} p_j \varepsilon_j \right). \tag{1.12}$$

$R^{md}(\mathcal{T}_n^{FB}, b^*)$ for all $n \in \{1, ..., N-1\}$, which is equivalent to $b^* \geq b_n$ for all $n \in \mathcal{L}$ and $b^* \leq b_n$ for all $n \in \mathcal{G}$. The conditions stated in the proposition guarantee the existence of such b^*. □

Remark. *A closer inspection of Equation 1.12 (see the footnote) reveals several interesting facts. First, the denominator*

$$\sum_{j=1}^{N} p_j(j-1) - \frac{\sum_{j=n+1}^{N} p_j[j-(n+1)]}{\sum_{j=n+1}^{N} p_j} = \sum_{j=1}^{N} p_j j - \frac{\sum_{j=1}^{N-n} p_{n+j} j}{\sum_{j=1}^{N-n} p_{n+j}},$$

has the first term as the weighted average of the set of integers $\{1, 2, ..., N\}$ *using the corresponding set of probability* $\{p_1, ..., p_N\}$ *and the second term as the weighted average of the set of integers* $\{1, 2, ..., N-n\}$ *using the corresponding set of probability* $\{\frac{p_{n+1}}{\sum_{j=1}^{N-n} p_{n+j}}, ..., \frac{p_N}{\sum_{j=1}^{N-n} p_{n+j}}\}$. *When the agent's ability is relatively evenly distributed, the first term is more likely to be greater than the second term. For example, this certainly holds true in case of discrete uniform distribution* $p_j = \frac{1}{N}$ *for all* j. *In such cases, the denominator of* b_n *is positive.*

Similarly, the nominator of b_n *can be re-written as*

$$\frac{\sum_{j=n+1}^{N} p_j(\varepsilon_j - \varepsilon_n)}{\sum_{j=n+1}^{N} p_j} - \sum_{j=1}^{N} p_j \varepsilon_j = \left(\mathbb{E}[\varepsilon_j | n < j \leq N] - \varepsilon_n\right) - \mathbb{E}[\varepsilon_j].$$

If the distribution of required effort[16] *is not too skewed (both in terms of value and probability), one should expect the nominator of* b_n *to be negative. For example, it would be so if the distribution of required effort is such that* $\varepsilon_{j+1} - \varepsilon_j = \delta$ *and* $p_j = \frac{1}{N}$ *for all* j. *When combined with the previous observation on the denominator of* b_n*, one can conclude that under certain circumstances in which the distribution of effort is relatively even,* b_n *would be negative. Given the equivalency between the denominator of* b_n *being positive and the set* $\mathcal{G}$ *being empty, one can*

[16] Given the one-to-one correspondence between ability (a_i) and required effort (ε_i), for any given probability distribution of ability, there is a corresponding probability distribution of required effort.

further expect that the condition for full surplus-extraction under $\mathcal{T}^{FB}$ stated in Proposition 9 to be satisfied. In such cases, the minimal-disclosing feedback policy can turn out to be quite attractive, even though it seems to face more constraints than its full-disclosing counterpart from the outset.

If full surplus-extraction is not feasible when $\mathcal{T}^{FB}$ is implemented, the optimal test fee is determined by the relative size of $\dfrac{dR^{md}(\mathcal{T}_0^{FB}, b)}{db}$ and that of the binding *interim* participation constraint. Within the interval $[0, \bar{B}^{md}(\mathcal{T}^{FB})]$ on which $\mathcal{T}^{FB}$ can be implemented, whenever $\dfrac{dR^{md}(\mathcal{T}_0^{FB}, b)}{db}$ is greater, the principal can decrease her expenditure by increasing b while increasing r at the same time in order to keep the *ex-ante* participation constraint satisfied. This observation is formalized in Proposition 10.

Proposition 10 (Minimum Expenditure to Implement $\mathcal{T}^{FB}$)**.** *For $b \in [0, \bar{B}^{md}(\mathcal{T}^{FB})]$, let*

$$n^*(b) := argmax_{n\in\{1,\ldots,N-1\}}\left\{R^{md}(\mathcal{T}_n^{FB}, b)\right\}$$

be the index of the test that requires the highest passing reward to satisfy the corresponding interim participation constraint for any given test fee b, and let

$$b^* = \begin{cases} \max\left\{0, \min\{b \,:\, n^*(b) \in \mathcal{G}\}\right\} & \text{if } \mathcal{G} \neq \emptyset \\ \bar{B}^{md}(\mathcal{T}^{FB}) & \text{if } \mathcal{G} = \emptyset \end{cases}.$$

Then, the minimum expenditure to implement $\mathcal{T}^{FB}$ can be achieved by choosing

$$\begin{cases} \tilde{b} = \min\left\{b^*, \bar{B}^{md}(\mathcal{T}^{FB})\right\} \\ \tilde{r} = \max_{n\in\{0,1,\ldots,N-1\}}\left\{R^{md}(\mathcal{T}_n^{FB}, \tilde{b})\right\}. \end{cases}$$

Proof. First, note that for all $b \in [0, \bar{B}^{md}(\mathcal{T}^{FB})]$, the net passing reward required to meet all the participation constraints are at least $\max_{n\in\{0,1,\ldots,N-1\}} \left\{R^{md}(\mathcal{T}_n^{FB}, b)\right\}$, and there is no need to increase it beyond this amount. Therefore, no matter what the optimal test fee $\tilde{b}$ is, the optimal net passing reward is always given by $\tilde{r} = \max_{n\in\{0,1,\ldots,N-1\}} \left\{R^{md}(\mathcal{T}_n^{FB}, \tilde{b})\right\}$.

As for the optimal test fee, when $\mathcal{G} = \emptyset$, there is no harm by setting the test fee as high as possible, given that it is inside the range that implements $\mathcal{T}^{FB}$. This is because if there exists $b' \in [0, \bar{B}^{md}(\mathcal{T}^{FB})]$ such that $n^*(b) = 0$ for $b \geq b'$, the principal's expenditure is the same by choosing any $\tilde{b} \in [b', \bar{B}^{md}(\mathcal{T}^{FB})]$. In such cases, FBME can be achieved. If such b' does not exist, as has been discussed above, it is beneficial for the principal to set the test fee as large as possible, and therefore $\tilde{b} = \bar{B}^{md}(\mathcal{T}^{FB})$ is optimal. Hence, when $\mathcal{G} = \emptyset$, $\tilde{b} = \bar{B}^{md}(\mathcal{T}^{FB})$ is always optimal.

When $\mathcal{G} \neq \emptyset$, it is not profitable to increase the test fee whenever $n^*(b) \in \mathcal{G}$, as the additional income brought in by increasing the test fee is not enough to compensate for the increase in the net passing reward needed to keep the *interim* participation constraint satisfied. Also note that if $n^*(b') \in \mathcal{G}$ for some b', then $n^*(b) \in \mathcal{G}$ for all $b \geq b'$. Thus, the principal would never raise the test fee above $\min\{b : n^*(b) \in \mathcal{G}\}$, unless it is too low to implement $\mathcal{T}^{FB}$. Hence, when $\mathcal{G} \neq \emptyset$, it is optimal to set $\tilde{b} = \max\left\{0, \min\{b : n^*(b) \in \mathcal{G}\}\right\}$, given that it does not exceed $\bar{B}^{md}(\mathcal{T}^{FB})$. If $\min\{b : n^*(b) \in \mathcal{G}\} > \bar{B}^{md}(\mathcal{T}^{FB})$, however, it is optimal to set $\tilde{b} = \bar{B}^{md}(\mathcal{T}^{FB})$. □

Remark. *The results presented in Proposition 9 and Proposition 10 apply to all possible discrete distributions of the agent's ability. Some interesting observations can be made when more assumptions are imposed on the distribution. For example, by assuming that all the ε_i's have the same probability $p_i = \dfrac{1}{N}$, it can be easily calculated from Equation 1.11 that* $\dfrac{dR^{md}(\mathcal{T}_n^{FB}, b)}{db} =$

$\dfrac{N-n-1}{2}$. *In this case,* $\dfrac{dR^{md}(\mathcal{T}^{FB},b)}{db} > \dfrac{dR^{md}(\mathcal{T}_n^{FB},b)}{db}$ *for all* $n = 1, ..., N-1$, *and thus* $\mathcal{G} = \emptyset$. *Proposition 10 then implies that it is optimal to set* $\tilde{b} = \bar{B}^{md}(\mathcal{T}^{FB}) = \min_{i\in\{1,...,N-1\}} \left\{ \dfrac{\varepsilon_{i+1}-\varepsilon_i}{N-i} \right\}$. *If one further assumes that* $\varepsilon_{i+1} - \varepsilon_i = \delta$ *for all* $i = 1, ..., N-1$, *then the optimal test fee would be* $\tilde{b} = \dfrac{\delta}{N-1}$. *The minimal net passing reward required at each* ε_n *would then be*

$$R^{md}(\mathcal{T}_n^{FB},\tilde{b}) = \frac{N^2-(n-1)N-2}{2(N-1)} \cdot \delta, \quad n = 1, ..., N-1,$$

whereas the minimal net passing reward required ex-ante is

$$R^{md}(\mathcal{T}_0^{FB},\tilde{b}) = \frac{N}{2} \cdot \delta + \varepsilon_1.$$

In this case, FBME is feasible under $\mathcal{T}^{FB}$ *if and only if* $R_0^{md}(\mathcal{T}^{FB},\tilde{b}) \geq R^{md}(\mathcal{T}_1^{FB},\tilde{b}) \Leftrightarrow \varepsilon_1 \geq \dfrac{N-2}{2(N-1)}\delta$. *Note that* $\dfrac{N-2}{2(N-1)}$ *is bounded above by* $\dfrac{1}{2}$. *One interpretation of this condition is that as long as the most talented agents do not have ability significantly higher than others (i.e. a very small* ε_1 *compared to* δ*), full surplus-extraction would be feasible under* $\mathcal{T}^{FB}$ *with the minimal-disclosing feedback rule.*

In contrast, with full-disclosing,

$$\beta^{fd}(1,i) = \frac{p_1(\varepsilon_i-\varepsilon_1)+\sum_{j=2}^{i} p_j(\varepsilon_i-\varepsilon_j)}{\sum_{j=2}^{i} p_j} = \frac{i}{2}\cdot\delta \ (i = 2, ..., N) \quad \Rightarrow \quad \bar{B}^{fd}(1) = \delta$$

and $\underline{R}^{fd}(1) = \varepsilon_N - \varepsilon_1 = (N-1)\delta$. *According to Proposition 2, FBME is achievable if and only if*

$$U^{fd}(1;(\underline{R}^{fd}(1),\bar{B}^{fd}(1))) = \frac{(N-1)(N-2)}{2N}\delta - \varepsilon_1 \leq 0 \quad \Leftrightarrow \quad \varepsilon_1 \geq \frac{(N-1)(N-2)}{2N}\delta.$$

Unlike the condition in the minimal-disclosing case, the condition above would be very hard to satisfy when N is big, which is intuitive, as the least talented agents demand very large compensation in this case.

The method discussed above can be readily generalized to cases in which another test schedule is implemented: For any test schedule that is implementable, the first step is to determine the range of b that implements that test schedule. After that, one should investigate the net passing reward required, as a function of b, at each point in the test schedule. By comparing the derivative of the binding *interim* participation constraint (if any) with respect to b and that of the *ex-ante* participation constraint under this test schedule, one can then decide what the optimal test fee should be, and thus the corresponding net passing reward required to satisfy all the participation constraints.

I now discuss the implementability of other test schedules. As I have briefly mentioned earlier, Lemma 2 alone is not sufficient to find all the implementable test schedules, as it offers no comparison between two test schedules unless one is a sub-schedule of the other. But the intuition it represents is valid – as the test fee is being raised from zero, more and more test will be skipped. It seems natural to deduce that this process generates a series of test schedules, one being a direct-sub-schedule of the preceding one, and therefore, Lemma 2 should indeed be able to identify all test schedules that are implementable. All one has to do is to find the order by which an agent skips his test, as a result of the increase in test fee. I now provide an example to show that this may not be the case.

Assume that $N = 3$ and $\{\varepsilon_1, p_1; \varepsilon_2, p_2; \varepsilon_3, p_3\} = \{1, 0.3; 8, 0.4; 10, 0.3\}$.[17] Figure 1.1 shows the *ex-ante* utility, as a function of b, associated with each acceptable test schedules. The passing reward r is set to be 20 in this example, although it has no impact on the ranking of acceptable test schedules. According to Figure 1.1, test schedule $\{1, 2, 3\}$ is optimal when the test fee is small. As b grows, it is more profitable for an agent to skip the test at effort ε_2. If Lemma 2 alone can be used to find all implementable test schedules, then the next test to be skipped should be the test at effort level ε_1. However, Figure 1.1 demonstrates that this is not true – instead of skipping the test at ε_1, it is optimal for the agents to switch from test schedule $\{1, 3\}$ to $\{2, 3\}$ at a certain range of b. Of course, if b is to be raised even higher, test schedule $\{3\}$ eventually becomes the best choice.

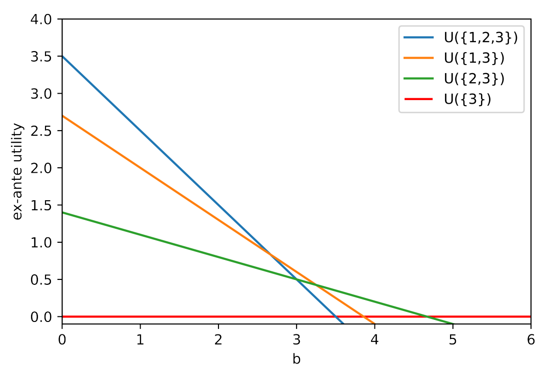

Figure 1.1. Example that Lemma 2 alone cannot identify all implementable test schedules

However, a slight modification to the example above results in a different set of implementable test schedules. Assume that $N = 3$ and $\{\varepsilon_1, p_1; \varepsilon_2, p_2; \varepsilon_3, p_3\} = \{1, 0.3; 11, 0.4; 13, 0.3\}$. As is shown in Figure 1.2, as b is raised from zero, the agents first choose test schedule $\{1, 2, 3\}$, and later change to test schedule $\{1, 3\}$. As b keeps increasing, they skip the test at ε_1 and

[17]For any given passing score x and the score function, there is clearly a one-to-one correspondence between an agent's ability a_i and the effort required of him to pass the test, ε_i. Therefore, there is an equivalency between the distribution of a_i's and ε_i's.

change their test schedule to $\{3\}$ directly, without ever choosing $\{2, 3\}$. In this case, it seems like Lemma 2 suffices to determine which test schedule is implementable. All one has to do is to find the optimal direct-sub-schedule and repeat this process until only one test remains, i.e. test schedule $\{N\}$.

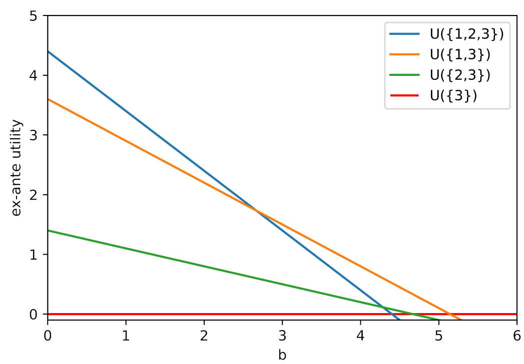

Figure 1.2. Example that Lemma 2 alone can identify all implementable test schedules

If such complication can arise when $N = 3$, it is almost impossible to find all implementable test schedules if N is large. Fortunately, the example above is more of the exception than the rule. To be more precise, assume that $N = 4$. If as b increases from zero, agents first choose test schedule $\{1, 2, 3, 4\}$. Assume that their next choice is $\{1, 2, 4\}$, then even though test schedule $\{1, 3, 4\}$ can become optimal at some point if b keeps increasing, test schedule $\{2, 3, 4\}$ never will. In contrast, if as b increases from zero, the agents first choose test schedule $\{1, 2, 3, 4\}$, then change to $\{1, 3, 4\}$. In this case, test schedule $\{1, 2, 4\}$ is never optimal, whereas $\{2, 3, 4\}$ may be at some point. The crucial factor to notice here is that when a test is skipped at a certain effort level, complication only arises regarding the test at the effort level immediate before the skipped one. In the previous example, if as b increases from zero, the agents first choose $\{1, 2, 3, 4\}$ and then $\{1, 2, 4\}$, one has to treat the test at ε_2 with special care, but not the one at ε_1. Similarly, if the optimal test schedule is $\{1, 3, 4\}$ for some b, then special attention needs to be paid to test schedule $\{2, 3, 4\}$.

Above is a concise exhibition of the complication that may arise as one tries to identify all acceptable test schedules that can be implemented. In what follows, I tackle this complication step by step. First, note that when N gets very large, the problem stays manageable only if one can focus "locally" without worrying that other parts of test schedule may also change as a result of any local changes. For example, assume that for a given test fee, test schedule $\{1, 2, 5, 7, 9, 10\}$ is optimal, and an agent is considering if he wants to skip the test at ε_2. His decision would be much more complicated if he had to simultaneously decide whether he should change his fourth test from ε_7 to ε_8 to make skipping the test at ε_2 more attractive. The following lemma states that the agent does not need to have such worries.

Lemma 3 (Irrelevance Beyond Local Level)**.** *Let $\mathcal{T}$ and $\mathcal{T}'$ be two acceptable test schedules. Assume that there exist $k, l \in \mathcal{T} \cap \mathcal{T}'$ $(k < l - 1)$ such that $i \notin \mathcal{T} \cup \mathcal{T}'$ for all $k < i < l$. Then for any $b > 0$ and $i \in (k, l)$, $\mathcal{T}$ is preferred to $\mathcal{T} \cup \{i\}$ if and only if $\mathcal{T}'$ is preferred to $\mathcal{T}' \cup \{i\}$.*

Proof. Note that[18]

$$
\begin{aligned}
V\big((\mathcal{T} \cup \{i\})_k)\big) - V\big(\mathcal{T}_k\big) &= \sum_{j=k+1}^{i} \tilde{p}_j(\varepsilon_l - \varepsilon_i) - \sum_{j=i+1}^{l} \tilde{p}_j b + \sum_{j=l+1}^{N} \tilde{p}_j \left[V\big((\mathcal{T} \cup \{i\})_l)\big) - V\big(\mathcal{T}_l\big) - b\right] \\
&= \left(\sum_{j=k+1}^{i} \tilde{p}_j\right) \cdot (\varepsilon_l - \varepsilon_i) - \left(\sum_{j=i+1}^{N} \tilde{p}_j\right) \cdot b,
\end{aligned}
$$

where $\tilde{p}_j = \dfrac{p_j}{\sum_{j=k+1}^{N} p_j}$.

[18] I denote $V\big((\mathcal{T} \cup \{i\})_k); (r, b)\big)$ and $V\big(\mathcal{T}_k; (r, b)\big)$ by $V\big((\mathcal{T} \cup \{i\})_k)\big)$ and $V\big(\mathcal{T}_k\big)$ for conciseness.

The second equality follows from $(\mathcal{T} \cup \{i\})_l = \mathcal{T}_l$. The same calculation applies to $V\left((\mathcal{T}' \cup \{i\})_k)\right) - V\left(\mathcal{T}'_k\right)$:

$$V\left((\mathcal{T}' \cup \{i\})_k)\right) - V\left(\mathcal{T}'_k\right) = \sum_{j=k+1}^{i} \tilde{p}_j(\varepsilon_l - \varepsilon_i) - \sum_{j=i+1}^{l} \tilde{p}_j b + \sum_{j=l+1}^{N} \tilde{p}_j \left[V\left((\mathcal{T}' \cup \{i\})_l)\right) - V\left(\mathcal{T}'_l\right) - b\right]$$
$$= \left(\sum_{j=k+1}^{i} \tilde{p}_j\right) \cdot (\varepsilon_l - \varepsilon_i) - \left(\sum_{j=i+1}^{N} \tilde{p}_j\right) \cdot b.$$

Therefore, $V\left((\mathcal{T} \cup \{i\})_k)\right) \geq V\left(\mathcal{T}_k\right)$ if and only if $V\left((\mathcal{T}' \cup \{i\})_k)\right) \geq V\left(\mathcal{T}'_k\right)$. What remains to be shown is the equivalency between $V\left((\mathcal{T} \cup \{i\})_k)\right) \geq V\left(\mathcal{T}_k\right)$ and $U\left((\mathcal{T} \cup \{i\}); (r, b)\right) \geq U\left(\mathcal{T}; (r, b)\right)$. Ditto for $V\left((\mathcal{T}' \cup \{i\})_k)\right) \geq V\left(\mathcal{T}'_k\right)$ and $U\left((\mathcal{T}' \cup \{i\}); (r, b)\right) \geq U\left(\mathcal{T}'; (r, b)\right)$. Let $k = t_n = t'_q$, i.e. the n^{th} element in $\mathcal{T}$ and the q^{th} in $\mathcal{T}'$. Then

$$U\left((\mathcal{T} \cup \{i\}); (r, b)\right) = \sum_{m=0}^{n-1} \sum_{j=t_m+1}^{t_{m+1}} p_j[r - mb - \varepsilon_{t_{m+1}}] + \sum_{j=k+1}^{N} p_j\left[V\left((\mathcal{T} \cup \{i\})_k\right) - (n-1)b - \varepsilon_k\right]$$

and

$$U\left(\mathcal{T}; (r, b)\right) = \sum_{m=0}^{n-1} \sum_{j=t_m+1}^{t_{m+1}} p_j[r - mb - \varepsilon_{t_{m+1}}] + \sum_{j=k+1}^{N} p_j\left[V\left(\mathcal{T}_k\right) - (n-1)b - \varepsilon_k\right].$$

Hence,

$$U\left((\mathcal{T} \cup \{i\}); (r, b)\right) - U\left(\mathcal{T}; (r, b)\right) = \left(\sum_{j=k+1}^{N} p_j\right) \cdot \left[V\left((\mathcal{T} \cup \{i\})_k\right) - V\left(\mathcal{T}_k\right)\right],$$

implying that $U\left((\mathcal{T} \cup \{i\}); (r, b)\right) \geq U\left(\mathcal{T}; (r, b)\right)$ if and only if $V\left((\mathcal{T} \cup \{i\})_k)\right) \geq V\left(\mathcal{T}_k\right)$. The same argument leads to $U\left((\mathcal{T}' \cup \{i\}); (r, b)\right) \geq U\left(\mathcal{T}'; (r, b)\right)$ if and only if $V\left((\mathcal{T}' \cup \{i\})_k)\right) \geq V\left(\mathcal{T}'_k\right)$. Therefore, $U\left((\mathcal{T} \cup \{i\}); (r, b)\right) \geq U\left(\mathcal{T}; (r, b)\right)$ if and only if $U\left((\mathcal{T}' \cup \{i\}); (r, b)\right) \geq U\left(\mathcal{T}'; (r, b)\right)$. $\square$

Essentially, Lemma 3 says that given any test fee $b > 0$, the decision of whether a test should be taken at ε_i is determined solely by the "latest past" and the "immediate future" when another test is taken. None of the other decisions is relevant. Indeed, whenever an agent takes a test and fails, only this newest failure matters in shaping the agent's belief. For instance, taking the first test at ε_3 and fails does not make an agent any smarter than someone who fails at ε_1, ε_2, and ε_3 consecutively. This explains why anything precedes the latest test does not matter. For the same reason, all accumulated uncertainty is resolved by the immediate future that a test is taken, anything lays beyond it is irrelevant. Using Lemma 3, the agent in the previous example does not need to consider whether moving his test on ε_7 to ε_8 would make skipping at ε_2 a better deal. Any part of the test schedule that is after ε_5 is irrelevant at this point. By the same token, if he indeed wonders if moving the test on ε_7 to ε_8 is more ideal, he should not worry if he needs to cancel the test at ε_2 at the same time.

The flip side of Lemma 3 is that if the decision of whether an agent should take a test at ε_i changes, it can potentially influence not only the decisions about the latest preceding test and the earliest subsequent test, but also everything in between. In the previous example where test schedule $\{1, 2, 5, 7, 9, 10\}$ is optimal for some given b, if the test fee inceases and the agent considers whether he would like to skip the test at ε_2, he should not think about ε_2 in isolation – his desicion regarding ε_1, ε_3, ε_4 and ε_5 may also require some adjustments in response.

Although the workload is greatly reduced by Lemma 3, it is still daunting. Thus, further simplification of the problem is desirable. Again, let $\mathcal{T}$ be an acceptable test schedule. Let $k < i < l$ be such that $k, l \in \mathcal{T}$ and $j \notin \mathcal{T}\ \forall\ k < j < i$ and $i < j < l$. That is, the agent takes tests at both ε_k and ε_l. Whether a test should be taken at ε_i is subject to further assumptions, but no other test is schedule between ε_k and ε_l. Using the result in Lemme 3, I denote the threshold test fee associated with ε_i by $B_{i|k,l}$ for clearness, as it only depends on k and l.

It is straightforward to verify that

$$B_{i|k,l} = \frac{\sum_{j=k+1}^{i} p_j(\varepsilon_l - \varepsilon_i)}{\sum_{j=i+1}^{N} p_j}. \tag{1.13}$$

According to Equation 1.13, it is clear that if k decreases, $B_{i|k,l}$ becomes bigger. This occurs if an agent decides to skip the test at ε_k, the net effect of which is equivalent to decrease k. Hence, if a test is taken at ε_i, i.e. $i \in \mathcal{T}$, then it remains optimal to do so. In contrast, if $\varepsilon_i \notin \mathcal{T}$, then the agent needs to check whether taking a test at ε_i would be a better option. Similarly, according to Equation 1.13, if l increases, $B_{i|k,l}$ becomes bigger as well. This happens when an agent decides to skip the test at ε_l, and thus effectively makes l bigger. The rest of the argument is exactly the same as before. That is, in case $i \in \mathcal{T}$, it remains optimal to do so. Otherwise, the agent needs to check whether taking a test at ε_i would be a better option. Since this argument applies to all $k < i < l$, a general observation can be reached as follows:

Observation 1. *Given the optimal test schedule under a certain test fee, if the test fee is raised to a level that the agent finds it more profitable to skip a test in the original test schedule, he needs to check whether he should add another test between:*

(1) the newly-skipped test and the test immediately before it in the original test schedule.

(2) the newly-skipped test and the test immediately after it in the original test schedule.

To make Observation 1 easier to understand, consider the following example. Assume that for a given test fee b, the optimal test schedule is $\{1, 2, 5, 7, 9, 10\}$, and as a result of an increase in b, the agent wants to skip the test at ε_5. To check if any further modification on the rest of the original test schedule is needed, the agent has to check whether he wants to take a test at ε_3, ε_4 or ε_6. If not, then the optimal test schedule becomes $\{1, 2, 7, 9, 10\}$. Observation 1 is intuitive, as in exchange for saving the test fee, skipping a test increases the risk of spending more effort than needed. Therefore, when a test has been taken out of the test schedule, it sometimes makes sense to consider adding another test to partially compensate for the increased risk of over-excerting effort. However, since skipping a test is the agent's response to an inflated test fee in the first place, one should not expect him to add another test that results in him paying more test fee. To put things more concretely, note that in Observation 1, case 1 is never optimal. That is, if the test fee is large enough so that the agent finds it more profitable to skip a test, he does not have to check if it makes more sense to add another test that precedes the one he wants to skip. In the previous example, if the agent decides to skip the test at ε_5, he only needs to check if adding a test at ε_6 makes him better-off. There is no need for him to contemplate whether a test should be added at ε_3 or at ε_4.

This conclusion is not obvious at first sight – although adding a test before the newly-skipped one results in more expected test fee being paid, which makes less sense given that it is the high test fee that makes the agent wants to skip one more test in the first place, it nonetheless prevents some agents from spending more effort than they have already over-spent, as their chance of passing the newly-skipped test has just been eliminated. To see where this balance lies, assume that the original test schedule that is being implemented is $\mathcal{T}$, and $\exists\ k, i, l \in \mathcal{T}$ such that $k + 1 < i < l$ and $j \notin \mathcal{T}\ \forall\ k < j < l\ (j \neq i)$. Take any $m \in (k, i)$ and note that

$$B_{i|k,l} = \frac{\sum_{j=k+1}^{i} p_j(\varepsilon_l - \varepsilon_i)}{\sum_{j=i+1}^{N} p_j} \quad \text{and} \quad B_{m|k,l} = \frac{\sum_{j=k+1}^{m} p_j(\varepsilon_l - \varepsilon_m)}{\sum_{j=m+1}^{N} p_j}.$$

Putting the previous reasoning into mathematical terms, it is not obvious whether $B_{m|k,l} < B_{i|k,l}$ is true, for $\frac{\sum_{j=k+1}^{m} p_j}{\sum_{j=m+1}^{N} p_j} < \frac{\sum_{j=k+1}^{i} p_j}{\sum_{j=i+1}^{N} p_j}$ but $\varepsilon_l - \varepsilon_m > \varepsilon_l - \varepsilon_i$. If it were not the case, then for b in the range of $\left(B_{i|k,l}, B_{m|k,l}\right]$, adding back the test at ε_m would deliver higher utility to the agent. As a result, the agent indeed has to check both cases mentioned in Observation 1. However, notice that

$$\beta^{md}\left((\mathcal{T} \setminus \{i\}) \cup \{m\}, \mathcal{T}\right) = \frac{\sum_{j=k+1}^{m} p_j(\varepsilon_l - \varepsilon_m) - \sum_{j=k+1}^{i} p_j(\varepsilon_l - \varepsilon_i)}{\sum_{j=m+1}^{i} p_j},$$

the nominator of which is the difference between the nominator of $B_{m|k,l}$ and that of $B_{i|k,l}$. The same is true for the denominator. As a general rule, consider $A, B, C, D > 0$ and $\frac{A}{B} > \frac{C}{D}$. It follows that $\frac{A}{B} > \frac{A+C}{B+D} > \frac{C}{D}$. This is because

$$\frac{A}{B} - \frac{C}{D} = \frac{AD - BC}{BD} > 0 \quad \text{implies that} \quad \frac{A}{B} - \frac{A+C}{B+D} = \frac{AD - BC}{B(B+D)} > 0$$

and similarly,

$$\frac{A+C}{B+D} - \frac{C}{D} = \frac{AD - BC}{D(B+D)} > 0.$$

In the current context, it implies that either

$$B_{i|k,l} > B_{m|k,l} > \beta^{md}\left((\mathcal{T} \setminus \{i\}) \cup \{m\}, \mathcal{T}\right) \quad \text{or} \quad B_{i|k,l} < B_{m|k,l} < \beta^{md}\left((\mathcal{T} \setminus \{i\}) \cup \{m\}, \mathcal{T}\right).$$

If $B_{m|k,l} > B_{i|k,l}$, then $\beta^{md}\left((\mathcal{T} \setminus \{i\}) \cup \{m\}, \mathcal{T}\right) > B_{i|k,l}$, meaning that test schedule $\mathcal{T}$ is dominated by $(\mathcal{T} \setminus \{i\}) \cup \{m\}$ for $b < B_{i|k,l}$, which implies that $\mathcal{T}$ cannot be implemented in the first place, contradicting the initial assumption. Therefore, if test schedule $\mathcal{T}$ is indeed implementable, then it must be that $B_{m|k,l} < B_{i|k,l}$. Hence, Observation 1 can be revised into the following:

Lemma 4. *Given the optimal test schedule, if the test fee is raised to a level that the agent finds it more profitable to skip a test in the original test schedule, he only needs to check whether he should add another test in response between the newly-skipped test and the test immediately after it in the original test schedule.*

Note that in case an increase in test fee results in removing a test and simultaneously adding back another one at a higher effort level, the net effect is equivalent to "postponing" a test. Due to this change of test schedule, the threshold test fees associated with the tests that were originally scheduled right before and after the test being rescheduled (regardless of being skipped or postponed), as well as those in between, all need to be recalculated. Fortunately, all other threshold test fees remain unchanged, according to Lemma 3. In addition, note that in case of "postponing" a test, the resulting new test schedule is not a sub-schedule of the original test schedule before the test fee increase. This is exactly the missing component of Lemma 2. Therefore, by starting with a test fee of zero and gradually increasing it, one can find all implementable test schedules by recording the change in the optimal test schedule along the way. The key to keep track of the required changes in the optimal test schedule is to focus on the threshold test fees associated with each effort level and make appropriate adjustments when necessary. This process is formalized in the following theorem:

Theorem 2 (All Implementable Test Schedules under Minimal-Disclosing)**.** *When the feedback policy is minimal-disclosing, the principal can find all implementable test schedules and the ranges of test fee that implement each of them by:*

I. ***Initiation***

(1) When $b = 0$, test schedule $\mathcal{T}^{FB}$ is implemented. Let $\tilde{\mathcal{T}} = \mathcal{T}^{FB}$ and $\underline{B}(\tilde{\mathcal{T}}) = 0$.

(2) Let $\mathbb{O} = \tilde{\mathcal{T}} \setminus \{N\} = \{1, ..., N-1\}$ *and* $\mathbb{X} = \{1, ..., N-1\} \setminus \mathbb{O} = \emptyset$ *be the set of effort levels at which a test is to be taken and the set of effort levels at which no test is scheduled, correspondingly.*

(3) Calculate[19]

$$B_{i|i-1,i+1} = \frac{p_i(\varepsilon_{i+1} - \varepsilon_i)}{\sum_{j=i+1}^{N} p_j}, \quad \forall\, i \in \mathbb{O}.$$

(4) Let $l = argmin_{i\in\{1,...,N-1\}}\left\{B_{i|i-1,i+1}\right\}$ *and let* $\bar{B}(\tilde{\mathcal{T}}) = B_{l|l-1,l+1}$.

(5) Test schedule $\tilde{\mathcal{T}} = \mathcal{T}^{FB}$ *is implemented with* $\tilde{b} \in [\underline{B}(\tilde{\mathcal{T}}), \bar{B}(\tilde{\mathcal{T}})]$.

(6) The test schedule that is implemented by $b = \bar{B}(\tilde{\mathcal{T}}) + \epsilon$ *is* $\tilde{\mathcal{T}} \setminus \{l\}$.

(7) Update the threshold test fees associated with ε_{l-1} *and* ε_{l+1}*, if applicable,*[20] *to:*

$$B_{l-1|l-2,l+1} = \frac{p_{l-1}(\varepsilon_{l+1} - \varepsilon_{l-1})}{\sum_{j=l}^{N} p_j}$$

and

$$B_{l+1|l-1,l+2} = \frac{(p_l + p_{l+1})(\varepsilon_{l+2} - \varepsilon_{l+1})}{\sum_{j=l+2}^{N} p_j}.$$

Update the threshold test fee associated with ε_l *to:*

$$\beta^{md}\left(\left(\tilde{\mathcal{T}} \setminus \{l\}, \tilde{\mathcal{T}} \setminus \{l-1\}\right)\right) = \frac{p_{l-1}(\varepsilon_l - \varepsilon_{l-1}) - p_l(\varepsilon_{l+1} - \varepsilon_l)}{p_l}.$$

[19] Recall that, in general, given test schedule $\mathcal{T} = \{t_1, ...t_{|\mathcal{T}|}\}$,

$$B_{t_i|t_{i-1},t_{i+1}} = \frac{\sum_{j=t_{i-1}+1}^{t_i} p_j(\varepsilon_{t_{i+1}} - \varepsilon_{t_i})}{\sum_{j=t_i+1}^{N} p_j}, \quad i = 1, ..., N-1.$$

[20] Only one of them is applicable if $l = 1$ or $l = N - 1$.

(8) Update $\tilde{\mathcal{T}}$, $\underline{B}(\tilde{\mathcal{T}})$, $\mathbb{O}$ and $\mathbb{X}$ to $\tilde{\mathcal{T}} = \tilde{\mathcal{T}} \backslash \{l\}$, $\underline{B}(\tilde{\mathcal{T}}) = B_{l|l-1,l+1}$, $\mathbb{O} = \mathbb{O} \setminus \{l\}$ and $\mathbb{X} = \mathbb{X} \cup \{l\}$.

II. ***Induction***

(1) Let $\tilde{\mathcal{T}} = \{\tilde{t}_1, ..., \tilde{t}_{|\tilde{\mathcal{T}}|}\}$ be the test schedule that is implemented by $b = \underline{B}(\tilde{\mathcal{T}}) + \epsilon$. The corresponding $\mathbb{O}$ and $\mathbb{X}$, as defined similarly as those in the Initialtion Steps, are $\mathbb{O} = \{\tilde{t}_1, ..., \tilde{t}_{|\tilde{\mathcal{T}}|-1}\}$ and $\mathbb{X} = \{1, ..., N-1\} \setminus \mathbb{O}$, respectively.

(2) For each $\tilde{t}_i \in \mathbb{O}$, there is a threshold test fee[21]

$$B_{\tilde{t}_i|\tilde{t}_{i-1},\tilde{t}_{i+1}} = \beta^{md}\left(\tilde{\mathcal{T}}, \tilde{\mathcal{T}} \setminus \{\tilde{t}_i\}\right) = \frac{\sum_{j=\tilde{t}_{i-1}+1}^{\tilde{t}_i} p_j(\varepsilon_{\tilde{t}_{i+1}} - \varepsilon_{\tilde{t}_i})}{\sum_{j=\tilde{t}_i+1}^{N} p_j}$$

above which it is better for the agent to skip the test at $\varepsilon_{\tilde{t}_i}$ when the other parts of the test schedule remains unchanged.

For each $m \in \mathbb{X}$, there exists $k = 0, ..., |\tilde{\mathcal{T}}| - 1$ such that $\tilde{t}_k < m < \tilde{t}_{k+1}$. When $k \neq 0$, there is a threshold test fee

$$\beta^{md}\left(\tilde{\mathcal{T}}, \left(\tilde{\mathcal{T}} \setminus \{\tilde{t}_k\}\right) \cup \{m\}\right) = \frac{\sum_{j=\tilde{t}_{k-1}+1}^{\tilde{t}_k} p_j(\varepsilon_m - \varepsilon_{\tilde{t}_k}) - \sum_{j=\tilde{t}_k+1}^{m} p_j(\varepsilon_{\tilde{t}_{k+1}} - \varepsilon_m)}{\sum_{j=\tilde{t}_k+1}^{m} p_j},$$

above which it is better for the agent to postpone the test from $\varepsilon_{\tilde{t}_k}$ to ε_m. Note that for those $m < \tilde{t}_1$, if exist, they will not reappear in an optimal test schedule once they have been skipped at some point.

All the $B_{\tilde{t}_i|\tilde{t}_{i-1},\tilde{t}_{i+1}}$'s and $\beta^{md}\left(\tilde{\mathcal{T}}, \left(\tilde{\mathcal{T}} \setminus \{\tilde{t}_k\}\right) \cup \{m\}\right)$'s mentioned above should already have been calculated in the previous steps.

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[21]Again, let $\tilde{t}_0 = 0$ for notational convenience.

(3) Let

$$l = argmin_{i=1,\ldots,|\tilde{\mathcal{T}}|-1} \left\{ B_{\tilde{t}_i|\tilde{t}_{i-1},\tilde{t}_{i+1}} \right\}$$

and

$$n = argmin_{m\in\mathbb{X},\, m>\tilde{t}_1} \left\{ \beta^{md}\left(\tilde{\mathcal{T}}, \left(\tilde{\mathcal{T}} \setminus \{\tilde{t}_k\}\right) \cup \{m\}\right) \,:\, \tilde{t}_k < m < \tilde{t}_{k+1} \right\}.$$

(4) If $B_{\tilde{t}_l|\tilde{t}_{l-1},\tilde{t}_{l+1}} < \beta^{md}\left(\tilde{\mathcal{T}}, \left(\tilde{\mathcal{T}} \setminus \{\tilde{t}_k\}\right) \cup \{n\}\right)$ $\left(\tilde{t}_k < n < \tilde{t}_{k+1},\ k \geq 1\right)$*, then let* $\bar{B}(\tilde{\mathcal{T}}) = B_{\tilde{t}_l|\tilde{t}_{l-1},\tilde{t}_{l+1}}$.

(i) Test schedule $\tilde{\mathcal{T}}$ *is implemented with* $\tilde{b} \in \left(\underline{B}(\tilde{\mathcal{T}}), \bar{B}(\tilde{\mathcal{T}})\right]$.

(ii) The test schedule that is implemented by $b = \bar{B}(\tilde{\mathcal{T}}) + \epsilon$ *is* $\tilde{\mathcal{T}} \setminus \{\tilde{t}_l\}$.

(iii) Update the threshold test fees associated with $\varepsilon_{\tilde{t}_{l-1}}$ *and* $\varepsilon_{\tilde{t}_{l+1}}$*, if applicable,*[22] *to:*

$$B_{\tilde{t}_{l-1}|\tilde{t}_{l-2},\tilde{t}_{l+1}} = \frac{\sum_{j=\tilde{t}_{l-2}+1}^{\tilde{t}_{l-1}} p_j(\varepsilon_{\tilde{t}_{l+1}} - \varepsilon_{\tilde{t}_{l-1}})}{\sum_{j=\tilde{t}_{l-1}+1}^{N} p_j}$$

and

$$B_{\tilde{t}_{l+1}|\tilde{t}_{l-1},\tilde{t}_{l+2}} = \frac{\sum_{j=\tilde{t}_{l-1}+1}^{\tilde{t}_{l+1}} p_j(\varepsilon_{\tilde{t}_{l+2}} - \varepsilon_{\tilde{t}_{l+1}})}{\sum_{j=\tilde{t}_{l+1}+1}^{N} p_j}.$$

For all $m \in (\tilde{t}_{l-1}, \tilde{t}_{l+1})$*, update the threshold test fee associated with* ε_m *to:*

$$\beta^{md}\left(\tilde{\mathcal{T}} \setminus \{\tilde{t}_l\}, \left(\tilde{\mathcal{T}} \setminus \{\tilde{t}_{l-1}, \tilde{t}_l\}\right) \cup \{m\}\right)$$
$$= \frac{\sum_{j=\tilde{t}_{l-2}+1}^{\tilde{t}_{l-1}} p_j(\varepsilon_{\tilde{t}_{l+1}} - \varepsilon_{\tilde{t}_{l-1}}) - \sum_{j=\tilde{t}_{l-2}+1}^{m} p_j(\varepsilon_{\tilde{t}_{l+1}} - \varepsilon_m)}{\sum_{j=\tilde{t}_{l-1}+1}^{m} p_j}.$$

[22]Only one of them is applicable if $l = 1$ or $l = |\tilde{\mathcal{T}}| - 1$.

(iv) Update $\tilde{\mathcal{T}}$, $\underline{B}(\tilde{\mathcal{T}})$, $\mathbb{O}$ and $\mathbb{X}$ to $\tilde{\mathcal{T}} = \tilde{\mathcal{T}} \backslash \{\tilde{t}_l\}$, $\underline{B}(\tilde{\mathcal{T}}) = B_{\tilde{t}_l|\tilde{t}_{l-1},\tilde{t}_{l+1}}$, $\mathbb{O} = \mathbb{O} \setminus \{\tilde{t}_l\}$ and $\mathbb{X} = \mathbb{X} \cup \{\tilde{t}_l\}$.

(5) If $B_{\tilde{t}_l|\tilde{t}_{l-1},\tilde{t}_{l+1}} \geq \beta^{md}\left(\tilde{\mathcal{T}}, \left(\tilde{\mathcal{T}} \setminus \{\tilde{t}_k\}\right) \cup \{n\}\right)$ $\left(\tilde{t}_k < n < \tilde{t}_{k+1},\ k \geq 1\right)$, then let $\bar{B}(\tilde{\mathcal{T}}) = \beta^{md}\left(\tilde{\mathcal{T}}, \left(\tilde{\mathcal{T}} \setminus \{\tilde{t}_k\}\right) \cup \{n\}\right)$.

(i) Test schedule $\tilde{\mathcal{T}}$ is implemented with $\tilde{b} \in \left(\underline{B}(\tilde{\mathcal{T}}), \bar{B}(\tilde{\mathcal{T}})\right]$.

(ii) The test schedule that is implemented by $b = \bar{B}(\tilde{\mathcal{T}}) + \epsilon$ is $\left(\tilde{\mathcal{T}} \setminus \{\tilde{t}_k\}\right) \cup \{n\}$.

(iii) Update the threshold test fees associated with $\varepsilon_{\tilde{t}_{k-1}}$ and $\varepsilon_{\tilde{t}_{k+1}}$, if applicable,[23] to:

$$B_{\tilde{t}_{k-1}|\tilde{t}_{k-2},n} = \frac{\sum_{j=\tilde{t}_{k-2}+1}^{\tilde{t}_{k-1}} p_j(\varepsilon_n - \varepsilon_{\tilde{t}_{k-1}})}{\sum_{j=\tilde{t}_{k-1}+1}^{N} p_j}$$

and

$$B_{\tilde{t}_{k+1}|n,\tilde{t}_{k+2}} = \frac{\sum_{j=n+1}^{\tilde{t}_{k+1}} p_j(\varepsilon_{\tilde{t}_{k+2}} - \varepsilon_{\tilde{t}_{k+1}})}{\sum_{j=\tilde{t}_{k+1}+1}^{N} p_j}.$$

Update the threshold test fee associated with ε_n to:

$$B_{n|\tilde{t}_{k-1},\tilde{t}_{k+1}} = \frac{\sum_{j=\tilde{t}_{k-1}+1}^{n} p_j(\varepsilon_{\tilde{t}_{k+1}} - \varepsilon_n)}{\sum_{j=n+1}^{N} p_j}.$$

For all $m \in (\tilde{t}_{k-1}, n)$, update the threshold test fee associated with ε_m to:

$$\beta^{md}\left(\left(\tilde{\mathcal{T}} \setminus \{\tilde{t}_k\}\right) \cup \{n\}, \left(\tilde{\mathcal{T}} \setminus \{\tilde{t}_{k-1}, \tilde{t}_k\}\right) \cup \{m, n\}\right)$$
$$= \frac{\sum_{j=\tilde{t}_{k-2}+1}^{\tilde{t}_{k-1}} p_j(\varepsilon_n - \varepsilon_{\tilde{t}_{k-1}}) - \sum_{j=\tilde{t}_{k-2}+1}^{m} p_j(\varepsilon_n - \varepsilon_m)}{\sum_{j=\tilde{t}_{k-1}+1}^{m} p_j}.$$

[23] Only one of them is applicable if $k = 1$ or $k = |\tilde{\mathcal{T}}| - 1$.

For all $m \in (n, \tilde{t}_{k+1})$, update the threshold test fee associated with ε_m to:

$$\beta^{md}\left(\left(\tilde{\mathcal{T}} \setminus \{\tilde{t}_k\}\right) \cup \{n\}, \left(\tilde{\mathcal{T}} \setminus \{\tilde{t}_k\}\right) \cup \{m\}\right)$$
$$= \frac{\sum_{j=\tilde{t}_{k-1}+1}^{n} p_j(\varepsilon_{\tilde{t}_{l+1}} - \varepsilon_n) - \sum_{j=\tilde{t}_{k-1}+1}^{m} p_j(\varepsilon_{\tilde{t}_{l+1}} - \varepsilon_m)}{\sum_{j=n+1}^{m} p_j}.$$

(iv) Update $\tilde{\mathcal{T}}$, $\underline{B}(\tilde{\mathcal{T}})$, $\mathbb{O}$ and $\mathbb{X}$ to

$$\tilde{\mathcal{T}} = \left(\tilde{\mathcal{T}} \setminus \{\tilde{t}_k\}\right) \cup \{n\},$$

$$\underline{B}(\tilde{\mathcal{T}}) = \beta^{md}\left(\tilde{\mathcal{T}}, \left(\tilde{\mathcal{T}} \setminus \{\tilde{t}_k\}\right) \cup \{n\}\right),$$

$$\mathbb{O} = \mathbb{O} \setminus \{\tilde{t}_k\} \cup \{n\}$$

and

$$\mathbb{X} = \left(\mathbb{X} \cup \{\tilde{t}_k\}\right) \setminus \{n\}.$$

III. ***Termination***

Repeat Step II (Induction) until $\tilde{\mathcal{T}} = \{N\}$.

Proof. Fixing the passing reward r at a level high enough to guarantee 100% passing rate, the *ex-ante* utility associated with each acceptable test schedule is linear in b. The problem of identifying all implementable test schedules is equivalent to finding the upper envelop of this collection of linear functions: 2^{N-1} in total. A brute force method is provided in the remark following this proof. The basic idea behind the method presented by Theorem 2 is to use the threshold test fees associated with each test in the test schedule that is being implemented by some b to divide the problem of finding implementable test schedules into sub-problems using Lemma 3, and then focus on the sub-problems one at a time. More specifically, during the process of raising the test fee from zero, some threshold that triggers adjustments to the optimal test schedule will be hit. According to Lemma 3, this adjustment is local, which means that one does not need to recalculate all threshold test fees after the adjustment.

To identify the influence of a particular local adjustment, first note that according to Lemma 4, the outcome of an adjustment is either (1) the test at some ε_i is skipped, or (2) the test at ε_i is postponed to ε_m, with ε_m being an effort level smaller than the next test scheduled directly after ε_i according to the previous optimal test schedule before the adjustment occurs. In case (1), the threshold test fee associated with the test immediately proceeds ε_i and that immediately follows ε_i should be recalculated, as skipping ε_i makes the information provided by them more valuable and thus makes them more costly to skip. For a similar reason, the threshold test fees associated with each effort level between the two tests mentioned above should also be recalculated, as adding them back to the test schedule can partially compensate for the information loss should the test immediately proceeding ε_i be skipped in the future. Aside from these recalculations, all the other threshold test fees are not influenced by skipping the test at ε_i. In case (2), postponing the test from ε_i to ε_m makes the information provided by the test immediately proceeding ε_i more valuable and less so for the test immediately follows ε_i. Therefore, the threshold test fees associated with these two tests should be recalculated. In addition, the threshold test fees attached to the formerly-skipped tests in between should also be recalculated, as the information gain from adding them back should the test proceeding them be skipped in the future have changed.

Once the adjustment to the optimal test schedule, together with the recalculation of the set of threshold test fees involved are complete, the test fee can be raised further without changing the optimal test schedule, until the next threshold is reached. At that point, based on the nature of the threshold, either case (1) – skipping a test, or case (2) – postponing a test, should be performed, thus complete the modification to the optimal test schedule. This modification in turn triggers some recalculation of the relevant threshold test fees on a local level, and this process goes on until the test fee gets so high that the optimal test schedule requires the agents not take any test until they are absolutely certain that they can pass the test. At the end of this process, the whole spectrum of implementable test schedules and the range of test fee that implements each of them can be identified. This process is guaranteed to come to an end at some point, since after each adjustment, a test is either taken out of the test schedule or postponed, both of which are one-way processes. Another way to see that the adjustment process must end is by simply noticing that if the test fee is extremely large, the optimal test schedule can only be $\{N\}$. □

Remark. *As noted in the proof of Theorem 2, the problem of identifying all implementable test schedules is equivalent to finding the upper envelop of the* 2^{N-1} *linear functions. A straightforward way of doing this is to start from the line with the smallest slope, i.e.* $U(\mathcal{T}^{FB})$*, and find its intersection with all the other lines. The intersection with the smallest b value is the first threshold test fee, below which test schedule* $\mathcal{T}^{FB}$ *is implemented. The line associated with this intersection is the next test schedule that is implementable. Next, find the intersection of this line with the other* $2^{N-1}-2$ *lines. Among those intersections whose b value is above the first threshold that has been identified earlier, take the one with the smallest b value, which offers the second threshold test fee. Repeat the above process until test schedule N is reached. This brute force calculation guarantees a solution, but lacks economic intuition. And when N is large, the calculation required becomes astronomical.*[24]

[24]Using the big O notion in computer science, the process described above of finding the upper envelop of n linear functions is $O(n^2)$, with $n = 2^{N-1}$. The fastest way of finding the upper envelop, to the best of my knowledge, is to use the divide-and-conquer algorithm, which has a speed of $O(n\log(n))$. The bottleneck is still $n = 2^{N-1}$, which grows exponentially by itself.

Identifying all implementable test schedules is the first step to solve for the principal's expenditure minimization problem. The next step is to calculate the minimum expenditure associated with each of such test schedules. Similar to the previous discussion on implementing $\mathcal{T}^{FB}$, full-surplus-extraction is possible for a given test schedule that can be implemented if and only if the *ex-ante* participation constraint is the most demanding constraint as an agent carries out the test schedule. Otherwise, the principal's surplus-extraction capacity would be limited by one of the *interim* participation constraints. Again, which constraint is most demanding depends both on the test fee and on the relationship between the participation constraint at each stage and the test fee.

More specifically, let $\tilde{\mathcal{T}} = \{\tilde{t}_1, ..., \tilde{t}_{|\tilde{\mathcal{T}}|}\}$ be the test schedule that is implemented by $b \in \left(\underline{B}^{md}(\tilde{\mathcal{T}}), \bar{B}^{md}(\tilde{\mathcal{T}})\right]$, where $\underline{B}^{md}(\tilde{\mathcal{T}})$ and $\bar{B}^{md}(\tilde{\mathcal{T}})$ are the threshold test fees that trigger test schedule adjustments described above – $\tilde{\mathcal{T}}$ is the resulting test schedule as the test fee is increased above $\underline{B}^{md}(\tilde{\mathcal{T}})$ whereas $\bar{B}^{md}(\tilde{\mathcal{T}})$ is the test fee beyond which the agents shift away from $\tilde{\mathcal{T}}$. Let

$$R^{md}(\tilde{\mathcal{T}}_0, b) = \sum_{m=0}^{|\tilde{\mathcal{T}}|-1} \sum_{j=\tilde{t}_m+1}^{\tilde{t}_{m+1}} p_j(mb + \varepsilon_{\tilde{t}_{m+1}}) \tag{1.14}$$

be the passing reward required to satisfy the *ex-anti* participation constraint and

$$R^{md}(\tilde{\mathcal{T}}_{\tilde{t}_i}, b) = \sum_{m=i}^{|\tilde{\mathcal{T}}|-1} \sum_{j=\tilde{t}_m+1}^{\tilde{t}_{m+1}} \frac{p_j}{\sum_{n=\tilde{t}_i+1}^{N} p_n} \left[(m-i)b + (\varepsilon_{\tilde{t}_{m+1}} - \varepsilon_{\tilde{t}_i})\right] \tag{1.15}$$

be the minimum passing reward required to satisfy the *interim* participation constraint after an agent fails at $\varepsilon_{\tilde{t}_i}$. Obviously, both $R^{md}(\tilde{\mathcal{T}}_0, b)$ and $R^{md}(\tilde{\mathcal{T}}_{\tilde{t}_i}, b)$ depend on b. If there exists $\tilde{b} \in \left(\underline{B}^{md}(\tilde{\mathcal{T}}), \bar{B}^{md}(\tilde{\mathcal{T}})\right]$ such that $R^{md}(\tilde{\mathcal{T}}_0, \tilde{b}) \geq \max_{i\in\{1,...,|\tilde{\mathcal{T}}|-1\}} R^{md}(\tilde{\mathcal{T}}_{\tilde{t}_i}, \tilde{b})$, then full-surplus-extraction is possible under $\tilde{\mathcal{T}}$. It can be achieved by setting $b = \tilde{b}$ and $r = R^{md}(\tilde{\mathcal{T}}_0, \tilde{b})$. In contrast, if $\forall\ \tilde{b} \in \left(\underline{B}^{md}(\tilde{\mathcal{T}}), \bar{B}^{md}(\tilde{\mathcal{T}})\right]$, $\exists\ i \in \{1, ..., |\tilde{\mathcal{T}}| - 1\}$ such that $R^{md}(\tilde{\mathcal{T}}_{\tilde{t}_i}, \tilde{b}) > R^{md}(\tilde{\mathcal{T}}_0, \tilde{b})$, then full-surplus-extraction is not possible. In such cases, the principal may still want to see how much surplus she has to leave to the agents. If not much, the test schedule may nonetheless be worth implementing. Proposition 11 provides the condition under which full-surplus-extraction is possible for a given implementable test schedule, whereas Proposition 12 calculates the minimum expenditure associated with each implementable test schedule.

Proposition 11 (Conditions for Full-Surplus-Extraction)**.** *Given an implementable test schedule $\tilde{\mathcal{T}} = \{\tilde{t}_1, ..., \tilde{t}_{|\tilde{\mathcal{T}}|}\}$ and $b \in \left(\underline{B}^{md}(\tilde{\mathcal{T}}), \bar{B}^{md}(\tilde{\mathcal{T}})\right]$ that implements it, divide the set $\tilde{\mathcal{T}} \setminus \{N\}$ into $\mathcal{L}(\tilde{\mathcal{T}})$ and $\mathcal{G}(\tilde{\mathcal{T}})$ such that*

$$\mathcal{L}(\tilde{\mathcal{T}}) = \left\{ n \in \left(\tilde{\mathcal{T}} \setminus \{N\}\right) : \frac{dR^{md}\left(\tilde{\mathcal{T}}_n, b\right)}{db} < \frac{dR^{md}\left(\tilde{\mathcal{T}}_0, b\right)}{db} \right\}$$

and

$$\mathcal{G}(\tilde{\mathcal{T}}) = \left\{ n \in \left(\tilde{\mathcal{T}} \setminus \{N\}\right) : \frac{dR^{md}\left(\tilde{\mathcal{T}}_n, b\right)}{db} \geq \frac{dR^{md}\left(\tilde{\mathcal{T}}_0, b\right)}{db} \right\}.$$

For each $n \in \left(\tilde{\mathcal{T}} \setminus \{N\}\right)$, let b_n be such that $R^{md}\left(\tilde{\mathcal{T}}_n, b_n\right) = R^{md}\left(\tilde{\mathcal{T}}_0, b_n\right)$.

When $\mathcal{L}(\tilde{\mathcal{T}}), \mathcal{G}(\tilde{\mathcal{T}}) \neq \emptyset$, full-surplus-extraction is feasible if and only if

$$\left[\max_{n\in\mathcal{L}(\tilde{\mathcal{T}})}\{b_n\}, \min_{n\in\mathcal{G}(\tilde{\mathcal{T}})}\{b_n\}\right] \bigcap \left(\underline{B}^{md}(\tilde{\mathcal{T}}), \bar{B}^{md}(\tilde{\mathcal{T}})\right] \neq \emptyset;$$

When $\mathcal{L}(\tilde{\mathcal{T}}) = \emptyset$, full-surplus-extraction is feasible if and only if $\min_{n\in\mathcal{G}(\tilde{\mathcal{T}})}\{b_n\} \geq \underline{B}^{md}(\tilde{\mathcal{T}})$;

When $\mathcal{G}(\tilde{\mathcal{T}}) = \emptyset$, full-surplus-extraction is feasible if and only if $\max_{n\in\mathcal{L}(\tilde{\mathcal{T}})}\{b_n\} \leq \bar{B}^{md}(\tilde{\mathcal{T}})$.

Proof. By the ways that $\mathcal{L}(\tilde{\mathcal{T}})$, $\mathcal{G}(\tilde{\mathcal{T}})$ and b_n are defined, it is clear that for $n \in \mathcal{L}(\tilde{\mathcal{T}})$, $R^{md}\left(\tilde{\mathcal{T}}_0, b_n\right) \geq R^{md}\left(\tilde{\mathcal{T}}_n, b_n\right)$ if $b \geq b_n$, whereas the reverse is true for $n \in \mathcal{G}(\tilde{\mathcal{T}})$. Thus, to realize full-surplus-extraction, it is necessary and sufficient that there exists $b^* \in \left(\underline{B}^{md}(\tilde{\mathcal{T}}), \bar{B}^{md}(\tilde{\mathcal{T}})\right]$ such that $R^{md}\left(\tilde{\mathcal{T}}_0, b^*\right) \geq R^{md}\left(\tilde{\mathcal{T}}_n, b^*\right)$ for all $n \in \left(\tilde{\mathcal{T}} \setminus \{N\}\right)$, which is equivalent to $b^* \geq b_n$ for all $n \in \mathcal{L}(\tilde{\mathcal{T}})$ and $b^* \leq b_n$ for all $n \in \mathcal{G}(\tilde{\mathcal{T}})$. The conditions presented in the proposition guarantee the existence of such b^*. □

Proposition 12 (Minimum Expenditure to Implement any Implementable Test Schedule)**.** *Given an implementable test schedule* $\tilde{\mathcal{T}} = \{\tilde{t}_1, ..., \tilde{t}_{|\tilde{\mathcal{T}}|}\}$ *and* $b \in \left(\underline{B}^{md}(\tilde{\mathcal{T}}), \bar{B}^{md}(\tilde{\mathcal{T}})\right]$ *that implements it, let*

$$n^*(b) = argmax_{n \in \left(\tilde{\mathcal{T}} \setminus \{N\}\right)} \left\{R^{md}\left(\tilde{\mathcal{T}}_n, b\right)\right\}$$

be the index of the test that requires the highest passing reward to satisfy the corresponding interim participation constraint for any given test fee b, and let

$$b^* = \begin{cases} \max\left\{\underline{B}^{md}(\tilde{\mathcal{T}}) + \epsilon, \min\{b : n^*(b) \in \mathcal{G}(\tilde{\mathcal{T}})\}\right\} & \text{if } \mathcal{G}(\tilde{\mathcal{T}}) \neq \emptyset \\ \bar{B}^{md}(\tilde{\mathcal{T}}) & \text{if } \mathcal{G}(\tilde{\mathcal{T}}) = \emptyset \end{cases},$$

with $\epsilon > 0$ *being infinitesimal. Then, the minimum expenditure to implement* $\tilde{\mathcal{T}}$ *can be achieved by choosing*

$$\begin{cases} \tilde{b} = \min\left\{b^*, \bar{B}^{md}(\tilde{\mathcal{T}})\right\} \\ \tilde{r} = \max_{n \in \left(\{0\} \cup \tilde{\mathcal{T}} \setminus \{N\}\right)} \left\{R^{md}\left(\tilde{\mathcal{T}}_n, b\right)\right\}. \end{cases}$$

Proof. First, note that for all $b \in \left(\underline{B}^{md}(\tilde{\mathcal{T}}), \bar{B}^{md}(\tilde{\mathcal{T}})\right]$, the net passing reward required to meet all the participation constraints are at least $\max_{n \in \left(\{0\} \cup \tilde{\mathcal{T}} \setminus \{N\}\right)} \left\{R^{md}\left(\tilde{\mathcal{T}}_n, b\right)\right\}$, and there is no need to increase it beyond this amount. Therefore, no matter what the optimal test fee $\tilde{b}$ is, the optimal net passing reward is always given by $\tilde{r} = \max_{n \in \left(\{0\} \cup \tilde{\mathcal{T}} \setminus \{N\}\right)} \left\{R^{md}\left(\tilde{\mathcal{T}}_n, b\right)\right\}$.

As for the optimal test fee, when $\mathcal{G}(\tilde{\mathcal{T}}) = \emptyset$, there is no harm by setting the test fee as high as possible, given that it is inside the range that implements $\tilde{\mathcal{T}}$. This is because if there exists $b' \in \left(\underline{B}^{md}(\tilde{\mathcal{T}}), \bar{B}^{md}(\tilde{\mathcal{T}})\right]$ such that $n^*(b) = 0$ for $b \geq b'$, the principal's expenditure is the same by choosing any $\tilde{b} \in [b', \bar{B}^{md}(\tilde{\mathcal{T}})]$. In such cases, full-surplus-extraction is feasible. If such b' does not exist, as has been discussed above, it is beneficial for the principal to set the test fee as large as possible, and therefore $\tilde{b} = \bar{B}^{md}(\tilde{\mathcal{T}})$ is optimal. Hence, when $\mathcal{G}(\tilde{\mathcal{T}}) = \emptyset$, $\tilde{b} = \bar{B}^{md}(\mathcal{T}^{FB})$ is always optimal.

When $\mathcal{G}(\tilde{\mathcal{T}}) \neq \emptyset$, it is not profitable to increase the test fee as long as $n^*(b) \in \mathcal{G}(\tilde{\mathcal{T}})$, as the additional income brought in by increasing test fee is not enough to compensate for the increase in the net passing reward needed to keep the *interim* participation constraint satisfied. Also note that if $n^*(b') \in \mathcal{G}(\tilde{\mathcal{T}})$ for some b', then $n^*(b) \in \mathcal{G}(\tilde{\mathcal{T}})$ for all $b \geq b'$. Thus, the principal would never raise the test fee above $\min\{b : n^*(b) \in \mathcal{G}(\tilde{\mathcal{T}})\}$, unless it is too low to implement $\tilde{\mathcal{T}}$. Hence, when $\mathcal{G}(\tilde{\mathcal{T}}) \neq \emptyset$, it is optimal to set $\tilde{b} = \max\left\{\underline{B}^{md}(\tilde{\mathcal{T}}) + \epsilon, \min\{b : n^*(b) \in \mathcal{G}(\tilde{\mathcal{T}})\}\right\}$, given that it does not exceed $\bar{B}^{md}(\tilde{\mathcal{T}})$. If $\min\{b : n^*(b) \in \mathcal{G}(\tilde{\mathcal{T}})\} > \bar{B}^{md}(\tilde{\mathcal{T}})$, however, it is optimal to set $\tilde{b} = \bar{B}^{md}(\tilde{\mathcal{T}})$. □

Now that with all implementable test schedules and the minimum expenditure associated with each of them at hand, the principal can simply choose the cheapest to implement.

1.5. Partial-Disclosing

In this section, I show how a partial-disclosing feedback system can be employed for the principal's benefit. However, as this kind of feedback system relies more heavily on model parameters ($\{a_i, p_i\}$'s), universal conclusions are hard to drawn. Therefore, the discussion that follows is limited in scope, and focuses mainly on how to improve the outcome for the principal, instead of identifying the optimal outcome itself.

When the feedback policy is partial-disclosing, a *grade* is assigned to the agent based on his current progress, and the agent would use this grade to refine his belief about his ability based on the amount of effort that he has spent, which remains to be the agent's private information. Here, no explicit distinction between a passing score and a failing score has to be made, as a special grade can be created to indicate that an agent has passed the test. Just as in the minimal-disclosing case, if an agent who has not passed the test yet decides to keep working and retake the test in the future, he may fail again and face a similar decision of whether to go on, and if so, how much more effort to spend before taking the next test. Hence, an agent's decision under the partial-disclosing feedback policy is a comprehensive test-taking-plan consisting of all the effort levels at which a test is to be taken, all of which are history-contingent. Clearly, both the full-disclosing feedback policy and the minimal-disclosing feedback policy are special cases of the partial-disclosing policy: In the former case, the grade enables an agent to uniquely pin down his ability, whereas in the latter case, the grade reveals no information in addition to the fact that the agent has not passed the test yet.

To see how a partial-disclosing feedback policy works, assume that an agent's ability is a, and after exerting effort e, the agent scores $s = S(a, e)$ according to the score function. Since the principal cannot directly observe a or e, the *grade* (denoted by g) has to solely depend on s. Two possibilities arise at this point: the grade can be either deterministic or random. In the former case, any s only generates a unique g, whereas in the latter case, a single s induces a distribution of possible g's. It is worth pointing out that even in the former case, the disclosing power of the feedback policy can still be partial instead of full. This is because there could be many s's that generate the same g. In other words, even though a single s cannot generate multiple g's, the reverse is not true – a single g can be generated by multiple s's. This is much like the letter grade system: a student gets an A no matter his/her grade is 95 or 97. In contrast, if the grade is random, not only a certain g can be generated by multiple s's, any s can also generate multiple g's.

Regardless of the randomness of the grade-generating process, one thing is clear – the agent is able to use the grade to update his belief about his ability using the Bayes rule. Using $P\left\{a_i|e,g\right\}$ to denote the probability that an agent's ability is a_i, given that the agent gets grade g after spending effort e, it follows that:

$$P\left\{a_i|e,g\right\} = \frac{P\left\{g|S(a_i,e)\right\} \cdot p_i}{\sum_{j=1}^{N} P\left\{g|S(a_j,e)\right\} \cdot p_j}, \tag{1.16}$$

where $P\left\{g|S(a_i,e)\right\}$ is the probability that grade g is generated by score $s = S(a_i,e)$. The difference between deterministic and random grade-generating processes lies in $P\left\{g|S(a_i,e)\right\}$. Let $G \neq \emptyset$ be the *grade set*. When the process is deterministic, for any given $i \in \{1,...,N\}$, $\exists! g \in G$ so that $P\left\{g|S(a_i,e)\right\} = 1$. When the process is random, in contrast, for any given $i \in \{1,...,N\}$, $P\left\{g|S(a_i,e)\right\} \geq 0$ for all $g \in G$ and $\sum_{g\in G} P\left\{g|S(a_i,e)\right\} = 1$.

As mentioned above, an agent's decision under the partial-disclosing feedback policy consists of a complete test-taking-plan including all the effort levels at which a test is to be taken, all of which are history-dependent. A representative test-taking-plan can be denoted by $\{e_1(\mathbf{h}^0)$, $e_2(\mathbf{h}^1)$,..., $e_M(\mathbf{h}^{M-1})\}$, with each e_i being the effort level at which the i^{th} test should be taken given that $\mathbf{h}^k = \{(e_1,g_1),...,(e_k,g_k)\}$ is the full history of the past test-and-grade combinations. As a special case, $\mathbf{h}^0 = \emptyset$, as no history exists when an agent takes his first test.

Similar to the minimal-disclosing case, if the test fee is zero, there is no downside for an agent to take as many tests as he likes. In fact, he may be tempted to take a "speculative" test at $e_1 < \varepsilon_1$, knowing that even thought there is no chance that he would pass the test, he could nonetheless get some information about his ability. Of course, this could cause trouble for the principal, as this type of information leaking forces her to offer a lucrative passing reward for those who are pessimistic about their abilities. From this aspect, the partial-disclosing feedback system is similar to its full-disclosing counterpart. Also similar to the full-disclosing case, "speculative" test-taking can be discouraged by imposing a huge fine on those whose score is below $S(a_N, \varepsilon_1)$. Or it can be eliminated more elegantly by increasing the test fee so that the agents no longer have any incentive to take a sure-to-fail test just to fish out some information. For simplicity, I adapt the first approach to resolve the "speculative" test-taking issue. The second approach is discussed in the Appendix.

Given a test-taking-plan $\{e_1(\mathbf{h}^0), e_2(\mathbf{h}^1), ..., e_M(\mathbf{h}^{M-1})\}$, there is a continuation payoff, which I temporarily denote by $V\left(\mathbf{h}^i\right)$,[25] associated with history $\mathbf{h}^i$ $(i = 1, ..., M-1)$, such that

(1.17)
$$V\left(\mathbf{h}^i\right) = \sum_{j=l}^{m} \tilde{p}_j \cdot r + \sum_{j=m+1}^{N} \tilde{p}_j \cdot \left(\sum_{g\in G} P\left\{g | S\left(a_j, e_{i+1}(\mathbf{h}^i)\right)\right\} \cdot \left[V\left(\mathbf{h}^i \cup \left\{\left(e_{i+1}(\mathbf{h}^i), g\right)\right\}\right) - b\right] \right) - \left(e_{i+1}(\mathbf{h}^i) - e_i\right),$$

where $l := \min\{k : \varepsilon_k > e_i\}$, $m := \max\{k : \varepsilon_k < e_{i+1}(\mathbf{h}^i)\}$, and $\tilde{p}_j := \dfrac{p_j}{\sum_{k=l}^{N} p_k}$. That is, based on history $\mathbf{h}^i$ and the test-taking-plan, the agent will choose to take the next test at effort level $e_{i+1}(\mathbf{h}^i)$. If the agent passes the test with effort $e_{i+1}(\mathbf{h}^i)$, he receives r and leaves the

[25]Note that the continuation payoff also depends on r and b. They are omitted from the notation for conciseness.

system. Otherwise, by the time the agent fails at $e_{i+1}(\mathbf{h}^i)$, he receives grade g with probability $P\left\{g|S\left(a, e_{i+1}(\mathbf{h}^i)\right)\right\}$ and the updated history becomes $\mathbf{h}^{i+1} = \mathbf{h}^i \cup \left\{\left(e_{i+1}(\mathbf{h}^i), g\right)\right\}$, which has a continuation payoff $V\left(\mathbf{h}^{i+1}\right)$ associated with it.

From an agent's perspective, taking tests serves two purposes. First, it offers him a chance to pass the test and get the passing reward. Second, it provides information about the agent's ability, and thus how the agent should proceed in the future. Under a minimal-disclosing feedback system, the second purpose does not exist, as no extra insight is to be gained other than the fact that the agent has not passed the test yet. Therefore, the agents are extra sensitive to the test fee, as in a sense, taking a test does not offer them much in return – even if the tests are free, they will not take one other than at each of the ϵ_i's. In contrast, under a full-disclosing feedback system, the second purpose of test-taking prevails. When speculative test-taking is not prohibited (by imposing a large fine for those who scores below $S(a_N, \epsilon_1)$) mentioned in the previous section of this paper), the agents will take a test after having spend a tiny amount of effort just to learn about their abilities, as long as the test fee is not too large.

Unlike a full-disclosing feedback system or a minimal-disclosing one, a partial-disclosing feedback system offers the principal another dimension of disclosing flexibility. Perhaps the best way to see this is to take a step back and re-consider the minimal-disclosing feedback policy. At each ϵ_i, a test serves its two purposes, as mentioned above, simultaneously. By introducing a grade system, as in the partial-disclosing case, the principal is able to separate the two purposes of test-taking. As an example, consider a simple grade system in which two deterministic grades are introduced. An agent gets "A" if $s \geq \frac{x}{2}$ and "B" otherwise.[26] Then by taking a test at $\mathcal{E}\left(a, \frac{x}{2}\right)$, an agent would be able to know whether his ability is above or below a. Or, the grade system

[26]Recall that x is the passing score.

can be made slightly more complicated by introducing three grades, say, "A" if $s \geq \frac{3x}{5}$, "B" if $\frac{x}{3} \leq s < \frac{3x}{5}$ and "C" otherwise. Now, a test result reveals more information to an agent. From this point of view, the full-disclosing feedback system can be seen as a grade system that has infinite number of grades that are continuously distributed. This point can be better illustrated using a graph:

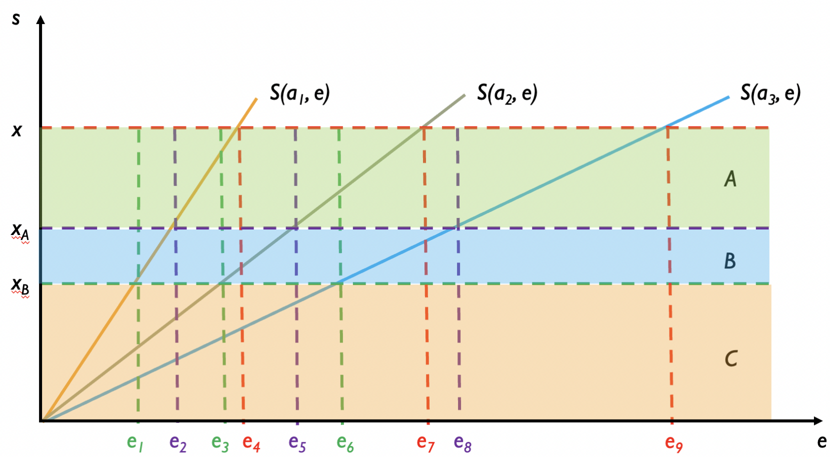

Figure 1.3. Grade System Illustrated

In Figure 1.3, an agent has something to gain at each of $e_1, ..., e_9$. Among them, e_4, e_7 and e_9 are different from the others as they offer the additional benefit of passing the test. However, in terms of information, e_4, e_7 and e_9 are not among the most informative group. To see this, consider effort level e_3. In this simple example, e_3 has full revealing power: If an agent takes a test after spending effort e_3 and gets "A", he knows that his ability is a_1; If he gets "B", then his ability is a_2; If he gets "C", then his ability is a_3. In fact, this is true for all effort levels between e_3 and e_4. This is reminiscent to the minimal-disclosing case in which it is never optimal for an agent to take a test on an effort level that does not coincide with any of the ε_i's. Here, even though an agent that fails a test with an effort between e_3 and e_4 gains the same insight regarding his ability, intuitively, he should prefer to take the test at either e_3 or e_4, but no anywhere in between. This intuition goes beyond this simple example – it also holds in a random grade generating environment, as is formally stated in the following Proposition:

Proposition 13. *Under a partial-disclosing feedback policy, if there exist two effort levels* $\hat{e}$ *and* $\hat{e}'$ $(\hat{e} < \hat{e}')$*, with* $P\{g|S(a_i, \hat{e})\} = P\{g|S(a_i, \hat{e}')\}$ *for all* $g \in G$ *and* $i \in \{1, \ldots N\}$*, then it is never optimal for an agent to take a test at* $\hat{e}'$*.*

Proof. Assume that the test-taking-plan $\{e_1(\mathbf{h}^0), e_2(\mathbf{h}^1), \ldots, e_M(\mathbf{h}^{M-1})\}$ is optimal, and $\hat{e}' = e_m(\mathbf{h}^{m-1})$ $(m \in \{1, \ldots, M\})$. Also assume that there exists $\hat{e} < \hat{e}'$ such that $P\{g|S(a_i, \hat{e})\} = P\{g|S(a_i, \hat{e}')\}$ for all $g \in G$ and $i \in \{1, \ldots N\}$.

First, note that the assumption $P\{g|S(a_i, \hat{e})\} = P\{g|S(a_i, \hat{e}')\}$ for all $g \in G$ and $i \in \{1, \ldots N\}$ implies that taking a test at $\hat{e}'$ or at $\hat{e}$ offer the same prospect of passing the test to agents with all ability levels. In other words, even though $\hat{e} < \hat{e}'$, there does not exist any ε_i $(i = 1, \ldots, N)$ such that $\hat{e} < \varepsilon_i \leq \hat{e}'$. Otherwise, $P\{g|S(a_i, \hat{e})\} = P\{g|S(a_i, \hat{e}')\}$ for all $g \in G$ cannot hold for this particular i.

Now, consider an alternative test-taking-plan by replacing $\hat{e}'$ with $\hat{e}$ while keeping the other part unaltered. Indeed, since each test in a test-taking-plan is history dependent, many modifications may be necessary to keep the other part unaltered. More specifically, if there exists $j \in \{1, \ldots, M\}$ such that $\hat{e} < e_j(\mathbf{h}^{j-1}) < \hat{e}'$, then $\mathbf{h}^{m-1}$ needs to be truncated accordingly so that $\hat{e}$ only depends on the test results that proceeds it. In this case, the test at $e_j(\mathbf{h}^{j-1})$ should still be taken, and its dependence on history does not incorporate the test result at $\hat{e}$. For all the tests after $\hat{e}'$, the original schedule remains unchanged except that now the outcome of the test at $\hat{e}$ replaces that at $\hat{e}'$. If there does not exist $j \in \{1, \ldots, M\}$ such that $\hat{e} < e_j(\mathbf{h}^{j-1}) < \hat{e}'$, then simply replace $\hat{e}'$ with $\hat{e}$ and keep the original schedule remains unchanged except that now the outcome of the test at $\hat{e}$ replaces that at $\hat{e}'$.

If $m = M$, that is, if there is no test scheduled after an agent fails at $\hat{e}$ or every agent is sure to pass at $\hat{e}$, then by switching to the modified test-taking-plan, the agents are able to save both test fee and efforts without sacrificing their chance of passing the test. Thus, the original test-taking-plan cannot be optimal. Similarly, if $m < M$, then by switching to the modified test-taking-plan, the agent's chance of passing the test at $\hat{e}$ does not change, while he is able to save effort in case he passes the test. If the agent is not able to pass the test at $\hat{e}'$, then by the way the modified test-taking-plan is constructed, the agent's expected payoff is the same under the two alternatives. Therefore, the proposed modification to the original test-taking-plan makes an agent strictly better off in case he is going to pass the test at $\hat{e}'$. Otherwise, the agent's payoff remains unchanged. Thus, the original test-taking-plan is not optimal. $\square$

Proposition 13 implies that an agent's problem can be simplified into choosing a set of effort levels at which the test results would change his belief about his ability. This, in turn, implies that the principal should design when these changes of beliefs should happen.

1.5.1. Letter Grade

In this subsection, I look at the letter grade system that is commonly used in the real world. According to Proposition 13, the agents take their first test at $t_1 \in \{\varepsilon_1, ..., \varepsilon_N\}$. Based on the test score, an agent either pass or gets a letter grade. These letter grades then divide the agents into sub-groups, each having a range of ability levels attached to it. The nature of letter grades implies that there is no overlap of ability levels between the sub-groups. More specifically, assume that $G = \{g_1, ..., g_L\}$. Let there be $0 < s_L < s_{L-1} < ... < s_1 \leq x$ such that an agent, whose score is

s, gets grade:

$$g = \begin{cases} g_l & \text{if } s_{l+1} \leq s < s_l,\ l = 1, ...L-1 \\ g_L & \text{if } s < s_L \end{cases} . \tag{1.18}$$

The grade set G and the cutoff scores $s_1 ..., s_L$, along with b and r, are what the principal has to decide under the letter grade setting. For ease of notation, let $s_{L+1} = 0$. Then Equation 1.18 simplifies into $g = g_l$ if $s_{l+1} \leq s < s_l,\ l = 1, ..., L$.

Just as before, $t_1 = \varepsilon_1$ is required to achieve FBME. First, assume that for all $l = 1, ..., L$, there exists at most one a_i $(i = 1, ..., N)$ such that $s_{l+1} \leq S(a_i, \varepsilon_1) < s_l$. In other words, assume that the letter grade system is so enriched that it has the same effect as a full-disclosing feedback system. Recall that from previous discussion, FBME requires:

$$U^{fd}(1; (\underline{R}^{fd}(1), \bar{B}^{fd}(1))) = \underline{R}^{fd}(1) - \sum_{j=2}^{N} p_j \bar{B}^{fd}(1) - \sum_{j=1}^{N} p_j \varepsilon_j \leq 0,$$

where $\underline{R}^{fd}(1) = \varepsilon_N - \varepsilon_1$ and $\bar{B}^{fd}(1) = \min_{i \in \{2,...,N\}} \ \beta^{fd}(1, i)$.

Consider truncating the grade set to $G = \{g_1, ..., g_{L'}\}$ so that $S(a_{N-1}, \varepsilon_1) < s_{L'} \leq S(a_{N-2}, \varepsilon_1)$. Then after taking the first test at ε_1, agents with ability a_{N-1} and a_N both get grade $g_{L'}$. Clearly, if the principal wants to achieve FBME, these two types of agents have to be willing to (1) continue, and (2) take the next test at ε_{N-1}. The willingness to continue can be guaranteed as long as:

$$r \geq \frac{p_{N-1}}{p_{N-1} + p_N}(\varepsilon_{N-1} - \varepsilon_1) + \frac{p_N}{p_{N-1} + p_N}(b + \varepsilon_N - \varepsilon_1), \tag{1.19}$$

while taking the next test at ε_{N-1} can be guaranteed as long as:

$$b \leq \frac{p_{N-1}}{p_N}(\varepsilon_N - \varepsilon_{N-1}). \tag{1.20}$$

Note that Equation 1.19 and Equation 1.20, if combined, indicates that the lower bound on the passing reward is relaxed under this regime. More specifically, given that $b \leq \frac{p_{N-1}}{p_N}(\varepsilon_N - \varepsilon_{N-1})$, the right-hand-side of Equation 1.19 is no greater than $\varepsilon_N - \varepsilon_1$, which is the passing reward needed to prevent the agents with ability a_N from quitting when a full-disclosing feedback policy is used. At the first glance, this change should be welcomed by the principal, as under the full-disclosing feedback policy, the obstacle for achieving FBME emerges when she cannot lower the passing reward below $\varepsilon_N - \varepsilon_1$ while the test fee is set at the upper bound. However, similar to the minimal-disclosing case discussed before, sometimes it may not be in the principal's best interest to raise the test fee even though it has not hit the upper bound yet. To see this, note that Equation 1.19 makes clear that as b increases, r has to be increased correspondingly to prevent the agents from quitting. This problem does not arise under the full-disclosing feedback policy because at the *interim* stage, the test fee is not relevant anymore since all agents know exactly how much more effort it takes for them to pass the test, and thus they do not risk paying the test fee again in the future. Using the same logic as before, if[27]

$$\frac{p_N}{p_{N-1} + p_N} > \sum_{i=2}^{N} p_i + p_N,$$

that is, if at the *interim* stage, the passing reward needs to be raised at a faster rate to compensate for an increase in the test fee when compared to the *ex-ante* stage, then it is not beneficial for the

[27]Note that by bundling a_{N-1} and a_N together, the *ex-ante* expected utility for the agents is $r - \left(\sum_{i=2}^{N} p_i + p_N\right) b - \sum_{i=1}^{N} p_i \varepsilon_i$.

principal to raise the test fee beyond the point that demands the same passing reward from the two stages. More concretely, solving for b^* such that

$$\left(\sum_{i=2}^{N} p_i + p_N\right) b^* + \sum_{i=1}^{N} p_i \varepsilon_i = \frac{p_{N-1}}{p_{N-1}+p_N}(\varepsilon_{N-1} - \varepsilon_1) + \frac{p_N}{p_{N-1}+p_N}(b^* + \varepsilon_N - \varepsilon_1),$$

then it is not optimal for the principal to increase b as long as $b \geq b^*$, even if the upper bound on b is not binding.[28] The mathematical details can be found in the Appendix, as they are not too important to understand the nature of the letter grade system.

There in no fundamental difference between bundling the a_{N-1} and a_N agents together or bundling any other types of agents instead, except that the former immediately relax the lower bound on r. It is not hard to see that under a letter grade system, the lower bound on r at the *interim* stage is always determined by the group of agents, be it single type of a bundle of types, who get the worst grade.

Proposition 14. *Given a grade set* $G = \{g_1, ..., g_L\}$ *and* $0 = s_{L+1} < s_L < ... < s_1 \leq x$ *in which an agent with score s gets grade* $g = g_l$ *if* $s_{l+1} \leq s < s_l$ $(l = 1, ..., L)$*. Let* $g_k \in G$ *be such that* $s_{k+1} \leq S(a_m, \varepsilon_1) < ... < S(a_n, \varepsilon_1) < s_k$ $(n < N)$*. Without loss of generality, assume that* $S(a_N, \varepsilon_1) < s_L$*. Then* $V(\{(\varepsilon_1, g_k)\}) > V(\{(\varepsilon_1, g_L)\})$*.*[29] *That is, all agents with grade better than* g_L *have higher continuation payoff than those who get* g_L*.*

Proof. Suppose $S(a_q, \varepsilon_1) < s_L \leq S(a_{q-1}, \varepsilon_1)$. Then $V(\{(\varepsilon_1, g_L)\}) \leq r - (\varepsilon_q - \varepsilon_1)$, as a_q is the highest possible ability among those who get grade g_L. Meanwhile, $V(\{(\varepsilon_1, g_k)\}) \geq$

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[28]It is straightforward to calculate that $b^* = \dfrac{\frac{p_{N-1}}{p_{N-1}+p_N}(\varepsilon_{N-1} - \varepsilon_1) + \frac{p_N}{p_{N-1}+p_N}(\varepsilon_N - \varepsilon_1) - \sum_{i=1}^{N} p_i \varepsilon_i}{\sum_{i=2}^{N} p_i + p_N - \frac{p_N}{p_{N-1}+p_N}}$.

[29]Again, 100% eventual passing rate is implicitly assume, as this can always be achieved by setting r large enough.

$r-(\varepsilon_m-\varepsilon_1)$. This is because a_m is the lowest possible ability among those who get grade g_k. For this group of agents, although they may retake the test at an earlier effort level than ε_m, they will only do so if this is more profitable than work until ε_m is reached so that they are sure to pass. The above two observations, when combined together, yields $V(\{(\varepsilon_1, g_L)\}) < V(\{(\varepsilon_1, g_k)\})$, since $\varepsilon_q > \varepsilon_m \Rightarrow r-(\varepsilon_q-\varepsilon_1) < r-(\varepsilon_m-\varepsilon_1)$. □

Proposition 14 is crucial to understand the advantage of the letter grade system. Essentially, it implies that bundling in some cases is almost as good as free lunch, if not better. To see this, assume again that the feedback system is full-disclosing. The discussion above shows that bundling a_{N-1} and a_N may not always be beneficial, although it relaxes the lower bound on r. Now, consider that the principal bundles a_i and a_{i+1} $(1 < i < N-1)$ together while trying to achieve FBME. Proposition 14 says that the continuation payoff is not going to be a problem, which means the only constraint that the principal faces from this bundling arrangement is that the agents are willing to retake the test at ε_i instead of ε_{i+1}, which requires $b \leq \frac{p_i}{p_{i+1}}(\varepsilon_{i+1}-\varepsilon_i)$. As long as this constraint on b is not binding, bundling a_i and a_{i+1} contributes to the pursuit of FBME in two ways.[30] First, it generates extra income for the principal, as those with ability a_{i+1} pays the test fee again when they fail at ε_i. Second, because of the possibility of paying the test fee again, the agents' *ex-ante* expected payoff deceases for any fixed (r, b). As has been illustrated before, FBME is not feasible if all (r, b) pairs required to satisfy $V(\{(\varepsilon_1, g_L)\}) \geq 0$ generate a positive *ex-ante* expected payoff for the agents. Since bundling a_i and a_{i+1} has no influence on $V(\{(\varepsilon_1, g_L)\})$, the fact that it can lower the *ex-ante* expected payoff is beneficial to the principal.

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[30]The assumption of $b \leq \frac{p_i}{p_{i+1}}(\varepsilon_{i+1}-\varepsilon_i)$ not binding is crucial. I discuss this point in more details later.

A new complication that is unique to the letter-grade system is that when agents take the first test at an effort level other than ε_1, aside from getting a higher probability of passing the test in one sitting, they may get different information regarding their abilities based on the how the letter grades are set. For example, assume that an agent's ability is a_5. If the agent takes the first test at ε_1, the letter grade he gets enable him to deduce that his ability is either a_5 or a_6. In contrast, if the agent were to take the first test at ε_2 instead, he is able to get a better letter grade, which is the same grade that someone with ability a_4 would get after spending the same amount of effort. In this situation, the continuation problems that the agent faces are not the same, which in turn, changes the *ex-ante* expected utility that the agent uses to make his decision of when to take the first test.

In general, an agent's *ex-ante* expected utility with $t_1 = \varepsilon_1$ is (assuming 100% passing rate):

$$U^{lg}(1) = r - \sum_{j=1}^{N} p_j \left(\varepsilon_j + n_j b\right),$$

where n_j is the number of failed attempts made by agents with ability a_j. Similarly, an agent's *ex-ante* expected utility with $t_1 = \varepsilon_k$ is (assuming 100% passing rate):

$$U^{lg}(k) = r - \sum_{j=1}^{k} p_j \left(\varepsilon_k + n'_j b\right) - \sum_{j=k+1}^{N} p_j \left(\varepsilon_j + n'_j b\right),$$

where n'_j is the number of failed attempts made by agents with ability a_j given the grade system and subsequent re-optimization once fail at ε_k. Then the threshold test fee above which $t_1 = \varepsilon_k$ is preferred to $t_1 = \varepsilon_1$ is:

$$\beta^{lg}(1,k) = \frac{\sum_{j=1}^{k} p_j \left(\varepsilon_k - \varepsilon_j\right)}{\sum_{j=2}^{N} p_j \left(n_j - n'_j\right)}. \tag{1.21}$$

Without any bundling, $n_j - n'_j = 1$ for all $j = 2, ..., k$, as these agents are able to pass the test with effort ε_k. For all other j's, $n_j - n'_j = 0$, since they either pass in both situations or fail in both situations. Therefore, the RHS of Equation 1.21 becomes $\frac{\sum_{j=1}^{k} p_j \left(\varepsilon_k - \varepsilon_j\right)}{\sum_{j=2}^{k} p_j}$, which is a special case of Equation 1.1.

In a nutshell, achieving FBME boils down to two aspects. First, make the cost of continuation for the group of agents with the lowest grade as low as possible. This cost includes both the expected effort and additional test fees to be paid. Second, extract as much test fee as possible from those with higher grades, given that the high test fee does not make them over-exerting effort. Another thing to keep in mind is the impact of the structure of letter grade system on the agents' choice of the first test. Under the letter grade system, postponing the first test does not only serve to save test fee, rather, it may come with an alternative set of information of a failing agent's ability, which has further implication regarding the continuation payoff.

To see why the last point can be important, consider the example of $N = 4$. Assume that the grade set contains two grades $G = \{g_1, g_2\}$. When the first test is taken with effort ε_1, g_2 implies that the agent's ability is a_4, whereas g_1 implies that the agent's ability is either a_2 or a_3. Apparently, $V\left(\{(\varepsilon_1, g_2)\}\right) \geq 0$ requires $r \geq \varepsilon_4 - \varepsilon_1$, which is the same as in the full-disclosing feedback system. In both cases, all participation constraints at the *interim* stages are satisfied as long as $r \geq \varepsilon_4 - \varepsilon_1$. Therefore, the other constraints that the principal faces are regarding the schedules of the tests, which depend solely on b. In particular, it is certain that

$$\beta^{fd}(1,2) = \frac{p_1(\varepsilon_2 - \varepsilon_1)}{p_2}, \quad \beta^{fd}(1,3) = \frac{p_1(\varepsilon_3 - \varepsilon_1) + p_2(\varepsilon_3 - \varepsilon_2)}{p_2 + p_3},$$
$$\text{and} \quad \beta^{fd}(1,4) = \frac{p_1(\varepsilon_4 - \varepsilon_1) + p_2(\varepsilon_4 - \varepsilon_2) + p_3(\varepsilon_4 - \varepsilon_3)}{p_2 + p_3 + p_4},$$

while

$$\beta^{lg}(1,3) = \frac{p_1(\varepsilon_3 - \varepsilon_1) + p_2(\varepsilon_3 - \varepsilon_2)}{p_2 + 2p_3} \quad \text{and} \quad \beta^{lg}(1,4) = \frac{p_1(\varepsilon_4 - \varepsilon_1) + p_2(\varepsilon_4 - \varepsilon_2) + p_3(\varepsilon_4 - \varepsilon_3)}{p_2 + 2p_3 + p_4}.$$

However, $\beta^{lg}(1,2)$ depends on $\{p_i, \varepsilon_i\}$'s and s_1. More specifically, using Equation 1.21, if $t_1 = \varepsilon_2$ results in agents with a_3 and a_4 getting different grades (a_3 gets g_1 and a_4 gets g_2), then

$$\beta^{lg}(1,2) = \frac{p_1(\varepsilon_2 - \varepsilon_1)}{p_2 + p_3}.$$

Compare $\beta^{lg}(1,2)$, $\beta^{lg}(1,3)$ and $\beta^{lg}(1,4)$ to $\beta^{fd}(1,2)$, $\beta^{fd}(1,3)$ and $\beta^{fd}(1,4)$, all $\beta^{lg}(1,i)$ and $\beta^{fd}(1,i)$ have the same numerator, while the difference in the denominator is always p_3. An immediate implication of this observation is that

$$(p_2 + 2p_3 + p_4) \cdot \bar{B}^{lg}(1) \leq (p_2 + p_3 + p_4) \cdot \bar{B}^{fd}(1),$$

where $\bar{B}^{fd}(1) = \min_{i=2,3,4} \left\{\beta^{fd}(1,i)\right\}$ and $\bar{B}^{lg}(1) = \min \left\{ \min_{i=2,3,4} \left\{\beta^{lg}(1,i)\right\}, \frac{p_2(\varepsilon_3 - \varepsilon_2)}{p_3}, \frac{p_3(\varepsilon_4 - \varepsilon_3)}{p_4} \right\}$. Given that $\underline{R}^{fd}(1) = \underline{R}^{lg}(1)$, this further implies that FBME is achievable with the letter-grade system only if it is achievable under full-disclosing, but not the other way around. Intuitively, although the letter-grade system makes some agents (those with a_3 in this example) pay the test fee more than once, it does not bring in more revenue for the principal, as the possibility of paying the test fee more than once results in a tighter upper bound on the test fee itself.

However, the situation changes if there exists s_1 such that $t_1 = \varepsilon_2$ results in agents with a_3 and a_4 both get g_1. In this case,

$$\beta^{lg}(1,2) = \frac{p_1(\varepsilon_2 - \varepsilon_1)}{p_2 + p_3 - p_4},$$

as those with a_4 must pay the test fee again when they fail at ε_3, given that $b \leq \frac{p_3(\varepsilon_4 - \varepsilon_3)}{p_4}$. The "$-p_4$" term in the denominator helps relax the upper bound on b in certain situations, and it becomes possible that FBME is indeed achievable with the letter-grade system while it is not so with full-disclosing. A numerical example is provided in the Appendix. Note that such s_1 may not exist with certain $\{p_i, \varepsilon_i\}$'s and some score function.

In the example of $N = 4$ and bundling a_2 and a_3 together, it is obvious that having to impose $r \geq \varepsilon_4 - \varepsilon_1$ is a major drawback. But even so, the extra flexibility in the information flow design gives it some advantages over the full-disclosing feedback policy under certain circumstances. This advantage becomes more clear when N is large. Loosely speaking, a letter grade system cuts agents into bundles, and within each bundle, the least talented agents bear most of the test fee payments (since they have to pay the test fee multiple times). In contrast, the agents who are less talented but who are "lucky" enough to be the smarted type within the another bundle do not have to pay the test fee many times over. If the FBME test schedule were to be achieve, these agents would spend just the right amount of effort to pass the test in the end regardless of which bundle they were initially put into. Therefore, the possibility of ending up in a bundle that is designed for higher ability agents deters an agent from postponing his first test in the first place. This is exactly the situations in which a letter grade system enables the principal to collect test fee more times without making too much compromise in setting a lower ceiling on the test fee itself.

Another possible letter grade system in the $N = 4$ example is to bundle a_3 and a_4, instead of a_2 and a_3. By doing so, aside from collecting the test fee for more times than under the full-disclosing feedback policy, the previous lower bound $r \geq \varepsilon_4 - \varepsilon_1$ can be relaxed. To see how this grade system works, assume that the grade set contains two grades $G = \{g_1, g_2\}$. When the first test is taken with effort ε_1, g_1 implies that the agent's ability is a_2 while g_2 implies that the agent's ability is either a_3 or a_4. As I have discussed earlier, $V\left(\left\{\varepsilon_1, g_1\right\}\right) \geq 0$ is never binding. In contrast, $V\left(\left\{\varepsilon_1, g_2\right\}\right) \geq 0$ requires that $r \geq \frac{p_3}{p_3 + p_4}(\varepsilon_3 - \varepsilon_1) + \frac{p_4}{p_3 + p_4}(\varepsilon_4 - \varepsilon_1 + b)$. As I have shown before, as long as $b \leq B^{lg}(3, 4|\{(\varepsilon_1, g_2)\}) = \frac{p_3(\varepsilon_4 - \varepsilon_3)}{p_4}$ is satisfied, this lower bound on r is lower than that of $r \geq \varepsilon_4 - \varepsilon_1$. Another complication that should be taken into account is whether choosing $t_1 = \varepsilon_2$ instead of $t_1 = \varepsilon_1$ provides the agents with different information about their abilities. There are two possibilities. The first possibility is that when $t_1 = \varepsilon_2$ is chosen, both a_3 and a_4 still get grade g_2. In this case, $\beta^{lg}(1, 2) = \frac{p_1(\varepsilon_2 - \varepsilon_1)}{p_2}$ is the same as in the full-disclosing setting. Another possibility is that when $t_1 = \varepsilon_2$ is chosen, agents with a_3 is able to get grade g_1 while agents with a_4 still get g_2. In this case, $\beta^{lg}(1, 2)$ decreases to $\frac{p_1(\varepsilon_2 - \varepsilon_1)}{p_2 + p_4}$, as this information bonus makes postponing the first test more attractive. Unlike the previous case in which a_2 and a_3 are bundled together, this extra restriction on b does not necessarily lead to the conclusion that under such circumstances, the letter-grade system has no advantage over the full-disclosing feedback system. This is because the relaxation of the lower bound on r is a unique advantage that is not present before. A numerical example is provided in the Appendix, showing FBME can be achieve by the letter-grade system while both fulll-disclosing and minimal-disclosing fail to do so. The fact that postponing $t_1 = \varepsilon_1$ to $t_1 = \varepsilon_2$ may bring information bonus does not change this conclusion.

The analysis of $N = 4$ can be generalized into the following:

Proposition 15 (Improving a Full-Disclosing Feedback System into a Letter-Grade System)**.** *Suppose that FBME is not feasible under the full-disclosing feedback system, then the principal can modify the full-disclosing feedback system into a letter-grade system to improve her expenditure by:*

(1) ***Starting from the full-disclosing feedback system***

   (a) Let the initial grade set be $G = \{g_1, ...g_{N-1}\}$ with cutoff scores $S(a_{i+1}, \varepsilon_1) < s_i \leq S(a_i, \varepsilon_1)$ for $i = 1, ..., N-1$. An agent whose score is s gets grade g_i if $s_{i+1} \leq s < s_i$.[31] *Use $\tilde{G}$ to denote the letter-grade system, including both the grade set, the cutoff scores, and how the grades in the grade sets are assigned based on the cutoff scores.*

   (b) Let $\bar{B}^{lg} = \bar{B}^{fd}(1) = \min_{i=2,...,N} \left\{\beta^{fd}(1, i)\right\}$ and $\underline{R}^{lg} = \underline{R}^{fd}(1) = \varepsilon_N - \varepsilon_1$.

(2) ***Constructing the lowest grade group***

   (a) Let $\tilde{G}^{temp}$ be the modified version of $\tilde{G}$ so that the associated grade set is $G^{temp} = \{g_1, ...g_{N-2}\}$, with $s_i^{temp} = s_i$ for $i = 1, ..., N-2$ and $s_{N-1}^{temp} = 0$.

   (b) *(i) If*

$$\frac{\partial V\left(\{(\varepsilon_1, g_{N-2})\}; (r, b)\right)}{\partial b} = \frac{p_N}{p_{N-1} + p_N} \leq \sum_{i=2}^{N} p_i + p_N = \frac{\partial U^{lg}\left(1; (r, b)|\tilde{G}^{temp}\right)}{\partial b},$$

then let

$$\bar{B}^{lg,temp} = \min\left\{ \min_{i=2,...,N} \left\{\beta^{lg}(1, i|\tilde{G}^{temp})\right\}, B^{lg}\left(N-1, N|\{(\varepsilon_1, g_{N-2})\}\right) \right\},$$

[31]Continue the convention of letting $s_N = 0$.

where

$$\beta^{lg}(1, i|\tilde{G}^{temp}) = \frac{\sum_{j=1}^{i-1} p_j(\varepsilon_i - \varepsilon_j)}{\sum_{j=2}^{i} p_j} \quad or \quad \beta^{lg}(1, i|\tilde{G}^{temp}) = \frac{\sum_{j=1}^{i-1} p_j(\varepsilon_i - \varepsilon_j)}{\sum_{j=2}^{i} p_j + p_N} \tag{1.22}$$

are the constraints on b at the ex-ante stage, depending on whether a_{N-1} and a_N get the same grade at $t_1 = \varepsilon_i$, and

$$B^{lg}\left(N-1, N|\{(\varepsilon_1, g_{N-2})\}\right) = \frac{p_{N-1}(\varepsilon_N - \varepsilon_{N-1})}{p_N}$$

is the constraint on b at the interim stage. Let

$$\underline{R}^{lg,temp} = \max\left\{-V\left(\{(\varepsilon_1, g_{N-2})\}; (0, \bar{B}^{lg,temp})\right), -V\left(\{(\varepsilon_{N-1}, *)\}; (0, \bar{B}^{lg,temp})\right)\right\},$$

with

$$V\left(\{(\varepsilon_1, g_{N-2})\}; (0, \bar{B}^{lg,temp})\right) = \frac{p_{N-1}}{p_{N-1} + p_N}(\varepsilon_{N-1} - \varepsilon_1) + \frac{p_N}{p_{N-1} + p_N}(\varepsilon_N - \varepsilon_1 + \bar{B}^{lg,temp})$$

and

$$V\left(\{(\varepsilon_{N-1}, *)\}; (0, \bar{B}^{lg,temp})\right) = \varepsilon_N - \varepsilon_{N-1}$$

being the continuation payoff with $r = 0$ and $b = \bar{B}^{lg,temp}$ (and thus effectively the continuation cost) at $t_1 = \varepsilon_1$ and $t_2 = \varepsilon_{N-1}$, respectively. The "$*$" symbol in $V\left(\{(\varepsilon_{N-1}, *)\}; (0, \bar{B}^{lg,temp})\right)$ means the specific grade is not important, as all grades narrow down to the only possible ability a_N at this point.

(ii) Otherwise, that is, if

$$\frac{\partial V\left(\{(\varepsilon_1, g_{N-2})\}; (r,b)\right)}{\partial b} > \frac{\partial U^{lg}\left(1; (r,b)|\tilde{G}^{temp}\right)}{\partial b},$$

then calculate $b^*(\varepsilon_1, g_{N-2})$ such that[32]

$$V\left(\{(\varepsilon_1, g_{N-2})\}; (0, b^*(\varepsilon_1, g_{N-2}))\right) = U^{lg}\left(1; (0, b^*(\varepsilon_1, g_{N-2}))|\tilde{G}^{temp}\right).$$

Let

$$\bar{B}^{lg,temp} = \min\left\{\min_{i=2,\ldots,N}\left\{\beta^{lg}(1,i|\tilde{G}^{temp})\right\}, B^{lg}\left(N-1, N|\{(\varepsilon_1, g_{N-2})\}\right), \max\left\{b^*(\varepsilon_1, g_{N-2}), 0\right\}\right\}$$

and

$$\underline{R}^{lg,temp} = \max\left\{-V\left(\{(\varepsilon_1, g_{N-2})\}; (0, \bar{B}^{lg,temp})\right), -V\left(\{(\varepsilon_{N-1}, *)\}; (0, \bar{B}^{lg,temp})\right)\right\}.$$

In either case, the minimum expenditure associated with this letter-grade system is $\underline{R}^{lg,temp} - \sum_{i=1}^N p_i\varepsilon_i - \left(\sum_{i=2}^N p_i + p_N\right)\bar{B}^{lg,temp}$.[33] *If $\underline{R}^{lg,temp} - \sum_{i=1}^N p_i\varepsilon_i - \left(\sum_{i=2}^N p_i + p_N\right)\bar{B}^{lg,temp} \le \underline{R}^{fd} - \sum_{i=1}^N p_i\varepsilon_i - \left(\sum_{i=2}^N p_i\right)\bar{B}^{fd}$, then bundling a_{N-1} and a_N is more profitable for the principal. Update $\tilde{G} = \tilde{G}^{temp}$, $\bar{B}^{lg} = \bar{B}^{lg,temp}$ and $\underline{R}^{lg} = \underline{R}^{lg,temp}$ for subsequent analysis.*

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[32] In this case,

$$b^*(\varepsilon_1, g_{N-2}) = \frac{\frac{p_{N-1}}{p_{N-1}+p_N}(\varepsilon_{N-1}-\varepsilon_1) + \frac{p_N}{p_{N-1}+p_N}(\varepsilon_N - \varepsilon_1) - \sum_{i=1}^N p_i\varepsilon_i}{\sum_{i=2}^N p_i + p_N - \frac{p_N}{p_{N-1}+p_N}}.$$

[33] If $\underline{R}^{lg,temp} - \sum_{i=1}^N p_i\varepsilon_i - \left(\sum_{i=2}^N p_i + p_N\right)\bar{B}^{lg,temp} \le 0$, then FBME is feasible. It can be implemented by $\tilde{b} = \bar{B}^{lg,temp}$ and $\tilde{r} = \sum_{i=1}^N p_i\varepsilon_i + \left(\sum_{i=2}^N p_i + p_N\right)\bar{B}^{lg,temp}$.

Record $\bar{B}^{lg} \in \{\beta^{lg}\}$ *if* $\bar{B}^{lg} = \min_{i=2,\dots,N}\left\{\beta^{lg}(1,i|\tilde{G}^{temp})\right\}$ *for some i, or* $\bar{B}^{lg} \in \{B^{lg}\}$ *if* $\bar{B}^{lg} = B^{lg}\left(N-1,N|\{(\varepsilon_1,g_{N-2})\}\right)$*, or* $\bar{B}^{lg} \in \{b^*\}$ *if* $\bar{B}^{lg} = b^*(\varepsilon_1,g_{N-2})$*. Otherwise, if* $\underline{R}^{lg,temp} - \sum_{i=1}^{N} p_i\varepsilon_i - \left(\sum_{i=2}^{N} p_i + p_N\right)\bar{B}^{lg,temp} > \underline{R}^{fd} - \sum_{i=1}^{N} p_i\varepsilon_i - \left(\sum_{i=2}^{N} p_i\right)\bar{B}^{fd}$*, do not bundle* a_{N-1} *with* a_N*; Keep* $\tilde{G}$*,* $\bar{B}^{lg}$ *and* $\underline{R}^{lg}$ *unchanged.*

(c) *If bundling* a_{N-1} *and* a_N *is indeed more profitable for the principal, then she should further explore whether to bundle* a_{N-2} *into the lowest grade group as well. The steps are similar those described above. However, the specific analysis may differ depending the model parameters. This is because even though a bundle of* a_{N-1} *and* a_N *can more or less be treated similar to a minimal-disclosing feedback system,*[34] *this may not be true once* a_{N-2} *is included, for at* $t_2 = \varepsilon_{N-2}$*, the agents' abilities may be separated, depending on the cutoff scores and the score function. The principal has limited control*[35] *over the former and no control over the latter. But the basic logic remains valid: find the relevant* $\bar{B}^{lg}$ *and* $\underline{R}^{lg}$*, then calculate the minimum expenditure and see if there is any improvement over the minimum expenditure before the bundling occurs. If so, then make the bundling happen and consider whether the bundle should be enlarged to include the agents with ability one level higher. Repeat this process until enlarging the bundle is not profitable anymore. After each change to the letter-grade system, record which category of constraints (*$\{\beta^{lg}\}$*,* $\{B^{lg}\}$ *or* $\{b^*\}$*) that* $\bar{B}^{lg}$ *belongs to.*

(3) ***Constructing other grade groups***

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[34] "More or less" in the sense that the sub-collection of $\{a_1, a_{N-1}, a_N\}$ is indeed like a minimal-disclosing feedback system, the analysis at the *ex-ante* stage differs between the two, as can be seen from $\beta^{lg}(1,i|\tilde{G})$.

[35] Limited control in the sense that for an agent with a_i and another with a_{i+1} to receive different grades upon taking the first test at $t_1 = \varepsilon_1$, the cutoff score has to lie between $S(a_i,\varepsilon_1)$ and $S(a_{i+1},\varepsilon_1)$, but within this range, the principal may be able to move the cutoff score to take some control over the information flow when agents take their second test.

(a) Suppose that the bundling process for the lowest grade group is completed with a_{k+1} being the highest possible ability within this group of agents.[36] *Then the current letter-grade system $\tilde{G}$ consists of the letter grade set $G = \{g_1, ..., g_k\}$ with cutoff scores $S(a_{i+1}, \varepsilon_1) < s_i \leq S(a_i, \varepsilon_1)$ for $i = 1, ..., k$ and $s_{k+1} = 0$. Now consider bundling a_k with a_{k-1}. Let $\tilde{G}^{temp}$ be the modified version of $\tilde{G}$ so that the associated grade set is $G^{temp} = \{g_1, ...g_{k-1}\}$, with $s_i^{temp} = s_i$ for $i = 1, ..., k-2$, $s_{k-1}^{temp} = s_k$ and $s_k^{temp} = 0$.*

(i) If $\bar{B}^{lg} \in \{\beta^{lg}\}$, then calculate and see if

$$\sum_{j=2}^{N} p_j \left[n^{lg}(j; 1|\tilde{G}^{temp}) - n^{lg}(j; i|\tilde{G}^{temp})\right] - \sum_{j=2}^{N} p_j \left[n^{lg}(j; 1|\tilde{G}) - n^{lg}(j; i|\tilde{G})\right] \geq p_k, \tag{1.23}$$

where $n^{lg}(j; i|\tilde{G})$ is the number of times the agent with ability a_j would have to pay the test fee under $\tilde{G}$ were he to take the first test at $t_1 = \varepsilon_i$. If Equation 1.23 is true, then bundling a_k with a_{k-1} is not profitable.

(ii) If $\bar{B}^{lg} \in \{\beta^{lg}\}$ but Equation 1.23 does not hold, then let

$$\bar{B}^{lg,temp} = \min\left\{ \min_{i=2,...,N} \left\{\beta^{lg}(1, i|\tilde{G}^{temp})\right\}, B^{lg}\left(k-1, k|\{(\varepsilon_1, g_{k-2})\}\right) \right\}$$

and recalculate $\underline{R}^{lg,temp}$ if needed, by substituting $b = \bar{B}^{lg,temp}$ in the continuation costs at all stages within the lowest grade group.

(iii) If $\bar{B}^{lg} \in \{B^{lg}\}$, let

$$\bar{B}^{lg,temp} = \min\left\{ \min_{i=2,...,N} \left\{\beta^{lg}(1, i|\tilde{G}^{temp})\right\}, \bar{B}^{lg}, B^{lg}\left(k-1, k|\{(\varepsilon_1, g_{k-2})\}\right) \right\}$$

[36] In case bundling is not optimal, $a_{k+1} = a_N$.

and recalculate $\underline{\boldsymbol{R}}^{lg,temp}$ *if needed, by substituting* $b = \bar{\boldsymbol{B}}^{lg,temp}$ *in the continuation costs at all stages within the lowest grade group.*

(iv) If $\bar{\boldsymbol{B}}^{lg} \in \{b^*\}$*, let*

$$\bar{\boldsymbol{B}}^{lg,temp} = \min\left\{ \min_{i=2,\ldots,N} \left\{ \beta^{lg}(1, i|\tilde{G}^{temp}) \right\}, \min\left\{ b^* |\tilde{G}^{temp} \right\}, \boldsymbol{B}^{lg}_{min}, \boldsymbol{B}^{lg}\left(k-1, k|\{(\varepsilon_1, g_{k-2})\}\right) \right\},$$

where $\left\{ b^* |\tilde{G}^{temp} \right\}$ *is the set of all the* b^**'s in the lowest grade group if the letter-grade system is* $\tilde{G}^{temp}$ *instead of* $\tilde{G}$*, and* $\boldsymbol{B}^{lg}_{min}$ *is the smallest* $\boldsymbol{B}^{lg}$*'s in the lowest grade group.* $\boldsymbol{B}^{lg}_{min}$ *has been calculated earlier and does not need to be recalculated at this point, since it does not change when* $\tilde{G}$ *is modified into* $\tilde{G}^{temp}$*. Recalculate* $\underline{\boldsymbol{R}}^{lg,temp}$ *if needed, by substituting* $b = \bar{\boldsymbol{B}}^{lg,temp}$ *in the continuation costs at all stages within the lowest grade group.*

In case (ii)-(iv), calculate the minimum expenditure associated with $\tilde{G}^{temp}$ *by substituting* $b = \bar{\boldsymbol{B}}^{lg,temp}$ *and* $r = \underline{\boldsymbol{R}}^{lg,temp}$*, and compare it with the minimum expenditure obtained under* $\tilde{G}$*. If the former is smaller, then bundling* a_k *with* a_{k-1} *is more profitable for the principal. Update* $\tilde{G} = \tilde{G}^{temp}$*,* $\bar{\boldsymbol{B}}^{lg} = \bar{\boldsymbol{B}}^{lg,temp}$ *and* $\underline{\boldsymbol{R}}^{lg} = \underline{\boldsymbol{R}}^{lg,temp}$ *for subsequent analysis.*

As before, record $\bar{\boldsymbol{B}}^{lg} \in \{\beta^{lg}\}$ *if* $\bar{\boldsymbol{B}}^{lg} = \min_{i=2,\ldots,N} \left\{ \beta^{lg}(1, i|\tilde{G}^{temp}) \right\}$ *for some i, or* $\bar{\boldsymbol{B}}^{lg} \in \{\boldsymbol{B}^{lg}\}$ *if* $\bar{\boldsymbol{B}}^{lg} = \boldsymbol{B}^{lg}\left(N-1, N|\{(\varepsilon_1, g_{N-2})\}\right)$*, or* $\bar{\boldsymbol{B}}^{lg} \in \{b^*\}$ *if* $\bar{\boldsymbol{B}}^{lg} = b^*(\varepsilon_1, g_{N-2})$*. Otherwise, do not bundle* a_k *with* a_{k-1}*; Keep* $\tilde{G}$*,* $\bar{\boldsymbol{B}}^{lg}$ *and* $\underline{\boldsymbol{R}}^{lg}$ *unchanged.*

(b) If bundling a_k *and* a_{k-1} *is not profitable for the principal, the next bundling candidates to consider is* a_{k-1} *and* a_{k-2}*. Let* $\tilde{G}^{temp}$ *be the modified version of* $\tilde{G}$ *so that*

the associated grade set is $G^{temp} = \{g_1, ...g_{k-1}\}$*, with* $s_i^{temp} = s_i$ *for* $i = 1, ..., k-3$, $s_{k-2}^{temp} = s_{k-1}$, $s_{k-1}^{temp} = s_k$ *and* $s_k^{temp} = 0$*. Repeat the steps above.*

(c) *If bundling* a_k *and* a_{k-1} *is indeed more profitable for the principal, then she should further explore whether to bundle* a_{k-2} *into this grade group as well. The steps are similar those described above. Repeat until enlarging the grade group is no longer optimal. Then move on to construct the next grade group using the same procedure.*

Proof. In achieving FBME, it is necessary that a test-taking-plan that does not involve any waste of effort is implemented. Such test-taking-plan can surely be implemented with an appropriate upper bound on the test fee. Within a letter-grade framework, there are two types of such constraints on the test fee that need to be taken into account. The first type is from the *ex-ante* point of view – the first test has to be taken at $t_1 = \varepsilon_1$, which can only be guaranteed if $b \leq \beta^{lg}(1, i|\tilde{G})$ for all $i = 2, ..., N$. The second type of constraint is from the *interim* point of view – after receiving his grade, an agent has to be willing to take the next test at the most optimistic effort level. For example, suppose that an agent takes a test at ε_k and gets grade g. Let $\left\{a_j : P\{a_j|\varepsilon_k, g\} > 0\right\}$ be the collection of all possible abilities that the agent may have, given ε_k and g. Let $l = \min\left\{j : P\{a_j|\varepsilon_k, g\} > 0\right\}$. Then the *interim* constraint on b can only be satisfied if $b \leq B^{lg}(l, m|\{(\varepsilon_k, g)\})$ for all $m \in \left\{j : P\{a_j|\varepsilon_k, g\} > 0\right\}$. While moving up to a higher grade to construct the letter-grade system, the *interim* constraints on b within the already constructed lower grade groups remain unchanged, the *ex-ante* constraint changes with $\tilde{G}$, and thus should be recalculated each time.

Aside from the constraints on b, the passing reward r should be high enough to first make the agents willing to participate then make them willing to keep going after each failure. Proposition

14 says to make sure that nobody will quit halfway, the principal only needs to focus on the group of agents who get the lowest possible grade after the first test. It is the relative size of the passing reward demanded by this group of agents at each *interim* stage and that demanded by all agents at the *ex-ante* stage that ultimately determines the feasibility of FBME. Sometime the passing reward demanded at an *interim* stage does not depend on b, such as $r \geq \varepsilon_N - \varepsilon_{N-1}$. Sometimes it does. In the latter case, there are two aspects to consider. First, as b increases, the lower bound on r rises both at the *ex-ante* stage and at the *interim* stage in concern. When the latter rises more rapidly than the former, it would not be optimal to do so as long as $b > b^*$, with b^* being the test fee that equalizes the two lower bounds on r. While moving up to a higher grade to construct the letter-grade system, the *interim* constraints on r, as well as their relations with b, do not change in the lower grade groups. But the relevant lower bound on r at the *ex-ante* stage does change, so does its relation with b. That is why $\dfrac{\partial U^{lg}(1|\tilde{G})}{\partial b}$ and b^*'s have to be recalculated each time a modification to $\tilde{G}$ is to be considered.

Once the basic principles outlined above are clearly understood, most parts of the Proposition is straightforward and do not require further explanation. The only part that may need to elaborate on is why $\underline{\boldsymbol{R}}^{lg}$ can be ignored at all *interim* stages within the grade groups that do not have the lowest grade, for Proposition 14 only deals with the continuation constraints at $t_1 = \varepsilon_1$. This is another major difference between the lowest grade group and other grade groups. Recall that within the lowest grade group, each *interim* stage can potentially add additional constraint on $\underline{\boldsymbol{R}}^{lg}$, for example, there is no guarantee that $\dfrac{p_{N-1}}{p_{N-1}+p_N}(\varepsilon_{N-1}-\varepsilon_1)+\dfrac{p_N}{p_{N-1}+p_N}(\varepsilon_N-\varepsilon_1+\bar{B}^{lg}) > \varepsilon_N - \varepsilon_{N-1}$ while deciding whether to bundle a_N and a_{N-1}. However, within other grade groups, the *interim* continuation cost is always less that $\varepsilon_i - \varepsilon_1$, with i being such that a_i is the lowest possible ability in that grade group. If one uses a_k to denote the highest possible ability in the lowest grade group, then $\varepsilon_i - \varepsilon_1 < \varepsilon_k - \varepsilon_1$ implies that no *interim* continuation constraints should be binding. This further implies that $\underline{R}^{lg}$ is irrelevant in deciding whether bundling is desirable for other grade groups. □

Remark. *Regarding Step 3.(a).i in Proposition 15, the general observation, putting loosely, is that it is not worthwhile to extract from some at the cost of loosing from all. For example, if* a_k *were to be bundled with* a_{k-1}*, and if the interim constraint* $b < B^{lg}(k-1, k|\{(\varepsilon_1, g_{k-2})\}) = \dfrac{p_{k-1}(\varepsilon_k - \varepsilon_{k-1})}{p_k}$ *is not binding, then the principal can gain* $p_{k-1} \cdot b$ *in test fee. However, if at the ex-ante stage, the competing* $t_1 = \varepsilon_i$*,*[37] *enables the agent with* a_{k-1} *to avoid paying this extra test fee without resulting in others paying more than they otherwise would with* $t_1 = \varepsilon_1$*, then the principal should not try to extract this extra payment from those whose ability is* a_{k-1}*, as it is not enough to make up for the lost in income from having to decrease the test fee. In abstract terms,* $x \cdot \dfrac{y}{z} > (x + w) \cdot \dfrac{y}{z + w}$ *if and only if* $x > z$*. Applying this relation to our context by substituting* $\beta^{lg}(1, i|\tilde{G}) = \dfrac{y}{z}$*,* $w = p_{k-1}$ *and* $x = \sum_{i=2}^{N} p_i n_i$*, it is clear that exploiting* a_{k-1} *only results in an overall loss of income. This situation is interesting because it has resemblance of the monopoly pricing policy – at some point, it is not profitable to try to sell to more people when one has to lower the price for everyone.*

Proposition 15 provides a systematic way of improving the outcome for the principal, given that the FBME test schedule is to be implemented. In case no bundling is ever found to be optimal, the principal should proceed with the full-disclosing feedback system. At the other extreme, the principal may find it more profitable to bundle all agents from a_2 to a_N together. In this case, the resulting minimal-disclosing feedback system is the best choice. Any bundling between these two extremes is the gain from this more flexible partial-disclosing approach.

[37] In the sense that $\beta^{lg}(1, i|\tilde{G})$ is the binding constraint.

1.6. Flexible Passing Reward and Test Fee

In this section, I assume that the principal cannot observe the agent's effort, but can keep track of how many tests he has taken. I explore the possibility of setting a flexible passing reward or test fee structure that is contingent on the number of tests that an agent has already taken.

1.6.1. Flexible Passing Reward with Full-Disclosing

With full-disclosing, the maximum number of tests that an agent ever needs to take is two. We construct a test system with passing reward (r_1, r_2) such that an agent gets r_1 if he passes the test at the first attempt and r_2 if he passes the test at the second attempt. Once having passed the test, an agent is not permitted to take the test again and re-claim the passing reward. In case of failure, the agent pays the test fee b.[38]

Theorem 3. *FBME is always feasible with* $((r_1, r_2), b)$ *and full-disclosing.*

Proof. The agent's expected payoff by choosing $t_1 = \varepsilon_i$ is

$$U^{fd}(i;(r_1, r_2, b)) = \sum_{j=1}^{i} p_j r_1 + \sum_{j=i+1}^{N} p_j \Big\{ \mathbb{I}\{r_2 \geq \varepsilon_j - \varepsilon_i\} \cdot [r_2 - (\varepsilon_j - \varepsilon_i)] - b \Big\} - \varepsilon_i.$$

Set $r_2 \geq \varepsilon_N - \varepsilon_1$ so that 100% passing rate is guaranteed. Let $r_1 = \tilde{r}$ and $b = r_2 + \tilde{b} - \tilde{r}$. The right-hand-side of the equation above becomes

$$\sum_{j=1}^{i} p_j \tilde{r} + \sum_{j=i+1}^{N} p_j \Big\{ [b + \tilde{r} - \tilde{b} - (\varepsilon_j - \varepsilon_i)] - b \Big\} - \varepsilon_i = \tilde{r} - \sum_{j=i+1}^{N} p_j \tilde{b} - \sum_{j=1}^{i} p_j \varepsilon_i - \sum_{j=i+1}^{N} p_j \varepsilon_j,$$

[38] A single test fee is sufficient, as no one will pay it twice.

which is the same as if the test fee is $\tilde{b}$ and passing reward is $\tilde{r}$.[39] Therefore, our previous results in Section 1.3 remain valid. In particular, $t_1 = \varepsilon_1$ can certainly be implemented by letting $\tilde{b} = 0$. Full surplus-extraction can thus be achieved by letting $r_1 = \tilde{r} = \sum_{j=1}^{N} p_j \varepsilon_j$. As for r_2 and b, we can simply set $r_2 = \varepsilon_N$, which implies $b = \varepsilon_N - \sum_{j=1}^{N} p_j \varepsilon_j$. To sum up, by setting

$$\begin{cases} r_1 = \sum_{j=1}^{N} p_j \varepsilon_j \\ r_2 = \varepsilon_N \\ b = \varepsilon_N - \sum_{j=1}^{N} p_j \varepsilon_j \end{cases},$$

$t_1 = \varepsilon_1$ can be implemented and FBME can be achieved. □

1.6.2. Flexible Test Fee with Minimal-Disclosing

I construct a test system with passing reward r and test fee sequence $(b_1, ..., b_{N-1})$ such that an agents pays b_i if he fails a test for the i^{th} time.

Theorem 4. *FBME is always feasible with* $(r, (b_1, ..., b_{N-1}))$ *and minimal-disclosing.*

Proof. I prove by construction. Adapting previous method, I first assume 100% passing rate, then make sure it is indeed so with our choice of the passing reward and test fee. Under $\mathcal{T}^{FB}$, an agent's continuation payoff at ε_n is

$$V(\mathcal{T}_n^{FB}; (r, (b_1, ..., b_{N-1}))) = r - \frac{1}{\sum_{j=n+1}^{N} p_j} \left(\sum_{j=n+1}^{N} p_j (\varepsilon_j - \varepsilon_n) + \sum_{j=n+2}^{N} p_j \sum_{k=n+1}^{j-1} b_k \right).$$

[39] Note that we do not need $\tilde{r} = r_1 \geq \varepsilon_N - \epsilon_i$ to maintain our previous conclusions. In Section **??**, such condition is required only because otherwise the expected utility would take another form (since some low ability agents will quit). For this reason, in the current construction, the relevant condition that guarantees the expected utility taking this particular form is $r_2 \geq \varepsilon_N - \epsilon_i$, which is already assumed by setting $r_2 \geq \varepsilon_N - \varepsilon_1$.

Participation constraints require $V(\mathcal{T}_n^{FB};(r,(b_1,...,b_{N-1}))) \geq 0$ for all $n = 0,...,N-1$[40] Parallel to Equation 1.11, define

$$R_n((b_1,...,b_{N-1})) = \frac{1}{\sum_{j=n+1}^{N} p_j}\left(\sum_{j=n+1}^{N} p_j(\varepsilon_j - \varepsilon_n) + \sum_{j=n+2}^{N} p_j \sum_{k=n+1}^{j-1} b_k\right)$$

as the minimal passing reward required to satisfy the participation constraint at ε_n for a given test fee sequence $(b_1,...,b_{N-1})$. Note that only $b_{n+1},...,b_{N-1}$ effectively enter the function $V(\cdot)$ and $R_n(\cdot)$.

(1) <u>*Step* 1</u>

Let[41]

$$n_1 = \text{argmax}_{n\in\{0,1,...,N-1\}} R_n((\underbrace{0,...,0}_{\text{all 0's}})).$$

If $n_1 = 0$, FBME is achievable by setting $b_n = 0$ for all n and $r = R_0((\underbrace{0,...,0}_{\text{all 0's}})) = \sum_{j=1}^{N} p_j\varepsilon_j$. Otherwise, set $b_n = 0$ for all $n = n_1+1,...,N-1$, and set $r_1 = R_{n_1}(0,...,0)$. In this way, we have $V(\mathcal{T}_{n_1}^{FB};(r_1,(0,...,0))) = 0$ and $V(\mathcal{T}_n^{FB};(r_1,(0,...,0))) \geq 0$ for all $n > n_1$. I next show that b_{n_1} can be chosen to make $V(\mathcal{T}_{n_1-1}^{FB};(r_1,(0,...,0,b_{n_1},0,...,0))) = 0$ without causing agents to skip the test at ε_{n_1}.

At ε_{n_1-1}, with test fee b_{n_1}, not skipping the test at ε_{n_1} yields a continuation payoff of

$$\frac{p_{n_1}}{\sum_{j=n_1}^{N} p_j} r_1 + \frac{\sum_{j=n_1+1}^{N} p_j}{\sum_{j=n_1}^{N} p_j} \underbrace{V(\mathcal{T}_{n_1}^{FB};(r_1,(0,...,0,b_{n_1},0,...,0)))}_{=V(\mathcal{T}_{n_1}^{FB};(r,(0,...,0)))=0} - \frac{\sum_{j=n_1+1}^{N} p_j}{\sum_{j=n_1}^{N} p_j} b_{n_1} - (\varepsilon_{n_1} - \varepsilon_{n_1-1}),$$

[40] Let $\varepsilon_0 = 0$.

[41] In case of tie, choose $N(b_1,...,b_{N-1})$ to be the smallest.

while skipping the test at ε_{n_1} yields a continuation payoff of

$$\frac{p_{n_1}+p_{n_1+1}}{\sum_{j=n_1}^{N} p_j} r_1 + \frac{\sum_{j=n_1+2}^{N} p_j}{\sum_{j=n_1}^{N} p_j} V(\mathcal{T}_{n_1+1}^{FB}; (r_1, (0, ..., 0, b_{n_1}, 0, ..., 0))) - \frac{\sum_{j=n_1+2}^{N} p_j}{\sum_{j=n_1}^{N} p_j} b_{n_1} - (\varepsilon_{n_1+1} - \varepsilon_{n_1-1}).$$

Hence, not skipping the test at ε_{n_1} requires

$$\frac{p_{n_1+1}}{\sum_{j=n_1}^{N} p_j} b_{n_1} \leq (\varepsilon_{n_1+1} - \varepsilon_{n_1}) - \frac{p_{n_1+1}}{\sum_{j=n_1}^{N} p_j} r_1 - \frac{\sum_{j=n_1+2}^{N} p_j}{\sum_{j=n_1}^{N} p_j} V(\mathcal{T}_{n_1+1}^{FB}; (r_1, (0, ..., 0, b_{n_1}, 0, ..., 0))). \tag{1.24}$$

Moreover, note that $V(\mathcal{T}_{n_1}^{FB}; (r_1, (0, ..., 0, b_{n_1}, 0, ..., 0))) = 0$ implies

$$\frac{p_{n_1+1}}{\sum_{j=n_1+1}^{N} p_j} r_1 + \frac{\sum_{j=n_1+2}^{N} p_j}{\sum_{j=n_1+1}^{N} p_j} V(\mathcal{T}_{n_1+1}^{FB}; (r_1, (0, ..., 0, b_{n_1}, 0, ..., 0))) - (\varepsilon_{n_1+1} - \varepsilon_{n_1}) = 0.$$

Using

$$V(\mathcal{T}_{n_1+1}^{FB}; (r_1, (0, ..., 0, b_{n_1}, 0, ..., 0))) = \frac{\sum_{j=n_1+1}^{N} p_j}{\sum_{j=n_1+2}^{N} p_j} \left[(\varepsilon_{n_1+1} - \varepsilon_{n_1}) - \frac{p_{n_1+1}}{\sum_{j=n_1+1}^{N} p_j} r_1 \right],$$

Equation 1.24 simplifies into

$$b_{n_1} \leq \frac{p_{n_1}}{p_{n_1+1}} (\varepsilon_{n_1+1} - \varepsilon_{n_1}).$$

The condition above sets the upper bound on the choice of b_{n_1}. I now demonstrate that we can always make $V(\mathcal{T}_{n_1-1}^{FB}; (r_1, (0, ..., 0, b_{n_1}, 0, ..., 0))) = 0$ without violating this upper bound. First, note that $R_{n_1}((0,...,0)) \geq R_{n_1+1}((0,...,0))$ implies

$$\frac{\sum_{j=n_1+1}^{N} p_j(\varepsilon_{n_1+1}-\varepsilon_{n_1})}{p_{n_1+1}} \geq \underbrace{\frac{\sum_{j=n_1+1}^{N} p_j(\varepsilon_j-\varepsilon_{n_1})}{\sum_{j=n_1+1}^{N} p_j}}_{=R_{n_1}((0,...,0))} \geq \underbrace{\frac{\sum_{j=n_1+2}^{N} p_j(\varepsilon_j-\varepsilon_{n_1+1})}{\sum_{j=n_1+2}^{N} p_j}}_{=R_{n_1+1}((0,...,0))},$$

as

$$\sum_{j=n_1+1}^{N} p_j(\varepsilon_j-\varepsilon_{n_1}) = \sum_{j=n_1+1}^{N} p_j(\varepsilon_{n_1+1}-\varepsilon_{n_1}) + \sum_{j=n_1+1}^{N} p_j(\varepsilon_j-\varepsilon_{n_1+1})$$

and

$$\sum_{j=n_1+1}^{N} p_j = p_{n_1+1} + \sum_{j=n_1+2}^{N} p_j.$$

Therefore, if we set $b_{n_1} = \dfrac{p_{n_1}}{p_{n_1+1}}(\varepsilon_{n_1+1}-\varepsilon_{n_1})$, then

$$\begin{aligned}
&V(\mathcal{T}_{n_1-1}^{FB};(r_1,(0,...,0,b_{n_1},0,...,0))) \\
=&\frac{p_{n_1}}{\sum_{j=n_1}^{N} p_j} \cdot R_{n_1}((0,...,0)) - \frac{\sum_{j=n_1+1}^{N} p_j}{\sum_{j=n_1}^{N} p_j} b_{n_1} - (\varepsilon_{n_1}-\varepsilon_{n_1-1}) \\
=&\frac{p_{n_1}}{\sum_{j=n_1}^{N} p_j} \cdot \frac{\sum_{j=n_1+1}^{N} p_j(\varepsilon_j-\varepsilon_{n_1})}{\sum_{j=n_1+1}^{N} p_j} - \frac{\sum_{j=n_1+1}^{N} p_j}{\sum_{j=n_1}^{N} p_j} \cdot \frac{p_{n_1}}{p_{n_1+1}}(\varepsilon_{n_1+1}-\varepsilon_{n_1}) - (\varepsilon_{n_1}-\varepsilon_{n_1-1}) \\
\leq&\frac{p_{n_1}}{\sum_{j=n_1}^{N} p_j} \cdot \frac{\sum_{j=n_1+1}^{N} p_j(\varepsilon_{n_1+1}-\varepsilon_{n_1})}{p_{n_1+1}} - \frac{\sum_{j=n_1+1}^{N} p_j}{\sum_{j=n_1}^{N} p_j} \cdot \frac{p_{n_1}}{p_{n_1+1}}(\varepsilon_{n_1+1}-\varepsilon_{n_1}) - (\varepsilon_{n_1}-\varepsilon_{n_1-1}) \\
=&-(\varepsilon_{n_1}-\varepsilon_{n_1-1}) < 0.
\end{aligned}$$

Therefore, the test fee b that is needed to make $V(\mathcal{T}_{n_1-1}^{FB};(r_1,(0,...,0,b,0,...,0))) = 0$ is less than $\dfrac{p_{n_1}}{p_{n_1+1}}(\varepsilon_{n_1+1}-\varepsilon_{n_1})$. In other words, we can make the participation constraint at ε_{n_1-1} binding without setting b_{n_1} too high so that agents would rather skip the test at ε_{n_1}. Now, let $b^*_{n_1}$ be implicitly defined by $V(\mathcal{T}^{FB}_{n_1-1}; (r_1, (0, ..., 0, b^*_{n_1}, 0, ..., 0))) = 0$.

(2) $\underline{Step}$ 2

For $n \in \{0, 1, ..., n_1 - 1\}$, let

$$n_2 = \text{argmax}_{n \in \{0,1,...,n_1-1\}} R_n((0, ..., 0, \underbrace{b^*_{n_1}}_{\text{the } n_1^{th}}, 0, ..., 0)).$$

If $n_2 = 0$, FBME is achievable by setting $b_{n_1} = b^*_{n_1}$, $b_n = 0$ for all $n \neq n_1$ and $r = R_0((0, ..., 0, b^*_{n_1}, 0, ..., 0)) = \sum_{j=1}^{N} p_j \epsilon_j + \sum_{j=n_1+1}^{N} p_j b^*_{n_1}$. Otherwise, set $b_n = 0$ for all $n = n_2 + 1, ..., n_1 - 1$, and set $r_2 = R_{n_2}((0, ..., 0, b^*_{n_1}, 0, ..., 0))$.[42] In this way, we have $V(\mathcal{T}^{FB}_{n_2}; (r_2, (0, ..., 0, b^*_{n_1}, 0, ..., 0))) = 0$ and $V(\mathcal{T}^{FB}_{n}; (r_2, (0, ..., 0, b^*_{n_1}, 0, ..., 0))) \geq 0$ for all $n > n_2$. Following the same procedure as in Step 1, it can be shown that we can choose $b_{n_2} = b^*_{n_2}$ to make $V(\mathcal{T}^{FB}_{n_2-1}; (r_2, (0, ..., 0, b^*_{n_2}, 0, b^*_{n_1}, 0, ..., 0))) = 0$ without worrying about b_{n_2} being so high that agents want to skip the test at ε_{n_2}.

(3) $\underline{Step}$ 3 Repeat the process above and move backward, until

$$0 = \text{argmax}_{n \in \{0,...,n^*_i-1\}} R_n((0, ..., 0, b_{n^*_i}, \underbrace{..., b_{n^*_{i-1}}, ..., b_{n^*_{i-2}}, ...}_{\text{optimal test fees constructed before}}))$$

for some i. Then FBME can be achieved by choosing the sequence of test fees $(0, ..., 0, b_{n^*_i}, ..., b_{n^*_{i-1}}, ..., b_{n^*_{i-2}}, ...)$ and $\tilde{r} = R_0((0, ..., 0, b_{n^*_i}, ..., b_{n^*_{i-1}}, ..., b_{n^*_{i-2}}, ...))$.

Note that whenever the sequence of test fees we have constructed does not make the *ex-ante* participation constraint the binding constraint, we add another non-zero test

[42] To remove any confusion, our design only involves one passing reward. r_1 and r_2 are the adjustments we make along the way to our choice of the passing reward. They should be viewed as temporary. Once we choose r_2, it replaces r_1, and the latter becomes obsolete.

fee to the sequence and at the same time shrink the set of participation constraints we need to consider. The process ends if and only if the *ex-ante* participation constraint becomes the binding constraint at some point. Therefore, FBME is always feasible.

□

1.7. Conclusion

In this paper I study the design of a feedback system that encourages a group of agents with different ability levels to willingly complete a task on the premise that they are sufficiently compensated for their effort. Although the principal and the agents both process private information – the former the progress each agent has made and the latter the effort they have spent – the characteristics of these two types of private information is inherently different. From the principal's perspective, her private information can be traded for a price, which serves to decrease her net expenditure. The quality of the information, i.e. whether it is precise or opaque, determines how much the agents are willing to pay for it. In contrast, an agent's information on the effort he has spent always remains private. Therefore, although both parties' private information are necessary to get a complete picture of the situation, the flow of information is one-way. The dilemma that the principal faces is that if her private information fails to be delivered to the agents on time, the resulting wasted effort does not only hurt the agents but also the principal herself. Therefore, the design of an optimal feedback system lies in controlling the information flow by properly balancing its disclosing power and price. This paper provides a systematic way to find the optimal feedback system that minimizes the principal's net expenditure with either full-disclosing or minimal-disclosing. I also demonstrate how partial-disclosing can be more optimal under certain circumstances. More importantly, the first-best outcome can always be achieve – either with full-disclosing or minimal-discoing – if the price for acquiring information from the principal can be made to be contingent on how many times an agent has already done so. One thing unique about this setting is the principal and the agents have aligned interest on one hand – no one benefit from an agent's over-spending effort – but contradicting interests on the other: an agent always try to gain more surplus whereas the principal always try not to leave surplus to the agent. The contradicting side of the incentives causes a coordination failure. In fact, as long as the principal is not permitted to "discriminate" the agents by offering different passing rewards to different agents depending on their ability, the principal's problem remains the same even if he knows all agents' ability outright. In a sense, the root of the inefficiency is not the principal's lack of knowledge, but the perceived fairness. This is very similar to a situation in which even if a monopoly knows each consumer's willingness to pay yet only a common price is allowed, the best the monopoly can do is to find the single price that maximized its profit in the meanwhile let go of some consumer and let go of some profit from the rest.

CHAPTER 2

Auction with Budget-Constrained Business Insiders and Deep-Pocketed Investor

2.1. Introduction

People have long realized that in many auctions, bidders' willingness to pay may not coincide with their ability to pay. Budget constrains arise under various circumstances due to credit limits and imperfect capital markets. Research on standard auctions with budget constrained bidders concentrates on broader symmetric settings in which all bidders face budget constraints and have similar type of valuations. This type of models, though appealing, fails to describe another important situation, in which the asset for sale in an auction attracts deep-pocket investors (and/or speculators) with an intention to take advantage of business insiders' potentially temporary budget constraints by winning the auction and reselling the asset later at a higher price once the business insiders' budget constraints are relaxed. For example, it has long been noted that in the cases of forced bankruptcy liquidations, credit constraints of industry buyers may result in the liquidated assets being sold at deep discount to industry outsiders. The following example provides a vivid picture of this situation. In the mid-1980s, cash flows from the oil shipping business temporarily plummeted and many tankers were forced to be sold for scrap value. Astute investors outside the oil shipping industry bought some tankers and mothballed them. These investors made a seven-fold return on their tanker investments over a five-year period.

To model this type of situations, it is necessary to make a clear distinction between the two types of potential bidders, namely, the business insiders and the outside investors. As an illustrative example, consider a heavily indebted farmer who is forced to liquidate his farm. A business insider could be a neighborhood farmer who would farm the land himself, or a big agricultural company who would like to expand its operation. One advantage that the business insiders have is that their expertise in farming allows them to put the farmland into its best use. This advantage becomes especially relevant when the liquidated asset is highly specialized. (The previous oil-tanker example falls into this category.) It also has significant implications in terms of allocation efficiency of the auction outcome. In addition, the business insiders are able to estimate the value that the farmland can generate for them fairly accurately. It is worth pointing out that these values do not necessarily coincide with each other – in fact, they usually do not: different business insiders may have different strengths in farming a specific type of farmland, face different costs, or even possess different market power, all of which contribute to their valuation for the farmland being private. Hence, the standard *independent private value model* (IPV) provides a reasonable characterization of the business insiders' valuations. However, in many circumstances, a business insider's valuation for the liquidated asset may surpass his or her ability to pay for the asset. In fact, if the factors that drove the farmer into bankruptcy are industry or economy wide, every business insider will be financially constrained with positive probability. In the farming industry example, such factors could be persistent bad weather, widespread plant diseases, or accelerating import competition – situations in which even big agricultural companies can be temporarily financially constrained. Outside investors, on the contrary, are firms or individuals who are not constrained by their budgets at the time of the auction. In the farmland example, these investors, upon winning the auction, could hire the current owner or some other farmer to farm the land, at least until they could resell it at a more attractive price in the future. These investors' lack of expertise in the farming industry makes the *common value model* more suitable for characterizing their valuations: on one hand, they might not be able to estimate the value of the farmland very accurately, and their own valuation could well be influenced by that of others (including the business insiders); on the other hand, at what price the deep-pocket investors could resell the farmland in the future might not depend on their identities, rather, it relies more heavily on the prospect of the macro economy or the farming industry. In general, these two factors also make a deep-pocket investor's valuation for the asset be less than that of a business insider, due to the embedded uncertainty of this investment, as well as the associated storage, management, and transaction costs.

Section 2.2 presents the model in its most general form. Section 2.3 looks into a special case of the model in which there is one business insider and one deep-pocket outsider participating in the auction. I first solve for the equilibrium bidding strategy for both parties and then discuss about the allocation efficiency and expected revenue of the auction. I show that when the business insider is less likely to be budget-constrained, the outsider bids less aggressively holding all other factors constant. This is the winner's curse associated with this type of auction, which is rooted in the fact that the less likely the insider is constrained by her budget, the less likely the outsider can successfully catch a true bargain. It supports the idea that the sole reason that the outsider participates in the auction is to take advantage of the insider's financial difficulty. As can be expected, the allocation efficiency is higher when the insider is less likely to be budget-constrained. Interestingly, when the probability that the insider is constrained by her budget is high, the seller's expected revenue increases if this probability gets lower. However, when this probability is low, the expected revenue decreases if the probability that the insider faces budget-constraint decreases even further. In Section 2.4, I extend the model to include multiple business insiders. Again, I first solve for the equilibrium bidding strategy and then look at the behavior of the allocation efficiency and expected revenue. Depending on the institutional details at the post-auction stage, I look at three types of formulation of the outsider's valuation. In one case, the outsider's valuation is determined by the highest private valuation among the insiders; In another case, the outsider's valuation is determined by the second highest private valuation among the insiders; In the final case, the outsider's valuation is determined by the average valuation of all the insiders. I show that in all three cases, the winner's curse (from the outsider's perspective) discussed in the previous section persists. However, it is alleviated to some extant by the presence of multiple insiders. As the number of insiders increases, the outsider does not shade his bid as much as he otherwise would. In terms of allocation efficiency, the probability of the insider with the highest valuation winning the auction decreases as the number of insider increases. However, when one views all the insiders as a single group, the probability that one of them wins the auction increases with the number of insiders. As for expected revenue, consistent with the previous conclusion, the outsider's contribution to the expected revenue decreases if the insiders become less likely to be budget-constrained. An increase in the number of insiders also makes the expected payment from the outsider become less important. However, the insiders' expected payments increase significantly as the number of insiders increases, with the overall effect being that the the seller benefits from a wider participation of insiders.

2.1.1. Related Work

The seminal paper in auction with financially constrained bidders traces back to Che and Gale (1998) [10], in which they analyze the revenue and efficiency performance of standard auctions when buyers have private information about both their willingness to pay and their ability to pay. They find that first-price auctions yield higher expected revenue and social surplus than second-price auctions. Another paper by Che and Gale (1996) [9] show that all-pay auctions dominate first-price sealed-bid auctions when bidders face budget constraints. Pai and Vohra (2014) [**?**] demonstrate that the symmetric revenue maximizing and constrained efficient auctions in this setting can be implemented via a modified all-pay auction in which the highest bidder need not win the good outright, or, stated differently, the auction has “pooling”. Under the same assumption that buyers with private value may be budget- or liquidity-constrained, Maskin (2000) [27] focuses on allocation efficiency and gives the characteristics of an efficient non-standard auction.

In Laffont (1996) [23] and Boulatov and Severinov (2021) [5], the buys’ budgets are commonly known before the auction whereas their valuations remain private. They then characterize the revenue-maximizing mechanism for allocating the good. Under the same assumptions, Gavious, Moldovanu and Sela (2002) [1] compare the revenues to be realized from the standard auction forms. In contrast, Bobkova (2020) [4] solves the first-price auction for two bidders with asymmetric budget distributions and known valuations for one object.

Ghosh (2021) [15] studies the sale of two units of a good through simultaneous sealed bid first-price auctions to bidders who are differentiated based on their valuations and budgets and have multi-unit demand. He finds that bidders with higher valuations (lower budgets) prefer more unequal splits of their budgets than bidders with lower valuations (higher budgets) and the same budget (valuation). Brusco and Lopomo (2009) [7] studies a similar setting, with the modification that the two objects are identical but bidders valuations exhibit complementarities. They find that bidders with higher budgets are more reluctant to bid, because opponents with lower budgets may end up pursuing a single object, thus preventing the realization of complementarities. In another work, Brusco and Lopomo (2008) [6] show that the possibility of binding budget constraints in simultaneous ascending bid auctions induces strategic demand reduction and generates significant inefficiencies. In Malakhov and Vohra (2008) [25]

Rhodes-Kropf and Viswanathan (2005) [31] takes a different approach by explicitly modeling multiple forms of financing and different levels of financial market competition and shows that most often, competitive financing is not efficient when bidders have different cash positions. In Zheng (2001) [32], budget-constrained bidders have access to financing as well as the option to declare bankruptcy. When the borrowing rate is above a threshold, high-budget bidders win, and the likelihood of bankruptcy is low. In contrast, when the borrowing rate is below the threshold, the winner is the most budget-constrained bidder and is most likely to declare bankruptcy.

The above literature assumes that budgets are exogenously determined. Others look into auctions with endogenous budget constraints. In Baisa and Rabinovich (2016) [3], budgets are determined endogenously to reflect either the financing cost of obtaining funding from a bank or opportunity cost of funds from diverting resources away from alternative profitable investments. In their paper, a bidder incurs borrowing costs regardless of whether she wins the auction or not. In contrast, Burkett (2015) [8] develops a model where the bidder's budget constraint is the endogenous result of an agency problem between the bidder and a principal responsible for funding the bidder's bid. Ausubel, Burkett and Filiz-Ozbay (2017) [24] performs laboratory experiments comparing auctions with endogenous budget constraints arising from a principal-agent problem and confirms the prediction that tighter constraints will be imposed in first-price auctions than in second-price auctions. Consequently, the second-price auction with an endogenous budget constraint generates exactly the same theoretical allocation as the first-price auction with an endogenous budget constraint—a restoration of the revenue equivalence theorem.

Another strand of literature that this paper is closely related to is auctions with resale. Some work on standard auctions with resale considers environments where resale is required to achieve an efficient allocation. For example, Haile (2000, 2003) [17] [18] evaluates standard auctions in symmetric environments where precise information about use values becomes available only after the initial auction. Gupta and Lebrun (1999) [16], in contrast, examine an asymmetric setting in which two private-value bidders observe each other's use value after the auction. Zheng (2002) [33] investigates the design of seller–optimal auctions when winning bidders can attempt to resell the good. Garratt, Tröger and Zheng (2009) [30] shows that when post-auction interbidder resale is allowed, the English auction is susceptible to tacit collusion where one bidder wins the auction without any competition and divides the spoils by optimally reselling the good to the other bidders. Similarly, Pagnozzi (2007) [29] shows that with post-auction resale, a strong bidder may prefer to lose the auction on purpose and acquire the object in the aftermarket, as allowing her rival to win at a relatively low price puts the strong bidder in a better bargaining position in the aftermarket. Finally, the closest work to this paper is done by Garratt and Tröger (2006) [14] where they consider an environment with symmetric independent private-value bidders and a speculator, who is commonly known to have no use value for the good on sale. They show that if an inter-bidder resale opportunity exists, the speculator can play an active role in any standard auction format, and, as a result, the final allocation is not efficient and revenue equivalence across standard auctions is no longer guaranteed.

To the best of my knowledge, there is no previous research on the auction setting presented in this paper, in which the prevalence of budget-constraints among business insiders may attract deep-pocketed outside investors to participate in the auction whose aim is to take advantage of the insiders financial difficulty and win the auction at a bargain price.

2.2. Model

A general description of the model is as follows. There is one single object for sale under the *second-price sealed bid auction* (SPSB), and ties are broken randomly. There are two groups of bidders – business insiders (group $\mathcal{I}$) and outside investors (group $\mathcal{O}$). Group $\mathcal{I}$ bidders are likely to be financially constrained. A group $\mathcal{I}$ bidder's 'type' is thus two-dimensional, with one dimension on her valuation for the object and the other on her budget. Formally, a group $\mathcal{I}$ bidder, say, bidder i, can be characterized by a 2-tuple $(X_i, W_i) = (x_i, w_i)$, with x_i being bidder i's private valuation for the object, and w_i her bidding budget. I carry through my analysis the assumption that all bidders' budget constraints are "hard", in the sense that a bidder would never be able to pay above her budget.[1] It is easy to see that with a hard budget constraint, a bidder will not bid above her budget. One can think of this situation as in the case that when a bidder bids above her budget and happens to win, she is forced to forfeit the object due to her inability to pay, and in addition pay a fine to compensate the auctioneer for the trouble she has caused. I also assume that although the particular realization of (X_i, W_i) is bidder i's private information, the joint distribution of (X_i, W_i), $F(\cdot, \cdot)$, is common knowledge to both groups of bidders. To focus on the difference between the two groups of bidders, I abstract away from the asymmetry among bidders of the same group. Therefore, there is no subscript i in the joint distribution function $F(\cdot, \cdot)$ of the valuation-budget pair. Furthermore, as in most auction literature, I assume that the corresponding joint density function $f(\cdot, \cdot)$ exists, and is greater than zero almost everywhere on the domain of (X_i, W_i), which is assumed to be

[1]When a bidder can use outside financing to relax her bidding budget, the budget constraint is said to be "soft".

$[0, \bar{x}] \times [0, \bar{w}]$ with $\bar{x}, \bar{w} > 0$. Whether $\bar{x} \geq \bar{w}$ or $\bar{x} < \bar{w}$ is not crucial for the general analysis, but it is important for comparative statics analysis in what follows. Under this distributional assumption, bidders in group $\mathcal{I}$ are likely, but not necessarily, constrained by their budgets. It is worth pointing out that although this specification of the joint distribution allows for correlation between X_i and W_i, for the purpose of this paper, allowing X_i and W_i to be correlated brings much complication in the analysis without a comparable benefit of additional insight. Therefore, I further assume that X_i and W_i are independently distributed. Let $F_X : [0, \bar{x}] \to [0, 1]$ and $F_W : [0, \bar{w}] \to [0, 1]$ be the cumulative distribution functions of X_i and W_i, respectively, with corresponding density functions denoted by $f_X(\cdot)$ and $f_W(\cdot)$. Unlike group $\mathcal{I}$ bidders, group $\mathcal{O}$ bidders are assumed to have pure common valuation for the object. This assumption serves to capture the investing/speculating nature of this group of bidders. In other words, as business outsiders, the group $\mathcal{O}$ bidders are less likely to have particular use of the object per se – their valuation for it purely comes from the expected future return from reselling. Moreover, I assume that the business outsiders are not constrained by any bidding budget. Formally, a group $\mathcal{O}$ bidder, say, bidder j, can be characterized by a single random variable S_j, which can be viewed as a signal bidder j receives that is informative of the profitability of this speculation opportunity. Denote the distribution function of the signal S_j's as $K(\cdot)$, with $k(\cdot)$ being the corresponding density function that exists and is greater than zero almost everywhere on the domain of S_j. Again, I assume away from any asymmetry among the group $\mathcal{O}$ bidders. The object's common value for all group $\mathcal{O}$ bidders is given by $v(X_1, ..., X_N, S_1, ..., S_M)$, where N is the number of group $\mathcal{I}$ bidders, and M is the number of group $\mathcal{O}$ bidders. Thus, the common value to group $\mathcal{O}$ bidders depends on each business insider's own valuation, as well as each outside investor's signal. I assume that the function v is non-decreasing in each argument.

2.3. Two Bidder Case ($N = 1$, $M = 1$)

To get a crude idea of certain aspects of the auction outcome, it is useful to first study a highly simplified version of the model. Assume that there are only two bidders – one from each group. Call the bidder of group $\mathcal{I}$ bidder 1, and the bidder of group $\mathcal{O}$ bidder 2. Both bidders are risk-neutral. Also, assume that bidder 1's valuation-budget pair, (X, W), has uniform density on $[0, \bar{x}]\times[0, \bar{w}]$ with $\bar{x} > 0$ and $\bar{w} > 0$. Assume the signal that bidder 2 receives, S, is independent of (X, W), and has uniform distribution on $[0, 1]$. The value of the object to bidder 2 is given by $V = v(X, S) = X \cdot S$. The signal S in this value function intends to fully summarize the investment risk (such as the possibility that bidder 1 never recovers from her financial difficulty), together with the storage, management, and transaction costs associated with this investment. In addition, this value function illustrates the potential efficiency loss resulting from the business insider's budget constraint – those who could make best use of the object might fail to win the auction due to liquidity issues.

2.3.1. Equilibrium

In the case with only one bidder in each group, there exists a trivial equilibrium in which bidder 2 always bids above $\bar{w}$ and bidder 1 always bids zero. This equilibrium does not emerge in models with more bidders, and can be eliminated in the two bidders case by simply imposing an entry fee. Hence, I will focus my analysis on other more interesting bidding equilibria.

Theorem 5. *It is a (weakly) dominant strategy for bidder 1 to bid* $Z \equiv \min\{X, W\}$.

Proof. Let b_1 and b_2 be bidder 1 and bidder 2's bids, respectively.

(1) If $X \leq W$, then the budget constraint is not binding, and the argument is similar to that in the standard SPSB auctions without budget constraints.

(a) Any bid $b_1 < X = \min\{X, W\} = Z$ is not optimal: if $b_2 < b_1$, bidder 1 gets the same payoff either by bidding b_1 or by bidding X; if $b_2 = b_1$, given that ties are broken randomly, bidder 1 makes a sure profit of $X - b_2$ by bidding X, in contrast to make the same profit with only $\frac{1}{2}$ chance by bidding b_1; if $b_1 < b_2 < X$, bidder 1 makes profit of $X - b_2$ by bidding X, whereas he loses the auction by bidding b_1; if $b_2 \geq X$, bidder 1 makes zero profit either by bidding b_1 or by bidding X. To summarize, bidding X instead of $b_1 < X$ makes bidder 1 weakly better-off.

(b) Any bid $b_1 > X = \min\{X, W\} = Z$ is not optimal: if $b_2 < X$, bidder 1 makes a profit of $X - b_2$ either by bidding X or b_1; if $b_2 = X$, bidder 1 makes zero profit either by bidding X or b_1; if $X < b_2 < b_1$, bidder 1 makes zero profit by bidding X but makes a loss of $b_2 - X$ by bidding b_1; if $b_2 = b_1$, bidder 1 makes zero profit by bidding X but makes a loss of $b_2 - X$ with a probability of $\frac{1}{2}$ by bidding b_1; if $b_2 > b_1$, bidder 1 loses the auction by either bidding b_1 or X. To summarize, bidding X instead of $b_1 < X$ makes bidder 1 weakly better-off.

(2) If $X > W$, the budget constraint is binding, which requires $b_1 \leq W$. However, $b_1 < W$ cannot be optimal: if $b_2 < b_1$, bidder 1 makes a profit of $X - b_1$ either by bidding b_1 or by bidding W; if $b_2 = b_1$, bidder 1 makes a sure profit of $X - b_2$ in case she bids W, in contrast to make the same profit with only $\frac{1}{2}$ probability when she bids b_1; if $b_1 < b_2 < W$, bidder 1 makes profit of $X - b_2$ in case she bids W but zero profit when she bids b_1; if $b_2 = W$, bidder 1 makes profit of $X - b_2$ with $\frac{1}{2}$ probability in case she

bids W but zero profit when she bids b_1. To summarize, bidding W instead of $b_1 < W$ makes bidder 1 weakly better-off.

□

Denote the cumulative distribution function of Z by $F_Z(\cdot)$, with the corresponding density function $f_Z(\cdot)$. Then

$$F_Z(z) = \begin{cases} 0 & z < 0 \\ F_X(z) + F_W(z) - F_X(z) \cdot F_W(z) & 0 \leq z < \min\{\bar{x}, \bar{w}\} \\ 1 & z \geq \min\{\bar{x}, \bar{w}\} \end{cases}, \tag{2.1}$$

and

$$f_Z(z) = \begin{cases} \left(1 - F_W(z)\right) \cdot f_X(z) + \left(1 - F_X(z)\right) \cdot f_W(z) & 0 \leq z < \min\{\bar{x}, \bar{w}\} \\ 0 & \text{otherwise} \end{cases}. \tag{2.2}$$

Given bidder 1's strategy, and the assumption that X and W are independently distributed, the expected net profit for bidder 2 with signal $S = s$ and bid b is thus[2][3]

$$\begin{aligned} \Pi(b, s) =& F_Z(b) \cdot (\mathbb{E}[X|Z < b] \cdot s - \mathbb{E}[Z|Z < b]) \\ =& \left(\int_0^b x f_X(x) dx + F_W(b) \int_b^{\bar{x}} x f_X(x) dx \right) \cdot s \\ &- \int_0^b z \cdot \left[\left(1 - F_W(z)\right) \cdot f_X(z) + \left(1 - F_X(z)\right) \cdot f_W(z) \right] dz. \end{aligned} \tag{2.3}$$

[2]Since in the SPSB environment, bidding $b > \min\{\bar{x}, \bar{w}\}$ always delivers the same result as bidding $b = \min\{\bar{x}, \bar{w}\}$ to bidder 2, I assume that bidder 2 will never bid strictly above $\min\{\bar{x}, \bar{w}\}$.
[3]Details of the derivation can be found in Appendix 4.1

For any given s, the first order necessary condition (FONC) requires that the optimal bid b^* satisfies $\frac{\partial \Pi}{\partial b}(b^*, s) = 0$. Using Equation 2.3 and taking derivative with respect to b yields[4]

$$\begin{aligned}\frac{\partial \Pi}{\partial b}(b, s) = &\left(b f_X(b) \left(1 - F_W(b)\right) + f_W(b) \int_b^{\bar{x}} x f_X(x) dx \right) s \\ &- b\left[\left(1 - F_W(b)\right) f_X(b) + \left(1 - F_X(b)\right) f_W(b)\right],\end{aligned} \tag{2.4}$$

which has the standard interpretation of the first term being the marginal gain and the second term being the marginal cost.

With the assumption that (X, W) has uniform density on $(0, \bar{x}) \times (0, \bar{w})$, the FONC then becomes:

$$\frac{\partial \Pi}{\partial b}(b^*, s) = \frac{s\bar{x}^2 - 2[(1-s)\bar{w} + \bar{x}]b^* + (4-3s)(b^*)^2}{2\bar{x}\bar{w}} = 0. \tag{2.5}$$

For Equation 2.5 to have solution, it is necessary that

$$\Big(- 2[(1-s)\bar{w} + \bar{x}]\Big)^2 - 4(4-3s)s\bar{x}^2 = 4(1-s)\left[(\bar{w}+\bar{x})^2 - (\bar{w}^2 + 3\bar{x}^2)\cdot s\right] \geq 0,$$

i.e.

$$s \leq \frac{(\bar{w}+\bar{x})^2}{\bar{w}^2 + 3\bar{x}^2}. \tag{2.6}$$

For now, I simply assume the expression above is true, and solve for b^*. I will then show that the expression above is indeed satisfied, given the optimal bidding strategy that is to be derived later. Assuming that the condition in Equation 2.6 holds, there are two solutions to Equation

[4]With the assumption that (X, W) has uniform density on $[0, \bar{x}] \times [0, \bar{w}]$, one has

$$\frac{\partial \Pi}{\partial b}(b, s) = \frac{4-3s}{2\bar{w}\bar{x}} \cdot b^2 - \frac{(1-s)\bar{w} + \bar{x}}{\bar{w}\bar{x}} \cdot b + \frac{s\bar{x}}{2\bar{w}}.$$

2.5:

$$b_1 = \frac{(1-s)\bar{w} + \bar{x} + \sqrt{[(1-s)\bar{w} + \bar{x}]^2 - (4-3s)s\bar{x}^2}}{4-3s}$$

and

$$b_2 = \frac{(1-s)\bar{w} + \bar{x} - \sqrt{[(1-s)\bar{w} + \bar{x}]^2 - (4-3s)s\bar{x}^2}}{4-3s}.$$

The second order condition (SOC) requires that

$$\frac{\partial^2}{\partial b^2}\Pi(b^*, s) = \frac{-[(1-s)\bar{w} + \bar{x}] + (4-3s)b^*}{\bar{x}\bar{w}} < 0 \quad \Rightarrow \quad b^* < \frac{(1-s)\bar{w} + \bar{x}}{4-3s},$$

which helps pin down the unique optimal bid as

$$b^* = \frac{(1-s)\bar{w} + \bar{x} - \sqrt{[(1-s)\bar{w} + \bar{x}]^2 - (4-3s)s\bar{x}^2}}{4-3s}. \tag{2.7}$$

Another caveat at this point is that b^* in Equation 2.7 may not satisfy $0 \leq b^* \leq \min\{\bar{x}, \bar{w}\}$ for all $s \in [0, 1]$. Theoretically, nothing prevents bidder 2 to submit a bid that is greater than $\min\{\bar{x}, \bar{w}\}$. Nonetheless, I assume that bidder 2 would never bid more than $\min\{\bar{x}, \bar{w}\}$, as anything above $\min\{\bar{x}, \bar{w}\}$ is a sure-to-win bid and thus delivers the same outcome as bidding $\min\{\bar{x}, \bar{w}\}$. Before formally stating bidder 2's optimal bidding strategy $\beta(s, \bar{x}, \bar{w})$, it is useful to get a brief summary of what properties $\beta(s, \bar{x}, \bar{w})$ should have on an intuitive level.

Property 1. *$\beta(s, \bar{x}, \bar{w})$ increases with s.*

This is very intuitive, since for any $s \in [0, 1]$, $\beta(s)$ is the bid that makes the marginal gain equal to the marginal cost. Given that S and (X, W) are independent, an increment in S has no effect on the marginal cost (holding fixed the bid), but increases the marginal gain .[5] Therefore, it

[5]See Equation 2.4.

is optimal for bidder 2 to increase his bid correspondingly. $\square$

Property 2. $\lim_{s\to 1} \beta(s, \bar{x}, \bar{w}) = \min\{\bar{x}, \bar{w}\}$. *That is, as* $s \to 1$*, bidder 2 should submit sure-to-win bid.*

As $s \to 1$, $V(x, s) \to x$ for all x. That is, bidder 2's valuation gets arbitrarily close to bidder 1's valuation. To see that it is always optimal for bidder 2 to lock up his profit by submitting a guaranteed winning bid, suppose that bidder 2 bids $b < \min\{\bar{x}, \bar{w}\}$. If $z \leq b$, then bidder 2 wins the auction with the same payment as if he bids $\min\{\bar{x}, \bar{w}\}$. Therefore, bidding $b < \min\{\bar{x}, \bar{w}\}$ is not a profitable deviation in this case. If $z > b$, then bidder 2 loses the auction with zero profit. However, note that $z = \min\{x, w\} \leq x = V(x, s)$, with strict inequality when $w < x$. This implies that bidder 2 could have won the auction with a possibly positive profit if he bids $\min\{\bar{x}, \bar{w}\}$ instead. Therefore, bidding $b < \min\{\bar{x}, \bar{w}\}$ is not a profitable deviation in this case, either. By the previous assumption that bidder 2 never submits any bid strictly greater than $\min\{\bar{x}, \bar{w}\}$, one can conclude that bidding $\min\{\bar{x}, \bar{w}\}$ is optimal in the limit situation of $s = 1$. $\square$

Property 3. $\lim_{s\to 0} \beta(s, \bar{x}, \bar{w}) = 0$. *That is, as* $s \to 0$*, bidder 2 should submit sure-to-lose bid.*

As $s \to 0$, $V(x, s) \to 0$ for all x. Namely, the object becomes (almost) worthless to bidder 2. In the limit of $s = 0$, any positive bid will result in a negative profit for bidder 2 whenever he wins.

$\square$

Property 4. *Fix any $s \in [0, 1]$ and $\bar{w} > 0$, bidder 2's optimal bid is non-decreasing in $\bar{x}$.*[6]

This observation follows the intuition that as bidder 1's valuation for the object becomes more likely to be high[7], bidder 2 bids more aggressively for any realization of his private signal S since it becomes more likely that his opponent loses the auction due to budget constraint rather than having low valuation. The same intuition can also be confirmed by investigating how bidder 2's optimal bid changes with $\bar{w}$ when $\bar{x}$ is held fixed, which is given by Property 5. □

Property 5. *Fix any $s \in [0, 1]$ and $\bar{x} > 0$, bidder 2's optimal bid is non-increasing in $\bar{w}$.*[8]

In words, Property 5 states that as bidder 1 becomes less likely to be budget constrained[9], bidder 2 bids less aggressively. This could seem counter-intuitive at first, since one might expect a deep-pocket investor to raise his bid in order to match his opponent's increased ability to submit higher bid. However, this argument fails to take into account that as bidder 1 becomes less likely to be financially constrained, it becomes more likely that bidder 2 only wins when bidder 1 does not value the object highly. Essentially, the fact that bidder 2 bids more cautiously when his opponent becomes less likely to be financially constrained, is simply a result from bidder 2's effort to avoid the *winner's curse* in this context. It is worth emphasizing that the *winner's curse* can play a role in this auction model even without incorporating the common value ingredient. □

Property 1 – 5 together provide a qualitative description of bidder 2's optimal bid as a function of his private signal: starting from $\beta(0) = 0$, $\beta(s)$ increases with s. Depending on the

[6]The formal proof can be found in the Appendix.
[7]For example, in the first-order stochastic dominance sense.
[8]The formal proof can be found in the Appendix.
[9]For example, in the first-order stochastic dominance sense.

relative sizes of $\bar{x}$ and $\bar{w}$, $\beta(s)$ either keeps increasing until $s = 1$ with $\beta(1) = \min\{\bar{x}, \bar{w}\}$, or it reaches $\beta(s^*) = \min\{\bar{x}, \bar{w}\}$ at some level $s^* < 1$, and stays constant at $\min\{\bar{x}, \bar{w}\}$ afterwards.

Theorem 6. *Bidder 2's optimal bidding function is*

$$\beta(s, \bar{x}, \bar{w}) = \begin{cases} \dfrac{(1-s)\bar{w} + \bar{x} - \sqrt{[(1-s)\bar{w} + \bar{x}]^2 - (4-3s)s\bar{x}^2}}{4-3s} & 0 \le s < s^*(\bar{x}, \bar{w}) \\ \min\{\bar{x}, \bar{w}\} & s^*(\bar{x}, \bar{w}) \le s \le 1 \end{cases}, \tag{2.8}$$

where

$$s^*(\bar{x}, \bar{w}) = \begin{cases} 1 & \bar{x} \le \bar{w} \\ \dfrac{2\bar{w}}{\bar{x} + \bar{w}} & \bar{w} < \bar{x} \end{cases}. \tag{2.9}$$

Proof. As has been shown above, when the condition in Equation 2.6 is satisfied, $\beta(s, \bar{x}, \bar{w}) = \dfrac{(1-s)\bar{w} + \bar{x} - \sqrt{[(1-s)\bar{w} + \bar{x}]^2 - (4-3s)s\bar{x}^2}}{4-3s}$ is the optimal bidding function for bidder 2, as long as $\beta(s, \bar{x}, \bar{w}) \le \min\{\bar{x}, \bar{w}\}$.

When $\bar{x} \le \bar{w}$, $\dfrac{(\bar{w} + \bar{x})^2}{\bar{w}^2 + 3\bar{x}^2} \ge 1$, and thus the condition in Equation 2.6 holds. All that remains to be shown is $\dfrac{(1-s)\bar{w} + \bar{x} - \sqrt{[(1-s)\bar{w} + \bar{x}]^2 - (4-3s)s\bar{x}^2}}{4-3s} < \bar{x}$. An intuitive explanation for why $\beta(s)$ should increase with s is presented together with Property 1, now I provide a formal proof for this conclusion. Note that neither the conclusion nor the proof relies on the assumption of $\bar{x} \le \bar{w}$.

Lemma 5. b^* *as given by Equation 2.7 increases with* s *for* $s \in [0, 1]$.

Proof. **(of Lemma 5)**

First, rewrite

$$b^* = \underbrace{\frac{(1-s)\bar{w}+\bar{x}}{4-3s}}_{A} - \underbrace{\frac{\sqrt{[(1-s)\bar{w}+\bar{x}]^2-(4-3s)s\bar{x}^2}}{4-3s}}_{B}.$$

Then

$$\frac{A}{B} = \frac{(1-s)\bar{w}+\bar{x}}{\sqrt{[(1-s)\bar{w}+\bar{x}]^2-(4-3s)s\bar{x}^2}} = \frac{1}{\sqrt{1-\underbrace{\frac{(4-3s)s\bar{x}^2}{[(1-s)\bar{w}+\bar{x}]^2}}_{C}}}$$

and

$$\frac{\partial C}{\partial s} = \frac{2[(1-s)\bar{w}+\bar{x}][(2-3s)\bar{x}^3+(2-s)\bar{x}^2\bar{w}]}{[(1-s)\bar{w}+\bar{x}]^4}.$$

When $\bar{x} \le \bar{w}$,

$$(2-3s)\bar{x}^3+(2-s)\bar{x}^2\bar{w} \ge 4(1-s)\bar{x}^3 \ge 0.$$

Similarly, when $\bar{x} > \bar{w}$,

$$(2-3s)\bar{x}^3+(2-s)\bar{x}^2\bar{w} > 4(1-s)\bar{w}^3 \ge 0.$$

Therefore, $\frac{\partial C}{\partial s} \ge 0$. This implies that $\frac{\partial}{\partial s}\left(\frac{A}{B}\right) \ge 0$, which further implies that $\frac{\partial b^*}{\partial s} > 0$. The last part of the induction, i.e. $\frac{\partial}{\partial s}\left(\frac{A}{B}\right) \ge 0 \;\Rightarrow\; \frac{\partial b^*}{\partial s} > 0$ is based on the fact that

$$0 \le \frac{\partial}{\partial s}\left(\frac{A}{B}\right) = \frac{\frac{\partial A}{\partial s}\cdot B - \frac{\partial B}{\partial s}\cdot A}{B^2} \overset{(\text{since } A>B>0)}{<} \frac{\frac{\partial A}{\partial s}\cdot A - \frac{\partial B}{\partial s}\cdot A}{B^2} = \left(\frac{\partial A}{\partial s} - \frac{\partial B}{\partial s}\right)\frac{A}{B^2}.$$

Therefore, $\frac{\partial b^*}{\partial s} = \frac{\partial A}{\partial s} - \frac{\partial B}{\partial s} > 0$. $\square$

According to Lemma 5,

$$\frac{(1-s)\bar{w}+\bar{x}-\sqrt{[(1-s)\bar{w}+\bar{x}]^2-(4-3s)s\bar{x}^2}}{4-3s}$$
$$<\underbrace{\left(\frac{(1-s)\bar{w}+\bar{x}-\sqrt{[(1-s)\bar{w}+\bar{x}]^2-(4-3s)s\bar{x}^2}}{4-3s}\right)\Bigg|_{s=1}}_{=\bar{x}}.$$

Hence, when $\bar{x}\leq\bar{w}$, $\beta(s,\bar{x},\bar{w})=\dfrac{(1-s)\bar{w}+\bar{x}-\sqrt{[(1-s)\bar{w}+\bar{x}]^2-(4-3s)s\bar{x}^2}}{4-3s}$ is bidder 2's optimal bidding function for all $s\in[0,1]$.

When $\bar{x}>\bar{w}$,

$$\left(\frac{(1-s)\bar{w}+\bar{x}-\sqrt{[(1-s)\bar{w}+\bar{x}]^2-(4-3s)s\bar{x}^2}}{4-3s}\right)\Bigg|_{s=\frac{2\bar{w}}{\bar{x}+\bar{w}}}=\bar{w}.$$

According to Lemma 5, $\dfrac{(1-s)\bar{w}+\bar{x}-\sqrt{[(1-s)\bar{w}+\bar{x}]^2-(4-3s)s\bar{x}^2}}{4-3s}\leq\min\{\bar{x},\bar{w}\}$ is thus satisfied for all $s\in[0,s^*(\bar{x},\bar{w})]$. It remains to be shown that the condition in Equation 2.6 is also valid. This is guaranteed by the fact that

$$s^*(\bar{x},\bar{w})-\frac{(\bar{w}+\bar{x})^2}{\bar{w}^2+3\bar{x}^2}=\frac{(\bar{w}-\bar{x})^3}{(\bar{w}+\bar{x})(\bar{w}^2+3\bar{x}^2)}<0,$$

under the assumption that $\bar{x}>\bar{w}$. Therefore, the condition in Equation 2.6 holds for $s\in[0,s^*(\bar{x},\bar{w})]$.

To sum up, both the condition in Equation 2.6 and the requirement of

$$\frac{(1-s)\bar{w}+\bar{x}-\sqrt{[(1-s)\bar{w}+\bar{x}]^2-(4-3s)s\bar{x}^2}}{4-3s}\leq\min\{\bar{x},\bar{w}\}$$

are satisfied for all $s \in [0, s^*(\bar{x}, \bar{w})]$, no matter $\bar{x} \leq \bar{w}$ or $\bar{x} > \bar{w}$. In either case,

$$\frac{(1 - s^*(\bar{x}, \bar{w}))\bar{w} + \bar{x} - \sqrt{[(1 - s^*(\bar{x}, \bar{w}))\bar{w} + \bar{x}]^2 - (4 - 3s^*(\bar{x}, \bar{w}))s^*(\bar{x}, \bar{w})\bar{x}^2}}{4 - 3s^*(\bar{x}, \bar{w})} = \min\{\bar{x}, \bar{w}\}.$$

Therefore, bidder 2's optimal bidding strategy is as stated in Theorem 6. □

Corollary 15.1. *Bidder 2's optimal bidding function given by Theorem 6 satisfies Property 1-5.*

The proof of Corollary 15.1 is in Appendix 4.2.

Figure 2.1 shows bidder 2's optimal bid as a function of his private signal. Since the optimal bidding function given by Equation 2.8 is homogeneous of degree one in $(\bar{x}, \bar{w})$, one can without loss of generality set $\bar{x} = 1$. Therefore, Figure 2.1 also shows how the relative sizes of $\bar{x}$ and $\bar{w}$ influence bidder 2's equilibrium bidding behavior. As the graph shows, bidder 2's optimal bid is indeed an increasing function of his private signal, and it strictly increases until it reaches $\min\{\bar{x}, \bar{w}\}$, and remains constant afterwards. Figure 2.1 also illustrates that to avoid the *winner's curse*, bidder 2 bids less aggressively as his opponent becomes less likely to be constrained by her budget (the optimal bidding curve shifts downward as $\bar{w}$ increases). In what follows, I routinely fix $\bar{x}$ and vary $\bar{w}$ to see how various aspects of the auction depend on the probability of bidder 1 being budget-constrained. The justification of this approach is the equivalency between the two measures. More concretely, notice that

$$\mathbb{P}\{w < x\} = \int_0^{\bar{x}} F_W(x) \cdot f_X(x) dx = \frac{\bar{x}}{2\bar{w}},$$

implying that the probability of bidder 1's budget being less than her valuation is proportional to $\frac{\bar{x}}{\bar{w}}$. Therefore, keeping $\bar{x}$ fixed at $\bar{x} = 1$ and varying $\bar{w}$ is a legitimate way to model the likelihood of bidder 1 being budget-constrained.

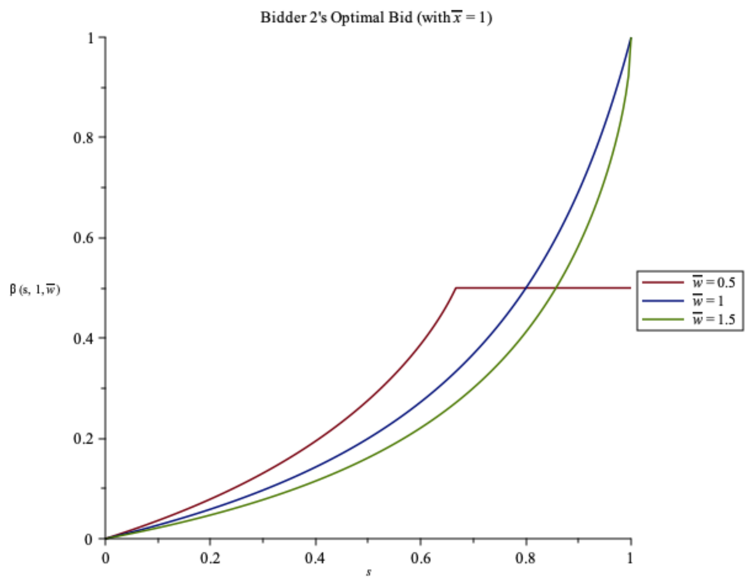


Figure 2.1. Bidder 2's Equilibrium Bidding Strategy as a Function of His Private Signal

Figure 2.2 further illustrates how the likelihood of bidder 1 being budget-constrained influences bidder 2's bidding strategy. When $S = 1$, bidder'2 has the same valuation of the object as bidder 1 does, which makes the winner's curse obsolete. Therefore, bidder 2 is not deterred by the potential improvement in bidder 1's financial standing. However, when $S < 1$, bidder 2 is willing to make sure-to-win bid only when $\bar{w}$ is small. When $\bar{w}$ is big, bidder 2 bids less for the same signal as he otherwise would were $\bar{w}$ smaller.

2.3.2. Efficiency

To learn about the efficiency of this auction, denoted by $E(s, \bar{x}, \bar{w})$, one can simply calculate the probability that bidder 1 wins the auction, since her valuation is always (weakly) greater than

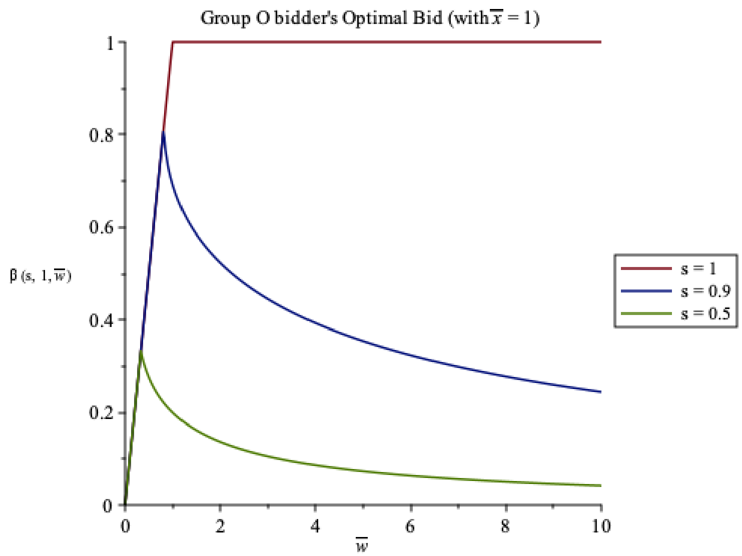


Figure 2.2. Influence of Bidder 1's Budget-Constrained Probability on Bidder 2's Equilibrium Bidding Strategy

that of bidder 2:

$$E(\bar{x}, \bar{w}) = \int_0^{s^*(\bar{x},\bar{w})} [1 - F_Z(\beta(s, \bar{x}, \bar{w}))]ds = s^*(\bar{x}, \bar{w}) - \int_0^{s^*(\bar{x},\bar{w})} F_Z(\beta(s, \bar{x}, \bar{w}))ds. \tag{2.10}$$

Intuitively, if bidder 1 is less likely to be budget constrained, she should be more likely to win the auction, which turns out to be exactly the case.

Theorem 7. *For any given $\bar{x}$, the auction efficiency increases with $\bar{w}$.*

Proof. (1) If $\bar{x} < \bar{w}$, then according to Equation 2.9, $s^*(\bar{x}, \bar{w}) = 1$. Equation 2.10 becomes

$$E(\bar{x}, \bar{w}) = 1 - \int_0^1 F_Z(\beta(s, \bar{x}, \bar{w}))ds.$$

Hence,

$$\frac{\partial E}{\partial \bar{w}}(\bar{x}, \bar{w}) = -\int_0^1 f_Z(\beta(s, \bar{x}, \bar{w})) \frac{\partial \beta(s, \bar{x}, \bar{w})}{\partial \bar{w}} ds > 0,$$

as

$$f_Z(\beta(s, \bar{x}, \bar{w})) > 0$$

and

$$\frac{\partial \beta(s, \bar{x}, \bar{w})}{\partial \bar{w}} = \frac{1-s}{4-3s}\left[1 - \frac{(1-s)\bar{w} + \bar{x}}{\sqrt{[(1-s)\bar{w} + \bar{x}]^2 - (4-3s)s\bar{x}^2}}\right] < 0.$$

(2) If $\bar{x} \geq \bar{w}$, then according to Equation 2.9, $s^*(\bar{x}, \bar{w}) = \dfrac{2\bar{w}}{\bar{x} + \bar{w}}$. Equation 2.10 becomes

$$E(\bar{x}, \bar{w}) = \frac{2\bar{w}}{\bar{x} + \bar{w}} - \int_0^{\frac{2\bar{w}}{\bar{x}+\bar{w}}} F_Z(\beta(s, \bar{x}, \bar{w})) ds$$

and

$$\frac{\partial E}{\partial \bar{w}}(\bar{x}, \bar{w}) = \frac{2\bar{x}}{(\bar{x} + \bar{w})^2}\left[1 - F_Z(\beta(s, \bar{x}, \bar{w}))\right] - \int_0^{\frac{2\bar{w}}{\bar{x}+\bar{w}}} f_Z(\beta(s, \bar{x}, \bar{w})) \frac{\partial \beta(s, \bar{x}, \bar{w})}{\partial \bar{w}} ds > 0.$$

Therefore, $\dfrac{\partial E}{\partial \bar{w}}(\bar{x}, \bar{w}) > 0$ and the auction efficiency increases with $\bar{w}$. □

Figure 2.3 shows the efficiency of this auction measured by the probability that bidder 1 wins. Again, I fix $\bar{x} = 1$ and vary $\bar{w}$ to learn how the allocation efficiency changes with the relative sizes of $\bar{x}$ and $\bar{w}$. It is clear from the graph that bidder 1's probability of winning strictly increases with $\bar{w}$.

Based on the previous analysis, two factors contribute to this result. One factor is that as $\bar{w}$ increases, bidder 1 becomes less likely to be constrained by her budget, and thus more capable of making higher bids in a SPSB. The other factor is that as $\bar{w}$ increases, bidder 2 further shades his bid to avoid the *winner's curse*, which also helps enhance the allocation efficiency.

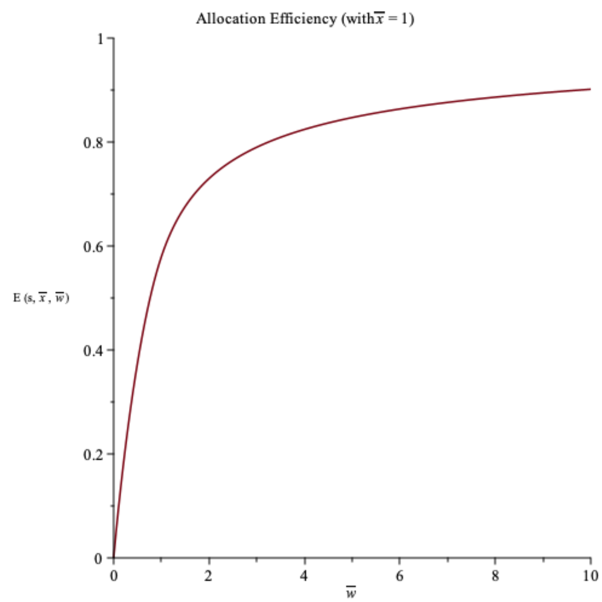


Figure 2.3. Allocation Efficiency Measured as Bidder 1's Winning Probability

2.3.3. Revenue

Although the allocation efficiency is an important attribute of an auction, the seller often cares more about the expected revenue. The seller's expected revenue can be derived using both bidders' equilibrium bidding strategies derived above. In the *ex-ante* sense, the seller's expected revenue is the sum of each bidder's expected payment. With any realized $(X, W) = (x, w)$, let

$z \equiv \min\{x, w\}$. Bidder 1's *interim* expected payment is[10][11]

$$\begin{aligned} EP_1^{interim}(z) &= \mathbb{P}\{\beta(S) < z\} \cdot \mathbb{E}[\beta(S)|\beta(S) < z] \\ &= \mathbb{P}\{S < \beta^{-1}(z)\} \cdot \mathbb{E}[\beta(S)|S < \beta^{-1}(z)] \\ &= \int_0^{\beta^{-1}(z)} \beta(s)ds. \end{aligned}$$

$\beta^{-1}(z)$ can be obtained from Equation 2.8 as

$$\beta^{-1}(z) = \frac{2(\bar{x} + \bar{w} - 2z)z}{\bar{x}^2 + 2\bar{w}z - 3z^2}, \quad z \in [0, \min\{\bar{x}, \bar{w}\}].$$

The *ex ante* expected payment made by bidder 1 is therefore

$$EP_1^{ex-ante}(\bar{x}, \bar{w}) = \int_0^{\min\{\bar{x},\bar{w}\}} \left(\int_0^{\beta^{-1}(z)} \beta(s)ds \right) dF_Z(z) \tag{2.11}$$

[10]In all subsequent sections, except for doing comparative statics analysis, I suppress $\bar{x}$ and $\bar{w}$ from the expression of $\beta(s, \bar{x}, \bar{w})$ for conciseness, as there is no randomness in $\bar{x}$ or $\bar{w}$ during the auction.

[11]Although bidder 2's bidding function is not invertible in the region $s \in [s^*(\bar{x}, \bar{w}), 1]$, given that bidder 1 cannot win if S is in that region, rewriting $\mathbb{P}\{\beta(S) < z\}$ as $\mathbb{P}\{S < \beta^{-1}(z)\}$ is without loss of generality.

It can be calculated that when $\bar{x} \le \bar{w}$, Equation 2.11 yields to[12]

(2.12)

$$
\begin{aligned}
EP_1^{ex-ante}(\bar{x},\bar{w})|_{\bar{x}\le\bar{w}} = {} & \frac{\left(\bar{w}^2-3\bar{w}\bar{x}+6\bar{x}^2\right)\left(\bar{w}+3\bar{x}+\sqrt{\bar{w}^2+3\bar{x}^2}\right)}{81\bar{w}\bar{x}}\ln\left(\frac{3\bar{x}-\bar{w}+\sqrt{\bar{w}^2+3\bar{x}^2}}{-\bar{w}+\sqrt{\bar{w}^2+3\bar{x}^2}}\right) \\
& +\frac{\left(\bar{w}^2-3\bar{w}\bar{x}+6\bar{x}^2\right)\left(\bar{w}+3\bar{x}-\sqrt{\bar{w}^2+3\bar{x}^2}\right)}{81\bar{w}\bar{x}}\ln\left(\frac{-3\bar{x}+\bar{w}+\sqrt{\bar{w}^2+3\bar{x}^2}}{\bar{w}+\sqrt{\bar{w}^2+3\bar{x}^2}}\right) \\
& +\frac{2\left(\bar{w}^4-9\bar{w}^2\bar{x}^2+18\bar{w}\bar{x}^3-90\bar{x}^4\right)}{81\bar{w}\bar{x}\sqrt{\bar{w}^2+3\bar{x}^2}}\left[\arctan\left(\frac{\bar{w}-3\bar{x}}{\sqrt{\bar{w}^2+3\bar{x}^2}}\right)-\arctan\left(\frac{\bar{w}}{\sqrt{\bar{w}^2+3\bar{x}^2}}\right)\right] \\
& -\frac{(\bar{w}-3\bar{x})\left(\bar{w}^2+15\bar{x}^2\right)}{81\bar{w}\bar{x}}\ln\left(\frac{2(\bar{w}-\bar{x})}{\bar{x}}\right)+\frac{\bar{x}(\bar{w}-15\bar{x})}{9\bar{w}}
\end{aligned}
$$

When $\bar{w} < \bar{x}$, Equation 2.11 yields to[13]

(2.13)

$$
\begin{aligned}
EP_1^{ex-ante}(\bar{x},\bar{w})|_{\bar{x}>\bar{w}} = {} & \frac{\left(\bar{w}^2-3\bar{w}\bar{x}+6\bar{x}^2\right)\left(\bar{w}+3\bar{x}+\sqrt{\bar{w}^2+3\bar{x}^2}\right)}{81\bar{w}\bar{x}}\ln\left(\frac{2\bar{w}+\sqrt{\bar{w}^2+3\bar{x}^2}}{-\bar{w}+\sqrt{\bar{w}^2+3\bar{x}^2}}\right) \\
& +\frac{\left(\bar{w}^2-3\bar{w}\bar{x}+6\bar{x}^2\right)\left(\bar{w}+3\bar{x}-\sqrt{\bar{w}^2+3\bar{x}^2}\right)}{81\bar{w}\bar{x}}\ln\left(\frac{-2\bar{w}+\sqrt{\bar{w}^2+3\bar{x}^2}}{\bar{w}+\sqrt{\bar{w}^2+3\bar{x}^2}}\right) \\
& -\frac{2\left(\bar{w}^4-9\bar{w}^2\bar{x}^2+18\bar{w}\bar{x}^3-90\bar{x}^4\right)}{81\bar{w}\bar{x}\sqrt{\bar{w}^2+3\bar{x}^2}}\left[\arctan\left(\frac{2\bar{w}}{\sqrt{\bar{w}^2+3\bar{x}^2}}\right)+\arctan\left(\frac{\bar{w}}{\sqrt{\bar{w}^2+3\bar{x}^2}}\right)\right] \\
& -\frac{(\bar{w}-3\bar{x})\left(\bar{w}^2+15\bar{x}^2\right)}{81\bar{w}\bar{x}}\ln\left(\frac{\bar{x}^2-\bar{w}^2}{\bar{x}^2}\right)-\frac{3\bar{w}^2-13\bar{w}\bar{x}+24\bar{x}^2}{9\bar{x}}
\end{aligned}
$$

[12]See Appendix 4.3 for details.

[13]See Appendix 4.3 for details.

To see how changes in the likelihood of bidder 1 being budget-constrained influence her expected payment, simply take derivative of $EP_1^{ex-ante}(\bar{x},\bar{w})$ with respect to $\bar{w}$ and obtain:

$$\begin{aligned}
\frac{\partial EP_1^{ex-ante}}{\partial \bar{w}}(\bar{x},\bar{w})|_{\bar{x}\leq\bar{w}} = &\frac{2\bar{w}^4-3\bar{w}^3\bar{x}+3\bar{w}^2\bar{x}^2-18\bar{x}^4+2\left(\bar{w}^3-9\bar{x}^3\right)\sqrt{\bar{w}^2+3\bar{x}^2}}{81\bar{w}^2\bar{x}\sqrt{\bar{w}^2+3\bar{x}^2}} \\
&\cdot\ln\left(\frac{3\bar{x}-\bar{w}+\sqrt{\bar{w}^2+3\bar{x}^2}}{-\bar{w}+\sqrt{\bar{w}^2+3\bar{x}^2}}\right) \\
&-\frac{2\bar{w}^4-3\bar{w}^3\bar{x}+3\bar{w}^2\bar{x}^2-18\bar{x}^4-2\left(\bar{w}^3-9\bar{x}^3\right)\sqrt{\bar{w}^2+3\bar{x}^2}}{81\bar{w}^2\bar{x}\sqrt{\bar{w}^2+3\bar{x}^2}} \\
&\cdot\ln\left(\frac{-3\bar{x}+\bar{w}+\sqrt{\bar{w}^2+3\bar{x}^2}}{\bar{w}+\sqrt{\bar{w}^2+3\bar{x}^2}}\right) \\
&+\frac{2\left(2\bar{w}^6+9\bar{w}^4\bar{x}^2-18\bar{w}^3\bar{x}^3+153\bar{w}^2\bar{x}^4+270\bar{x}^6\right)}{81\bar{w}^2\bar{x}\left(\bar{w}^2+3\bar{x}^2\right)^{\frac{3}{2}}} \\
&\cdot\left[\arctan\left(\frac{\bar{w}-3\bar{x}}{\sqrt{\bar{w}^2+3\bar{x}^2}}\right)-\arctan\left(\frac{\bar{w}}{\sqrt{\bar{w}^2+3\bar{x}^2}}\right)\right] \\
&-\frac{2\bar{w}^3-3\bar{w}^2\bar{x}+45\bar{x}^3}{81\bar{w}^2\bar{x}}\ln\left(\frac{2\left(\bar{w}-\bar{x}\right)}{\bar{x}}\right)+\frac{\bar{x}\left(\bar{w}^3+15\bar{w}^2\bar{x}+15\bar{w}\bar{x}^2+45\bar{x}^3\right)}{9\bar{w}^2\left(\bar{w}^2+3\bar{x}^2\right)},
\end{aligned}$$

while

$$\frac{\partial EP_1^{ex-ante}}{\partial \bar{w}}(\bar{x},\bar{w})|_{\bar{x}>\bar{w}} = \frac{2\bar{w}^4 - 3\bar{w}^3\bar{x} + 3\bar{w}^2\bar{x}^2 - 18\bar{x}^4 + 2\left(\bar{w}^3 - 9\bar{x}^3\right)\sqrt{\bar{w}^2+3\bar{x}^2}}{81\bar{w}^2\bar{x}\sqrt{\bar{w}^2+3\bar{x}^2}}$$
$$\cdot \ln\left(\frac{2\bar{w}+\sqrt{\bar{w}^2+3\bar{x}^2}}{-\bar{w}+\sqrt{\bar{w}^2+3\bar{x}^2}}\right)$$
$$-\frac{2\bar{w}^4 - 3\bar{w}^3\bar{x} + 3\bar{w}^2\bar{x}^2 - 18\bar{x}^4 - 2\left(\bar{w}^3 - 9\bar{x}^3\right)\sqrt{\bar{w}^2+3\bar{x}^2}}{81\bar{w}^2\bar{x}\sqrt{\bar{w}^2+3\bar{x}^2}}$$
$$\cdot \ln\left(\frac{-2\bar{w}+\sqrt{\bar{w}^2+3\bar{x}^2}}{\bar{w}+\sqrt{\bar{w}^2+3\bar{x}^2}}\right)$$
$$-\frac{2\left(2\bar{w}^6 + 9\bar{w}^4\bar{x}^2 - 18\bar{w}^3\bar{x}^3 + 153\bar{w}^2\bar{x}^4 + 270\bar{x}^6\right)}{81\bar{w}^2\bar{x}\left(\bar{w}^2+3\bar{x}^2\right)^{\frac{3}{2}}}$$
$$\cdot\left[\arctan\left(\frac{2\bar{w}}{\sqrt{\bar{w}^2+3\bar{x}^2}}\right) + \arctan\left(\frac{\bar{w}}{\sqrt{\bar{w}^2+3\bar{x}^2}}\right)\right]$$
$$-\frac{2\bar{w}^3 - 3\bar{w}^2\bar{x} + 45\bar{x}^3}{81\bar{w}^2\bar{x}}\ln\left(\frac{\bar{x}^2-\bar{w}^2}{\bar{x}^2}\right) - \frac{6\bar{w}^5 - 7\bar{w}^4\bar{x} + 5\bar{w}^3\bar{x}^2 - 33\bar{w}^2\bar{x}^3 - 51\bar{w}\bar{x}^4 - 72\bar{x}^5}{9\bar{w}\bar{x}\left(\bar{w}+\bar{x}\right)\left(\bar{w}^2+3\bar{x}^2\right)}$$

It can be shown that $\frac{\partial EP_1^{ex-ante}}{\partial \bar{w}}(\bar{x},\bar{w}) > 0$ for all $\bar{w} \in [0,\bar{x})$, meaning that when bidder 1 is very likely to be budget-constrained, her expected payment increases as she becomes less prone to be constrained by her budget. However, with $\bar{w} \geq \bar{x}$, $\frac{\partial EP_1^{ex-ante}}{\partial \bar{w}}(\bar{x},\bar{w})$ remains positive as long as $\bar{w} < 1.9\bar{x}$ and then turns negative as $\bar{w}$ keeps increasing. In other words, when bidder 1 is very likely to be budget-constrained (in the sense that $\bar{w}$ is not too large in comparison to $\bar{x}$), her expected payment increases as she becomes less prone to be constrained by her budget. However, when bidder 1 is not likely to be budge-constrained (in the sense that $\bar{w}$ is considerably larger than $\bar{x}$), her expected payment decreases with the probability that she would be constrained by her budget.

This conclusion makes intuitive sense, since bidder 1's expected payment equals bidder 2's bid, given that bidder 1 wins the auction. When bidder 1 is very likely to be budget-constrained, her winning probability is low, due to the fact that she is less likely to afford to make high bids. Furthermore, in case that she wins the auction with a low bid, it implies that bidder 2's bid is even lower, which results in bidder 1 not paying much for her win. Therefore, as bidder 1's financial condition improves, her expected payment increases correspondingly. However, based on Theorem 6, bidder 2 shades his bid as the likelihood of bidder 1 being budget-constrained decreases in fear of the winner's curse. Consistent with the previous analysis on the auction's allocation efficiency, bidder 1's increased ability to make higher bids and bidder 2's bid-shading tendency both contribute to the increase in bidder 1's probability of winning. However, since in case of winning, bidder 1's payment equals bidder 2's bid, the latter's bid-shading behavior as $\bar{w}$ becomes large relative to $\bar{x}$ results in bidder 1 paying less. Thus, the overall effect is that as the probability of bidder 1 being constrained by her budget decreases, her expected payment first rises and then declines.

Bidder 2's *interim* expected payment with any realized signal $S = s$ is

$$EP_2^{interim}(s) = \mathbb{P}\{Z < \beta(s)\} \cdot \mathbb{E}[Z|Z < \beta(s)] = \int_0^{\beta(s)} z dF_Z(z),$$

and his *ex-ante* expected payment is thus

$$EP_2^{ex-ante}(\bar{x},\bar{w}) = \int_0^1 \left(\int_0^{\beta(s)} z dF_Z(z)\right) ds$$

$$= \int_0^{s^*(\bar{x},\bar{w})} \left(\int_0^{\frac{(1-s)\bar{w}+\bar{x}-\sqrt{[(1-s)\bar{w}+\bar{x}]^2-(4-3s)s\bar{x}^2}}{4-3s}} z dF_Z(z)\right) ds$$

$$+ \int_{s^*(\bar{x},\bar{w})}^1 \left(\int_0^{\min\{\bar{x},\bar{w}\}} z dF_Z(z)\right) ds.$$

It can be calculated that

$$\int_0^{\frac{(1-s)\bar{w}+\bar{x}-\sqrt{[(1-s)\bar{w}+\bar{x}]^2-(4-3s)s\bar{x}^2}}{4-3s}} z dF_Z(z)$$

(2.14)
$$= \frac{1}{6\bar{w}\bar{x}(4-3s)^3} \cdot \left[(1-s)\bar{w} + \bar{x} - \sqrt{(1-s)\left[(1-s)\bar{w}^2 + 2\bar{w}\bar{x} + \bar{x}^2(1-3s)\right]}\right]^2$$
$$\cdot \left[8\bar{w} + 8\bar{x} - (5\bar{w} + 9\bar{x})s + 4\sqrt{(1-s)\left[(1-s)\bar{w}^2 + 2\bar{w}\bar{x} + \bar{x}^2(1-3s)\right]}\right].$$

When $\bar{x} \leq \bar{w}$, $s^*(\bar{x},\bar{w}) = 1$, and therefore

$$EP_2^{ex-ante}(\bar{x},\bar{w})\big|_{\bar{x}\leq\bar{w}} = \int_0^1 \int_0^{\frac{(1-s)\bar{w}+\bar{x}-\sqrt{[(1-s)\bar{w}+\bar{x}]^2-(4-3s)s\bar{x}^2}}{4-3s}} z dF_Z(z) ds.$$

Substituting 2.14 for the integrand, it can be calculated that when $\bar{w} \geq 3\bar{x}$,

(2.15)
$$EP_2^{ex-ante}(\bar{x},\bar{w})\big|_{\bar{w}\geq 3\bar{x}}$$
$$= \frac{1}{162\bar{w}\bar{x}} \left[3\bar{x}\left(4\bar{w}^2 - 21\bar{w}\bar{x} + 51\bar{x}^2\right) + 4\left(\bar{w}^3 - 6\bar{w}^2\bar{x} + 15\bar{w}\bar{x}^2 - 18\bar{x}^3\right) \cdot \ln\left(\frac{2(\bar{w}-\bar{x})}{\bar{x}}\right)\right.$$
$$\left. - \frac{2\left(2\bar{w}^4 - 12\bar{w}^3\bar{x} + 33\bar{w}^2\bar{x}^2 - 54\bar{w}\bar{x}^3 + 63\bar{x}^4\right)}{\sqrt{\bar{w}^2 + 3\bar{x}^2}} \cdot \arctan\left(\frac{(\bar{w}+\bar{x})\sqrt{\bar{w}^2+3\bar{x}^2}}{\bar{w}^2 + \bar{w}\bar{x} + 2\bar{x}^2}\right)\right]$$

and when $\bar{x} \leq \bar{w} < 3\bar{x}$,

$$
\begin{aligned}
& EP_2^{ex-ante}(\bar{x},\bar{w})\big|_{\bar{x}\leq\bar{w}<3\bar{x}} \\
&= \frac{1}{162\bar{w}\bar{x}}\Big[3\bar{x}\left(4\bar{w}^2 - 21\bar{w}\bar{x} + 51\bar{x}^2\right) + 4\left(\bar{w}^3 - 6\bar{w}^2\bar{x} + 15\bar{w}\bar{x}^2 - 18\bar{x}^3\right)\cdot\ln(2) \\
&\quad + 4\left(\bar{w}^3 + 56\bar{w}\bar{x}^2 - 18\bar{x}^3\right)\ln\left(\frac{\bar{w}-\bar{x}}{\bar{x}}\right) - 8\bar{w}\bar{x}\left(6\bar{w} + 41\bar{x}\right)\arctan\left(\frac{\bar{w}-2\bar{x}}{\bar{w}}\right) \\
&\quad - \frac{2\left(2\bar{w}^4 - 12\bar{w}^3\bar{x} + 33\bar{w}^2\bar{x}^2 - 54\bar{w}\bar{x}^3 + 63\bar{x}^4\right)}{\sqrt{\bar{w}^2+3\bar{x}^2}}\cdot\arctan\left(\frac{(\bar{w}+\bar{x})\sqrt{\bar{w}^2+3\bar{x}^2}}{\bar{w}^2+\bar{w}\bar{x}+2\bar{x}^2}\right)\Bigg].
\end{aligned}
\tag{2.16}
$$

It can be further shown that when $\bar{w} \geq 3\bar{x}$,

$$
\begin{aligned}
& \frac{\partial}{\partial\bar{w}}EP_2^{ex-ante}(\bar{x},\bar{w})\big|_{\bar{w}\geq3\bar{x}} \\
&= \frac{1}{162\bar{w}^2\bar{x}\left(\bar{w}^2+3\bar{x}^2\right)}\Bigg[3\bar{x}\left(8\bar{w}^4 - 18\bar{w}^3\bar{x} - 3\bar{w}^2\bar{x}^2 - 54\bar{w}\bar{x}^3 - 153\bar{x}^4\right) \\
&\quad + 8\left(\bar{w}^5 - 3\bar{w}^4\bar{x} + 3\bar{w}^3\bar{x}^2 + 27\bar{x}^5\right)\cdot\ln\left(\frac{2(\bar{w}-\bar{x})}{\bar{x}}\right) \\
&\quad - \frac{2\left(4\bar{w}^6 - 12\bar{w}^5\bar{x} + 18\bar{w}^4\bar{x}^2 - 18\bar{w}^3\bar{x}^3 - 27\bar{w}^2\bar{x}^4 - 189\bar{x}^6\right)}{\sqrt{\bar{w}^2+3\bar{x}^2}} \\
&\quad \cdot\arctan\left(\frac{(\bar{w}+\bar{x})\sqrt{\bar{w}^2+3\bar{x}^2}}{\bar{w}^2+\bar{w}\bar{x}+2\bar{x}^2}\right)\Bigg]
\end{aligned}
$$

and when $\bar{x} \leq \bar{w} < 3\bar{x}$,

$$
\begin{aligned}
&\frac{\partial}{\partial \bar{w}} EP_2^{ex-ante}(\bar{x}, \bar{w})|_{\bar{x}\leq\bar{w}<3\bar{x}} \\
=&\frac{1}{162\bar{w}^2\bar{x}\left(\bar{w}^2+3\bar{x}^2\right)}\left[3\bar{x}\left(8\bar{w}^4-18\bar{w}^3\bar{x}-3\bar{w}^2\bar{x}^2-54\bar{w}\bar{x}^3-153\bar{x}^4\right)\right. \\
&+8\left(\bar{w}^3+9\bar{x}^3\right)\left(\bar{w}^2+3\bar{x}^2\right)\cdot\ln\left(\frac{\bar{w}-\bar{x}}{\bar{x}}\right)-48\bar{w}^2\left(\bar{w}^2\bar{x}+3\bar{x}^3\right)\arctan\left(\frac{\bar{w}-2\bar{x}}{\bar{w}}\right) \\
&+8\left(\bar{w}^5-3\bar{w}^4\bar{x}+3\bar{w}^3\bar{x}^2+27\bar{x}^5\right)\ln(2) \\
&-\frac{2\left(4\bar{w}^6-12\bar{w}^5\bar{x}+18\bar{w}^4\bar{x}^2-18\bar{w}^3\bar{x}^3-27\bar{w}^2\bar{x}^4-189\bar{x}^6\right)}{\sqrt{\bar{w}^2+3\bar{x}^2}} \\
&\left.\cdot\arctan\left(\frac{(\bar{w}+\bar{x})\sqrt{\bar{w}^2+3\bar{x}^2}}{\bar{w}^2+\bar{w}\bar{x}+2\bar{x}^2}\right)\right]
\end{aligned}
$$

When $\bar{x} > \bar{w}$, $s^*(\bar{x}, \bar{w}) = \dfrac{2\bar{w}}{\bar{w}+\bar{x}}$, and therefore

(2.17)

$$
\begin{aligned}
&EP_2^{ex-ante}(\bar{x}, \bar{w})|_{\bar{x}>\bar{w}} \\
=&\int_0^{\frac{2\bar{w}}{\bar{w}+\bar{x}}}\left(\int_0^{\frac{(1-s)\bar{w}+\bar{x}-\sqrt{[(1-s)\bar{w}+\bar{x}]^2-(4-3s)s\bar{x}^2}}{4-3s}} z\,dF_Z(z)\right)ds+\int_{\frac{2\bar{w}}{\bar{w}+\bar{x}}}^1\left(\int_0^{\bar{w}} z\,dF_Z(z)\right)ds \\
=&\frac{1}{162\bar{w}\bar{x}}\left[3\bar{w}\left(19\bar{w}^2-69\bar{w}\bar{x}+84\bar{x}^2\right)+4\left(\bar{w}-3\bar{x}\right)\left(\bar{w}^2-3\bar{w}\bar{x}+6\bar{x}^2\right)\ln\left(\frac{\bar{x}^2-\bar{w}^2}{\bar{x}^2}\right)\right. \\
&+\frac{2\bar{w}^4-12\bar{w}^3\bar{x}+33\bar{w}^2\bar{x}^2-54\bar{w}\bar{x}^3+63\bar{x}^4}{\sqrt{\bar{w}^2+3\bar{x}^2}} \\
&\left.\cdot\ln\left(\frac{2\bar{w}^4+5\bar{w}^2\bar{x}^2+\bar{x}^4-2\bar{w}^3\sqrt{\bar{w}^2+3\bar{x}^2}-2\bar{w}\bar{x}^2\sqrt{\bar{w}^2+3\bar{x}^2}}{2\bar{w}^4+5\bar{w}^2\bar{x}^2+\bar{x}^4+2\bar{w}^3\sqrt{\bar{w}^2+3\bar{x}^2}+2\bar{w}\bar{x}^2\sqrt{\bar{w}^2+3\bar{x}^2}}\right)\right],
\end{aligned}
$$

with

$$\frac{\partial}{\partial \bar{w}} EP_2^{ex-ante}(\bar{x}, \bar{w})|_{\bar{x}>\bar{w}} = \frac{1}{162\bar{w}^2\bar{x}} \left[8\left(\bar{w}^3 - 3\bar{w}^2\bar{x} + 9\bar{x}^3\right) \ln\left(\frac{\bar{x}^2 - \bar{w}^2}{\bar{x}^2}\right) \right.$$
$$+3\bar{w}\left(-74\bar{w}^2 + 51\bar{w}\bar{x} - 84\bar{x}^2 + \frac{108\bar{w}^3}{\bar{w}+\bar{x}} + \frac{4\bar{w}^3(\bar{w}-3\bar{x})}{\bar{w}^2+3\bar{x}^2}\right)$$
$$+ \frac{4\bar{w}^6 - 12\bar{w}^5\bar{x} + 18\bar{w}^4\bar{x}^2 - 18\bar{w}^3\bar{x}^3 - 27\bar{w}^2\bar{x}^4 - 189\bar{x}^6}{\left(\bar{w}^2+3\bar{x}^2\right)^{\frac{3}{2}}}$$
$$\left. \cdot \ln\left(\frac{2\bar{w}^4 + 5\bar{w}^2\bar{x}^2 + \bar{x}^4 - 2\bar{w}^3\sqrt{\bar{w}^2+3\bar{x}^2} - 2\bar{w}\bar{x}^2\sqrt{\bar{w}^2+3\bar{x}^2}}{2\bar{w}^4 + 5\bar{w}^2\bar{x}^2 + \bar{x}^4 + 2\bar{w}^3\sqrt{\bar{w}^2+3\bar{x}^2} + 2\bar{w}\bar{x}^2\sqrt{\bar{w}^2+3\bar{x}^2}}\right)\right].$$

It can be shown that $\frac{\partial EP_2^{ex-ante}}{\partial \bar{w}}(\bar{x}, \bar{w}) > 0$ when $\bar{w} < 0.54\bar{x}$ and $\frac{\partial EP_2^{ex-ante}}{\partial \bar{w}}(\bar{x}, \bar{w}) < 0$ otherwise. That is, as bidder 1 becomes less likely to be budget-constrained, bidder 2's *ex-ante* expected payment only increases when the probability that bidder 1 faces budget-constraint is very high. This is intuitive, as when bidder 1 is extremely likely to be constrained by her budget, her ability to make competitive bids is low. This contributes to bidder 2's expected payment in two ways. First, knowing that bidder 1's bid is most likely a reflection of her budget instead of her valuation, bidder 2 worries less about the winner's curse. Therefore, he is more prone to make higher bids that result in him winning the auction, and thus paying for his winning. Since bidder 2's payment in case of win equals to bidder 1's bid, which in turn is most likely equals to the latter's budget, any increase in the budget results in bidder 1 paying more. Thus, in the region of low $\bar{w}$ values relative to that of $\bar{x}$, $\frac{\partial EP_2^{ex-ante}}{\partial \bar{w}}(\bar{x}, \bar{w}) > 0$. However, as $\bar{w}$ gets bigger, bidder 2's fear of the winner's curse takes effect and his bid-shading behavior leads him to make lower bids. At the same time, bidder 1's ability to make higher bids increases. Both factors result in bidder 2 winning less often, which is shown in the discussion of the auction efficiency. And in case bidder 2 indeed wins, bidder 1's bid must be very low, which leads to a small payment from bidder 2. Hence, as $\bar{w}$ grows larger relative to $\bar{x}$, $\frac{\partial EP_2^{ex-ante}}{\partial \bar{w}}(\bar{x}, \bar{w}) < 0$.

The seller's *ex-ante* expected revenue is the sum of the the *ex ante* expected payments from the two bidders:

$$Rev^{ex-ante}(\bar{x}, \bar{w}) = EP_1^{ex-ante}(\bar{x}, \bar{w}) + EP_2^{ex-ante}(\bar{x}, \bar{w}),$$

with $EP_1^{ex-ante}(\bar{x}, \bar{w})$ given by Equation 2.12 or Equation 2.13, and $EP_2^{ex-ante}(\bar{x}, \bar{w})$ given by Equation 2.15, Equation 2.16, or Equation 2.17, depending on the relation between $\bar{w}$ and $\bar{x}$.

Taking derivative of $Rev^{ex-ante}(\bar{x}, \bar{w})$ with respect to $\bar{w}$ reveals that $\frac{\partial Rev^{ex-ante}}{\partial \bar{w}}(\bar{x}, \bar{w}) > 0$ when $\bar{w} < 0.97\bar{x}$ and $\frac{\partial Rev^{ex-ante}}{\partial \bar{w}}(\bar{x}, \bar{w}) < 0$ otherwise. This is the combined effect of both bidder 1 and bidder 2's expected payments rise with $\bar{w}$ when it is relatively small compared to $\bar{x}$ and then decline as $\bar{w}$ gets bigger. This results provides an interesting insight from the seller's point of view. To maximize the revenue of the auction, it is most desirable to have the business insider (bidder 1) budget-constrained to a certain degree, for if the insider is too constrained, the outsider can acquire the object at a very low price. On the contrary, if the insider is unlikely to be budget-constrained, the outsider would be concerned about the winner's curse and bid more conservatively. As a result, either the outsider wins or loses, the seller's revenue would be small. Therefore, it is optimal for the seller when the two forces balance each other – the insider can afford to make bids of reasonable sizes and the outsider is not too concerned about suffering from the winner's curse to shade his bid considerably. In some forced bankruptcy auctions[14], it has been observed that the major creditor of an insolvent firm sometimes choose to sponsor another firm in the same industry to compete in the auction, in the hope of increasing

[14]For example, 'Chapter 7' in the USA and Liquidation in the UK.

the auction revenue, and therefore the amount of debt that can be recovered. The prediction of this model indicates that this strategy can be effective, but only when the correct balance is found. When overdone, too much financial backup to an insider firm can be counterproductive as it discourages competition.

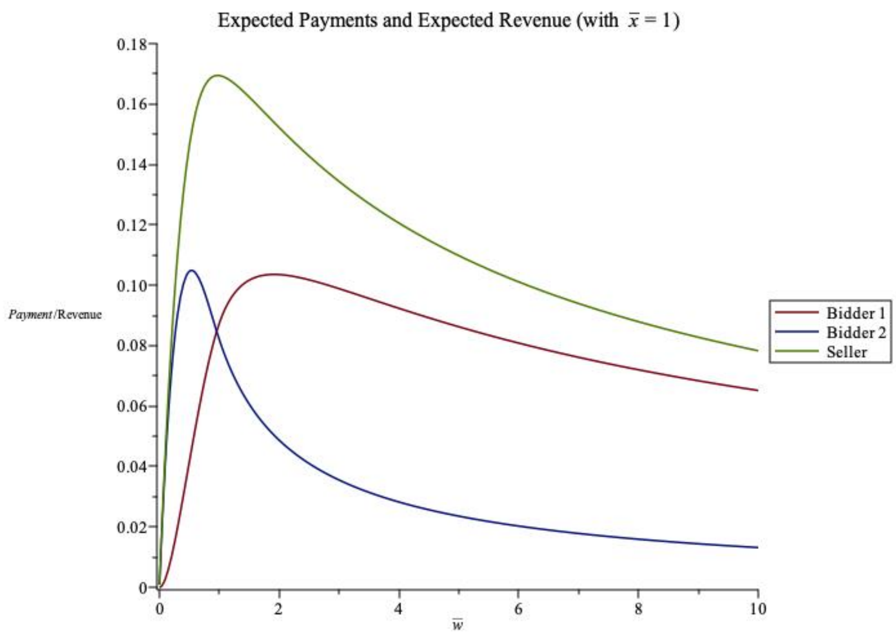


Figure 2.4. Bidders' *Ex Ante* Expected Payments and Seller's Expected Revenue

2.4. Multiple Bidder Case ($N > 1$, $M = 1$)

The previous analysis of the case with one bidder in each group provides some interesting observations. However, to further understand how the existence of competitors, not only from the other group but also from his or her own group, changes a bidder's behavior as well as the auction outcome, it is necessary to extend the analysis to multiple bidders situation. In this section, I examine the case where there are multiple bidders in group $\mathcal{I}$. I maintain the assumption that there is only 1 bidder from group $\mathcal{O}$ in order to isolate the effects of increasing group $\mathcal{I}$ bidders.

Formally, suppose that there are $N > 1$ bidders from group $\mathcal{I}$, and $M = 1$ bidder from group $\mathcal{O}$. All bidders are risk-neutral. Assume that for $i = 1, ..., N$, (X_i, W_i) is independent and identically distributed, with a uniform density on $[0, \bar{x}] \times [0, \bar{w}]$. The private signal S that the group $\mathcal{O}$ bidder receives is independent of the (X_i, W_i)'s, and is uniformly distributed on $[0, 1]$. Again, S serves to fully capture the risks and costs associated with the group O bidder's investment. Both the distribution of (X_i, W_i)'s and that of S are common knowledge to all bidders. Group $\mathcal{O}$ bidder's valuation for the object is given by $V = v(X_1, ...X_N) \cdot S$, where $v(X_1, ...X_N)$ is non-decreasing in all arguments.

For a group $\mathcal{I}$ bidder, say, bidder i, let $Z_i \equiv \min\{X_i, W_i\}$ and $Y \equiv \max_{i=1,...,N}\{Z_i\}$. The cumulative distribution function of Y, $F_Y(\cdot)$, can easily be derived as

$$F_Y(y) = \begin{cases} 0 & y < 0 \\ [F_Z(y)]^N & 0 \le y < \min\{\bar{x}, \bar{w}\} \\ 1 & y \ge \min\{\bar{x}, \bar{w}\} \end{cases},$$

with the corresponding density function

$$f_Y(y) = \begin{cases} N \cdot [F_Z(y)]^{N-1} \cdot f_Z(y) & 0 < y \le \min\{\bar{x}, \bar{w}\} \\ 0 & \text{otherwise} \end{cases},$$

where $F_Z(y)$ and $f_Z(y)$ are given by Equation 2.1 and Equation 2.2.

2.4.1. Equilibrium

Theorem 8. *It is a (weakly) dominant strategy for each group $\mathcal{I}$ bidder to bid $Z_i \equiv \min\{X_i, W_i\}$.*

Proof. Let i be a group $\mathcal{I}$ bidder, and let b_i be this bidder's bid. Let B be the highest bid among all other bidders, group $\mathcal{I}$ and group $\mathcal{O}$ included.

(1) If $X_i \leq W_i$, then the budget constraint is not binding, and the argument is similar to that in the standard SPSB auctions without budget constraints.
   (a) Any bid $b_i < X_i = \min\{X_i, W_i\} = Z_i$ is not optimal: if $B < b_i$, bidder i gets the same payoff either by bidding b_i or by bidding X_i; if $B = b_i$, given that ties are broken randomly, bidder i makes a sure profit of $X_i - B$ by bidding X_i, in contrast to make the same profit with some probability by bidding b_i; if $b_i < B < X_i$, bidder i makes profit of $X_i - B$ by bidding X_i, whereas he loses the auction by bidding b_i; if $B \geq X_i$, bidder i makes zero profit either by bidding b_i or by bidding X_i. To summarize, bidding X instead of $b_i < X$ makes bidder i weakly better-off.
   (b) Any bid $b_i > X_i = \min\{X_i, W_i\} = Z_i$ is not optimal: if $B < X_i$, bidder i makes a profit of $X_i - B$ either by bidding X_i or b_i; if $B = X_i$, bidder i makes zero profit either by bidding X_i or b_i; if $X_i < B < b_i$, bidder i makes zero profit by bidding X_i but makes a loss of $B - X_i$ by bidding b_i; if $B = b_i$, bidder i makes zero profit by bidding X_i but makes a loss of $B - X_i$ with positive probability by bidding b_i; if $B > b_i$, bidder i loses the auction by either bidding b_i or X_i. To summarize, bidding X_i instead of $b_i < X_i$ makes bidder i weakly better-off.

(2) If $X_i > W_i$, the budget constraint is binding, which requires $b_i \leq W_i$. However, $b_i < W_i$ cannot be optimal: if $B < b_i$, bidder i makes a profit of $X_i - b_i$ either by bidding b_i or

by bidding W_i; if $B = b_i$, bidder i makes a sure profit of $X_i - B$ in case she bids W, in contrast to make the same profit with some probability when she bids b_i; if $b_i < B < W_i$, bidder i makes profit of $X_i - B$ in case she bids W_i but zero profit when she bids b_i; if $B = W_i$, bidder i makes profit of $X_i - B$ with positive probability in case she bids W_i but zero profit when she bids b_i. To summarize, bidding W_i instead of $b_i < W_i$ makes bidder i weakly better-off.

□

With a private signal $S = s$, the expected payoff of the group $\mathcal{O}$ bidder when he bids $0 \leq b \leq \min\{\bar{x}, \bar{w}\}$ is[15]

$$\begin{aligned} \Pi(b,s) &= F_Y(b) \cdot \left(\mathbb{E}[v(X_1,...X_N) \cdot s | Y < b] - \mathbb{E}[Y|Y<b]\right) \\ &= F_Y(b) \cdot \left(\mathbb{E}[v(X_1,...X_N)|Y<b] \cdot s - \mathbb{E}[Y|Y<b]\right). \end{aligned} \tag{2.18}$$

The term $\mathbb{E}[Y|Y<b]$ in Equation 2.18 can be calculated straightforwardly as

$$\mathbb{E}[Y|Y<b] = \frac{1}{F_Y(b)} \int_0^b y dF_Y(y) = \frac{1}{F_Y(b)} \int_0^b y \cdot N \cdot [F_Z(y)]^{N-1} f_Z(y) dy.$$

[15]Again, I assume that the group $\mathcal{O}$ bidder never bothers to bid above $\min\{\bar{x}, \bar{w}\}$, since any bid above $\min\{\bar{x}, \bar{w}\}$ delivers the same payoff as bidding $\min\{\bar{x}, \bar{w}\}$.

Let $f_{X_1,...,X_N|Y<b}(x_1,...,x_N)$ be the joint density function of $(X_1,...,X_N)$ conditional on $Y < b$. Then

$$
\begin{aligned}
f_{X_1,...,X_N|Y<b}(x_1,...,x_N) = & \frac{\prod_{i=1}^{N}\left(\mathbb{P}\left\{Z_i < b \mid X_i = x_i\right\} \cdot f_X(x_i)\right)}{F_Y(b)} \\
= & \frac{\prod_{i=1}^{N}\left(\mathbb{I}\left\{x_i < b\right\} \cdot f_X(x_i) + \mathbb{I}\left\{x_i \geq b\right\} \cdot f_X(x_i) \cdot F_W(b)\right)}{F_Y(b)},
\end{aligned}
$$

which leads to

$$
\mathbb{E}[v(X_1,...X_N)|Y < b] = \int_0^{\bar{x}} ... \int_0^{\bar{x}} v(x_1,...x_N) f_{X_1,...,X_N|Y<b}(x_1,...,x_N) dx_1...dx_N
$$

$$
= \frac{1}{F_Y(b)} \int_0^{\bar{x}} ... \int_0^{\bar{x}} v(x_1,...x_N) \prod_{i=1}^{N} \left[\mathbb{I}\left\{x_i < b\right\} f_X(x_i) + \mathbb{I}\left\{x_i \geq b\right\} f_X(x_i) F_W(b)\right] dx_1...dx_N.
$$

Depending on institutional details in the post-auction stage, three special forms of $v(X_1,...X_N)$ are of particular interest. They are:

$$
v_I(X_1,...X_N) \equiv X_{(1)},
$$

$$
v_{II}(X_1,...,X_N) \equiv X_{(2)}
$$

and

$$
v_M(X_1,...,X_N) \equiv \frac{1}{N}\sum_{i=1}^{N} X_i,
$$

where $X_{(1)}$ and $X_{(2)}$ are the highest-order statistic and second-highest-order statistic, respectively. The underlining assumption of using $V = X_{(1)} \cdot S$ as the group $\mathcal{O}$ bidder's valuation is that once the insiders' budget constraints are relaxed in the future, the asset can be allocated to the group $\mathcal{I}$ bidder who values it most. The formulation of $V = X_{(2)} \cdot S$ is most appropriate when the group $\mathcal{I}$ bidder relies on standard auctions to resell the asset in the future. When the group $\mathcal{O}$ bidder is not certain about the post-auction stage, his valuation can roughly be assumed to depend on the average of all group $\mathcal{I}$ bidders' valuations, and in this case, $V = \frac{1}{N}\sum_{i=1}^{N} X_i \cdot S$ is most appropriate. It is worth pointing out that although the uncertainty about the post-auction stage can be summarized in S,[16] I choose to use S simply for the asset-specific aspects of the investment, and let different formulations of V to further capture different institutional details.

2.4.1.1. $V = X_{(1)} \cdot S$. In this case,

$$\mathbb{E}[v_I(X_1, ...X_N)|Y < b] = \mathbb{E}[X_{(1)}|Y < b] \cdot s.$$

Plugging $\mathbb{E}[v_I(X_1, ...X_N)|Y < b]$ and $\mathbb{E}[Y|Y < b]$ back into Equation 2.18 yields[17]

(2.19)

$$\begin{aligned}
&\Pi_I(b, s)\\
=&F_Y(b) \cdot \left(\mathbb{E}[X_{(1)}|Y < b] \cdot s - \mathbb{E}[Y|Y < b]\right)\\
=&N\left(s\int_0^b x f_X(x)[F_X(x)]^{N-1}dx + s\int_b^{\bar{x}} x f_X(x)F_W(b)[F_X(b) + (F_X(x) - F_X(b))F_W(b)]^{N-1}dx\right.\\
&\left.-\int_0^b z\left[F_X(z) + F_W(z) - F_X(z)F_W(z)\right]^{N-1}\left[\left(1 - F_W(z)\right) f_X(z) + \left(1 - F_X(z)\right) f_W(z)\right] dz\right).
\end{aligned}$$

[16]For example, if an investor is very uncertain about the resell stage, he is likely to have a very low realization of S.

[17]See Appendix 4.4 for details.

Taking derivative with respect to b in Equation 2.19 yields

$$
\begin{aligned}
&\frac{\partial}{\partial b}\Pi_I(b,s)\\
=&sNf_X(b)\left[F_X(b)\right]^{N-1}\left[1-F_W(b)\right]b\\
&+sN\int_b^{\bar{x}}\left\{x\left[(N-1)f_X(x)F_W(b)\left[F_X(b)+(F_X(x)-F_X(b))F_W(b)\right]^{N-2}\right.\right.\\
&\qquad\cdot\left(f_X(b)+(F_W(x)-F_X(b))f_W(b)-f_X(b)F_W(b)\right)\\
&\left.\left.+\ f_X(x)f_W(b)\left[F_X(b)+(F_X(x)-F_X(b))F_W(b)\right]^{N-1}\right]\right\}dx\\
&-N\left[\left(1-F_W(b)\right)f_X(b)+\left(1-F_X(b)\right)f_W(b)\right]\left[F_X(b)+F_W(b)-F_X(b)F_W(b)\right]^{N-1}b.
\end{aligned}
$$

Reorganizing the derivative above delivers the economic interpretation of each term:

$$
\begin{aligned}
&\frac{\partial}{\partial b}\Pi_I(b,s)\\
=&s\left(\underbrace{Nf_X(b)\left[F_X(b)\right]^{N-1}\left[1-F_W(b)\right]b}_{\text{Marginal gain if } X_{(1)}=Y=b}+\underbrace{\int_b^{\bar{x}}x\frac{\partial f_{X_{(1)}|Y<b}(x)}{\partial b}dx}_{\text{Marginal gain if } X_{(1)}>Y=b}\right)-\underbrace{N\left[F_Z(b)\right]^{N-1}f_Z(b)b}_{\text{Marginal cost}}.
\end{aligned}
$$

For any given s, the FONC requires that the optimal bid b^* satisfies $\frac{\partial}{\partial b}\Pi_I\left(b^*,s\right)=0$. With the assumption that (X_i,W_i)'s have uniform density on $(0,\bar{x})\times(0,\bar{w})$, the FONC simplifies into

$$
\begin{aligned}
&Ns\bar{w}^{N+1}-(N+1)s\bar{w}^Nb^*+N\left(\bar{x}+\bar{w}-b^*\right)^{N-1}\left[(N-s+1)b^*+s(\bar{w}-N\bar{x})\right]b^*\\
&-\left(\bar{x}+\bar{w}-b^*\right)^N\left[(N+1)(N-s)b^*+Ns(\bar{w}-N\bar{x})\right]=0.
\end{aligned}
\tag{2.20}
$$

According to Equation 2.20, if the group $\mathcal{O}$ bidder were to submit a sure-to-win bid based on his signal s, his bid would be $\bar{w}$ if $\bar{w}<\bar{x}$ and $\bar{x}$ if $\bar{x}\le\bar{w}$. Plugging $b^*=\bar{w}$ or $b^*=\bar{x}$ back into Equation 2.20, one gets:

$$s^*(\bar{x}, \bar{w}, N) = \begin{cases} 1 & \bar{x} \le \bar{w} \\ \dfrac{N(N+1)\bar{w}\bar{x}^{N-1}(\bar{x}-\bar{w})}{\bar{w}(\bar{x}^N - \bar{w}^N) + N^2\bar{x}^N(\bar{x}-\bar{w})} & \bar{w} < \bar{x} \end{cases}. \tag{2.21}$$

The case that $s^*(\bar{x}, \bar{w}, N) = 1$ when $\bar{x} \le \bar{w}$ is the same as before – when the probability of that the insiders are constrained by their budgets is low, the outsider will not submit a bid that guarantees a win unless his valuation exactly equals to that of the insider who values the object the highest. As for the case with $\bar{w} < \bar{x}$, it is not obvious from Equation 2.21 whether $s^*(\bar{x}, \bar{w}, N) < 1$. To make sure this is indeed the case, note that $\dfrac{\partial s^*(\bar{x}, \bar{w}, N)}{\partial \bar{w}} > 0$ for $\bar{w} \in [0, \bar{x})$[18] and $\lim_{\bar{w}\to\bar{x}} \dfrac{N(N+1)\bar{w}\bar{x}^{N-1}(\bar{x}-\bar{w})}{\bar{w}(\bar{x}^N - \bar{w}^N) + N^2\bar{x}^N(\bar{x}-\bar{w})} = 1$. Thus, $s^*(\bar{x}, \bar{w}, N) < 1$ for all $\bar{w} < \bar{x}$.

Theorem 9. ***(Group $\mathcal{O}$ bidder's Optimal Bidding Strategy with Valuation $V = X_{(1)} \cdot S$)***
When $\bar{w} < \bar{x}$ and $s > \dfrac{N(N+1)\bar{w}\bar{x}^{N-1}(\bar{x}-\bar{w})}{\bar{w}(\bar{x}^N - \bar{w}^N) + N^2\bar{x}^N(\bar{x}-\bar{w})}$, the group $\mathcal{O}$ bidder's optimal bid is $\beta_{(I)}(s) = \bar{w}$.

Otherwise, $\beta_{(I)}(s)$ is defined implicitly by:

$$\begin{aligned} & N s\bar{w}^{N+1} - (N+1)s\bar{w}^N \cdot \beta_{(I)}(s) \\ & + N\left(\bar{x} + \bar{w} - \beta_{(I)}(s)\right)^{N-1}\left[(N-s+1)\cdot\beta_{(I)}(s) + s(\bar{w} - N\bar{x})\right]\cdot\beta_{(I)}(s) \\ & - \left(\bar{x} + \bar{w} - \beta_{(I)}(s)\right)^{N}\left[(N+1)(N-s)\cdot\beta_{(I)}(s) + Ns(\bar{w} - N\bar{x})\right] = 0. \end{aligned} \tag{2.22}$$

A general solution to Equation 2.22 for all $N \ge 2$ is not obtainable. Figure 2.5a and 2.5b present the results obtained by solving Equation 2.22 implicitly for specific parameters. In Figure

[18]To be more precise, $\dfrac{\partial s^*(\bar{x}, \bar{w}, N)}{\partial \bar{w}} > 0$ for $\bar{w} \in \left[0, \dfrac{N\bar{x}}{N-1}\right]$.

2.5a, $\bar{w}$ is fixed as the same value as $\bar{x}$. It shows that as the number of group $\mathcal{I}$ bidders increases, the group $\mathcal{O}$ bidder bids more aggressively for the same signal S. This can be expected, as the winner's curse is alleviated by the assumption that he can resell the object to whoever values it the most. Figure 2.5b demonstrates the same effect. It shows how the group $\mathcal{O}$ bidder's optimal bid, holding his private signal fixed at $S = 0.8$, changes with $\bar{w}$ and the number of group $\mathcal{I}$ bidders. For any fixed number of group $\mathcal{I}$ bidders, the group $\mathcal{O}$ bidder bids more conservatively as $\bar{w}$ increases relative to $\bar{x}$ (after $\bar{w}$ surpasses a certain threshold). Again, this directly results from the group $\mathcal{O}$ bidder's effort to avoid the *winner's curse*. For any fixed $\bar{w}$, however, the group $\mathcal{O}$ bidder bids more aggressively as the number of group $\mathcal{I}$ bidders increases. Intuitively, $\max_{i=1,\dots,N}\left\{X_i\right\}$ increases with N faster than $\max_{i=1,\dots N}\left\{\min\{X_i, W_i\}\right\}$ does. As a result, when more group $\mathcal{I}$ bidders are present, by increasing his bid by a small amount, the group $\mathcal{O}$ bidder would get a higher expected valuation upon winning while the expected payment does not increase as much. Thus, it is optimal for the group $\mathcal{O}$ bidder to bid more aggressively to enhance his probability of winning, as the number of group $\mathcal{I}$ bidders increases. Another related observation is that, as $\bar{w}$ increases, the group $\mathcal{O}$ bidder does not shade his bid as much as he otherwise would when the number of group $\mathcal{I}$ bidders is large. In Figure 2.5b, the degree that the group $\mathcal{O}$ bidder shades his bid, as $\bar{w}$ increases, is to a lesser extend when $N = 10$ than when $N = 2$. Figure 2.5b also shows the influence of N on s^* – the threshold signal above which the group $\mathcal{O}$ bidder is willing to submit sure-to-win bids. As can be seen from Equation 2.21, when $\bar{w} < \bar{x}$, the group $\mathcal{O}$ bidder bids $\bar{w}$, which secures a win, when $s < s^*$. This corresponds to the linear portion of the graph. As can be seen from the graph, the turning points gets larger, in terms of $\bar{w}$, as N increases. In other words, when there are more group $\mathcal{I}$ bidders, the outsider is willing to stick to sure-to-win bids for higher $\bar{w}$. All the observations above support the intuition that the presence of more group $\mathcal{I}$ bidders help alleviate the winner's curse to the group $\mathcal{O}$ bidder.

(a) Group $\mathcal{O}$ Bidder's Optimal Bid (with $\bar{x} = 1$ and $\bar{w} = 1$)

(b) Group $\mathcal{O}$ Bidder's Optimal Bid (with $S = 0.8$ and $\bar{x} = 1$)

2.4.1.2. $V = X_{(2)} \cdot S$. In this case,

$$\mathbb{E}[v_{II}(X_1, ...X_N)|Y < b] = \mathbb{E}[X_{(2)}|Y < b].$$

Plugging $\mathbb{E}[v_{II}(X_1, ...X_N)|Y < b]$ and $\mathbb{E}[Y|Y < b]$ back into Equation 2.18 yields[19]

(2.23)

$$\begin{aligned}\Pi_{II}(b, s) =& F_Y(b) \cdot \left(\mathbb{E}[X_{(2)}|Y < b] \cdot s - \mathbb{E}[Y|Y < b]\right)\\ =& N(N-1)s \left\{ \int_0^b x \left[F_X(x)\right]^{N-2} \left(F_X(b) - F_X(x) + F_W(b)\left[1 - F_X(b)\right]\right) f_X(x)dx \right.\\ &\left. + \int_b^{\bar{x}} x \left[F_X(b) + F_W(b)\left(F_X(x) - F_X(b)\right)\right]^{N-2} \left[F_W(b)\right]^2 \left[1 - F_X(x)\right] f_X(x)dx \right\}\\ &- N \int_0^b z \left[F_Z(z)\right]^{N-1} f_Z(z)dz\end{aligned}$$

[19]See Appendix 4.5 for details.

Taking derivative with respect to b of Equation 2.23 delivers the economic interpretation of each term.

$$\frac{\partial}{\partial b}\Pi_{II}(b,s)|_{N=2}$$

$$=2s\Bigg\{ \underbrace{\left[1-F_X(b)\right] F_W(b)\left[1-F_W(b)\right] f_X(b)b}_{\text{Marginal gain if } X_{(2)}=Y=b} + \underbrace{2F_W(b)f_W(b)\int_b^{\bar{x}} x\left[1-F_X(x)\right] f_X(x)dx}_{\text{Marginal gain if } X_{(2)}>Y=b}$$

$$+\underbrace{\left[\left(1-F_W(b)\right) f_X(b)+\left(1-F_X(b)\right) f_W(b)\right]\int_0^b x f_X(x)dx}_{\text{Marginal gain if } X_{(2)}<Y=b}\Bigg\} - \underbrace{2F_Z(b)f_Z(b)b}_{\text{Marginal cost}}$$

and

$$\frac{\partial}{\partial b}\Pi_{II}(b,s)|_{N\geq 3} = sN(N-1)\Bigg\{ \underbrace{\left[1-F_X(b)\right] F_W(b)\left[1-F_W(b)\right]\left[F_X(b)\right]^{N-2} f_X(b)b}_{\text{Marginal gain if } X_{(2)}=Y=b}$$

$$+\underbrace{\left[\left(1-F_W(b)\right) f_X(b)+\left(1-F_X(b)\right) f_W(b)\right]\int_0^b x\left[F_X(x)\right]^{N-2} f_X(x)dx}_{\text{Marginal gain if } X_{(2)}<Y=b}$$

$$+\underbrace{2F_W(b)f_W(b)\int_b^{\bar{x}} x\left[F_X(b)+F_W(b)\left(F_X(x)-F_X(b)\right)\right]^{N-2}\left[1-F_X(x)\right] f_X(x)dx}_{\text{Marginal gain if } X_{(2)}>Y=b}$$

$$+\underbrace{(N-2)\left[F_W(b)\right]^2\int_b^{\bar{x}} x\left[F_X(b)+F_W(b)\left(F_X(x)-F_X(b)\right)\right]^{N-3}\left[f_W(b)\left(F_X(x)-F_X(b)\right)\right.}_{\text{Marginal gain if } X_{(2)}>Y=b}$$

$$\left.+\underbrace{f_X(b)\left(1-F_W(b)\right)\right]\left[1-F_X(x)\right] f_X(x)dx}_{\text{Marginal gain if } X_{(2)}>Y=b}\Bigg\} - \underbrace{N\left[F_Z(b)\right]^{N-1} f_Z(b)b}_{\text{Marginal cost}}.$$

For any given s, the FONC requires that the optimal bid b^* satisfies $\frac{\partial}{\partial b}\Pi_{II}(b^*, s) = 0$. With the assumption that (X_i, W_i)'s have uniform density on $(0, \bar{x})\times(0, \bar{w})$, the FONC simplifies into[20]

$$\begin{aligned}
&(\bar{x} + \bar{w} - b^*)^{N-1}\Big[N^2(\bar{x} + \bar{w} - 2(b^*))(\bar{x}s - (b^*)) + 2s(\bar{x} + \bar{w} - (b^*))(b^*) \\
&-N\big[(1-2s)(b^*) + (2\bar{w} + \bar{x})s\big](\bar{x} + \bar{w} - 2(b^*))\Big] + s\bar{w}^{N-1}\Big[(N+1)(N+2)(b^*)^2 \\
&-(N+1)\big[(N+1)\bar{x} + (N+3)\bar{w}\big](b^*) + N\big[(N+1)\bar{x} + 2\bar{w}\big]\bar{w}\Big] = 0.
\end{aligned} \tag{2.24}$$

According to Equation 2.24, if the group $\mathcal{O}$ bidder were to submit a sure-to-win bid based on his signal s, his bid would be $\bar{w}$ if $\bar{w} < \bar{x}$ and $\bar{x}$ if $\bar{x} \leq \bar{w}$. Recall that in case the group $\mathcal{O}$ bidder's valuation is $V = X_{(1)} \cdot S$, which is discussed in the previous section, the threshold signal is quite similar to that in the single group $\mathcal{I}$ bidder case. Namely, when $\bar{x} \leq \bar{w}$, the group $\mathcal{O}$ bidder only submits a sure-to-win bid when $S = 1$, whereas when $\bar{w} < \bar{x}$, there is a threshold signal $s^*(\bar{x}, \bar{w}, N) \in (0, 1)$ above which a sure-to-win bid is optimal. Moreover, $\lim_{\bar{w}\to\bar{x}} s^*(\bar{x}, \bar{w}, N) = 1$. With $V = X_{(2)} \cdot S$, the results are different. First, note that when $\bar{x} \leq \bar{w}$, a sure-to-win bid requires $b = \bar{x}$. Using Equation 2.24, $b = \bar{x}$ requires $s^*(\bar{x}, \bar{w}, N) = \frac{N}{N-1} > 1$[21], which contradicts with the assumption of $s \in [0, 1]$. Therefore, when $V = X_{(2)} \cdot S$, the group $\mathcal{O}$ bidder never submits a bid large enough to guarantee a win when $\bar{x} \leq \bar{w}$. When $\bar{w} < \bar{x}$, a sure-to-win bid requires $b = \bar{w}$. Recall that previously with $V = X_{(1)} \cdot S$, when the group $\mathcal{I}$ bidders are more likely to be budget-constrained, as in the case with $\bar{w} < \bar{x}$, there exists $s^*(\bar{x}, \bar{w}, N) < 1$ such that the group $\mathcal{O}$ bidder bids $b = \bar{w}$ as long as $S \geq s^*(\bar{x}, \bar{w}, N)$. Here, plug $b = \bar{w}$ into

[20]The case with $N = 2$ also fits this equation.

[21]Obtained by plugging $b = \bar{x}$ into Equation 2.24 and simplify.

Equation 2.24 and one gets:

$$s^*(\bar{x}, \bar{w}, N) = \frac{N(N+1)\bar{w}\,(\bar{w}-\bar{x})\,\bar{x}^{N-1}}{(N+1)\left[\bar{w}^N\bar{x} + (N-2)\bar{w}\bar{x}^N\right] - (N-1)\left(\bar{w}^{N+1} + N\bar{x}^{N+1}\right)}. \tag{2.25}$$

It can be shown that $s^*(\bar{x}, \bar{w}, N)$ decreases as N increases, and $s^*(\bar{x}, \bar{w}, N) \to \dfrac{\bar{w}}{\bar{x}}$ as $N \to \infty$. Furthermore, when comparing $s^*\left(\bar{x}, \bar{w}, N\right)_{|V=X_{(1)}\cdot S}$ given by Equation 2.21 to $s^*\left(\bar{x}, \bar{w}, N\right)_{|V=X_{(2)}\cdot S}$ given by Equation 2.25, it can be shown that $s^*\left(\bar{x}, \bar{w}, N\right)_{|V=X_{(2)}\cdot S} - s^*\left(\bar{x}, \bar{w}, N\right)_{|V=X_{(1)}\cdot S} > 0$ for $\bar{w} < \bar{x}$ and $\lim_{N\to\infty}\left(s^*\left(\bar{x}, \bar{w}, N\right)_{|V=X_{(2)}\cdot S} - s^*\left(\bar{x}, \bar{w}, N\right)_{|V=X_{(1)}\cdot S}\right) = 0$. All are intuitive, since $V = X_{(1)} \cdot S$ promises higher valuation for the group $\mathcal{O}$ bidder than $V = X_{(2)} \cdot S$, which leads to the group $\mathcal{O}$ bidder more willing to submit a sure-to-win bid. This difference vanishes as $N \to \infty$.

Theorem 10. ***(Group $\mathcal{O}$ bidder's Optimal Bidding Strategy with Valuation $V = X_{(2)} \cdot S$)***

When $\bar{w} < \bar{x}$ and $s > \dfrac{(N+1)N(\bar{x}-\bar{w})\bar{x}^{N-1}\bar{w}}{(N-1)\bar{w}^{N+1} - (N+1)\bar{x}\bar{w}^N + N(N-1)(\bar{x}-\bar{w})\bar{x}^N + 2\bar{x}^N\bar{w}}$,[22] the group $\mathcal{O}$ bidder's optimal bid is $\beta_{(II)}(s) = \bar{w}$.

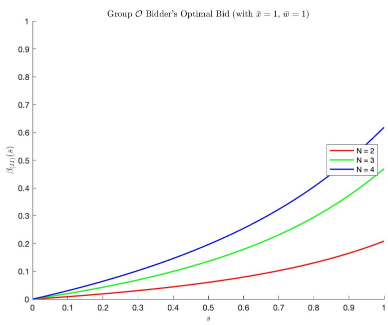
Otherwise, $\beta_{(II)}(s)$ is defined implicitly by:

(2.26)

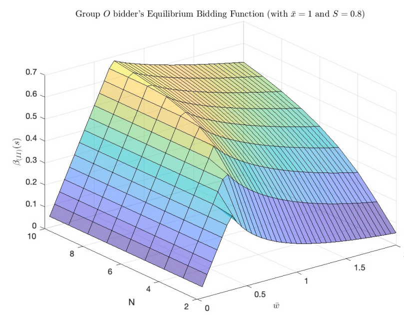
$$\begin{aligned}
&(\bar{x} + \bar{w} - \beta_{(II)}(s))^{N-1}\Bigg[N^2(\bar{x} + \bar{w} - 2\cdot\beta_{(II)}(s))(\bar{x}s - \beta_{(II)}(s)) + 2s(\bar{x} + \bar{w} - \beta_{(II)}(s))\cdot\beta_{(II)}(s) \\
&- N\left[(1-2s)\cdot\beta_{(II)}(s) + (2\bar{w} + \bar{x})s\right](\bar{x} + \bar{w} - 2\cdot\beta_{(II)}(s))\Bigg] + s\bar{w}^{N-1}\Bigg[(N+1)(N+2)\left[\beta_{(II)}(s)\right]^2 \\
&- (N+1)\left[(N+1)\bar{x} + (N+3)\bar{w}\right]\cdot\beta_{(II)}(s) + N\left[(N+1)\bar{x} + 2\bar{w}\right]\bar{w}\Bigg] = 0.
\end{aligned}$$

[22]Note that $\dfrac{(N+1)N(\bar{x}-\bar{w})\bar{x}^{N-1}\bar{w}}{(N-1)\bar{w}^{N+1} - (N+1)\bar{x}\bar{w}^N + N(N-1)(\bar{x}-\bar{w})\bar{x}^N + 2\bar{x}^N\bar{w}} < 1$ does not always hold.

A general solution to Equation 2.26 for all $N \geq 2$ is not obtainable. Figure 2.6a and 2.6b present the results obtained by solving Equation 2.26 implicitly for specific parameters. In Figure 2.6a, $\bar{w}$ is fixed as the same value as $\bar{x}$. It shows that as the number of group $\mathcal{I}$ bidders increases, the group $\mathcal{O}$ bidder bids more aggressively for the same signal S. Similar to the previous case, this can be expected, as the second-highest valuation among the group $\mathcal{O}$ bidders increases with the number of such bidders. Figure 2.6b demonstrates the same effect. It shows how the group $\mathcal{O}$ bidder's optimal bid, holding his private signal fixed at $S = 0.8$, changes with $\bar{w}$ and the number of group $\mathcal{I}$ bidders. For any fixed number of group $\mathcal{I}$ bidders, the group $\mathcal{O}$ bidder bids more conservatively as $\bar{w}$ increases relative to $\bar{x}$ (after $\bar{w}$ surpasses a certain threshold). Again, this directly results from the group $\mathcal{O}$ bidder's effort to avoid the *winner's curse*. For any fixed $\bar{w}$, however, the group $\mathcal{O}$ bidder bids more aggressively as the number of group $\mathcal{I}$ bidders increases. This trend is more significant than the setting where $V = X_{(1)} \cdot S$. This reflects the fact that when the group $\mathcal{O}$ bidder relies on future auction to resell the asset, an adequate level of competition in that future auction is crucial for the group $\mathcal{O}$ bidder's willingness to bid in the current auction. Another related observation is that, as $\bar{w}$ increases, the group $\mathcal{O}$ bidder does not shade his bid as much as he otherwise would when the number of group $\mathcal{I}$ bidders is large. In Figure 2.6b, the degree that the group $\mathcal{O}$ bidder shades his bid, as $\bar{w}$ increases, is to a lesser extend when $N = 10$ than when $N = 2$. Figure 2.6a also shows the influence of N on s^* – the threshold signal above which the group $\mathcal{O}$ bidder is willing to submit sure-to-win bids. As can be seen from the graph, the turning points gets larger, in terms of $\bar{w}$, as N increases. In other words, when there are more group $\mathcal{I}$ bidders, the outsider is willing to stick to sure-to-win bids for higher $\bar{w}$. All conclusions are similar to those from the previous case of $V = X_{(1)} \cdot S$, as the natural of the two cases are the same, especially when N is large.



(a) Group $\mathcal{O}$ Bidder's Optimal Bid (with $\bar{x} = 1$ and $\bar{w} = 1$)



(b) Group $\mathcal{O}$ Bidder's Optimal Bid (with $S = 0.8$ and $\bar{x} = 1$)

2.4.1.3. $V = \frac{1}{N}\sum_{i=1}^{N} X_i \cdot S$**.** In this case,

$$\mathbb{E}[v_{III}(X_1, ...X_N)|Y < b] = \mathbb{E}[\frac{1}{N}\sum_{i=1}^{N} X_i|Y < b] = \frac{1}{N}\sum_{i=1}^{N}\mathbb{E}[X_i|Y < b] = \mathbb{E}[X_1|Y < b].$$

Plugging $\mathbb{E}[v_{III}(X_1, ...X_N)|Y < b]$ and $\mathbb{E}[Y|Y < b]$ back into Equation 2.18 yields

$$\begin{aligned} \Pi_{III}(b, s) =& F_Y(b) \cdot \left(\mathbb{E}[\frac{1}{N}\sum_{i=1}^{N} X_i|Y < b] \cdot s - \mathbb{E}[Y|Y < b]\right) \\ =& s\left[F_Z(b)\right]^{N-1}\int_0^b x f_X(x)dx + sF_W(b)\left[F_Z(b)\right]^{N-1}\int_b^{\bar{x}} x f_X(x)dx \\ &- N\int_0^b z\left[F_Z(z)\right]^{N-1} f_Z(z)dz. \end{aligned} \tag{2.27}$$

Taking derivative with respect to b of Equation 2.27:

$$\frac{\partial}{\partial b}\Pi_{III}(b,s) = \underbrace{s\left[1-F_W(b)\right]\left[F_Z(b)\right]^{N-1} f_X(b)b}_{\text{Marginal gain if } X_1 = Y = b} + \underbrace{(N-1)s\left[F_Z(b)\right]^{N-2} f_Z(b)\int_0^b x f_X(x)dx}_{\text{Marginal gain if } X_1 < Y = b}$$

$$+\underbrace{(N-1)sF_W(b)\left[F_Z(b)\right]^{N-2} f_Z(b)\int_b^{\bar{x}} x f_X(x)dx + s f_W(b)\left[F_Z(b)\right]^{N-1}\int_b^{\bar{x}} x f_X(x)dx}_{\text{Marginal gain if } X_1 > Y = b}$$

$$-\underbrace{N\left[F_Z(b)\right]^{N-1} f_Z(b)b}_{\text{Marginal cost}}.$$

For any given s, the FONC requires that the optimal bid b^* satisfies $\frac{\partial}{\partial b}\Pi_{III}(b^*, s) = 0$. With the assumption that (X_i, W_i)'s have uniform density on $(0,\bar{x})\times(0,\bar{w})$, the FONC simplifies into:[23]

$$\begin{aligned}(2.28)\quad &\left[2N(s-2)+s\right](b^*)^3 + \left[N\left[6\bar{x}+6\bar{w}-(\bar{x}+3\bar{w})s\right]-2(\bar{x}+\bar{w})s\right](b^*)^2\\ &+\left\{N\left[(2\bar{x}+\bar{w})(\bar{w}-\bar{x})s-2(\bar{x}+\bar{w})^2\right]+(\bar{x}^2+\bar{x}\bar{w}+\bar{w}^2)s\right\}b^* + N(\bar{x}+\bar{w})\bar{x}^2 s = 0.\end{aligned}$$

The same as before, when $\bar{x} \le \bar{w}$, a sure-to-win bid from the group $\mathcal{O}$ bidder is $\bar{x}$, whereas $\bar{w} < \bar{x}$ implies a sure-to-win bid of $\bar{w}$. Using Equation 2.28, $b^* = \bar{x}$ can only result from $s^*(\bar{x},\bar{w},N) = \frac{2N}{N+1}$. Since $\frac{2N}{N+1} > 1$ for $N \ge 2$, the group $\mathcal{O}$ bidder never submits a bid that guarantees a win in case $\bar{x} \le \bar{w}$. When $\bar{w} < \bar{x}$, setting $b^* = \bar{w}$ in Equation 2.28 leads to $s^*(\bar{x},\bar{w},N) = \frac{2N\bar{w}}{N\bar{x}+\bar{w}}$. Unlike the previous two cases in which $s^*(\bar{x},\bar{w},N)$ decreases as N increases, here, $s^*(\bar{x},\bar{w},N) = \frac{2N\bar{w}}{N\bar{x}+\bar{w}}$ increases with N. In other words, when more group $\mathcal{I}$ bidders are present, the group $\mathcal{O}$ bidder needs a higher private signal to be willing to secure a win in the auction. This is because when the group $\mathcal{O}$ bidder's valuation depends on the average of those of the group $\mathcal{I}$ bidders, the group $\mathcal{O}$ bidder's valuation does not benefit from an increase

[23]See Appendix 4.6 for details.

in the number group $\mathcal{I}$ bidders, since $\lim_{N\to\infty} \frac{1}{N}\sum_{i=1}^{N} X_i = \mathbb{E}[X]$ is constant. However, the group $\mathcal{O}$ bidder's expected payment $\mathbb{E}\left[\max_{i=1,\ldots,N}\left\{\min\left(X_i, W_i\right)\right\}\right]$ does increase as N gets bigger.

Theorem 11. ***(Group $\mathcal{O}$ bidder's Optimal Bidding Strategy with Valuation $V = \frac{1}{N}\sum_{i=1}^{N} X_i \cdot S$)***

When $\bar{w} < \bar{x}$ and $s > \dfrac{2N\bar{w}}{N\bar{x}+\bar{w}}$,[24] the group $\mathcal{O}$ bidder's optimal bid is $\beta_{(III)}(s) = \bar{w}$.

Otherwise, $\beta_{(III)}(s)$ is defined implicitly by:

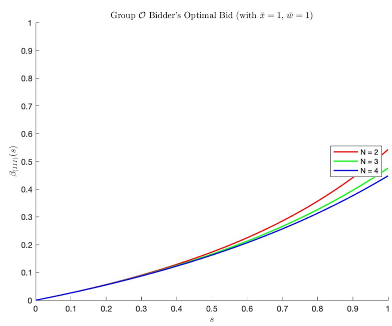
(2.29)

$$\left[2N(s-2)+s\right]\left[\beta_{(III)}(s)\right]^3 + \left[N\left[6\bar{x}+6\bar{w}-(\bar{x}+3\bar{w})s\right]-2(\bar{x}+\bar{w})s\right]\left[\beta_{(III)}(s)\right]^2$$
$$+\left\{N\left[(2\bar{x}+\bar{w})(\bar{w}-\bar{x})s-2(\bar{x}+\bar{w})^2\right]+(\bar{x}^2+\bar{x}\bar{w}+\bar{w}^2)s\right\}\beta_{(III)}(s)+N(\bar{x}+\bar{w})\bar{x}^2 s = 0.$$

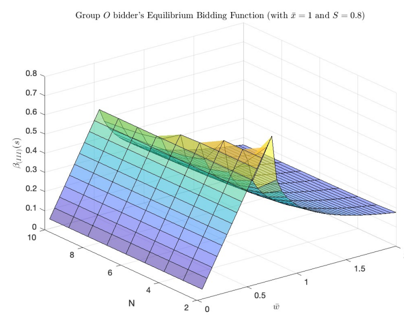
Unlike the previous two cases, Equation 2.29 can be solved explicitly, as the highest order on $\beta_{(III)}(s)$ does not depend on N. However, the solution to Equation 2.29, by itself, is of no extra help in providing additional insight. Therefore, I choose to use graphs for better illustration. Figure 2.7a and 2.7b present the results obtained by solving Equation 2.29 implicitly for specific parameters. In Figure 2.7a, $\bar{w}$ is fixed as the same value as $\bar{x}$. In contrast to the previous two cases in which the group $\mathcal{O}$ bidder bids more aggressively when the number of group $\mathcal{I}$ bidders increases, the group $\mathcal{O}$ bidder actually becomes more conservative with his bid in this case. This is because with $V = \frac{1}{N}\sum_{i=1}^{N} X_i \cdot S$, the group $\mathcal{O}$ bidder's expected valuation does not increase with the number of group $\mathcal{I}$ bidders, while his expected payment does. Therefore, a large number of group $\mathcal{I}$ bidders participating in the auction is not good news for the group $\mathcal{O}$ bidder. This point if further illustrated in Figure 2.7b. Figure 2.7b demonstrates how the group $\mathcal{O}$ bidder's

[24]Note that $\dfrac{2N\bar{w}}{N\bar{x}+\bar{w}} < 1$ does not always hold.

optimal bid, holding his private signal fixed at $S = 0.8$, changes with $\bar{w}$ and the number of group I bidders. Again, for any fixed number of group $\mathcal{I}$ bidders, the group $\mathcal{O}$ bidder bids more conservatively as $\bar{w}$ increases relative to $\bar{x}$ (after $\bar{w}$ surpasses a certain threshold), which directly results from the group $\mathcal{O}$ bidder's effort to avoid the *winner's curse*. Unlike the other two settings with $V = X_{(1)} \cdot S$ or $V = X_{(2)} \cdot S$, for any fixed $\bar{w}$, the group $\mathcal{O}$ bidder does not bid more aggressively as the number of group I bidders increases.



(a) Group $\mathcal{O}$ Bidder's Optimal Bid (with $\bar{x} = 1$ and $\bar{w} = 1$)



(b) Group $\mathcal{O}$ Bidder's Optimal Bid (with $S = 0.8$ and $\bar{x} = 1$)

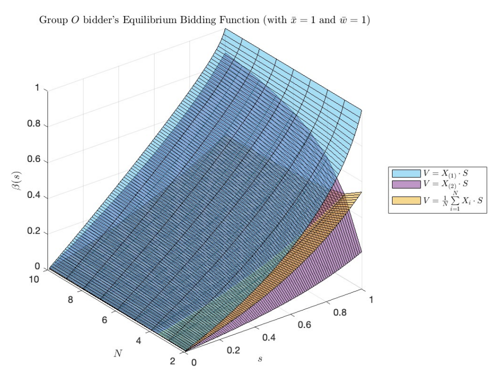


Figure 2.8. Group $\mathcal{O}$ Bidder's Optimal Bid (with $\bar{x} = 1$ and $\bar{w} = 1$)

Figure 2.8 further demonstrates how different assumptions regarding the group $\mathcal{O}$ bidder's valuation influence his equilibrium bidding strategies (with $\bar{w} = \bar{x} = 1$). Not surprisingly, for any given N and S, the group $\mathcal{O}$ bidder always bids higher when his valuation is given by $V = X_{(1)} \cdot S$. For any fixed S, his bid increases dramatically with N when his valuation is given by $V = X_{(2)} \cdot S$. Although this is also true for the case of $V = X_{(1)} \cdot S$, the difference in the magnitudes of this trend is quite notable. One possible explanation is as follows: when $V = X_{(1)} \cdot S$, the group $\mathcal{O}$ bidder engages in negotiation with the group $\mathcal{I}$ bidder with the highest valuation. When $V = X_{(2)} \cdot S$, the resell stage can be viewed as another auction. The number of group $\mathcal{I}$ bidders has a much larger impact in the later case, via increased competition. But the situation in $V = \frac{1}{N}\sum_{i=1}^{N} X_i \cdot S$ is just the opposite – an increase in the number of group $\mathcal{I}$ bidders does not contribute to the expected average valuation, while it does increase the expected highest bid. Therefore, the different behavior of the outsider under different valuation assumptions reflects the fact that even thought the increase in N has the same effect on the outsider's expected payment in case of winning, it has very different effect on the outsider's valuation.

2.4.2. Efficiency

Let $m \equiv \text{argmax}_{i=1,\ldots,N}\{X_i\}$, and define efficiency as the probability that bidder m wins the auction. In other words, define the auction efficiency as the probability that the object is allocated to the bidder who holds the highest valuation for it. For bidder m to win, two conditions must be satisfies.

First, all other group I bidders must bid less than $Z_m \equiv \min\{X_m, W_m\}$, that is, $Z_i < Z_m \, \forall\, i \in \{1, \ldots, N\} \setminus \{m\}$. When $W_m \geq X_m$, it follows that $Z_m = X_m$. Since by definition, X_m is the largest among all the X_i's, and $Z_i = \min\{X_i, W_i\} \leq X_i < X_m$, the condition $Z_i < Z_m \, \forall\, i \neq m$ is automatically satisfied. When $W_m < X_m$, it follows that $Z_m = W_m$. Bidder i ($i \neq m$) bids less than W_m either because $X_i < W_m$ or because $W_m \leq X_i < X_m$ but $W_i < W_m$. Therefore,

$$
\begin{aligned}
&\mathbb{P}\{(Z_i < Z_m) \cap (X_i < X_m) \ \forall\, i \neq m\} \\
=&\mathbb{I}\{W_m \geq X_m\} \cdot \left(\underbrace{\mathbb{P}\{Z_i < Z_m \mid X_i < X_m\}}_{=1} \cdot \underbrace{\mathbb{P}\{X_i < X_m\}}_{=F_X(X_m)} \right)^{N-1} \\
&+ \mathbb{I}\{W_m < X_m\} \cdot \left(\underbrace{\mathbb{P}\{Z_i < Z_m \mid X_i < X_m\}}_{=\frac{F_X(W_m)+[F_W(X_m)-F_X(W_m)]\cdot F_W(W_m)}{F_X(X_m)}} \cdot \underbrace{\mathbb{P}\{X_i < X_m\}}_{=F_X(X_m)} \right)^{N-1} \\
=&\mathbb{I}\{W_m \geq X_m\} \cdot \left[F_X(X_m)\right]^{N-1} \\
&+ \mathbb{I}\{W_m < X_m\} \cdot \left\{F_X(W_m) + [F_W(X_m) - F_X(W_m)] \cdot F_W(W_m)\right\}^{N-1}.
\end{aligned}
$$

The other condition necessary for the allocation to be efficient is that the group $\mathcal{O}$ bidder must also bid less than Z_m. Denote the group $\mathcal{O}$ bidder's equilibrium bidding function as $\beta(s)$, with $S = s$ being the realization of his private signal. Note that although when $\bar{w}$ is small, the group $\mathcal{O}$ bidder would submit "sure to win" bid as long as his signal is above a threshold $s^*(\bar{x}, \bar{w}, N)$, which makes the bidding function not invertible on the region $s \in [s^*(\bar{x}, \bar{w}, N), 1]$, it does not matter for the bidding efficiency analysis, since bidder m loses for sure in this case. Therefore, one can simply focus on the domain of S on which $\beta(\cdot)$ is invertible. Denote its inverse function by $\beta^{-1}(\cdot)$. Note that optimality in the relevant domain requires

$$\frac{\partial}{\partial b}\Pi(b, s) \mid_{s=\beta^{-1}(b)} = 0.$$

The probability that the group $\mathcal{O}$ bidder bids less than Z_m is simply

$$\mathbb{P}\{\beta(S) < Z_m\} = \mathbb{P}\{S < \beta^{-1}(Z_m)\} = \beta^{-1}(Z_m),$$

where the second equality builds on the assumption that $S \sim U[0,1]$.

Combining the above analysis together, the *ex ante* expected allocation efficiency is thus

(2.30)

$$\begin{aligned}
\text{Efficienty}^{ex\ ante} =& \mathbb{I}\{\bar{x} \geq \bar{w}\} \cdot \left(N \int_0^{\bar{w}} \left\{ \int_0^{w} \left[F_X(x)\right]^{N-1} \beta^{-1}(x) dF_X(x) \right.\right. \\
&+ \left.\left. \int_w^{\bar{x}} \left\{F_X(w) + \left[F_X(x) - F_X(w)\right] F_W(w)\right\}^{N-1} \beta^{-1}(w) dF_X(x) \right\} dF_W(w) \right) \\
&+ \mathbb{I}\{\bar{x} < \bar{w}\} \cdot \left(N \int_0^{\bar{x}} \left\{ \int_x^{\bar{w}} \left[F_X(x)\right]^{N-1} \beta^{-1}(x) dF_W(w) \right.\right. \\
&+ \left.\left. \int_0^{x} \left\{F_X(w) + \left[F_X(x) - F_X(w)\right] F_W(w)\right\}^{N-1} \beta^{-1}(w) dF_W(w) \right\} dF_X(x) \right).
\end{aligned}$$

Next, I discuss the allocation efficiency case-by-case, based on the different assumptions on the group $\mathcal{O}$ bidder's valuation.

2.4.2.1. $V = X_{(1)} \cdot S$**.** Substituting $s = \beta_{(I)}^{-1}(b)$ into Equation 2.20 and solving for $\beta_{(I)}^{-1}(b)$ yields:

$$\begin{aligned}
\beta_{(I)}^{-1}(b) = & \left(N(N+1)(\bar{x} + \bar{w} - 2b)(\bar{x} + \bar{w} - b)^{N-1} b\right) \\
& \cdot \left([(N+1)b + N^2\bar{x} - N\bar{w}](\bar{x} + \bar{w} - b)^N - (N+1)\bar{w}^N b \right. \\
& \left. + N\bar{w}^{N+1} - N(N\bar{x} - \bar{w} + b)(\bar{x} + \bar{w} - b)^{N-1} b\right)^{-1}.
\end{aligned}$$

Substituting $\beta_{(I)}^{-1}(\cdot)$ for $\beta^{-1}(\cdot)$ in Equation 2.30 gives the *ex ante* allocation efficiency when the group $\mathcal{O}$ bidder's value function is $V = X_{(1)} \cdot S$.

Figure 2.9 illustrates how the allocation efficiency changes with $\bar{w}$ and N (with $\bar{x} = 1$). Similar to the $N = 1$ case discussed in the previous section, the allocation efficiency increases with $\bar{w}$ for any fixed N. However, for any given $\bar{w}$, it becomes less likely for the highest-valuation bidder to win when the number of group $\mathcal{I}$ competitors increases. Expectedly, when $\bar{w}$ is large relative to $\bar{x}$, the probability that the highest valuation bidder wins the auction does not decrease much as N increases. This is intuitive because in this case, the auction is similar to a standard SPSB auction without budget constraint, which is always allocation the object most efficiently.

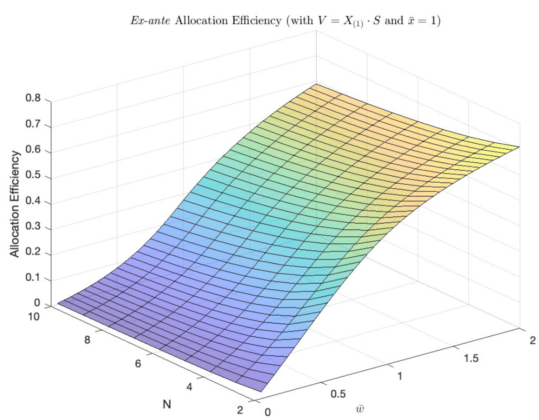


Figure 2.9. Highest Valuation Bidder's Winning Probability (When $V = X_{(1)} \cdot S$)

2.4.2.2. $V = X_{(2)} \cdot S$**.** Substituting $s = \beta^{-1}_{(II)}(b)$ into Equation 2.24 and solving for $\beta^{-1}_{(II)}(b)$ yields:

$$\beta^{-1}_{(II)}(b) = \frac{A}{B},$$

with

$$A = N(N+1)\left(\bar{w} + \bar{x} - 2b\right)\left(\bar{w} + \bar{x} - b\right)^{N} b$$

and

$$B = (N+2)(N+1)\bar{w}^N b^3$$
$$+ \left\{2(2N+1)\bar{w}\,(\bar{w}+\bar{x}-b)^N - (N+1)\bar{w}\,[(2N+5)\bar{w}+(2N+3)\bar{x}]\right\} b^2$$
$$+ \bar{w}^N \left[(N^2+6N+3)\bar{w}^2 + (3N+4)(N+1)\bar{w}\bar{x} + (N+1)^2\bar{x}^2\right] b$$
$$+ N\bar{w}\,(\bar{w}+\bar{x})\left\{(\bar{w}+\bar{x}-b)^N\,[2\bar{w}-(N-1)\bar{x}] - \bar{w}^N\,[2\bar{w}+(N+1)\bar{x}]\right\}.$$

Substituting $\beta^{-1}_{(II)}(\cdot)$ for $\beta^{-1}(\cdot)$ in Equation 2.30 gives the *ex ante* allocation efficiency when the group $\mathcal{O}$ bidder's value function is $V = X_{(2)} \cdot S$.

Figure 2.10 illustrates how the allocation efficiency changes with $\bar{w}$ and N (with $\bar{x} = 1$). Similar to the case of $V = X_{(1)} \cdot S$, the allocation efficiency increases with $\bar{w}$ for any fixed N. And, for any given $\bar{w}$, it becomes less likely for the highest-valuation bidder to win when the number of group $\mathcal{I}$ competitors increases.

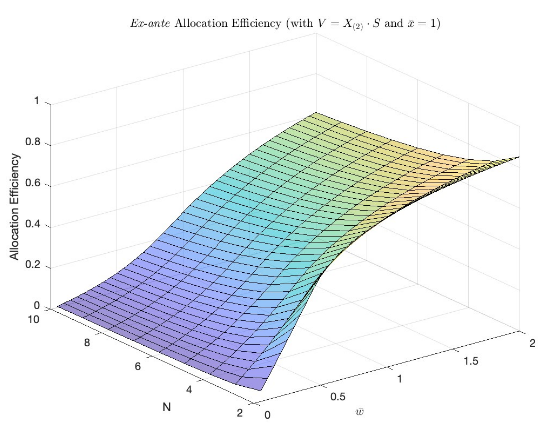


Figure 2.10. Highest Valuation Bidder's Winning Probability (When $V = X_{(2)} \cdot S$)

2.4.2.3. $V = \frac{1}{N}\sum_{i=1}^{N} X_i \cdot S$**.** Substituting $s = \beta_{(III)}^{-1}(b)$ into Equation 2.28 and solving for $\beta_{(III)}^{-1}(b)$ yields:

$$\beta_{(III)}^{-1}(b) =$$

$$\frac{2N(\bar{x}+\bar{w}-b)(\bar{x}+\bar{w}-2b)b}{(2N+1)b^3 - [N(\bar{x}+3\bar{w}) + 2\bar{x} + 2\bar{w}]b^2 + [N(2\bar{x}+\bar{w})(\bar{w}-\bar{x}) + \bar{w}^2 + \bar{x}\bar{w} + \bar{x}^2]b + N\bar{x}^2(\bar{x}+\bar{w})}.$$

Substituting $\beta_{(III)}^{-1}(\cdot)$ for $\beta^{-1}(\cdot)$ in Equation 2.30 gives the *ex ante* allocation efficiency when the group $\mathcal{O}$ bidder's value function is $V = \frac{1}{N}\sum_{i=1}^{N} X_i \cdot S$.

Figure 2.11 illustrates how the allocation efficiency changes with $\bar{w}$ and N (with $\bar{x} = 1$). Similar to the other two cases, the allocation efficiency increases with $\bar{w}$ for any fixed N. And, for any given $\bar{w}$, it becomes less likely for the highest-valuation bidder to win when the number of group $\mathcal{I}$ competitors increases.

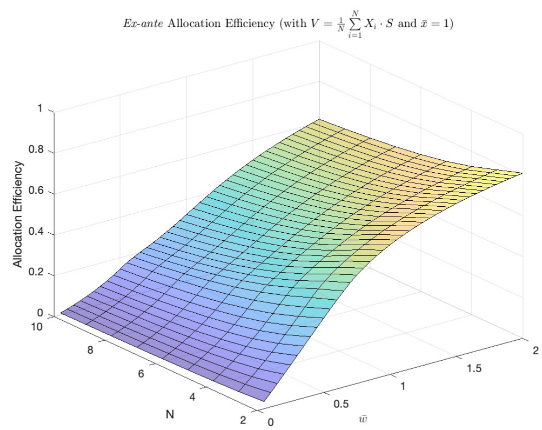


Figure 2.11. Highest Valuation Bidder's Winning Probability (When $V = \frac{1}{N}\sum_{i=1}^{N} X_i \cdot S$)

Figure 2.12 illustrates how the highest-valuation bidder's probability of winning in the presence of one speculator compares to that without speculators.[25] Consistent with the analysis

[25]The efficiency calculation without the group $\mathcal{O}$ bidder can be found in Appendix 4.7.

above, when N and $\bar{w}$ are both large, the presence of the group $\mathcal{O}$ bidder barely influences the allocation efficiency.

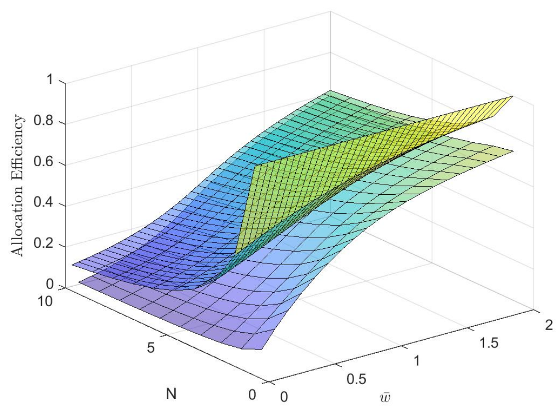


Figure 2.12. Efficiency Comparison With and Without the Group $\mathcal{O}$ Bidder ($\bar{x} = 1$)

A natural question to ask at this point is how does the probability that ANY group $\mathcal{I}$ bidder wins the object changes with N. On one hand, it seems intuitive that the probability that the object is allocated to a group $\mathcal{I}$ bidder should increase when more group $\mathcal{I}$ bidder participate in the auction. On the other hand, however, the group $\mathcal{O}$ bidder gets less concerned about the *winner's curse* and thus bids more aggressively as the number of group $\mathcal{I}$ bidder increases, when his valuation is given by $V = X_{(1)} \cdot S$ or $V = X_{(2)} \cdot S$. The following analysis shows that the first effect outweighs the second, and the probability that the object is allocated to a group I bidder increases with N.

Consider the scenario that the group $\mathcal{O}$ bidder submits a bid $b > 0$. For him to win, all the group $\mathcal{I}$ bidders must bid less than b, that is, $Z_i = \min\{X_i, W_i\} < b \; \forall \; i \in \{1, ..., N\}$. Mathematically,

$$\mathbb{P}\{\text{group } \mathcal{O} \text{ bidder wins by bidding } b > 0\} = \mathbb{P}\{Z_i < b \; \forall \; i \in \{1, ..., N\}\} = \left[F_Z(b)\right]^N .$$

In the *ex ante* sense, the group $\mathcal{O}$ bidder's probability of winning is

$$\mathbb{P}\{\text{group } \mathcal{O} \text{ bidder wins}\} = \int_0^1 \left[F_Z(\beta(s))\right]^N ds.$$

Consequently, the probability that the object is allocated to a group $\mathcal{I}$ bidder is simply

$$\mathbb{P}\{\text{group } \mathcal{I} \text{ bidder wins}\} = 1 - \mathbb{P}\{\text{group } \mathcal{O} \text{ bidder wins}\} = 1 - \int_0^1 \left[F_Z(\beta(s))\right]^N ds.$$

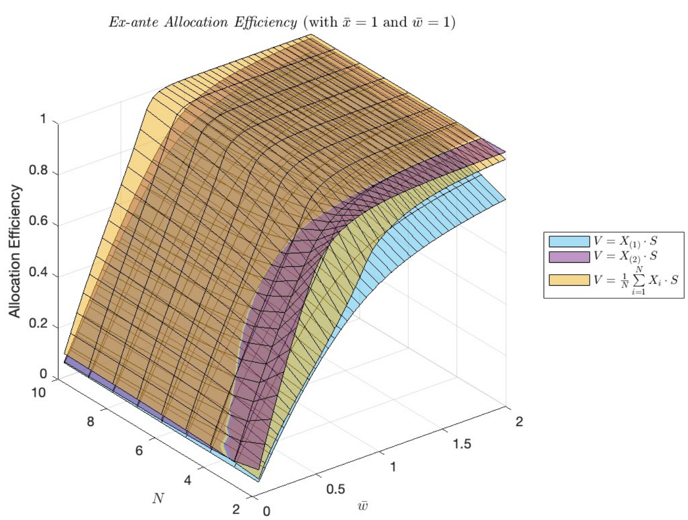


Figure 2.13. Group $\mathcal{I}$ Bidder's Winning Probability

Figure 2.13 illustrates how the probability that the object is allocated to a group $\mathcal{I}$ bidder varies with $\bar{w}$ and N while holding $\bar{x} = 1$ fixed. In general, the probability that ANY group $\mathcal{I}$ bidder wins the auction indeed increases with the number of group $\mathcal{I}$ bidders. However, this effect is much less significant compared with that caused by an increase in $\bar{w}$. Intuitively, this is because the effect of increasing $\bar{w}$ has two folds that work in the same direction – one being that a group $\mathcal{I}$ bidder's ability to compete is higher, and the other being that the group

$\mathcal{O}$ bidder increasingly shades his bid to avoid the *winner's curse*. On the contrary, the effect of increasing N also has two folds, but they work in opposite direction. On one hand, as N increases, it becomes more likely that there exists a group $\mathcal{I}$ bidder that is able to make high bid, which contributes to enhancing the group $\mathcal{I}$'s probability of winning as a whole. On the other hand, the group $\mathcal{O}$ bidder bids more aggressively as N increases, which diminishes the group $\mathcal{I}$'s probability of winning as a whole. Consistent with the previous observation that the group $\mathcal{O}$ bidder systematically bids higher when his valuation is $V = X_{(1)} \cdot S$, the probability that any group $\mathcal{I}$ bidder wins the auction is systematically lower in this case.

2.4.3. Revenue

Again, in the *ex ante* sense, the seller's expected revenue is the sum of the expected payment made by each group I bidder and that made by the group $\mathcal{O}$ bidder.

For any bidder i ($i \in \{1, ..., N\}$) with realized valuation-budget pair $(X_i, W_i) = (x_i, w_i)$, her expected payment can be calculated as the probability that she wins the auction, times the expected payment she faces when she indeed wins. Using a similar argument as before, in order for bidder i to win, two conditions must be met. First, all other group I bidders must bid less than $z_i = \min\{x_i, w_i\}$. Second, the group $\mathcal{O}$ bidder must bid less than z_i as well.

The probability that all other group $\mathcal{I}$ bidder bid less than z_i is

$$\mathbb{P}\{Z_j < z_i \ \forall j \in \{1, ..., N\} \setminus \{i\}\} = \left[F_Z(z_i)\right]^{N-1}.$$

The probability that the group $\mathcal{O}$ bidder bids less than z_i is[26]

$$\mathbb{P}\{\beta(S) < z_i\} = \mathbb{P}\{S < \beta^{-1}(z_i)\} = \beta^{-1}(z_i).$$

Therefore, bidder i's probability of winning is

$$\mathbb{P}\{\text{bidder } i \text{ with } (x_i, w_i) \text{ wins}\} = \beta^{-1}(z_i) \cdot \left[F_Z(z_i)\right]^{N-1}.$$

Define $H \equiv \max\{Z_1, ..., Z_{i-1}, Z_{i+1}, ...Z_N, \beta(S)\}$. That is, let H be the highest bid among all of bidder i's competitors – group $\mathcal{I}$ and group $\mathcal{O}$ alike. Then the cumulative distribution function of H conditional on bidder i wins the auction, $F_H(\cdot \mid H < z_i)$, is

$$F_H(h|H < z_i) = \begin{cases} 0 & h < 0 \\ \dfrac{\beta^{-1}(h) \cdot \left[F_Z(h)\right]^{N-1}}{\beta^{-1}(z_i) \cdot \left[F_Z(z_i)\right]^{N-1}} & h \in [0, z_i] \\ 1 & h > z_i \end{cases}.$$

Therefore, with realized valuation-budget pair (x_i, w_i), bidder i's expected payment in case of winning is

$$EP_i^{interim}(x_i, w_i) = \mathbb{P}\{H < z_i\} \cdot \mathbb{E}\left[H \mid H < z_i\right] = \int_0^{z_i} h dF_H(h),$$

with $F_H(h) = \beta^{-1}(h)\left[F_Z(h)\right]^{N-1}$.

[26]Again, since only when s falls in the domain in which $\beta(s)$ is invertible, can the group $\mathcal{O}$ bidder lose the auction, one can thus take the inverse of the bidding function $\beta(\cdot)$ for the relevant analysis without compromising rigorousness.

The *ex ante* expected payment made by any bidder i is therefore

$$EP_i^{ex\ ante} = \int_0^{\min\{\bar{x},\bar{w}\}} \left(\int_0^z h dF_H(h) \right) dF_Z(z).$$

Analogously, the group $\mathcal{O}$ bidder's *ex ante* expected payment is the probability that he wins the auction, times his expected payment in case of winning. For a group $\mathcal{O}$ bidder with realized signal $S = s$, he wins the auction when all group $\mathcal{I}$ bidders bid less than $\beta(s)$. Hence,

$$\mathbb{P}\{\text{group } \mathcal{O} \text{ bidder with } s \text{ wins}\} = \mathbb{P}\{Z_i < \beta(s)\ \forall i \in \{1, ..., N\}\} = \left[F_Z\left(\beta(s)\right)\right]^N.$$

As before, define Y to be the highest bid among all group $\mathcal{I}$ bidders, i.e., $Y \equiv \max_{i=1,...,N}\{Z_i\}$. Then the group $\mathcal{O}$ bidder's expected payment when he wins the auction is

$$EP_O^{interim}(s) = \mathbb{P}\{Y < \beta(s)\} \cdot \mathbb{E}\left[Y \mid Y < \beta(s)\right] = \int_0^{\beta(s)} y dF_Y(y).$$

The *ex ante* expected payment made by the group $\mathcal{O}$ bidder is therefore

$$EP_O^{ex\ ante} = \int_0^1 \left(\int_0^{\beta(s)} y dF_Y(y) \right) ds.$$

The seller's expected revenue as the sum of all bidders' *ex ante* expected payments is therefore

(2.31)

$$
\begin{aligned}
&\mathrm{Rev}^{ex\ ante}\\
=&N\cdot EP_i^{ex\ ante}+EP_O^{ex\ ante}\\
=&N\cdot\int_0^{\min\{\bar{x},\bar{w}\}}\int_0^z h\cdot dF_H(h)\cdot dF_Z(z)+\int_0^1\int_0^{\beta(s)}y\cdot dF_Y(y)\cdot ds\\
=&N\cdot\int_0^{\min\{\bar{x},\bar{w}\}}\left[\int_0^z h\left(\frac{\left[F_Z(h)\right]^{N-1}}{\beta'\left(\beta^{-1}(h)\right)}+(N-1)\beta^{-1}(h)\left[F_Z(h)\right]^{N-2}f_Z(h)\right)dh\right]f_Z(z)dz\\
&+\int_0^1\left(\int_0^{\beta(s)}Ny\left[F_Z(y)\right]^{N-1}f_Z(y)dy\right)ds
\end{aligned}
$$

2.4.3.1. $V = X_{(1)} \cdot S$**.** Figure 2.14 shows how the seller's expected revenue changes with $\bar{w}$ and N. Again, I perform this analysis with $\bar{x} = 1$. Similar to the previous $N = 1$ case, for any given N, the seller's expected revenue increases with $\bar{w}$, but at a decreasing rate. However, this decrease in rate is not very significant when N is relatively large. Two factors contribute to this result. First, the magnitude that the group $\mathcal{O}$ bidder shades his bid is much smaller when N is large, which benefits the seller. Second, the competition among the group $\mathcal{I}$ bidders also gets more severe as N increases. These two factors can also be seen when one investigates how the seller's revenue changes with N for any given $\bar{w}$. Again, holding $\bar{w}$ fixed, the group $\mathcal{O}$ bidder bids more aggressively as N increases, which contributes to an increase in seller's expected revenue. At the same time, the highest bid from group I bidders also increase as N gets larger. Following this logic, it is not hard to perceive that the effects of both factors are stronger when $\bar{w}$ is large – otherwise, the seller's expected revenue is bounded by $\bar{w}$. It is worth noting that when

$N = 2$, if $\bar{w}$ is considerably larger than $\bar{x}$, at some point the seller's expected revenue starts to decline slightly as $\bar{w}$ gets larger. This is because in this case, the group $\mathcal{O}$ bidder shades his bid dramatically while the lack of competition among the group $\mathcal{I}$ bidders due to the small group size negatively affects the seller's income. This problem does not arise when N is large, as the group $\mathcal{O}$ bidder is less relevant when compared to the group $\mathcal{I}$ bidders. The implication of these results is that the seller cannot effectively increase his expected revenue by relaxing the group $\mathcal{I}$ bidders' budgets when the number of group $\mathcal{I}$ bidder is low (for example, by offering financing to the business insiders), nor can he effectively increase his expected revenue by encouraging more group $\mathcal{I}$ bidders to participate in the auction when they are all severely constrained by their budgets. The most effective way to increase expected revenue is to offer some financing to the group $\mathcal{I}$ bidders if they are severely budget-constrained and at the same time make sure there are enough group $\mathcal{I}$ bidders coming to the auction.

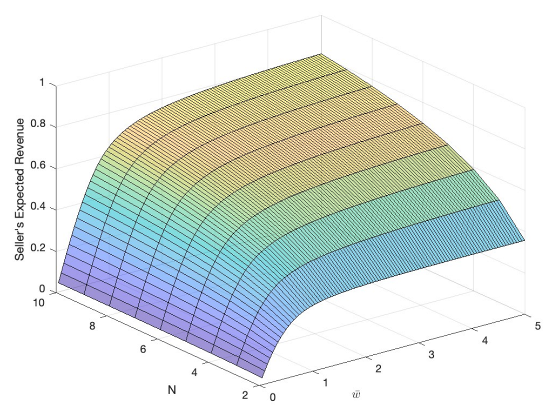


Figure 2.14. Seller's Expected Revenue (when $V = X_{(1)} \cdot S$)

Figure 2.15 illustrates how the seller's expected revenue in the presence of the $\mathcal{O}$ bidder compares to that without the $\mathcal{O}$ bidder. Consistent with the previous analysis, the group $\mathcal{O}$ bidder contributes more to the expected revenue when $\bar{w}$ is in an moderately low range, and his contribution decreases as N increases.

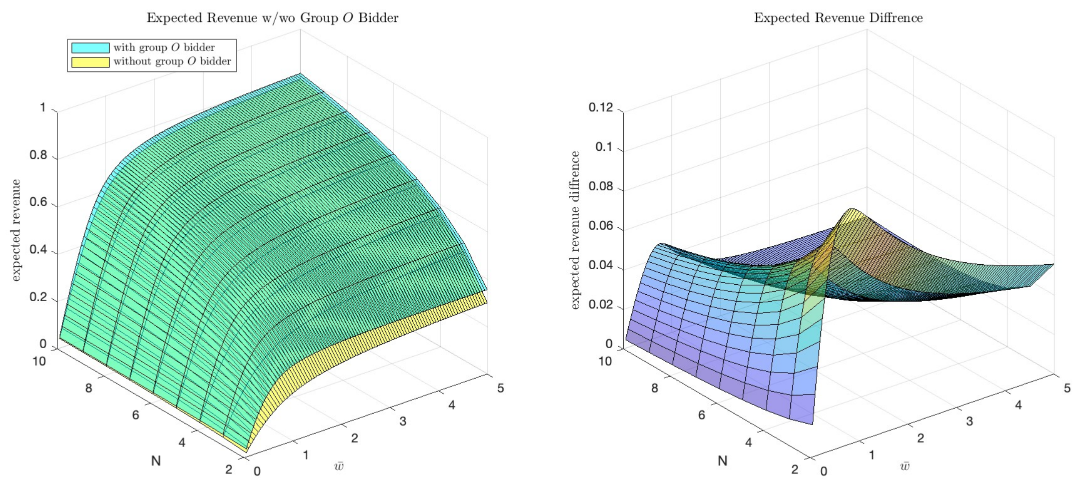


Figure 2.15. Expected Revenue with and Without the Group $\mathcal{O}$ Bidder

Figure 2.16 further breaks down the seller's expected revenue into the group $\mathcal{O}$ bidder's *ex ante* expected payment and the group $\mathcal{I}$ bidders' aggregated *ex ante* expected payment. The group $\mathcal{I}$ bidders' aggregated expected payment increases with $\bar{w}$ and N. The pattern of this increase is very similar to that of the seller's expected revenue in Figure 2.14 – the increase in competition (larger N) and the increase in financial strength (larger $\bar{w}$), only when combined together, cause the group $\mathcal{I}$ bidder's aggregated expected payment to increase dramatically.

The pattern of the group $\mathcal{O}$ bidder's expected payment is similar to that in the $N = 1$ case. Holding N fixed, the group $\mathcal{O}$ bidder's expected payment increases with $\bar{w}$ when $\bar{w}$ is below a certain threshold, and then starts to decrease as $\bar{w}$ gets bigger. Again, the increase in the group $\mathcal{O}$ bidder's expected payment in the lower region of $\bar{w}$ is mainly due to the increase in group $\mathcal{I}$

bidders' ability to pay. This increase in group $\mathcal{I}$ bidders' ability to pay, however, is not significant enough to lower the group $\mathcal{O}$ bidder's probability of winning by much. The overall effect is thus an increase in the group $\mathcal{O}$ bidder's *ex ante* expected payment. The fact that the group $\mathcal{O}$ bidder's *ex ante* expected payment eventually starts to decrease as $\bar{w}$ further increases, is again due to the group $\mathcal{O}$ bidder shading his bid to alleviate the *winner's curse* as the group $\mathcal{I}$ bidders get less likely to be financially constrained. As discussed in the previous sections, this shading behavior is minor when N is large. This explains why the group $\mathcal{O}$ bidder's *ex ante* expected payment increase with N in a moderate range of $\bar{w}$. When $\bar{w}$ is large enough, however, the group $\mathcal{I}$ bidders' abilities to compete in the auction is so high that the group $\mathcal{O}$ bidder's probability of winning decreases significantly when N gets large enough, despite the fact that the group $\mathcal{O}$ bidder himself also bids more aggressively. Therefore, for the high range of $\bar{w}$, the group $\mathcal{O}$ bidder's *ex ante* expected payment decreases as N gets large.

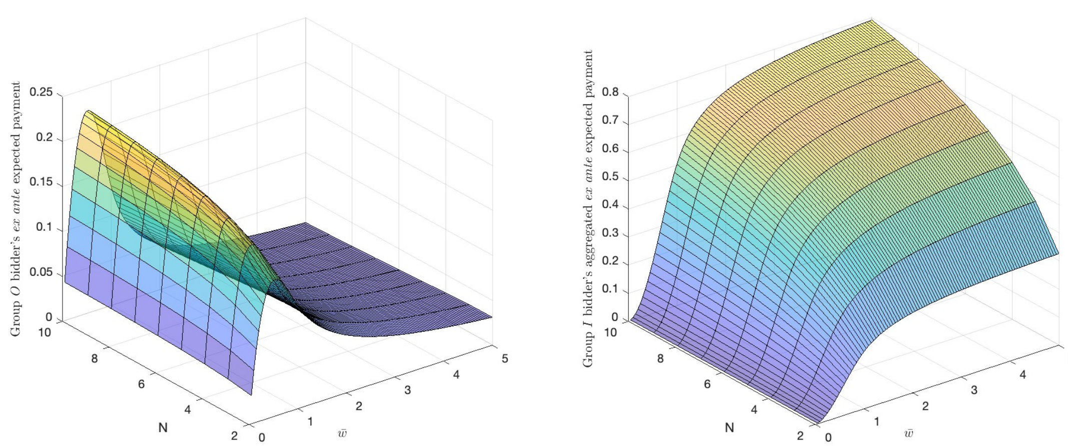


Figure 2.16. *Ex Ante* Expected Payments by both Types

2.4.3.2. $V = X_{(2)} \cdot S$**.** Figure 2.17 shows how the seller's expected revenue changes with $\bar{w}$ and N when $\bar{x} = 1$ is fixed. The general pattern and the reasons behind this pattern are similar to the previous case with $V = X_{(1)} \cdot S$. However, one thing worth noting is that under the assumption of $V = X_{(2)} \cdot S$, the group $\mathcal{O}$ bidder's valuation is less favorable, which makes him bidding more conservatively. As a result, his bid, and thus the expected payment from him, decreases dramatically as $\bar{w}$ gets even a little bit higher than zero. This effect is so large in magnitude that the increased expected payments from the group $\mathcal{I}$ bidders fail to keep up, which leads to the kink on the graph.

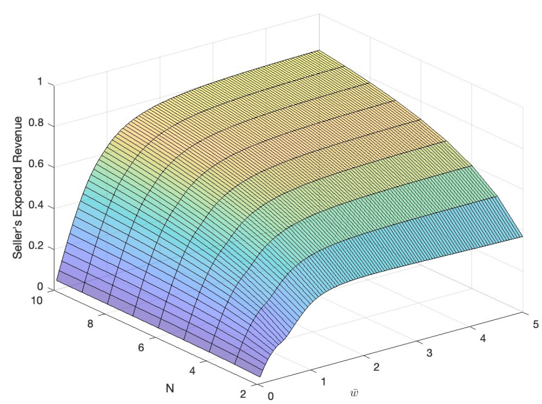


Figure 2.17. Seller's Expected Revenue (when $V = X_{(2)} \cdot S$)

Figure 2.18 illustrates how the seller's expected revenue in the presence of the $\mathcal{O}$ bidder compares to that without the $\mathcal{O}$ bidder. The kink mentioned in the previous paragraph is more obvious in Figure 2.18.

Figure 2.19 further breaks down the seller's expected revenue into the group $\mathcal{O}$ bidder's *ex ante* expected payment and the group $\mathcal{I}$ bidders' aggregated *ex ante* expected payment. The pattern is similar to the previous case with $V = X_{(1)} \cdot S$. The difference is that under the assumption of $V = X_{(2)} \cdot S$, the group $\mathcal{O}$ bidder's expected payment drops more quickly as $\bar{w}$ gets larger, and is more so when N is small.

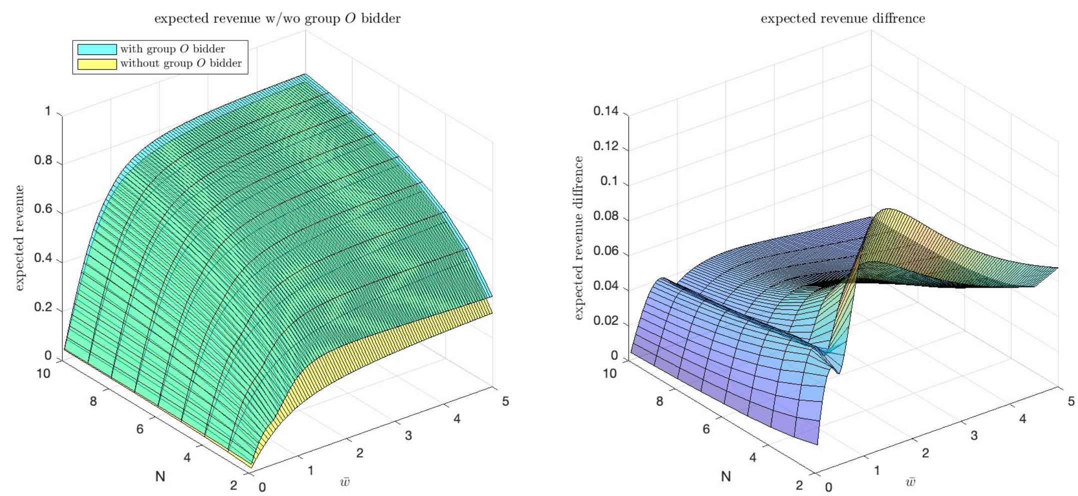


Figure 2.18. Expected Revenue with and Without the Group $\mathcal{O}$ Bidder

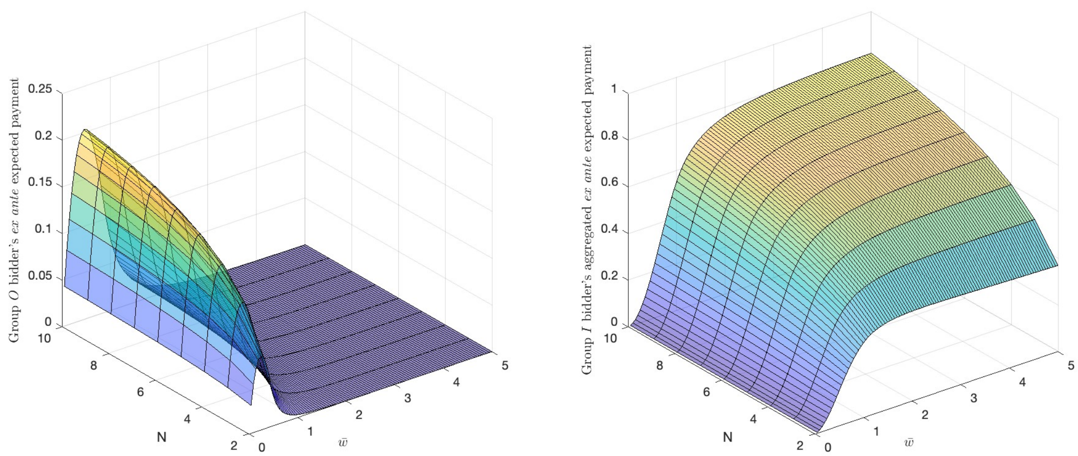


Figure 2.19. *Ex Ante* Expected Payments by both Types

2.4.3.3. $V = \frac{1}{N}\sum_{i=1}^{N} X_i \cdot S$**.** Figure 2.21 shows how the seller's expected revenue changes with $\bar{w}$ and N when $\bar{x} = 1$ is fixed. The general pattern is similar to the previous two cases. However, as has been discussed above, when $V = \frac{1}{N}\sum_{i=1}^{N} X_i \cdot S$, the group $\mathcal{O}$ bidder does not respond to the increase in the number of group $\mathcal{I}$ bidders in the same way as he does in the other two cases.

Therefore, any changes in the seller's expected revenue resulting from an increase in the number of group $\mathcal{I}$ bidders mainly comes from the competition among the group $\mathcal{I}$ bidders. This point if further illustrated in Figure 2.21 – the difference between the expected revenue with and without the group $\mathcal{O}$ bidder barely respond to a change in the number of group $\mathcal{I}$ bidders.

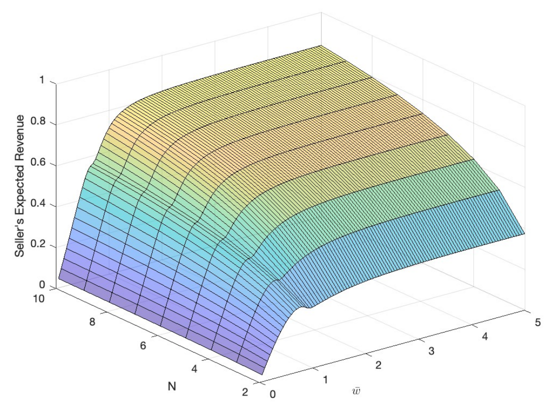


Figure 2.20. Seller's Expected Revenue (when $V = \frac{1}{N}\sum_{i=1}^{N} X_i \cdot S$)

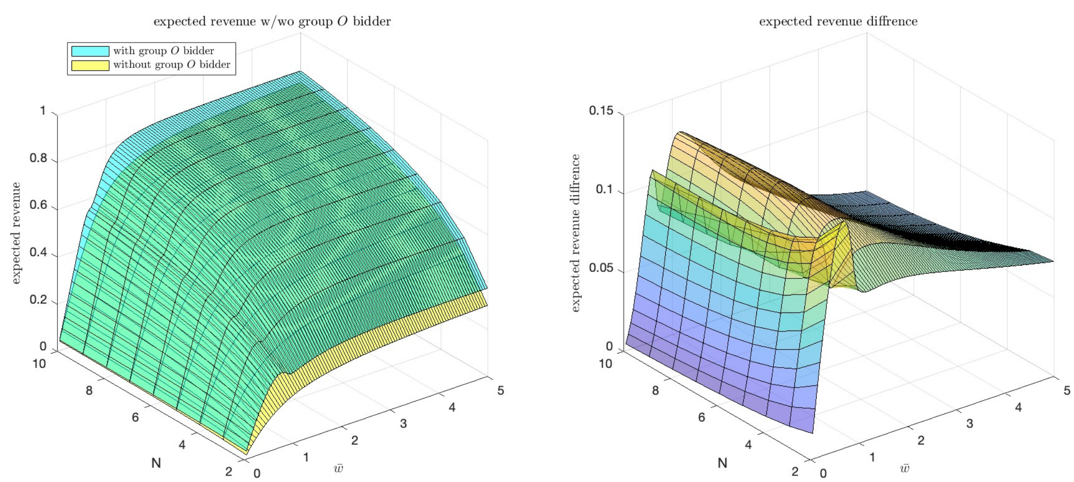


Figure 2.21. Expected Revenue with and Without the Group $\mathcal{O}$ Bidder

Similar as the other two cases, the group $\mathcal{O}$ bidder's expected payment first rises with $\bar{w}$ and then plummets if $\bar{w}$ increases even further. The major difference, again, is that increasing the number of $\mathcal{I}$ bidder does not help alleviate the group $\mathcal{O}$ bidder's concern. In fact, it only makes him more cautious.

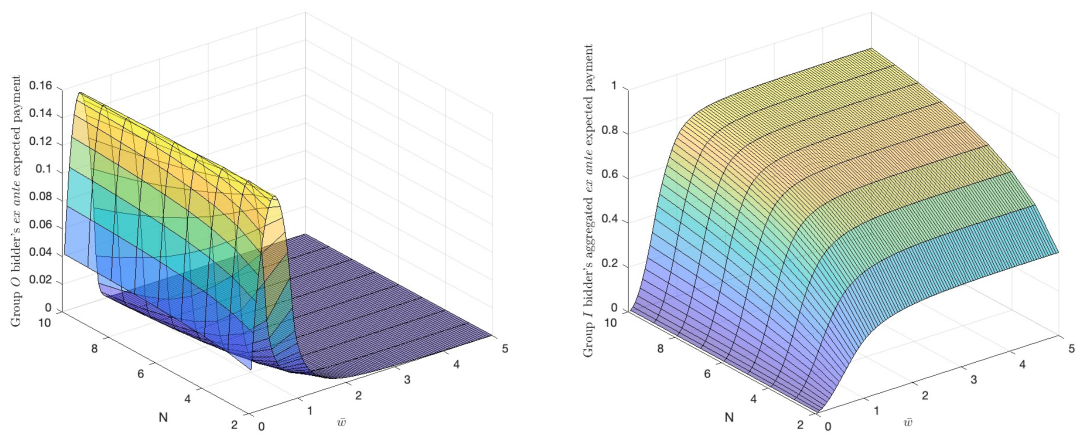


Figure 2.22. *Ex Ante* Expected Payments by both Types

2.5. Reserve Price

2.5.1. Equilibrium

In this section, I explore whether the seller will benefit from a reserve price. For simplicity, I only consider the case with one group $\mathcal{I}$ bidder (bidder 1) and one group $\mathcal{O}$ bidder (bidder 2). Assume that the reserve price is $0 < r < \min\{\bar{x}, \bar{w}\}$,[27] it is straightforward to verify that bidder 1's dominant strategy is to bid $Z \equiv \min\{X, W\}$ if $Z \geq r$ and drop out otherwise. In case bidder 1 drops out, bidder 2 wins the auction at the reserve price only if his bid is above the reservation

[27]If $r > \min\{\bar{x}, \bar{w}\}$, then bidder 1 is sure to be excluded from the auction, and the problem turns into a take it or leave it offer for bidder 2, which completely deviates from the main purpose of this paper. Therefore, I only consider $0 < r < \min\{\bar{x}, \bar{w}\}$.

price. In case neither bidder bids above the reservation price, the seller fails to sell the asset. No further interaction between the seller and the bidders is allowed to happen in the future, so that during the auction, the bidders do not take the post-auction stage into account to form their optimal bidding strategy.

Under this setting, the expected profit for bidder 2 with signal $S = s$ and bid $b \geq r$ is:

$$
\begin{aligned}
\Pi(b, s) =& F_Z(r) \cdot (\mathbb{E}[X|Z < r] \cdot s - r) \\
& + \left(F_Z(b) - F_Z(r)\right) \cdot (\mathbb{E}[X|r \leq Z < b] \cdot s - \mathbb{E}[Z|r \leq Z < b]) \\
=& \left(\int_0^b x f_X(x) dx + F_W(b) \int_b^{\bar{x}} x f_X(x) dx\right) \cdot s - F_Z(r) \cdot r - \int_r^b z \cdot f_Z(z) dz \\
=& \frac{(4 - 3s)b^3 - 3\left[(1 - s)\bar{w} + \bar{x}\right] b^2 + 3\bar{x}^2 sb - (3\bar{x} + 3\bar{w} - 2r)\, r^2}{6\bar{x}\bar{w}}.
\end{aligned}
\tag{2.32}
$$

For any given s, the FONC requires that the optimal bid b^* satisfies $\frac{\partial \Pi}{\partial b}(b^*, s) = 0$. Using Equation 2.32 and taking derivative with respect to b yields:

$$
\begin{aligned}
\frac{\partial \Pi}{\partial b}(b, s) = & \left(b f_X(b)\left(1 - F_W(b)\right) + f_W(b) \int_b^{\bar{x}} x f_X(x) dx\right) s \\
& - b\left[\left(1 - F_W(b)\right) f_X(b) + \left(1 - F_X(b)\right) f_W(b)\right],
\end{aligned}
$$

which is exactly the same as Equation 2.4. This is because the existence of a reserve price does not change either the marginal gain or the marginal cost of winning for bidder 2. However, since any bid below the reserve price is not allowed, bidder 2's private signal needs to be large enough for him to participate in the auction. This lower bound $\underline{s}\,(\bar{x}, \bar{w}, r)$ can be calculated by solving

$b^* = r$, i.e.

$$\text{(2.33)} \qquad \begin{aligned} &\frac{(1-\underline{s})\bar{w}+\bar{x}-\sqrt{[(1-\underline{s})\bar{w}+\bar{x}]^2-(4-3\underline{s})\underline{s}\bar{x}^2}}{4-3\underline{s}} = r \\ &\Rightarrow \quad \underline{s}\,(\bar{x},\bar{w},r) = \frac{2r\,(\bar{x}+\bar{w}-2r)}{\bar{x}^2+2\bar{w}r-3r^2}. \end{aligned}$$

It seems, then, bidder 2's optimal bidding function given by Equation 2.8 should be modified into:

(2.34)

$$\beta(s,\bar{x},\bar{w}) = \begin{cases} 0 & 0 \le s < \underline{s}\,(\bar{x},\bar{w},r) \\ \dfrac{(1-s)\bar{w}+\bar{x}-\sqrt{[(1-s)\bar{w}+\bar{x}]^2-(4-3s)s\bar{x}^2}}{4-3s} & \underline{s}\,(\bar{x},\bar{w},r) \le s < s^*(\bar{x},\bar{w}) \\ \min\{\bar{x},\bar{w}\} & s^*(\bar{x},\bar{w}) \le s \le 1 \end{cases},$$

with $s^*(\bar{x},\bar{w})$ and $\underline{s}\,(\bar{x},\bar{w},r)$ being defined in Equation 2.9 and Equation 2.33, respectively.

However, this optimal bidding function does not guarantee that bidder 2's expected profit is positive. For example, assume that $\bar{x} = 1$, $\bar{w} = 1$ and $r = 0.5$. Then $\underline{s} = 0.8$. According to Equation 2.34, if $s = 0.9$, then it would be optimal for bidder 2 to bid $b^* = 0.69$, which, by Equation 2.32, results in $\Pi = -0.089$. Thus, a profit-maximizing bid gives bidder 2 a negative expected payoff.

In fact, this is not coincidence. To see this, suppose that $s = \underline{s}\,(\bar{x},\bar{w},r)$, that is, suppose bidder 2's signal is just large enough so that he is willing to participate in the auction. Then, by definition of $\underline{s}\,(\bar{x},\bar{w},r)$, the optimal bid for bidder 2 is $b^* = r$. Substituting in Equation 2.32, one gets $\Pi = -\dfrac{\left[r^2-2\,(\bar{x}+\bar{w})\,r+\bar{x}^2+\bar{x}\bar{w}+\bar{w}^2\right]r^3}{\bar{x}\bar{w}\left(\bar{x}^2+2\bar{w}r-3r^3\right)}$. It becomes apparent that when $r = 0$, the previous expression yields $\Pi = 0$. Hence, since $\dfrac{d\Pi}{dr} < 0$, bidder 2's expected payoff is negative for all $r > 0$.

Therefore, Equation 2.8 cannot be bidder 2's optimal bidding function, for it fails to incorporate the extra constraint of non-negative expected payoff. The root of this issue is that $\underline{s}\,(\bar{x}, \bar{w}, r)$ defined by Equation 2.33 is the threshold signal that makes $b = r$ the bid that maximizes bidder 2's expected payoff, but in and of itself, is not large enough so that this maximized expected payoff is non-negative. The real threshold signal, call it $\tilde{s}$, should be the solution of:

$$\frac{(4 - 3\tilde{s})\tilde{\beta}^3 - 3\left[(1 - \tilde{s})\bar{w} + \bar{x}\right]\tilde{\beta}^2 + 3\bar{x}^2\tilde{s}\tilde{\beta} - (3\bar{x} + 3\bar{w} - 2r)\,r^2}{6\bar{x}\bar{w}} = 0, \tag{2.35}$$

where

$$\tilde{\beta} = \frac{(1 - \tilde{s})\bar{w} + \bar{x} - \sqrt{[(1 - \tilde{s})\bar{w} + \bar{x}]^2 - (4 - 3\tilde{s})\tilde{s}\bar{x}^2}}{4 - 3\tilde{s}}.$$

Unfortunately, a general solution of $\tilde{s}$ is not feasible. However, it is certain that $\tilde{s} > \underline{s}$, as $\dfrac{d\Pi}{ds} > 0$ and $\Pi\left(\beta(\underline{s}, \bar{x}, \bar{w}), \underline{s}\right) < 0$ for $r > 0$, as shown above. But as the profit maximizing bid for any $\tilde{s} \le s < s^*(\bar{x}, \bar{w})$ is always given by the FONC, it can be concluded that $\tilde{\beta} > r$. This interesting observation is summarized below:

Lemma 6. *When a reserve price $0 < r < \min\{\bar{x}, \bar{w}\}$ is imposed, bidder 2 either not take part in the auction or submit a bid that is higher than r. Stated differently, bidder 2 will never bid the reserve price.*

To illustrate, consider a numerical example with $\bar{x} = \bar{w} = 1$ and $r = 0.2$. The optimal bid corresponding to $\tilde{s}$ is thus $\tilde{\beta} = \dfrac{\tilde{s}}{4 - 3\tilde{s}}$. Substitute $\tilde{\beta} = \dfrac{\tilde{s}}{4 - 3\tilde{s}}$ into Equation 2.35 and solve for $\tilde{s}$ gives $\tilde{s} = 0.6$. Compared to $\underline{s}\,(\bar{x}, \bar{w}, r) = \dfrac{2r\,(\bar{x} + \bar{w} - 2r)}{\bar{x}^2 + 2\bar{w}r - 3r^2} = 0.5$, it is indeed that bidder 2 needs a higher signal to break even with the presence of a reserve price. Note that in this example,

$\tilde{\beta} = \dfrac{\tilde{s}}{4 - 3\tilde{s}} = 0.27 > r$ – bidder 2 bids above the reserve price even if his signal is just large enough to make him indifferent between participating or not.

Another point to note is that if the reserve price is too large, bidder 2 will never participate in the auction no matter how large his signal is, as his expected payoff is always negative. To see this, consider the earlier example in which $\bar{x} = \bar{w} = 1$ and $r = 0.5$. Again, the optimal bid corresponding to $\tilde{s}$ is $\tilde{\beta} = \dfrac{\tilde{s}}{4 - 3\tilde{s}}$ since it does not depend on r. When $\tilde{\beta} = \dfrac{\tilde{s}}{4 - 3\tilde{s}}$ is substituted into Equation 2.35, it turns out that there does not exist any $\tilde{s} \in [0, 1]$ that satisfies Equation 2.35. In fact, even with $s = 1$ – the highest signal possible – bidder 2's expected profit is $-0.04 < 0$.

Theorem 12. *For any given $(\bar{x}, \bar{w})$, there exists an upper bound on the reserve price $R(\bar{x}, \bar{w})$, above which bidder 2 will not participate in the auction, no matter how large his private signal is.*

When $\bar{x} \leq \bar{w}$, $R(\bar{x}, \bar{w})$ is given by one of:

$$r_k = \frac{1}{2}(\bar{x} + \bar{w})\left[2\cos\left(\frac{\theta + 2k\pi}{3}\right) + 1\right], \quad k = 0, 1, 2,$$

where

$$\theta = \arctan\left(\frac{2\bar{x}^{\frac{3}{2}}\bar{w}^{\frac{1}{2}}\left(3\bar{x}^2 + 3\bar{x}\bar{w} + \bar{w}^2\right)^{\frac{1}{2}}}{\bar{w}^3 + 3\bar{x}^2\bar{w} + 3\bar{x}\bar{w}^2 - \bar{x}^3}\right).$$

When $\bar{x} > \bar{w}$, $R(\bar{x}, \bar{w})$ is given by one of:

$$r_k = \frac{1}{2}(\bar{x} + \bar{w})\left[2\cos\left(\frac{\theta + 2k\pi}{3}\right) + 1\right], \quad k = 0, 1, 2,$$

where

$$\theta = \arctan\left(\frac{2\bar{x}^{\frac{1}{2}}\bar{w}^{\frac{1}{2}}\left(3\bar{x}^2 - 3\bar{x}\bar{w} + \bar{w}^2\right)^{\frac{1}{2}}\left(\bar{x}^2 + 6\bar{w}^2\right)^{\frac{1}{2}}}{\bar{x}^3 - 3\bar{x}^2\bar{w} + 9\bar{x}\bar{w}^2 - \bar{w}^3}\right).$$

PROOF. By Equation 2.32, it is clear that bidder'2 expected payoff is decreasing in r but increasing in s. Therefore, if r is so large that even getting the highest possible signal $s = 1$ cannot generate a non-negative expected payoff to bidder 2, he would forfeit his bidding right. Hence, $R(\bar{x}, \bar{w})$ is such that:

$$\Pi(\min\{\bar{x}, \bar{w}\}, 1) = \begin{cases} \dfrac{\bar{x}^3 - [3\bar{x} + 3\bar{w} - 2R(\bar{x}, \bar{w})]\,[R(\bar{x}, \bar{w})]^2}{6\bar{x}\bar{w}} & (\bar{x} \leq \bar{w}) \\ \dfrac{\bar{w}^3 - 3\bar{x}\bar{w}^2 + 3\bar{x}^2\bar{w} - [3\bar{x} + 3\bar{w} - 2R(\bar{x}, \bar{w})]\,[R(\bar{x}, \bar{w})]^2}{6\bar{x}\bar{w}} & (\bar{x} > \bar{w}) \end{cases} = 0$$

Solving the third degree polynomials leads to the conclusions presented in the theorem above. $\square$

Bidder 2's equilibrium bidding strategy can thus summarized as follows:

Theorem 13. *Given $(\bar{x}, \bar{w})$, if $r > R(\bar{x}, \bar{w})$, where $R(\bar{x}, \bar{w})$ is as defined in Theorem 12, then bidder 2 does not participate in the auction. Otherwise, bidder 2's equilibrium bidding function is:*

(2.36)

$$\beta(s, \bar{x}, \bar{w}) = \begin{cases} 0 & 0 \leq s < \tilde{s}(\bar{x}, \bar{w}, r) \\ \dfrac{(1-s)\bar{w} + \bar{x} - \sqrt{[(1-s)\bar{w} + \bar{x}]^2 - (4-3s)s\bar{x}^2}}{4-3s} & \tilde{s}(\bar{x}, \bar{w}, r) \leq s < s^*(\bar{x}, \bar{w}) \\ \min\{\bar{x}, \bar{w}\} & s^*(\bar{x}, \bar{w}) \leq s \leq 1 \end{cases},$$

where $\tilde{s}(\bar{x}, \bar{w}, r)$ is defined explicitly as the solution to Equation 2.35 and $s^*(\bar{x}, \bar{w})$ is given by Equation 2.9.

Figure 2.23 shows how $\underline{s}(\bar{x}, \bar{w}, r)$ and $\tilde{s}(\bar{x}, \bar{w}, r)$ change with r.

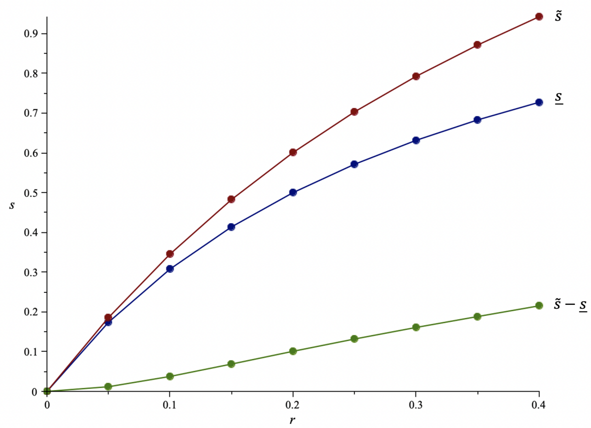


Figure 2.23. Changes in $\tilde{s}$ and $\underline{s}$ against r (with $\bar{x} = 1$ and $\bar{w} = 1$)

2.5.2. Efficiency

When $r < R(\bar{x}, \bar{w})$, there are two ways that bidder 1 wins the auction. The first way is bidder 2 does not have a signal large enough to convince him to submit a positive bid, i.e. $s < \tilde{s}(\bar{x}, \bar{w}, r)$, while bidder 1 is able to bid above the reserve price, i.e. $\min\{\bar{x}, \bar{w}\} \geq r$. The second way is both bidders submit a positive bid and bidder 1 outbid bidder 2. Hence, the auction efficiency in this case is:[28][29]

$$
\begin{aligned}
&E(\bar{x},\bar{w})\\
&=\underbrace{\int_0^{\tilde{s}(r)}[1-F_Z(r)]ds}_{\text{bidder 2 not bid}}+\underbrace{\int_{\tilde{s}(r)}^{s^*}[1-F_Z(\beta(s,\bar{x},\bar{w}))]ds}_{\text{bidder 1 bids higher than bidder 2}}\\
&=\underbrace{\int_0^{s^*}[1-F_Z(\beta(s,\bar{x},\bar{w}))]ds}_{\text{Efficiency without reserve price}}+\int_0^{\tilde{s}(r)}[1-F_Z(r)]ds-\int_0^{\tilde{s}(r)}[1-F_Z(\beta(s,\bar{x},\bar{w}))]ds\\
&=\underbrace{\int_0^{s^*}[1-F_Z(\beta(s,\bar{x},\bar{w}))]ds}_{\text{Efficiency without reserve price}}+\underbrace{\int_0^{\underline{s}(r)}\left[F_Z(\beta(s,\bar{x},\bar{w}))-F_Z(r)\right]ds}_{<0}\\
&\quad+\underbrace{\int_{\underline{s}(r)}^{\tilde{s}(r)}\left[F_Z(\beta(s,\bar{x},\bar{w}))-F_Z(r)\right]ds}_{>0}.
\end{aligned}
\tag{2.37}
$$

By imposing a reserve price, there are two forces that work in opposite direction. On the one hand, the reserve price has a deterrence effect on bidder 2 so that he does not bid unless $s > \tilde{s}$, which increases the chance that the asset is allocated to bidder 1. On the other hand, the reserve price also negatively affects bidder 1 since she is likely to drop out herself, whereas without the reserve price, she still has some winning probability no matter how small Z turns out to be. Therefore, whether imposing a reserve price increases the allocation efficiency depends which of these two forces dominates. Putting in another way, when $s < \tilde{s}(\bar{x},\bar{w},r)$, without a reserve price, bidder 1 fights against bidder 2; with a reserve price, bidder 1 fights against the reserve

[28]For simplicity, I denote $\tilde{s}(\bar{x},\bar{w},r)$ by $\tilde{s}(r)$, $\underline{s}(\bar{x},\bar{w},r)$ by $\underline{s}(r)$ and $s^*(\bar{x},\bar{w})$ by s^*.

[29]One may argue that now with a reserve price, it is possible that neither of the two bidders submit a bid above the reserve price and the asset ends up unsold. Without the reserve price, however, at least someone ends up with the asset. I nonetheless keep defining the efficiency as the probability that the asset is sold to the party that can make the best use of it – bidder 1 in this case.

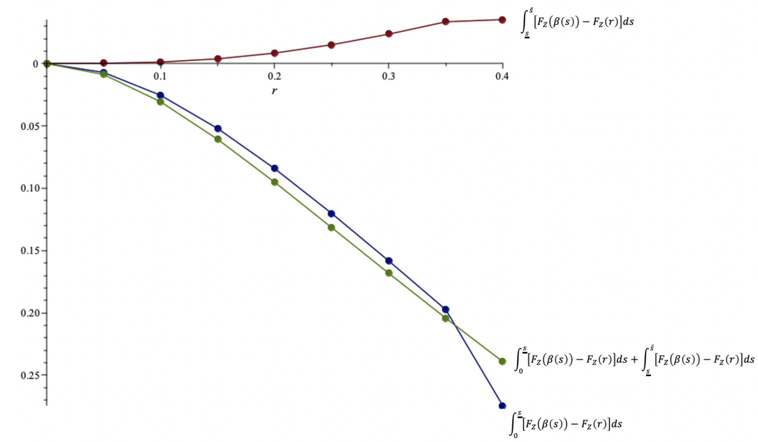


Figure 2.24. Change in Efficiency Compared to Without Reserve Price (with $\bar{x} = 1$ and $\bar{w} = 1$)

price. For $s < \underline{s}(\bar{x}, \bar{w}, r)$, the reserve price is harder to fight than bidder 2, while the reverse is true for $\underline{s}(\bar{x}, \bar{w}, r) < s < \tilde{s}(\bar{x}, \bar{w}, r)$. Figure 2.24 shows that the combined effect is negative on allocation efficiency. The higher the reserve price, the larger this negative effect. This can be further understood by looking back at Figure 2.23. In Figure 2.23, it is clear that as r increases, $\underline{s}(\bar{x}, \bar{w}, r)$ grows much faster than the difference between $\tilde{s}(\bar{x}, \bar{w}, r)$ and $\underline{s}(\bar{x}, \bar{w}, r)$. This results in bidder 1 having to "fight against the reservation price" more often, which dominates the benefit she gets from avoiding to fight against bidder 2 in case $\underline{s}(\bar{x}, \bar{w}, r) < s < \tilde{s}(\bar{x}, \bar{w}, r)$. Figure 2.25 shows how the allocation efficiency changes with r.

When $r \geq R(\bar{x}, \bar{w})$, as shown previously, the reserve price is too large for bidder 2 to bid. Therefore, bidder 1 wins the auction as long as she is able to bid above the reserve price:

$$E(\bar{x}, \bar{w}) = 1 - F_Z(r). \tag{2.38}$$

Clearly, the above expression reaches its maximum when $r = R(\bar{x}, \bar{w})$. Setting such a high reserve price can be viewed as a defense mechanism to prevent outsiders from getting the asset

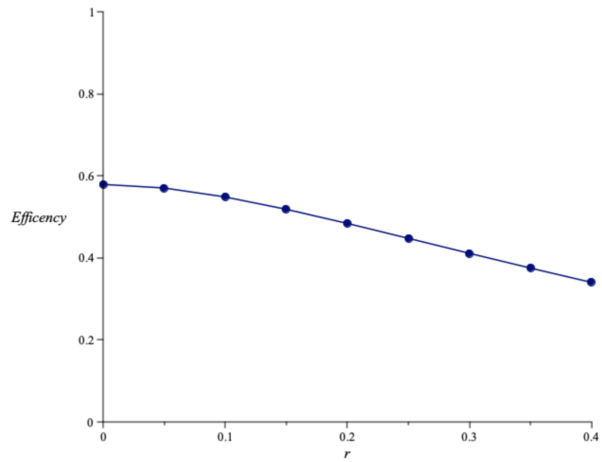


Figure 2.25. Allocation Efficiency as r Changes (with $\bar{x} = 1$ and $\bar{w} = 1$)

when no rules can be imposed directly to exclude the outsider outright. For comparison, when $\bar{x} = 1$ and $\bar{w} = 1$, as is in Figure 2.25, the efficiency at $r \geq R(\bar{x}, \bar{w})$ is 0.311, worse than not to exclude bidder 2, and much worse than not having a reservation price in the first place.

2.5.3. Revenue

Bidder 1's *ex-ante* expected payment is:

(2.39)

$$EP_1^{ex-ante} = \int_r^{\beta(\tilde{s}(r))} \left(\int_0^{\tilde{s}(r)} r ds \right) dF_Z(z) + \int_{\beta(\tilde{s}(r))}^{\min\{\bar{x},\bar{w}\}} \left(\int_0^{\tilde{s}(r)} r ds + \int_{\tilde{s}(r)}^{\beta^{-1}(z)} \beta(s) ds \right) dF_Z(z).$$

Bidder 2's *ex-ante* expected payment is:

$$EP_2^{ex-ante} = \int_{\tilde{s}(r)}^{1} \left(F_Z(r) \cdot r + \int_r^{\beta(\tilde{s}(r))} z dF_Z(z) \right) ds. \tag{2.40}$$

Figure 2.26 shows how the seller's expected revenue and the bidders' expected payments change with the reserve price. Unlike allocation efficiency, which decreases as the reserve price increases, the seller can actually benefit from imposing a moderate reserve price if he cares more about revenue than efficiency. The primary contributor is bidder 1. As the graph shows, bidder 1's expected payment rises with r[30]. As r increases from zero, there is some probability that bidder 1 has to drop out, which decreases her expected payment. However, this decrease in expected payment cannot be too large, as the very reason that bidder 1 drops out is she is not able to bid much. Therefore, even if she wins, her payment would be very small. However, as r increases, its deterrence effect on bidder 2 makes the chance of winning more favorable to bidder 1. As long as bidder 2 does not bid, bidder 1 pays the reserve price. As the reserve price increases, both the probability of bidder 2 not bidding and the payment from bidder 1 in case she does not drop out increase, which increases bidder 1's expected payment. Although there are situations in which bidder 1 wins at a bargain, in the sense that were bidder 2 not deterred by the reservation price, bidder 1 would have to pay a higher price in case of winning (this happens when $s \in [\underline{s}(r), \tilde{s}(r)]$), this potential decrease in bidder 1's expected payment is not large enough to redirect the general trend. According to Figure 2.26, having a reserve price does not have much of an impact on bidder 2's expected payment when the reserve price is small. This is because the deterrence effect is relatively small, and in case bidder 2 wins, there is some chance that he has to pay more than he otherwise would when bidder 1 drops out. The two factors roughly cancel out and thus bidder 2's expected payment stays relatively stable. However, if the reserve price is not too small, the deterrence effect dominates and bidder 2's expected payment decreases dramatically. As r getting closer to $R(\bar{x}, \bar{w})$, the revenue contribution from bidder 2

[30]Obviously this observation is only valid for a certain range of r. When r gets large, the probability of bidder 1 dropping out increases. As a crude approximation, one can view this case as setting a monopoly price for bidder 1. This approximation makes sense when r is not to small, since in this situation there is a great chance that bidder 2 does not bid, which results in bidder 1 paying the reserve price. The approximated revenue is then $\left(1 - F_Z(r)\right) \cdot r$. With $\bar{x} = 1$ and $\bar{w} = 1$, $\left(1 - F_Z(r)\right) \cdot r$ is an decreasing function in r for $r > \frac{1}{3}$.

is close to zero. Since the seller's expected revenue is the sum of the expected payments from both bidders, it first increases with r then decreases.

One thing worth noting is the contrast between allocation efficiency and expected revenue. As is shown previously, the allocation efficiency strictly decreases with r, while the expected revenue increase with r when r is not too large. This seems to be a contradiction, as the main reason that the expected revenue increases with r is from bidder 1's increasing contribution. But as the allocation efficiency is the measure of bidder 1's winning probability, it is natural to assume her expected payment would decrease when she is less likely to win. This can be understood by viewing the expected payment as a form of weighted average of the probability of winning. The reason that bidder 1 wins less often when r increases is there are situations in which she has to drop out. But in those situations, the weighting factor – her payment – is small. Therefore, an increase in r has very asymmetric impacts on allocation efficiency and bidder 1's expected payment.

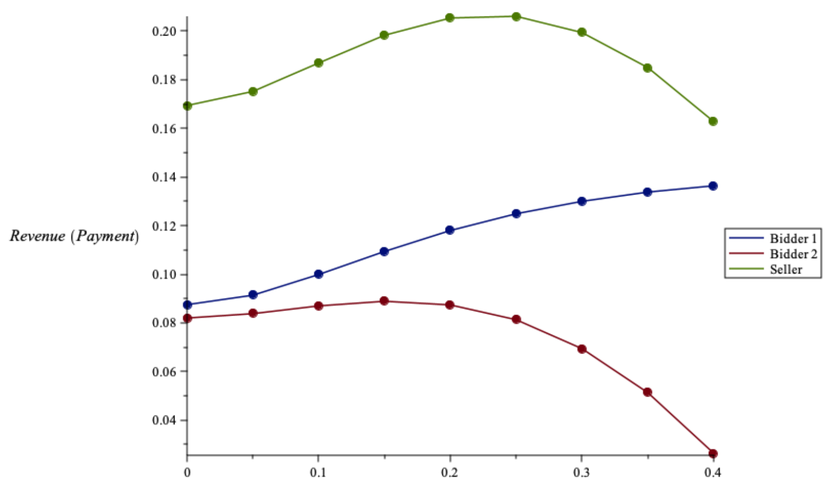


Figure 2.26. Expected Revenue and Payments as r Changes (with $\bar{x} = 1$ and $\bar{w} = 1$)

2.6. Conclusion

This paper studies a second price auction with a group of potentially budget-constrained business insiders and a deep-pocketed investor whose limited knowledge has a negative impact on his valuation. I find that the when the business insiders are very likely to be financially constrained, the deep-pocketed investor is more eager to compete. Consequently, the expected revenue rises as the insiders' budgets slightly improve. However, if the insiders' budgets improve significantly and thus the insiders become less likely to be financially constrained, the deep-pocketed investor bids more conservatively in fear of the winner's curse. This is beneficial to the allocation efficiency but detrimental to the seller's expected revenue. The latter is extremely severe when the number of business insiders is small, as the existence of more business insiders can alleviate the winner's curse to some extend. One exception is when the outsider's valuation depends on the average of those of the insiders'. In this case, the presence of more insider only makes the outsider bid more cautiously. This implies that aside from insiders' budget, how the outsider's valuation is determined by the insiders' is equally if not more important. The conclusions above apply to the auction setting without a reserve price. If a reserve price is imposed, the allocation efficiency decreases – the higher the reserve price, the lower the allocation efficiency. However, the seller can expect an improvement in revenue by requiring a moderate reserve price. The reason is that a reserve price has a deterrence effect on the deep-pocketed investor so that he would not bid in the auction even if his optimal bid is equal to or slightly above the reserve price level were the reserve price not exist. As a result, the reserve price has two contradicting effects on the business insider. On the one hand, it is probable that she has to drop out because she is not able to meet the reserve price. On the other hand, her chance of winning is improved in case she can bid above the reserve price, as her opponent is deterred.

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CHAPTER 3

Appendix for Chapter One

3.1. Lifting restriction on "speculative" test taking.

In this section, we drop the assumption of $t_1 \geq \varepsilon_1$. Although it may be hard to justify any possibility of learning about one's ability by exerting only an infinitesimal amount of effort, this discussion is nonetheless relevant, as it is common for people to try out their ability before full committing themselves to a task. This type of "speculative" test taking need not to be discouraged in many settings in which the completion of the task is voluntary by nature. For example, no country desires all its citizens to be NBA-qualified in their basketball skills. However, when the completion of the task by *everyone* is essential, "speculative" test taking can be problematic to the principal.

With $r \geq \varepsilon_N - \varepsilon_1$,

$$U^{fd}(1;(r,b)) - U^{fd}(0;(r,b)) = p_1 b + \sum_{j=M+1}^{N} p_j(r - \varepsilon_j)$$

where $M = \max\{i : \varepsilon_i \leq r\}$. Hence, $U^{fd}(1;(r,b)) > U^{fd}(0;(r,b))$ if and only if

$$b > \frac{\sum_{j=M+1}^{N} p_j(\varepsilon_j - r)}{p_1}$$

First, note that if $r \geq \varepsilon_N$, the right-hand-side of the inequality above becomes 0, meaning that any $b > 0$ will deter the "speculative" test taking. This is intuitive because the passing reward in this case is large enough so that everyone will choose to pass the test anyway even if they take the first test with infinitesimal amount of effort. They lose nothing by choosing $t_1 = \varepsilon_1$ instead – only with the potential gain of saving the test fee if their ability is a_1. Therefore, it is tempting to modify the conditions in Proposition 2 and state that FBME is achievable if

$$U^{fd}(1; (\varepsilon_N, \bar{B}^{fd}(1))) = \varepsilon_N - \sum_{j=2}^{N} p_j \bar{B}^{fd}(1) - \sum_{j=1}^{N} p_j \varepsilon_j \leq 0, \tag{3.1}$$

with $\bar{B}^{fd}(1) = \min_{i \in \{2,\ldots,N\}} \beta^{fd}(1, i)$ just as before. Notice that the condition above is sufficient but not necessary for FBME to be achievable, since to implement $t_1 = \varepsilon_1$, we do not really need $r \geq \varepsilon_N$. Instead, obverse that for $r \in [\varepsilon_N - \varepsilon_1, \varepsilon_N)$, the smaller r is, the bigger b is needed to deter speculative test taking. But this is exactly the same situation that we face when we try to implement $t_1 = \varepsilon_i > \varepsilon_1$. Increasing b to the largest extent without altering agents' choice of t_1 is both beneficial for decreasing expenditure directly and for leaving more flexibility to the choice of r. Therefore, $b = \bar{B}^{fd}(1)$ is the (weakly) dominating choice of test fee for the principal. With $b = \bar{B}^{fd}(1)$, the passing reward has to satisfy

$$r \geq \underline{\hat{R}}^{fd}(1) := \max\left\{\varepsilon_N - \varepsilon_1, \max\{r : U^{fd}(1; (r, \bar{B}^{fd}(1))) = U^{fd}(0; (r, \bar{B}^{fd}(1)))\}\right\}.$$

With these observations, the following conclusion is immediate:

Proposition 16. *With full-disclosing feedback policy and without restriction of $t_1 \geq \varepsilon_1$, FBME is achievable if and only if*

$$U^{fd}(1; (\underline{\hat{R}}^{fd}(1), \bar{B}^{fd}(1))) = \underline{\hat{R}}^{fd}(1) - \sum_{j=2}^{N} p_j \bar{B}^{fd}(1) - \sum_{j=1}^{N} p_j \varepsilon_j \leq 0.$$

Remark. *Note that the condition stated in Proposition 16 is both necessary and sufficient. In fact,* $\underline{\hat{R}}^{fd}(1) < \varepsilon_N$ *for sure. That is, the principal never has to set the passing reward so large that agents find it pointless to take a speculative test with infinitesimal amount of effort. To see this, note that as long as* $r \leq \varepsilon_N$, $M < N$ *for sure, and we need* $r > \frac{\sum_{j=M+1}^{N} p_j \varepsilon_j - p_1 b}{\sum_{j=M+1}^{N} p_j}$ *to make* $U^{fd}(1;(r,b)) > U^{fd}(0;(r,b))$*. Notice that*

$$\frac{\sum_{j=M+1}^{N} p_j \varepsilon_j - p_1 b}{\sum_{j=M+1}^{N} p_j} = \frac{\sum_{j=M+1}^{N} p_j \varepsilon_j}{\sum_{j=M+1}^{N} p_j} - \frac{p_1}{\sum_{j=M+1}^{N} p_j} b < \varepsilon_N - \frac{p_1}{\sum_{j=M+1}^{N} p_j} b < \varepsilon_N,$$

which implies $\underline{\hat{R}}^{fd}(1) = \max\left\{\varepsilon_N - \varepsilon_1, \max\{r : U^{fd}(1;(r,\bar{B}^{fd}(1))) = U^{fd}(0;(r,\bar{B}^{fd}(1)))\}\right\} < \varepsilon_N$*. Thus, the condition we find earlier (Equation 3.1), is indeed not necessary.*

3.2. Example of $t_1 = \varepsilon_1$ cheaper to implement than $t_1 = \varepsilon_2$ even though full-surplus-extraction is not possible with the former but possible with the latter.

Let $N = 3$ and

$$\mathcal{E}(a,x) = \begin{cases} 1 & \text{w.p. } 0.3 \\ 9 & \text{w.p. } 0.4 \cdot \\ 14 & \text{w.p. } 0.3 \end{cases}$$

Then

$$\beta^{fd}(1,2) = \frac{p_1(\varepsilon_2 - \varepsilon_1)}{p_2} = 6,$$

$$\beta^{fd}(1,3) = \frac{p_1(\varepsilon_3 - \varepsilon_1) + p_2(\varepsilon_3 - \varepsilon_2)}{p_2 + p_3} = \frac{59}{7},$$

$$\beta^{fd}(2,3) = \frac{(p_1 + p_2)(\varepsilon_3 - \varepsilon_2)}{p_3} = \frac{35}{3}.$$

Then $\bar{B}^{fd}(1) = 6$, $\underline{R}^{fd}(1) = 13$, and FBME is not feasible, since

$$U^{fd}(1; (\underline{R}^{fd}(1), \bar{B}^{fd}(1))) = \underline{R}^{fd}(1) - (p_2 + p_3)\bar{B}^{fd}(1) - \sum_{j=1}^{3} p_j \varepsilon_j = 0.7 > 0.$$

The minimum expenditure associated with $t_1 = \varepsilon_1$ is $EX(1; (\underline{R}^{fd}(1), \bar{B}^{fd}(1))) = 8.8$.

According to Proposition 3, $t_1 = \varepsilon_2$ is implementable, as $\beta^{fd}(1,2) < \beta^{fd}(2,3)$. Proposition 1.6 implies that full-surplus-extraction is possible with $t_1 = \varepsilon_2$, which means the minimum expenditure associated with $t_1 = \varepsilon_2$ equals to $(p_1 + p_2)\varepsilon_2 + p_3\varepsilon_3 = 10.5$, which is larger than that is required when $t_1 = \varepsilon_1$ is implemented.

3.3. Example of $t_1 = \varepsilon_2$ cheaper to implement than $t_1 = \varepsilon_1$.

Let $N = 3$ and

$$\mathcal{E}(a, x) = \begin{cases} 1 & \text{w.p. } 0.4 \\ 6 & \text{w.p. } 0.4 \cdot \\ 14 & \text{w.p. } 0.2 \end{cases}$$

Then

$$\beta^{fd}(1,2) = \frac{p_1(\varepsilon_2 - \varepsilon_1)}{p_2} = 5,$$
$$\beta^{fd}(1,3) = \frac{p_1(\varepsilon_3 - \varepsilon_1) + p_2(\varepsilon_3 - \varepsilon_2)}{p_2 + p_3} = 14,$$
$$\beta^{fd}(2,3) = \frac{(p_1 + p_2)(\varepsilon_3 - \varepsilon_2)}{p_3} = 32.$$

Then $\bar{B}^{fd}(1) = 5$, $\underline{R}^{fd}(1) = 13$, and FBME is not feasible, since

$$U^{fd}(1; (\underline{R}^{fd}(1), \bar{B}^{fd}(1))) = \underline{R}^{fd}(1) - (p_2 + p_3)\bar{B}^{fd}(1) - \sum_{j=1}^{3} p_j \varepsilon_j = 4.4 > 0.$$

The minimum expenditure associated with $t_1 = \varepsilon_1$ is $EX(1; (\underline{R}^{fd}(1), \bar{B}^{fd}(1))) = 10$.

According to Proposition 3, $t_1 = \varepsilon_2$ is implementable, as $\beta^{fd}(1,2) < \beta^{fd}(2,3)$. Proposition 1.6 implies that full-surplus-extraction is possible with $t_1 = \varepsilon_2$, which means the minimum expenditure associated with $t_1 = \varepsilon_2$ equals to $(p_1 + p_2)\varepsilon_2 + p_3\varepsilon_3 = 7.6$, which is smaller than that is required when $t_1 = \varepsilon_1$ is implemented.

3.4. Finding all implementable test schedules

Let $N = 5$ and

$$\mathcal{E}(a, x) = \begin{cases} 1 & \text{w.p. } 0.1 \\ 3 & \text{w.p. } 0.25 \\ 5 & \text{w.p. } 0.3 \\ 8 & \text{w.p. } 0.25 \\ 9 & \text{w.p. } 0.1 \end{cases}.$$

(1) Calculate which test is to be skipped first, as b increases from zero.

$$B_{1|0,2} = 0.22;$$

$$B_{2|1,3} = 0.77;$$

$$B_{3|2,4} = 2.57;$$

$$B_{4|3,5} = 2.50.$$

When $0 \le b \le 0.22$, $\{1, 2, 3, 4, 5\}$ is implemented.

(2) Since $B_{1|0,2}$ is the smallest, the next test schedule that can be implemented is $\{2, 3, 4, 5\}$. As b keeps increasing from 0.22, another test should be skipped. According to Lemma 3, having skipped ε_1 has no influence on the test on ε_3 or the test on ε_4, as they are not adjacent to the skipped test. Therefore, one only needs to calculate

$$B_{2|0,3} = 1.08.$$

When $0.22 < b \le 1.08$, $\{2, 3, 4, 5\}$ is implemented.

(3) Since $B_{2|0,3}$ is the smallest among $B_{2|0,3}$, $\beta^{md}_{3|2,4}$ and $B_{4|3,5}$, as b increases above 1.08, the next test to be skipped is ε_2. However, based on Proposition **??**, the agent has to check whether he wants to take a test on ε_1, as a result of skipping the one on ε_2. However, as

$$B_{1|0,3} = 0.44 < 1.08,$$

it is not profitable to add back the test at ε_1. Therefore, the next test that can be implemented is $\{3, 4, 5\}$. As b increases from 1.08, to find the test that is to be skipped next, in combination of Lemma 3, one only needs to calculate:

$$B_{3|0,4} = 5.57,$$

and compare it with $B_{4|3,5}$. As $B_{4|3,5} < B_{3|0,4}$, the next test to be skipped is ε_4, and test schedule $\{3, 4, 5\}$ is implemented by $1.08 < b \leq 2.50$.

(4) As b raises above 2.50, the test at ε_4 is skipped. To see what happens next, calculate:

$$B_{3|0,5} = 7.43.$$

Note that when ε_3 is skipped, the following calculations are required, according to Proposition **??**:

$$B_{1|0,5} = 0.89;$$

$$B_{2|0,5} = 3.23;$$

$$B_{4|0,5} = 9.00.$$

Since $B_{4|0,5} > B_{3|0,5} = 7.43$, the agent needs to solve for the test fee that makes him indifferent between test schedule $\{3, 5\}$ and test schedule $\{4, 5\}$:

$$\beta^{md}(\{3, 5\}, \{4, 5\}) = 6.80.$$

Hence, test schedule $\{3, 5\}$ is implemented for $2.50 < b \leq 6.80$, while test schedule $\{4, 5\}$ is implemented for $6.80 < b \leq 9.00$. When $b > 9.00$, test schedule $\{5\}$ becomes optimal.

The whole set of implementable test schedules is summarized below:

$$\mathcal{T}^* = \begin{cases} \{1,2,3,4,5\} & 0 \le b \le 0.22 \\ \{2,3,4,5\} & 0.22 < b \le 1.08 \\ \{3,4,5\} & 1.08 < b \le 2.50 \\ \{3,5\} & 2.50 < b \le 6.80 \\ \{4,5\} & 6.80 < b \le 9.00 \\ \{5\} & b > 9 \end{cases}.$$

This conclusion can be confirmed by graphing the *ex-ante* utility associated with each possible test schedule as a function of the test fee. The upper envelop consists of all the test schedules that can be implemented.

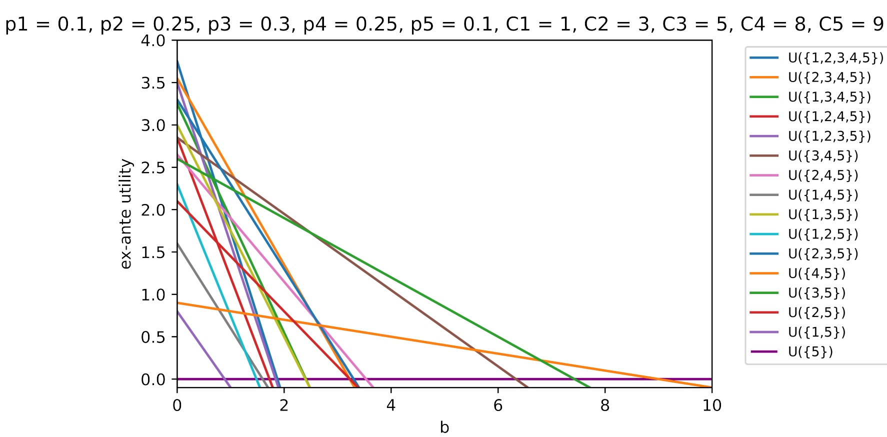

Figure 3.1. Illustration of Implementable Test Schedules

3.5. Mathematical Details of Bundling a_{N-1} and a_N.

Ex-ante expected utility of taking the first test at ε_i, and retake the test at ε_{N-1} in case the grade is $g_{L'}$ is

$$U_0(t_1 = \varepsilon_i, t_{2|g_{L'}} = \varepsilon_{N-1}) = r - \left(\sum_{j=i+1}^{N-1} p_j + 2p_N\right) b - \left(\sum_{j=1}^{i} p_j\right) \varepsilon_i - \sum_{j=i+1}^{N} p_j \varepsilon_j.$$

Ex-ante expected utility of taking the first test at ε_i, and the agents with ability a_{N-1} and those with a_N get different grades at that point, assuming that the letter grade system is fine enough so that no other agents get bundled together, is

$$U_0(t_1 = \varepsilon_i, \text{no bundling}) = r - \left(\sum_{j=1}^{i} p_j\right) \varepsilon_i - \sum_{j=i+1}^{N} p_j \left(b + \varepsilon_j\right).$$

The threshold test fee above which $t_1 = \varepsilon_i$ is preferred to $t_1 = \varepsilon_1$ is

$$\beta\left(1, i|_{t_{2|g_{L'}} = \varepsilon_{N-1}}\right) = \frac{\sum_{j=1}^{i} p_j \left(\varepsilon_i - \varepsilon_j\right)}{\sum_{j=2}^{i} p_j}$$

in case that $t_{2|g_{L'}} = \varepsilon_{N-1}$, or

$$\beta\left(1, i|_{\text{no bundling}}\right) = \frac{\sum_{j=1}^{i} p_j \left(\varepsilon_i - \varepsilon_j\right)}{\sum_{j=2}^{i} p_j + p_N}$$

in case that no failing agents are bundled with others.

As for the feasibility of FBME,

(1) If $\sum_{j=2}^{N-1} p_j + 2p_N \geq \dfrac{p_N}{p_{N-1} + p_N}$, FBME is feasible when

$$\left(\sum_{j=2}^{N-1} p_j + 2p_N\right) \tilde{b} + p_1 \varepsilon_1 + \sum_{j=2}^{N} p_j \varepsilon_j \geq \frac{p_{N-1}}{p_{N-1} + p_N}(\varepsilon_{N-1} - \varepsilon_1) + \frac{p_N}{p_{N-1} + p_N}(\tilde{b} + \varepsilon_N - \varepsilon_1),$$

where $\tilde{b} = \min_{i=2,\ldots,N} \beta(1,i)$.[1]

(2) If $\sum_{j=2}^{N-1} p_j + 2p_N < \dfrac{p_N}{p_{N-1}+p_N}$, FBME is feasible when

$$p_1\varepsilon_1 + \sum_{j=2}^{N} p_j\varepsilon_j \geq \frac{p_{N-1}}{p_{N-1}+p_N}(\varepsilon_{N-1}-\varepsilon_1) + \frac{p_N}{p_{N-1}+p_N}(\varepsilon_N-\varepsilon_1),$$

or there exist $\tilde{b} \in \left[0, \min\left\{b^*, \min_{i=2,\ldots,N} \beta(1,i)\right\}\right)$ such that

$$\left(\sum_{j=2}^{N-1} p_j + 2p_N\right)\tilde{b} + p_1\varepsilon_1 + \sum_{j=2}^{N} p_j\varepsilon_j \geq \frac{p_{N-1}}{p_{N-1}+p_N}(\varepsilon_{N-1}-\varepsilon_1) + \frac{p_N}{p_{N-1}+p_N}(\tilde{b}+\varepsilon_N-\varepsilon_1),$$

$$\text{where } b^* = \frac{\dfrac{p_{N-1}}{p_{N-1}+p_N}\varepsilon_{N-1} + \dfrac{p_N}{p_{N-1}+p_N}\varepsilon_N - \varepsilon_1 - \sum_{i=1}^{N} p_i\varepsilon_i}{\sum_{i=2}^{N-1} p_i + 2p_N - \dfrac{p_N}{p_{N-1}+p_N}}.$$

3.6. Example of FBME Being Feasible with Letter-Grade but Not Feasible with Full-Disclosing ($N = 4$, a_2&a_3 Bundle)

Let $N = 4$ and

$$\mathcal{E}(a,x) = \begin{cases} 2 & \text{w.p. } 0.3 \\ 3 & \text{w.p. } 0.25 \\ 4 & \text{w.p. } 0.15 \\ 7 & \text{w.p. } 0.3 \end{cases}.$$

[1]Whether $\beta(1,i) = \beta\left(1, i|_{t_{2|g_{L'}}=\varepsilon_{N-1}}\right)$ or $\beta(1,i) = \beta\left(1, i|_{\text{no bundling}}\right)$ depends on i.

Then

$$\beta^{fd}(1,2) = \frac{p_1(\varepsilon_2 - \varepsilon_1)}{p_2} = 1.2,$$

$$\beta^{fd}(1,3) = \frac{p_1(\varepsilon_3 - \varepsilon_1) + p_2(\varepsilon_3 - \varepsilon_2)}{p_2 + p_3} = 2.125,$$

$$\beta^{fd}(1,4) = \frac{p_1(\varepsilon_4 - \varepsilon_1) + p_2(\varepsilon_4 - \varepsilon_2) + p_3(\varepsilon_4 - \varepsilon_3)}{p_2 + p_3 + p_4} = 4.214,$$

$$\bar{B}^{fd}(1) = \min\{\beta^{fd}(1,2), \beta^{fd}(1,3), \beta^{fd}(1,4)\} = 1.2,$$

$$\underline{R}^{fd}(1) = \varepsilon_4 - \varepsilon_1 = 5,$$

$$U^{fd}\left(1; (\underline{R}^{fd}(1), \bar{B}^{fd}(1))\right) = \underline{R}^{fd}(1) - \sum_{i=1}^{4} p_i \varepsilon_i - (p_2 + p_3 + p_4) \cdot \bar{B}^{fd}(1) = 0.11 > 0,$$

while

$$\beta^{lg}(1,2) = \frac{p_1(\varepsilon_2 - \varepsilon_1)}{p_2 + p_3 - p_4} = 3,$$
$$\beta^{lg}(1,3) = \frac{p_1(\varepsilon_3 - \varepsilon_1) + p_2(\varepsilon_3 - \varepsilon_2)}{p_2 + 2p_3} = 1.545,$$
$$\beta^{lg}(1,4) = \frac{p_1(\varepsilon_4 - \varepsilon_1) + p_2(\varepsilon_4 - \varepsilon_2) + p_3(\varepsilon_4 - \varepsilon_3)}{p_2 + 2p_3 + p_4} = 3.4706,$$
$$\beta^{lg}(2,3|\{\varepsilon_1, g_1\}) = \frac{p_2(\varepsilon_3 - \varepsilon_2)}{p_3} = 1.667$$
$$\beta^{lg}(3,4|\{\varepsilon_2, g_1\}) = \frac{p_3(\varepsilon_4 - \varepsilon_3)}{p_4} = 1.5$$
$$\frac{\partial U^{lg}(1;(r,b))}{\partial b} = p_2 + 2p_3 + p_4 = 0.85$$
$$\frac{\partial V^{lg}\left(\{\varepsilon_1, g_1\}\right)}{\partial b} = \frac{p_3}{p_2 + p_3} = 0.375 < 0.85$$
$$\bar{\boldsymbol{B}}^{lg}(1) = \min\{\beta^{lg}(1,2), \beta^{lg}(1,3), \beta^{lg}(1,4), \beta^{lg}(2,3|\{\varepsilon_1, g_1\}), \beta^{lg}(3,4|\{\varepsilon_2, g_1\})\} = 1.5,$$
$$\underline{\boldsymbol{R}}^{lg}(1) = \varepsilon_4 - \varepsilon_1 = 5,$$
$$U^{lg}\left(1;(\underline{\boldsymbol{R}}^{lg}(1), \bar{\boldsymbol{B}}^{lg}(1))\right) = \underline{\boldsymbol{R}}^{lg}(1) - \sum_{i=1}^{4} p_i\varepsilon_i - (p_2 + 2p_3 + p_4) \cdot \bar{\boldsymbol{B}}^{lg}(1) = -0.325 < 0.$$

Thus, FBME is possible with a letter-grade system but not possible with full-disclosing.

3.7. Example of FBME Being Feasible with Letter-Grade but Not Feasible with Full-Disclosing nor Minimal-Disclosing ($N = 4$, $a_3\&a_4$ Bundle)

Let $N = 4$ and

$$\mathcal{E}(a,x) = \begin{cases} 1.5 & \text{w.p. } 0.2 \\ 2 & \text{w.p. } 0.3 \\ 3 & \text{w.p. } 0.3 \\ 6.25 & \text{w.p. } 0.2 \end{cases}.$$

Then

$$\beta^{fd}(1,2) = \frac{p_1(\varepsilon_2 - \varepsilon_1)}{p_2} = 0.333,$$

$$\beta^{fd}(1,3) = \frac{p_1(\varepsilon_3 - \varepsilon_1) + p_2(\varepsilon_3 - \varepsilon_2)}{p_2 + p_3} = 1,$$

$$\beta^{fd}(1,4) = \frac{p_1(\varepsilon_4 - \varepsilon_1) + p_2(\varepsilon_4 - \varepsilon_2) + p_3(\varepsilon_4 - \varepsilon_3)}{p_2 + p_3 + p_4} = 4,$$

$$\bar{B}^{fd}(1) = \min\{\beta^{fd}(1,2), \beta^{fd}(1,3), \beta^{fd}(1,4)\} = 0.333,$$

$$\underline{R}^{fd}(1) = \varepsilon_4 - \varepsilon_1 = 4.75,$$

$$U^{fd}\left(1; (\underline{R}^{fd}(1), \bar{B}^{fd}(1))\right) = \underline{R}^{fd}(1) - \sum_{i=1}^{4} p_i\varepsilon_i - (p_2 + p_3 + p_4) \cdot \bar{B}^{fd}(1) = 1.433 > 0,$$

and

$$\beta^{md}(\{1,2,3,4\},\{2,3,4\}) = \frac{p_1(\varepsilon_2 - \varepsilon_1)}{p_2 + p_3 + p_4} = 0.125,$$

$$\beta^{md}(\{1,2,3,4\},\{1,3,4\}) = \frac{p_2(\varepsilon_3 - \varepsilon_2)}{p_3 + p_4} = 0.6,$$

$$\beta^{md}(\{1,2,3,4\},\{1,2,4\}) = \frac{p_3(\varepsilon_4 - \varepsilon_3)}{p_4} = 4.875,$$

$$\frac{\partial U^{md}\left(\{1,2,3,4\};(r,b)\right)}{\partial b} = p_2 + 2p_3 + 3p_4 = 1.5$$

$$\frac{\partial V^{md}\left(\{2,3,4\};(r,b)\right)}{\partial b} = \frac{p_3}{p_2 + p_3 + p_4} + \frac{2p_4}{p_2 + p_3 + p_4} = 0.875 < 1.5$$

$$\frac{\partial V^{md}\left(\{3,4\};(r,b)\right)}{\partial b} = \frac{p_4}{p_3 + p_4} = 0.4 < 1.5$$

$$\bar{B}^{md}(1) = \min\{\beta^{md}(1,2), \beta^{md}(1,3), \beta^{md}(1,4)\} = 0.125,$$

$$\underline{R}^{md}(1) = 3.25$$

$$U^{md}\left(1;(\underline{R}^{md}(1), \bar{B}^{md}(1))\right) = \underline{R}^{md}(1) - \sum_{i=1}^{4} p_i\varepsilon_i - (p_2 + 2p_3 + 3p_4)\cdot \bar{B}^{md}(1) = 0.0125 > 0.$$

Assuming that for all s_1 such that $S(a_3, \varepsilon_1) < s_1 \leq S(a_2, \varepsilon_1)$, one has $S(a_4, \varepsilon_2) < s_1 \leq S(a_3, \varepsilon_2)$, then

$$\beta^{lg}(1,2) = \frac{p_1(\varepsilon_2 - \varepsilon_1)}{p_2 + p_4} = 0.2,$$

$$\beta^{lg}(1,3) = \frac{p_1(\varepsilon_3 - \varepsilon_1) + p_2(\varepsilon_3 - \varepsilon_2)}{p_2 + p_3 + p_4} = 0.75,$$

$$\beta^{lg}(1,4) = \frac{p_1(\varepsilon_4 - \varepsilon_1) + p_2(\varepsilon_4 - \varepsilon_2) + p_3(\varepsilon_4 - \varepsilon_3)}{p_2 + p_3 + 2p_4} = 3.2,$$

$$\beta^{lg}(3,4|\{\varepsilon_1, g_2\}) = \frac{p_3(\varepsilon_4 - \varepsilon_3)}{p_4} = 4.875$$

$$\frac{\partial U^{lg}\,(1;(r,b))}{\partial b} = p_2 + p_3 + 2p_4 = 1$$

$$\frac{\partial V^{lg}\,\left(\{\varepsilon_1, g_2\};(r,b)\right)}{\partial b} = \frac{p_4}{p_3 + p_4} = 0.4 < 1$$

$$\bar{B}^{lg}(1) = \min\{\beta^{lg}(1,2), \beta^{lg}(1,3), \beta^{lg}(1,4), \beta^{lg}(3,4|\{\varepsilon_1, g_2\})\} = 0.2,$$

$$\underline{R}^{lg}(1) = 3.25,$$

$$U^{lg}\,\left(1;(\underline{R}^{lg}(1), \bar{B}^{lg}(1))\right) = \underline{R}^{lg}(1) - \sum_{i=1}^{4} p_i\varepsilon_i - (p_2 + p_3 + 2p_4)\cdot \bar{B}^{lg}(1) = 0.$$

If there exists s_1 such that $S(a_3, \varepsilon_2) < s_1 \leq S(a_2, \varepsilon_1)$, then

$$\beta^{lg}(1,2) = \frac{p_1(\varepsilon_2 - \varepsilon_1)}{p_2} = 0.333,$$

$$\bar{B}^{lg}(1) = 0.333,$$

$$\underline{R}^{lg}(1) = 3.25,$$

$$U^{lg}\,\left(1;(\underline{R}^{lg}(1), \bar{B}^{lg}(1))\right) = \underline{R}^{lg}(1) - \sum_{i=1}^{4} p_i\varepsilon_i - (p_2 + p_3 + 2p_4)\cdot \bar{B}^{lg}(1) = -0.133 < 0.$$

In both cases, FBME is feasible under a letter-grade system.

3.8. Brief Discussion on Random Grade Generating System.

As has been shown in Proposition 13, any effort level that triggers a belief-updating once a test is taken is a candidate for the agent's test taking agenda. This inevitably introduces extra constraints on the test fee that are hard to keep track of. However, these constrains only emerge when the principal tries to raise the test fee. If the test fee is set to be zero, then there is no downside for an agent to take as many tests as he wants. The principal's job transforms into designing the grade generating system that induces the agents to keep working by simply reshaping their beliefs along the way.

CHAPTER 4

Appendix for Chapter Two

4.1. Derivation of Equation 2.3

The expected net profit for bidder 2 with signal $S = s$ and bid b is

$$\begin{aligned}\Pi(b,s) =& F_Z(b)\cdot(\mathbb{E}[v(X,S)|Z<b,S=s]-\mathbb{E}[Z|Z<b])\\ =& F_Z(b)\cdot(\mathbb{E}[XS|Z<b,S=s]-\mathbb{E}[Z|Z<b])\\ =& F_Z(b)\cdot(\mathbb{E}[X|Z<b]\cdot s-\mathbb{E}[Z|Z<b])\end{aligned} \tag{4.1}$$

with $F_Z(b)$ being bidder 2's probability of winning with bid b, $\mathbb{E}[v(X,S)|Z < b, S = s]$ being his expected valuation for the object conditional on winning, and $\mathbb{E}[Z|Z < b]$ his expected payment when he wins. The last equality of Equation 2.3 uses the assumption that S and (X, W) are independently distributed.

To derive $F_X(x|Z < b)$, the cumulative distribution function of X conditional on $Z < b$, note that if $0 \leq x < b$, the condition $Z < b$ is automatically satisfied for all $w \in [0, \bar{w}]$. Hence, for $0 \leq x < b$,

$$F_X(x|Z<b) = \frac{1}{F_Z(b)}\int_0^{\bar{w}}\int_0^x f(x,w)dx,dw = \frac{F_X(x)}{F_Z(b)}.$$

If $b \leq x < \bar{x}$, the condition of $Z < b$ is equivalent to $W < b$. Hence, for $b \leq x < \bar{x}$,

$$\begin{aligned} F_X(x|Z < b) =& F_X(b|Z < b) + \frac{1}{F_Z(b)} \int_b^x \int_0^b f(x, w) dw dx \\ =& \frac{F_X(b) + \left[F_X(x) - F_X(b)\right] F_W(b)}{F_Z(b)}. \end{aligned}$$

The cases with $x < 0$ and $x > \bar{x}$ are trivial. Therefore,

$$F_X(x|Z < b) = \begin{cases} 0 & x < 0 \\ \frac{F_X(x)}{F_Z(b)} & 0 \leq x < b \\ \frac{F_X(b) + \left[F_X(x) - F_X(b)\right] F_W(b)}{F_Z(b)} & b \leq x < \bar{x} \\ 1 & x \geq \bar{x} \end{cases}.$$

$\mathbb{E}[X|Z < b]$ can then be calculated as

$$\mathbb{E}[X|Z < b] = \int_0^{\bar{x}} x dF_X(x|Z < b) = \frac{1}{F_Z(b)} \left(\int_0^b x f_X(x) dx + F_W(b) \int_b^{\bar{x}} x f_X(x) dx \right).$$

The term $\mathbb{E}[Z|Z < b]$ in Equation 4.1 can be easily computed as

$$\begin{aligned} \mathbb{E}[Z|Z < b] =& \frac{1}{F_Z(b)} \int_0^b z dF_Z(z) \\ =& \frac{1}{F_Z(b)} \int_0^b z \cdot \left[\left(1 - F_W(z)\right) \cdot f_X(z) + \left(1 - F_X(z)\right) \cdot f_W(z) \right] dz. \end{aligned}$$

Using the results above, bidder 2's expected net profit can be written as[1]

$$
\begin{aligned}
\Pi(b,s) =& F_Z(b)\cdot\left\{\frac{1}{F_Z(b)}\left(\int_0^b x f_X(x)dx + F_W(b)\int_b^{\bar{x}} x f_X(x)dx\right)\cdot s\right.\\
&\left.-\frac{1}{F_Z(b)}\int_0^b z\cdot\left[\left(1-F_W(z)\right)\cdot f_X(z)+\left(1-F_X(z)\right)\cdot f_W(z)\right]dz\right\}\\
=&\left(\int_0^b x f_X(x)dx + F_W(b)\int_b^{\bar{x}} x f_X(x)dx\right)\cdot s\\
&-\int_0^b z\cdot\left[\left(1-F_W(z)\right)\cdot f_X(z)+\left(1-F_X(z)\right)\cdot f_W(z)\right]dz.
\end{aligned}
\tag{4.2}
$$

4.2. Proof of Corollary 15.1

Property 1. *See Lemma 5.*

Property 2. *When $\bar{w} < \bar{x}$, $\beta(s,\bar{x},\bar{w}) = \min\{\bar{x},\bar{w}\}$ for all $s \in \left[\frac{2\bar{w}}{\bar{x}+\bar{w}}, 1\right]$, and therefore the property holds. When $\bar{w} \geq \bar{x}$,*

$$\lim_{s\to 1}\beta(s,\bar{x},\bar{w}) = \lim_{s\to 1}\frac{(1-s)\bar{w}+\bar{x}-\sqrt{[(1-s)\bar{w}+\bar{x}]^2-(4-3s)s\bar{x}^2}}{4-3s} = \bar{x} = \min\{\bar{x},\bar{w}\}.$$

Property 3.

$$\lim_{s\to 0}\beta(s,\bar{x},\bar{w}) = \lim_{s\to 0}\frac{(1-s)\bar{w}+\bar{x}-\sqrt{[(1-s)\bar{w}+\bar{x}]^2-(4-3s)s\bar{x}^2}}{4-3s} = 0.$$

Property 4. *When $\bar{x} \leq \bar{w}$,*

$$\beta(s,\bar{x},\bar{w}) = \frac{(1-s)\bar{w}+\bar{x}-\sqrt{[(1-s)\bar{w}+\bar{x}]^2-(4-3s)s\bar{x}^2}}{4-3s} = \bar{x} = \min\{\bar{x},\bar{w}\} \quad \textit{for } s \in [0,1].$$

[1]With the assumption that (X,W) has uniform density on $[0,\bar{x}]\times[0,\bar{w}]$, one has

$$\Pi(b,s) = \frac{4-3s}{6\bar{w}\bar{x}}\cdot b^3 - \frac{(1-s)\bar{w}+\bar{x}}{2\bar{w}\bar{x}}\cdot b^2 + \frac{s\bar{x}}{2\bar{w}}\cdot b.$$

And

$$
\begin{aligned}
\frac{\partial\beta(s,\bar{x},\bar{w})}{\partial\bar{x}} =& \frac{1}{4-3s}\left(1-\frac{[(1-s)\bar{w}+\bar{x}]-(4-3s)s\bar{x}}{\sqrt{[(1-s)\bar{w}+\bar{x}]^2-(4-3s)s\bar{x}^2}}\right)\\
=& \frac{1}{4-3s}\left(1-\frac{1}{(1-s)\bar{w}+\bar{x}}\cdot\frac{[(1-s)\bar{w}+\bar{x}]^2-(4-3s)s\bar{x}^2-(4-3s)(1-s)s\bar{x}\bar{w}}{\sqrt{[(1-s)\bar{w}+\bar{x}]^2-(4-3s)s\bar{x}^2}}\right)\\
>& \frac{1}{4-3s}\left(1-\frac{1}{(1-s)\bar{w}+\bar{x}}\cdot\frac{[(1-s)\bar{w}+\bar{x}]^2-(4-3s)s\bar{x}^2}{\sqrt{[(1-s)\bar{w}+\bar{x}]^2-(4-3s)s\bar{x}^2}}\right)\\
=& \frac{1}{4-3s}\left(1-\underbrace{\frac{\sqrt{[(1-s)\bar{w}+\bar{x}]^2-(4-3s)s\bar{x}^2}}{(1-s)\bar{w}+\bar{x}}}_{<1}\right) > 0.
\end{aligned}
$$

When $\bar{x} > \bar{w}$*,* $\frac{\partial\beta(s,\bar{x},\bar{w})}{\partial\bar{x}} > 0$ *for* $s \in \left[0, \frac{2\bar{w}}{\bar{x}+\bar{w}}\right)$ *using the same calculation as above, and* $\beta(s,\bar{x},\bar{w}) = \min\{\bar{x},\bar{w}\}$ *remains the same for* $s \in \left[\frac{2\bar{w}}{\bar{x}+\bar{w}}, 1\right]$ *as* $\bar{x}$ *increases.*

Property 5. *When* $\bar{w} < \bar{x}$*,* $\beta(s,\bar{x},\bar{w}) = \min\{\bar{x},\bar{w}\}$ *remains the same for* $s \in \left[\frac{2\bar{w}}{\bar{x}+\bar{w}}, 1\right]$ *as* $\bar{w}$ *increases (note that* $s^*(\bar{x},\bar{w}) = \frac{2\bar{w}}{\bar{x}+\bar{w}}$ *changes with* $\bar{w}$*.), whereas for* $s \in \left[0, \frac{2\bar{w}}{\bar{x}+\bar{w}}\right)$*,*

$$
\frac{\partial\beta(s,\bar{x},\bar{w})}{\partial\bar{w}} = \frac{1-s}{4-3s}\left[1-\frac{(1-s)\bar{w}+\bar{x}}{\sqrt{[(1-s)\bar{w}+\bar{x}]^2-(4-3s)s\bar{x}^2}}\right] < 0.
$$

When $\bar{w} \geq \bar{x}$*,* $\frac{\partial\beta(s,\bar{x},\bar{w})}{\partial\bar{w}} < 0$ *based on the same calculation as above, for all* $s \in [0,1]$*. In fact, it is easy to verify that* $\lim_{\bar{w}\to\infty}\beta(s,\bar{x},\bar{w}) = 0$*, which is intuitive, as* $\bar{w} \to \infty$ *while maintaining the assumption of* w *has uniform distribution on* $[0,\bar{w}]$ *implies that the probability that the group* $\mathcal{I}$ *is budget-constrained is zero. Any winning by the group* $\mathcal{O}$ *brings a loss to him.*

4.3. Derivation of Equation 2.12 and Equation 2.13

Plugging $\beta(s)$ into Equation 2.11 directly yields:

$$
\begin{aligned}
&EP_1^{ex\ ante} \\
&= \int_0^{\min\{\bar{x},\bar{w}\}} \left(\int_0^{\beta^{-1}(z)} \beta(s) ds \right) dF_Z(z) \\
&= \int_0^{\min\{\bar{x},\bar{w}\}} \left(\int_0^{\frac{2(\bar{x}+\bar{w}-2z)z}{\bar{x}^2+2\bar{w}z-3z^2}} \frac{(1-s)\bar{w}+\bar{x}-\sqrt{[(1-s)\bar{w}+\bar{x}]^2-(4-3s)s\bar{x}^2}}{4-3s} ds \right) \frac{\bar{x}+\bar{w}-2z}{\bar{x}\bar{w}} dz.
\end{aligned}
$$

However, the integration above is hard to solve. Fortunately, the special relation between the integrand and the upper limit enables one to transform the integral into a workable form:

$$
\begin{aligned}
EP_1^{ex\ ante} &= \int_0^{\min\{\bar{x},\bar{w}\}} \left(\int_0^{\beta^{-1}(z)} \beta(s) ds \right) dF_Z(z) \\
&= \int_0^{\min\{\bar{x},\bar{w}\}} \left(\int_0^{z} \left[\beta^{-1}(z) - \beta^{-1}(y)\right] dy \right) dF_Z(z) \\
&= \int_0^{\min\{\bar{x},\bar{w}\}} \left[\int_0^{z} \left(\frac{2(\bar{x}+\bar{w}-2z)z}{\bar{x}^2+2\bar{w}z-3z^2} - \frac{2(\bar{x}+\bar{w}-2y)y}{\bar{x}^2+2\bar{w}y-3y^2} \right) dy \right] \frac{\bar{x}+\bar{w}-2z}{\bar{x}\bar{w}} dz
\end{aligned}
$$

It can be calculated that:

$$
\begin{aligned}
&\int_0^{z} \left(\frac{2(\bar{x}+\bar{w}-2z)z}{\bar{x}^2+2\bar{w}z-3z^2} - \frac{2(\bar{x}+\bar{w}-2y)y}{\bar{x}^2+2\bar{w}y-3y^2} \right) dy \\
&= \frac{2z^2(\bar{w}+\bar{x}-2z)}{\bar{x}^2+2\bar{w}z-3z^2} - \frac{4z}{3} + \frac{1}{9}(\bar{w}-3\bar{x}) \log\left(\frac{\bar{x}^2}{\bar{x}^2+2\bar{w}z-3z^2} \right) \\
&+ \frac{2\left(\bar{w}^2-3\bar{w}\bar{x}+6\bar{x}^2\right)}{9\sqrt{\bar{w}^2+3\bar{x}^2}} \left[\arctan\left(\frac{\bar{w}}{\sqrt{\bar{w}^2+3\bar{x}^2}} \right) - \arctan\left(\frac{\bar{w}-3z}{\sqrt{\bar{w}^2+3\bar{x}^2}} \right) \right].
\end{aligned}
$$

Further calculation shows that if $\bar{x} \le \bar{w}$,

$$\int_0^{\min\{\bar{x},\bar{w}\}} \int_0^z \left(\frac{2(\bar{x}+\bar{w}-2z)z}{\bar{x}^2+2\bar{w}z-3z^2} - \frac{2(\bar{x}+\bar{w}-2y)y}{\bar{x}^2+2\bar{w}y-3y^2} \right) dy f_Z(z) dz$$

$$= \frac{\left(\bar{w}^2-3\bar{w}\bar{x}+6\bar{x}^2\right)\left(\bar{w}+3\bar{x}+\sqrt{\bar{w}^2+3\bar{x}^2}\right)}{81\bar{w}\bar{x}} \ln\left(\frac{3\bar{x}-\bar{w}+\sqrt{\bar{w}^2+3\bar{x}^2}}{-\bar{w}+\sqrt{\bar{w}^2+3\bar{x}^2}} \right)$$

$$+ \frac{\left(\bar{w}^2-3\bar{w}\bar{x}+6\bar{x}^2\right)\left(\bar{w}+3\bar{x}-\sqrt{\bar{w}^2+3\bar{x}^2}\right)}{81\bar{w}\bar{x}} \ln\left(\frac{-3\bar{x}+\bar{w}+\sqrt{\bar{w}^2+3\bar{x}^2}}{\bar{w}+\sqrt{\bar{w}^2+3\bar{x}^2}} \right)$$

$$+ \frac{2\left(\bar{w}^4-9\bar{w}^2\bar{x}^2+18\bar{w}\bar{x}^3-90\bar{x}^4\right)}{81\bar{w}\bar{x}\sqrt{\bar{w}^2+3\bar{x}^2}} \left[\arctan\left(\frac{\bar{w}-3\bar{x}}{\sqrt{\bar{w}^2+3\bar{x}^2}} \right) - \arctan\left(\frac{\bar{w}}{\sqrt{\bar{w}^2+3\bar{x}^2}} \right) \right]$$

$$- \frac{(\bar{w}-3\bar{x})\left(\bar{w}^2+15\bar{x}^2\right)}{81\bar{w}\bar{x}} \ln\left(\frac{2(\bar{w}-\bar{x})}{\bar{x}} \right) + \frac{\bar{x}(\bar{w}-15\bar{x})}{9\bar{w}}.$$

If $\bar{w} < \bar{x}$, then

$$\int_0^{\min\{\bar{x},\bar{w}\}} \int_0^z \left(\frac{2(\bar{x}+\bar{w}-2z)z}{\bar{x}^2+2\bar{w}z-3z^2} - \frac{2(\bar{x}+\bar{w}-2y)y}{\bar{x}^2+2\bar{w}y-3y^2} \right) dy f_Z(z) dz$$

$$= \frac{\left(\bar{w}^2-3\bar{w}\bar{x}+6\bar{x}^2\right)\left(\bar{w}+3\bar{x}+\sqrt{\bar{w}^2+3\bar{x}^2}\right)}{81\bar{w}\bar{x}} \ln\left(\frac{2\bar{w}+\sqrt{\bar{w}^2+3\bar{x}^2}}{-\bar{w}+\sqrt{\bar{w}^2+3\bar{x}^2}} \right)$$

$$+ \frac{\left(\bar{w}^2-3\bar{w}\bar{x}+6\bar{x}^2\right)\left(\bar{w}+3\bar{x}-\sqrt{\bar{w}^2+3\bar{x}^2}\right)}{81\bar{w}\bar{x}} \ln\left(\frac{-2\bar{w}+\sqrt{\bar{w}^2+3\bar{x}^2}}{\bar{w}+\sqrt{\bar{w}^2+3\bar{x}^2}} \right)$$

$$- \frac{2\left(\bar{w}^4-9\bar{w}^2\bar{x}^2+18\bar{w}\bar{x}^3-90\bar{x}^4\right)}{81\bar{w}\bar{x}\sqrt{\bar{w}^2+3\bar{x}^2}} \left[\arctan\left(\frac{2\bar{w}}{\sqrt{\bar{w}^2+3\bar{x}^2}} \right) + \arctan\left(\frac{\bar{w}}{\sqrt{\bar{w}^2+3\bar{x}^2}} \right) \right]$$

$$- \frac{(\bar{w}-3\bar{x})\left(\bar{w}^2+15\bar{x}^2\right)}{81\bar{w}\bar{x}} \ln\left(\frac{\bar{x}^2-\bar{w}^2}{\bar{x}^2} \right) - \frac{3\bar{w}^2-13\bar{w}\bar{x}+24\bar{x}^2}{9\bar{x}}.$$

4.4. Derivation of Equation 2.19 and 2.20

$$
\begin{aligned}
&\mathbb{E}[v_I(X_1, ...X_N)|Y < b] \\
=&\mathbb{E}[X_{(1)}|Y < b] \\
=&\frac{N}{F_Y(b)} \int_0^{\bar{x}} \int_0^{x_1} ... \int_0^{x_1} x_1 \left[\prod_{i=1}^{N} \left(\mathbb{I}\left\{x_i < b\right\} \cdot f_X(x_i) + \mathbb{I}\left\{x_i \geq b\right\} \cdot f_X(x_i) \cdot F_W(b) \right) \right] \\
& dx_2...dx_N \cdot dx_1 \\
=&\frac{N}{F_Y(b)} \left\{ \int_0^{b} x_1 \left[\prod_{i=2}^{N} \left(\int_0^{x_1} f_X(x_i)dx_i \right) \right] f_X(x_1)dx_1 \right. \\
& \left. + \int_b^{\bar{x}} x_1 \left[\prod_{i=2}^{N} \left(\int_0^{b} f_X(x_i)dx_i + \int_b^{x_1} f_X(x_i)F_W(b)dx_i \right) \right] f_X(x_1)F_W(b)dx_1 \right\} \\
=&\frac{N}{F_Y(b)} \left\{ \int_0^{b} x_1 \left[F_X(x_1) \right]^{N-1} f_X(x_1)dx_1 \right. \\
& \left. + \int_b^{\bar{x}} x_1 \left[F_X(b) + \left(F_X(x_1) - F_X(b) \right) F_W(b) \right]^{N-1} f_X(x_1)F_W(b)dx_1 \right\}.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&\Pi_I(b, s) \\
=&F_Y(b) \cdot \left(\mathbb{E}[X_{(1)}|Y < b] \cdot s - \mathbb{E}[Y|Y < b] \right) \\
=&N \left(s \int_0^{b} x f_X(x)[F_X(x)]^{N-1}dx + s \int_b^{\bar{x}} x f_X(x)F_W(b)[F_X(b) + (F_X(x) - F_X(b))F_W(b)]^{N-1}dx \right. \\
& \left. - \int_0^{b} z[F_Z(z)]^{N-1} f_Z(z)dz \right),
\end{aligned}
$$

and

$$
\begin{aligned}
&\frac{\partial}{\partial b}\Pi_I(b,s)\\
=&sNbf_X(b)\left[F_X(b)\right]^{N-1}\left[1-F_W(b)\right]\\
&+sN\int_b^{\bar{x}} x\Big((N-1)f_X(x)F_W(b)\left[F_X(b)+(F_X(x)-F_X(b))F_W(b)\right]^{N-2}\\
&\cdot\left[f_X(b)+(F_W(x)-F_X(b))f_W(b)-f_X(b)F_W(b)\right]\\
&+f_X(b)f_W(b)\left[F_X(b)+(F_X(x)-F_X(b))F_W(b)\right]^{N-1}\Big)\,dx-Nb\left[F_Z(b)\right]^{N-1}f_Z(b)\\
=&s\left(\underbrace{Nbf_X(b)\left[F_X(b)\right]^{N-1}\left[1-F_W(b)\right]}_{\text{Marginal gain if } X_{(1)}=Y=b}+\underbrace{\int_b^{\bar{x}} x\frac{\partial f_{X_{(1)}|Y<b}(x)}{\partial b}dx}_{\text{Marginal gain if } X_{(1)}>Y=b}\right)-\underbrace{Nb\left[F_Z(b)\right]^{N-1}f_Z(b)}_{\text{Marginal cost}}\\
=&\frac{Ns\bar{w}^{N+1}b^{N-1}-(N+1)s\bar{w}^Nb^N+(\bar{x}+\bar{w}-b)^{N-1}\left[N(N-s+1)b^{N+1}+Ns(\bar{w}-N\bar{x})b^N\right]}{(N+1)\bar{x}^N\bar{w}^N}\\
&-\frac{(\bar{x}+\bar{w}-b)^N\left[(N+1)(N-s)b^N+Ns(\bar{w}-N\bar{x})b^{N-1}\right]}{(N+1)\bar{x}^N\bar{w}^N}.
\end{aligned}
$$

4.5. Derivation of Equation 2.23 and 2.24

$$\mathbb{E}[v_{II}(X_1,...X_N)|Y<b]$$

$$=\mathbb{E}[X_{(2)}|Y<b]$$

$$=\frac{N}{F_Y(b)}\int_0^{\bar{x}}(N-1)\left\{\int_{x_1}^{\bar{x}}\left[\int_0^{x_1}...\int_0^{x_1}x_1\left(\prod_{i=1}^{N}\left(\mathbb{I}\left\{x_i<b\right\}\cdot f_X(x_i)\right.\right.\right.\right.$$

$$\left.\left.\left.\left.+\mathbb{I}\left\{x_i\geq b\right\}\cdot f_X(x_i)\cdot F_W(b)\right)\right)\cdot dx_3...dx_N\right]\cdot dx_2\right\}dx_1$$

$$=\frac{N(N-1)}{F_Y(b)}\left\{\int_0^b x_1\left[\int_{x_1}^b\prod_{i=3}^{N}\left(\int_0^{x_1}f_X(x_i)dx_i\right)f_X(x_2)dx_2\right.\right.$$

$$\left.+\int_b^{\bar{x}}\prod_{i=3}^{N}\left(\int_0^{x_1}f_X(x_i)dx_i\right)f_X(x_2)F_W(b)dx_2\right]f_X(x_1)dx_1$$

$$+\int_b^{\bar{x}}x_1\int_{x_1}^{\bar{x}}\prod_{i=3}^{N}\left(\int_0^b f_X(x_i)dx_i+\int_b^{x_1}F_W(b)f_X(x_i)dx_i\right)f_X(x_2)F_W(b)dx_2f_X(x_1)F_W(b)dx_1$$

$$=\frac{N}{F_Y(b)}\int_0^{\bar{x}}\int_0^{x_1}...\int_0^{x_1}x_1\left[\prod_{i=1}^{N}\left(\mathbb{I}\left\{x_i<b\right\}\cdot f_X(x_i)+\mathbb{I}\left\{x_i\geq b\right\}\cdot f_X(x_i)\cdot F_W(b)\right)\right]$$

$$dx_2...dx_N\cdot dx_1$$

$$=\frac{N(N-1)}{F_Y(b)}\left\{\int_0^b x\left[F_X(x)\right]^{N-2}\left(F_X(b)-F_X(x)+F_W(b)\left[1-F_X(b)\right]\right)f_X(x)dx\right.$$

$$\left.+\int_b^{\bar{x}}x\left[F_X(b)+F_W(b)\left(F_X(x)-F_X(b)\right)\right]^{N-2}\left[F_W(b)\right]^2\left[1-F_X(x)\right]f_X(x)dx\right\}.$$

Therefore,

$$\Pi_{II}(b,s) = F_Y(b) \cdot \left(\mathbb{E}[X_{(2)}|Y<b] \cdot s - \mathbb{E}[Y|Y<b]\right)$$

$$= N(N-1)s \left\{ \int_0^b x \left[F_X(x)\right]^{N-2} \left(F_X(b) - F_X(x) + F_W(b)\left[1 - F_X(b)\right]\right) f_X(x)dx \right.$$

$$\left. + \int_b^{\bar{x}} x \left[F_X(b) + F_W(b)\left(F_X(x) - F_X(b)\right)\right]^{N-2} \left[F_W(b)\right]^2 \left[1 - F_X(x)\right] f_X(x)dx \right\}$$

$$- N \int_0^b z \left[F_Z(z)\right]^{N-1} f_Z(z)dz$$

When $N = 2$,

$$\frac{\partial}{\partial b}\Pi_{II}(b,s)\mid_{N=2}$$

$$= 2s\left\{ \underbrace{\left[1 - F_X(b)\right] F_W(b)\left[1 - F_W(b)\right] f_X(b)b}_{\text{Marginal gain if } X_{(2)} = Y = b} + \underbrace{2F_W(b)f_W(b)\int_b^{\bar{x}} x\left[1 - F_X(x)\right] f_X(x)dx}_{\text{Marginal gain if } X_{(2)} > Y = b} \right.$$

$$\left. + \underbrace{\left[\left(1 - F_W(b)\right) f_X(b) + \left(1 - F_X(b)\right) f_W(b)\right] \int_0^b x f_X(x)dx}_{\text{Marginal gain if } X_{(2)} < Y = b} \right\} - \underbrace{2F_Z(b)f_Z(b)b}_{\text{Marginal cost}}$$

$$= \frac{b}{\bar{x}^2\bar{w}^2}\left[4(\bar{x}+\bar{w})\left(\frac{3}{2} - s\right)b^2 - \left(4 - \frac{10}{3}s\right)b^3 - \left[\bar{w}^2(2-s) + \bar{x}\bar{w}(4-3s) + 2\bar{x}^2\right]b + \frac{2}{3}\bar{x}^3 s\right].$$

When $N \geq 3$,

$$
\begin{aligned}
&\frac{\partial}{\partial b}\Pi_{II}(b,s)\mid_{N\geq 3} \\
&= sN(N-1)\Bigg\{ \underbrace{\left[1-F_X(b)\right]F_W(b)\left[1-F_W(b)\right]\left[F_X(b)\right]^{N-2}f_X(b)b}_{\text{Marginal gain if } X_{(2)}=Y=b} \\
&+ \underbrace{\left[\left(1-F_W(b)\right)f_X(b)+\left(1-F_X(b)\right)f_W(b)\right]\int_0^b x\left[F_X(x)\right]^{N-2}f_X(x)dx}_{\text{Marginal gain if } X_{(2)}<Y=b} \\
&+ \underbrace{2F_W(b)f_W(b)\int_b^{\bar{x}} x\left[F_X(b)+F_W(b)\left(F_X(x)-F_X(b)\right)\right]^{N-2}\left[1-F_X(x)\right]f_X(x)dx}_{\text{Marginal gain if } X_{(2)}>Y=b} \\
&+ \underbrace{(N-2)\left[F_W(b)\right]^2\int_b^{\bar{x}} x\left[F_X(b)+F_W(b)\left(F_X(x)-F_X(b)\right)\right]^{N-3}\left[f_W(b)\left(F_X(x)-F_X(b)\right)\right.}_{\text{Marginal gain if } X_{(2)}>Y=b} \\
&+ \underbrace{\left.f_X(b)\left(1-F_W(b)\right)\right]\left[1-F_X(x)\right]f_X(x)dx}_{\text{Marginal gain if } X_{(2)}>Y=b}\Bigg\} - \underbrace{N\left[F_Z(b)\right]^{N-1}f_Z(b)b}_{\text{Marginal cost}} \\
&= \frac{(\bar{x}+\bar{w}-b)^{N-1}b^{N-1}}{(N+1)\bar{x}^N\bar{w}^N}\Bigg[N^2(\bar{x}+\bar{w}-2b)(\bar{x}s-b)+2s(\bar{x}+\bar{w}-b)b \\
&- N\left[(1-2s)b+(2\bar{w}+\bar{x})s\right](\bar{x}+\bar{w}-2b)\Bigg] + \frac{sb^{N-1}}{(N+1)\bar{x}^N\bar{w}}\Bigg[(N+1)(N+2)b^2 \\
&- (N+1)\left[(N+1)\bar{x}+(N+3)\bar{w}\right]b + N\left[(N+1)\bar{x}+2\bar{w}\right]\bar{w}\Bigg]
\end{aligned}
$$

4.6. Derivation of Equation 2.27 and 2.28

$$
\begin{aligned}
&\mathbb{E}[v_{III}(X_1, ...X_N)|Y < b] \\
=&\mathbb{E}[\frac{1}{N}\sum_{i=1}^{N} X_i|Y < b] \\
=&\mathbb{E}[X_1|Y < b] \\
=&\int_0^b x \cdot \frac{f_X(x)}{F_Z(b)} dx + \int_b^{\bar{x}} x \cdot \frac{F_W(b) f_X(x)}{F_Z(b)} dx.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\Pi_{III}(b, s) =&F_Y(b) \cdot \left(\mathbb{E}[\frac{1}{N}\sum_{i=1}^{N} X_i|Y < b] \cdot s - \mathbb{E}[Y|Y < b] \right) \\
=&s\left[F_Z(b)\right]^{N-1} \int_0^b x f_X(x) dx + sF_W(b)\left[F_Z(b)\right]^{N-1} \int_b^{\bar{x}} x f_X(x) dx \\
&- N\int_0^b z\left[F_Z(z)\right]^{N-1} f_Z(z) dz,
\end{aligned}
$$

and

$$
\begin{aligned}
&\frac{\partial}{\partial b}\Pi_{III}(b,s)\\
=&\underbrace{s\left[1-F_W(b)\right]\left[F_Z(b)\right]^{N-1} f_X(b)b}_{\text{Marginal gain if } X_1 = Y = b}\\
&+\underbrace{(N-1)sF_W(b)\left[F_Z(b)\right]^{N-2} f_Z(b)\int_b^{\bar{x}} x f_X(x)dx + s f_W(b)\left[F_Z(b)\right]^{N-1}\int_b^{\bar{x}} x f_X(x)dx}_{\text{Marginal gain if } X_1 > Y = b}\\
&+\underbrace{(N-1)s\left[F_Z(b)\right]^{N-2} f_Z(b)\int_0^{b} x f_X(x)dx}_{\text{Marginal gain if } X_1 < Y = b} - \underbrace{N\left[F_Z(b)\right]^{N-1} f_Z(b)b}_{\text{Marginal cost}}\\
=&\frac{(\bar{x}+\bar{w}-b)^{N-2}b^{N-1}}{2\bar{x}^N\bar{w}^N}\Bigg\{\left[2N(s-2)+s\right]b^3+\Big[N\left[6\bar{x}+6\bar{w}-(\bar{x}+3\bar{w})s\right]-2(\bar{x}+\bar{w})s\Big]b^2\\
&+\Big[N\left[(2\bar{x}+\bar{w})(\bar{w}-\bar{x})s-2(\bar{x}+\bar{w})^2\right]+(\bar{x}^2+\bar{x}\bar{w}+\bar{w}^2)s\Big]b+N(\bar{x}+\bar{w})\bar{x}^2 s\Bigg\}.
\end{aligned}
$$

4.7. Efficiency without the Group *O* Bidder

$$
\begin{aligned}
&\mathbb{P}\{(Z_i < Z_m) \cap (X_i < X_m)\ \forall\, i \neq m\} \\
=&\mathbb{I}\{W_m \geq X_m\} \cdot \left(\underbrace{\mathbb{P}\{Z_i < Z_m \mid X_i < X_m\}}_{=1} \cdot \underbrace{\mathbb{P}\{X_i < X_m\}}_{=F_X(X_m)} \right)^{N-1} \\
&+ \mathbb{I}\{W_m < X_m\} \cdot \left(\underbrace{\mathbb{P}\{Z_i < Z_m \mid X_i < X_m\}}_{=\frac{F_X(W_m)+[F_W(X_m)-F_X(W_m)]\cdot F_W(W_m)}{F_X(X_m)}} \cdot \underbrace{\mathbb{P}\{X_i < X_m\}}_{=F_X(X_m)} \right)^{N-1} \\
=&\mathbb{I}\{W_m \geq X_m\} \cdot \left[F_X(X_m)\right]^{N-1} \\
&+ \mathbb{I}\{W_m < X_m\} \cdot \left\{F_X(W_m) + [F_W(X_m) - F_X(W_m)] \cdot F_W(W_m)\right\}^{N-1}.
\end{aligned}
$$

The *ex ante* expected allocation efficiency is thus

$$
\begin{aligned}
&\text{Efficienty}^{ex\ ante} \\
=&\mathbb{I}\{\bar{x} \geq \bar{w}\} \cdot \Bigg(N \int_0^{\bar{w}} \Bigg\{ \int_0^{w} \left[F_X(x)\right]^{N-1} dF_X(x) \\
&+ \int_w^{\bar{x}} \left\{F_X(w) + \left[F_X(x) - F_X(w)\right] F_W(w)\right\}^{N-1} dF_X(x) \Bigg\} dF_W(w) \Bigg) \\
&+ \mathbb{I}\{\bar{x} < \bar{w}\} \cdot \Bigg(N \int_0^{\bar{x}} \Bigg\{ \int_x^{\bar{w}} \left[F_X(x)\right]^{N-1} dF_W(w) \\
&+ \int_0^{x} \left\{F_X(w) + \left[F_X(x) - F_X(w)\right] F_W(w)\right\}^{N-1} dF_W(w) \Bigg\} dF_X(x) \Bigg).
\end{aligned}
$$