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Mustafa Akan

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ABSTRACT

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Mustafa Akan

This thesis encompasses the work on bid-price controls for network revenue management and a dynamic model of revenue management with strategic consumers. The first line of research studies the circumstances under which bid-price controls are optimal or near optimal with minimal assumptions on the network topology and the stochastic structure of demand. In this general setting, in a discrete time model, I propose several novel bid-price control mechanisms and study their properties, proving that optimal bid prices form a martingale. To explore the martingale property further, I also consider a continuous time, rate-based model of network revenue management and show how an ε-optimal bid-price control and the corresponding bookings can be characterized as a solution to a Forward-Backward Stochastic Differential Equation (FBSDE). The analysis provides a new methodological approach to study revenue management problems by defining them as stochastic control problems and deriving the associated dual stochastic control problems. In the important special case of continuous information, machinery of FBSDE's and Ito calculus can be used to solve the network revenue management problem. Using the

FBSDE connection, Malliavin calculus and Monte Carlo methods for solving FBSDE's can be utilized to compute near optimal bid prices.

The second line of research incorporates the strategic consumer behavior in revenue management. I consider a dynamic model of revenue management with strategic consumers, where unlike in the classic revenue management literature, demand learning is the underlying process that leads to arrivals. In this setting, consumers learn their true valuations sequentially and a monopolist system manager tries to maximize her profits by sequentially screening the consumers. I identify the conditions under which the system manager can achieve the first-best solution. If these conditions are not satisfied, then the optimal mechanism is a menu of expiring refund contracts.

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CHAPTER 1

Bid-Price Controls for Network Revenue Management (joint with Barış Ata)

The defining feature of *network* revenue management is that the products being sold consume the capacities of multiple resources. In this complex setting, bid-price controls represent a popular and intuitively appealing approach to quantity-based revenue management. Such a control mechanism associates a threshold price (called a *bid price*) with each resource dynamically over time, and a booking request is accepted if the following two conditions are met: the remaining capacities of the various resources are adequate to meet the request; and the revenue generated by accepting the request exceeds the sum of the bid prices associated with the resources consumed. Bid-price controls are motivated by the price interpretation of optimal dual variables in deterministic linear programming, and a bid price is often described as the "opportunity cost" of consuming a resource's remaining capacity; see for example [**10**].

Bid-price controls are now widely used in practice, but their justification remains incomplete. Chapter 3 of [**60**] summarizes the literature of network revenue management through roughly 2003, including a thorough discussion of bid-price control mechanisms, which began with [**58**] and [**65**]; also see [**30**] for an elaborate survey. Bid-price controls continue to attract the attention of university and industry researchers, but we are not aware of any recent work that bears upon the issues discussed in this paper. The initial impetus for our research was a comment by Robert Phillips several years ago, conjecturing that optimal bid prices form a martingale. His conjecture was advanced in the context of ongoing discussions with colleagues in the airline industry, some of whom asserted that the bid prices associated with capacity of a flight leg tends to increase as the departure date approaches, while others made the reverse claim.

An obvious impediment to proving this conjecture is the negative result by [**59**], who showed by example that an optimal bid-price control mechanism need not exist. The authors produced a 2-period counter example in which the optimal sequence of accept-reject decisions cannot be achieved by means of the specific bid-price control mechanism they propose. In this paper, we identify the circumstances under which bid-price controls are optimal or near optimal. In particular, we report encouraging results regarding the optimality properties of bid-price controls. These results are proved without making *any* assumptions on the stochastic structure of demand; our model allows non-stationary demands with an arbitrary dependence structure, including both inter-temporal and cross-product dependencies which enables us to capture demand substitution across products and over time.

As we shall see in Section 1.1, the controls in network revenue management problem (P) are adapted and use all the information available at the time of decision making. Hence, if we require bid prices to be predictable, that is, bid prices can depend on information available only up to a point before the booking decisions are being made, there will be an optimality gap in general. In other words, the bookings resulting from a predictable bid-price control may not be optimal for the network revenue management problem in general. Therefore, we first adopt a generous definition of bid-price controls and allow the policy parameters to be adapted, i.e. they can depend on all the information available at the time of decision making. In this context, we identify a class of adapted generalized bid-price controls, and show that there *does* exist an optimal control within that class, cf. Theorem 1. Moreover, there exists an optimal control within that class such that the bid prices form a martingale, cf. Theorem 2.

In the course of establishing Theorem 1, we see that bid-price controls implemented in the ordinary sense, cf. [**60**], are not optimal in general even if they are adapted. As can be seen from our formulation (P), cf. Section 1.1, the network revenue management problem is a linear program and hence, its objective is piecewise linear and concave in the vector of remaining capacities. Intuitively, the basic idea behind bid-price controls is that they capture the displacement cost or the opportunity cost of capacity associated with booking decisions. Taking a dynamic programming point of view, the system manager wishes to assess how the value-to-go function changes as she makes booking decisions at each decision point. If the value-to-go function was affine, then the bid-pricing approach in the ordinary sense would work. However, at each decision point, the value-to-go function is a piecewise linear and concave function because the network revenue management problem is a linear program. Therefore, if the current capacity and remaining capacity after each booking decision are on the same facet of the value-to-go function, then the bid-pricing approach in the ordinary sense would work, but problems arise if they are on different facets. Moreover, if the two capacity vectors lie on an edge, then the gradient of the value-to-go function is not well defined. In these cases, it is not clear how to define and implement bid-price controls. Thus, the bid-prices implemented in the ordinary sense are not optimal in general even if they can depend on all the information available at the time of decision making. Nonetheless, we show in Theorem 1 that using bid-price in conjunction with capacity usage limits gives rise to optimal bookings. The capacity usage limit processes also address the concerns raised by [**65**] that without any explicit booking limits, the bid-price control mechanisms would book more than optimal requests in case of a burst in demand. Such capacity usage limits are typically used in practice as part of bid-price systems for network revenue management in order to handle unanticipated demand surges, cf. [**49**].

Consider the usual case where the incremental information arriving during a small time window is also small. Then we expect intuitively that the extra latitude of adaptedness we allow in formulating and applying bid-prices is not very significant so that the predictable version of the bid-price control should be near optimal as the updating frequency of the policy parameters increases. To explore this, we consider predictable generalized bid-price controls in Section 1.3, whose parameters at a decision point are required to depend only on the information available at a point strictly before decision making. While predictable bid-price controls are sub-optimal in general, we construct a near-optimal one in Section 1.3 under mild assumptions. To be specific, we show that the expected revenue resulting from the proposed predictable bid-price control converges to the objective function value of the network revenue management problem as the updating frequency of the bid-price and the capacity usage limits increases. Moreover, the predictable bid prices we construct form a martingale, too. The near optimality of the predictable bid-price controls shows their robustness to small information distortions. In the same vein, martingale property is also robust to informational distortions and still holds for predictable bid-prices.

Bid-price controls are practically appealing because they are easy to implement. The generalized bid-price controls we propose decompose the complex network revenue management problem across time. Nonetheless, they do *not* decompose decision making across products. In particular, one needs to solve a linear program at each decision point. Fortunately, we prove that there exists a bid-price control which uses only the bid prices associated with various resources and decomposes decision making across products. Better yet, we show that there exists a predictable such bid-price control which is near optimal in the usual case when periods are small provided that either demand during each period is small or the incremental information arriving during each period is small. The bid-prices form a martingale in this case, too. Moreover, the bid-prices used at each decision point are last updated in the previous period reflecting the way bid-prices are used in practice. Moreover, the bid-price control we propose can be viewed as a perturbation of those implemented in the ordinary sense. In particular, given bid prices, the two booking mechanisms result in the same bookings except when the fare of a product is only slightly larger than the sum of the bid prices of the resources that product uses, cf. (1.4). In that case, the bid-price control we propose books only a fraction of demand while the ordinary implementation suggests booking all demand.

In establishing the near optimality of the latter bid-price control, we consider an asymptotic regime where the number of periods grow to infinity and period lengths tend to zero, while the planning horizon $[0, T]$ and the underlying probabilistic primitives (cumulative demand and fare processes) remain unchanged. We believe preserving the uncertainty in the limit is necessary to maintain the key trade-offs in the network revenue management problem and the novel feature of this asymptotic regime is that it does preserve the stochastic nature of demand unlike the asymptotic regimes considered previously in the literature. Our analysis also provides bounds on the optimality gap associated with predictable bid-price control we propose, which do vanish in this asymptotic regime, cf. Corollaries 10 and 12.

To elaborate further on the practical significance these results, note that anyone using bid-price controls in practice will update bid-prices less often than required for exact optimality. Thus, if there existed an optimal bid-price control in the ordinary sense, the practical significance of this would have been to provide rough assurance that by refreshing policy parameters often enough, one could approach optimality. We believe that our results do equally well in that regard. Moreover, our results provide the additional insight that to assure good performance, bid-prices should either be used in conjunction with capacity usage limits, cf. Theorems 1 and 5, or one must take additional care (if one uses only bid prices in decision making) when the fare of a product only slightly exceeds the sum of the bid prices of the resources it uses, in which case one should not book all the demand contrary to the ordinary implementation of the bid-price controls.

Finally, our paper is the first to establish the martingale property of (near) optimal bid prices. The martingale property provides us with the understanding that if the system manager makes the optimal accept/reject decisions, a decrease in the option value of capacity should be balanced by the increase in the opportunity cost of capacity. The martingale representation of optimal bid-prices also leads to a promising connection to the literature on the pricing of American options where one tries to pick the best martingale to optimize a certain objective. This connection is more transparent if the capacity of a resource in the network revenue management problem is thought of as an option that could be exercised in each period. The martingale property of (near) optimal bid prices is further explored in Chapter 2 using a continuous-time rate based model of network revenue management.

1.1. The model

We consider a network revenue management model consisting of K resources and J products. In an airline setting, a resource corresponds to a flight leg and a product corresponds to a particular itinerary. A primitive of our model is a $K \times J$ non-negative capacity consumption matrix A, where the entry A_{kj} denotes the amount of resource k capacity consumed by one unit of product j. There are N periods to the terminal time. The periods need not have equal length. To be more specific, the end of each period is denoted by t_n for $n = 1, \ldots, N$; and we let $t_0 = 0$ and $t_N = T$. In each period n, first the demand during that period is realized, then the number of bookings for each product is determined at time t_n so that the capacity constraints are not violated. Then revenue resulting from the bookings is realized and the capacity vector is updated. The booking vectors are continuous[1] and the fares are set exogenously. The capacity not utilized until the terminal time has no value. The objective is to determine the optimal controls (booking decisions) in each period to maximize total expected revenue subject to capacity constraints.

The evolution of information (or uncertainty) is described by the filtered complete probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t : 0 \leq t \leq T\}, \mathbb{P})$, where $\mathcal{F}_t$ denotes the information available at time t. The system manager observes the evolution of information continuously starting

[1]That is, we may choose to fulfill any fraction of demand by allowing the booking vectors to take values in $\mathbb{R}^J_+$. The case with integrality requirements on booking controls can be analyzed using a similar framework.

at time $t_0 = 0$, although she exerts control at the end of each period, that is, at times $\{t_n : n = 1, \dots, N\}$. These decision times are exogenous but the time difference between the decision times may be arbitrary as well as the number of decision times. To facilitate the analysis to follow let $\Gamma = \{t_1, \dots, t_N\}$ denote the set of decision times where t_n also denotes the end of period n for $n = 1, \dots, N$.

The cumulative demand is modelled as a non-decreasing, J-dimensional stochastic process $\{D(t) : 0 \leq t \leq T\}$ with finite mean, adapted to the underlying information structure $\{\mathcal{F}_t : 0 \leq t \leq T\}$, where $D_j(t)$ denotes the cumulative demand for product j up to time t. No assumption of the independence of demand across products or across time is made; indeed, we allow for dependent demand. In our formulation, the system state at decision time t_n is described by a K-dimensional vector $x(t_n)$ of remaining capacities; $x_k(t_n)$ denotes the remaining capacity of resource k at time t_n. The demand observed by the system manager at time t_n is the J-dimensional vector $D(t_n) - D(t_{n-1})$, which is the demand accumulated since the last decision time t_{n-1}, that is, during period n. Upon observing demand realization $D(t_n) - D(t_{n-1})$ during period n, the system manager chooses a J-dimensional vector $u(t_n)$ of booking levels, where $u_j(t_n)$ denotes the booking level for product j at decision time t_n. Given the initial capacity vector $x(t_0) = C$, and the booking levels $u(t_n)$ for $n = 1, \dots, N$, the evolution of the system state (the capacity process) is governed by the following system dynamics equation:

$$x(t_n) = x(t_{n-1}) - A\,u(t_n) \quad \text{for} \quad n = 1, \dots, N. \tag{1.1}$$

A booking vector $u(t_n)$ at decision time t_n results in a revenue of $f(t_n) \cdot u(t_n)$, where $f(t_n)$ is the exogenously set vector of fares at decision time t_n. The process of fares is a bounded, non-negative, adapted (i.e. $f(t) \in \mathcal{F}_t$, $t \in [0, T]$) continuous-time stochastic process, which need not be constant nor stationary, allowing us to model randomness in product fares and incorporate the time value of money into the analysis[2]. The fare process f is continuous-time stochastic process and we are concerned with its value only at decision times t_n for $n = 1, \ldots, T$. The fare and demand processes can be dependent as well. Clearly, D, x, u, f are all stochastic processes, but their dependence on the sample path will be suppressed for notational brevity. The objective is to choose adapted booking controls $u(t_n)$ for $n = 1, \ldots, N$ to maximize total expected revenue subject to feasibility constraints. That is, choose $u(t_n)$, which can depend on all the information $\mathcal{F}_{t_n}$ available at time t_n, for $n = 1, \ldots, N$ so as to

$$\begin{aligned}
&\text{Maximize } \sum_{n=1}^{N} \mathbb{E}[f(t_n) \cdot u(t_n)] \\
&\text{subject to} \\
&x(t_0) = C, \\
&x(t_n) = x(t_{n-1}) - Au(t_n), \quad n = 1, \ldots, N, \\
&Au(t_n) \leq x(t_{n-1}), \quad n = 1, \ldots, N, \\
&0 \leq u(t_n) \leq D(t_n) - D(t_{n-1}), \quad n = 1, \ldots, N,
\end{aligned} \tag{P}$$

where the first two constraints describe how capacity evolves over time and the last two impose the capacity and demand restrictions on booking levels in each period. The formulation (P) will be referred to as the network revenue management problem, and an optimal control refers to the set of controls $\{u(t_n) : n = 1, \ldots, N\}$ that maximizes the

[2]We allow constant and/or deterministic fares as special cases.

expected revenues $\sum_{n=1}^{N} \mathbb{E}[f(t_n) \cdot u(t_n)]$ while satisfying the constraints of (P). We denote the optimal objective function value (that is, the maximum expected revenues that can be generated in the network revenue management problem) by P*.

1.2. An Adapted Generalized Bid-Price Control and Its Optimality

In this subsection we introduce a class of adapted bid-price control mechanisms and show that there exists an optimal bid-price control within that class. We also prove that the optimal bid-prices form a martingale. The adapted bid-price control we propose involves the pair of adapted vector valued stochastic processes (π, λ); we will suppress the dependence of π and λ on the sample path for notational brevity. We allow $\pi(t_n)$ and $\lambda(t_n)$ to depend on all the information $\mathcal{F}_{t_n}$ available at time t_n for $n = 1, \ldots, N$. Each component of $\pi(t_n)$ is associated with a particular resource and $\pi_k(t_n)$ is be interpreted as the bid price for resource k at time t_n for $k = 1, \ldots, K$. Similarly, $\lambda_k(t_n)$ is associated with resource k, and is interpreted as a capacity usage limit on resource k at decision time t_n. The proposed bid-price control is executed as follows: At each decision time t_n, the system manager first observes the demand $D(t_n) - D(t_{n-1})$ for period n. Then, she solves the following linear program denoted by (P(t_n)) to determine the booking levels:

$$
(P(t_n)) \qquad
\begin{aligned}
&\mathrm{Max}_u (f(t_n) - A'\pi(t_n)) \cdot u + \eta(Au - \lambda(t_n)) \cdot \mathbf{e} \\
&\text{subject to} \\
&Au \leq \lambda(t_n), \\
&0 \leq u \leq D(t_n) - D(t_{n-1}),
\end{aligned}
$$

where $\eta > 0$ is arbitrarily small and $\mathbf{e}$ is the K-dimensional vector of ones. This linear program is lexicographic in the following sense. The system manager first sets $\eta = 0$ and solves $(\mathrm{P}(t_n))$. If there are multiple solutions, then she picks the one that maximizes the second term in the objective. Ties can be broken arbitrarily. Ideally, the system manager strives to choose a "maximal" solution that has $Au = \lambda(t_n)$. Here, the constraint $Au \leq \lambda(t_n)$ creates "usage limits" or booking limits on the resources in some sense. However, these limits differ from the booking limits in the literature in the sense that the latter refers to the limits on the amount we can sell of each product, cf. [**49**].

Hereafter, we will refer to the process λ as the capacity usage limit process and π as the bid price process. The following theorem establishes the optimality of the adapted bid-price controls and a constructive proof is provided in Appendix A.1.

Theorem 1. *There exists an optimal adapted bid-price control* (π, λ)*. That is, the booking controls* u *resulting from the adapted bid-price control* (π, λ) *constitute an optimal solution to the network revenue management problem (P).*

Next, we further investigate the structural properties of the bid prices associated with an optimal adapted bid-price control. The following theorem establishes the martingale property of optimal adapted bid prices and it is proved in Appendix A.1.

Theorem 2. *There exists an optimal adapted bid-price control* (π, λ) *such that the optimal bid-price process* $\{\pi(t_n) : n = 1, \ldots, N\}$ *is a martingale adapted to* $(\{\mathcal{F}_{t_n} : n = 1, \ldots, N\}, \mathbb{P})$.

To elaborate further on the implications of the martingale property of bid-prices, consider the two effects on the valuation of capacity in the network revenue management problem. The first effect is that the option value of capacity goes down as we get closer to the terminal time. The second effect is that the opportunity cost of capacity goes up as time advances and we sell capacity. Even though it is not obvious which one of these two effects will dominate over time, the martingale property implies that if the system manager makes the optimal booking decisions at each time point, then these two effects will balance each other and the optimal adapted bid-prices, which can be thought as the opportunity cost of capacity, remains constant in a stochastic sense. In other words, there is no upward or downward trend for bid prices.

Capacity usage limit process λ is essential for the optimality since it ensures that we follow an optimal trajectory of remaining capacities. The capacity usage limits are necessary for optimality only if there are multiple solutions to the network revenue management problem. This suggests that multiplicity of solutions can cause bid prices to be non-optimal. Indeed, the capacity usage limit process λ also addresses the concerns raised by [**65**] that without any explicit booking limits, the bid-price control mechanisms would book more than optimal requests in case of a burst in demand. Such permissible capacity restrictions are typically used in practice as part of bid-price systems for network revenue management in order to handle unanticipated demand surges, cf. [**49**].

The martingale property of optimal bid prices also leads to a notable connection to the literature on the pricing of American options where one tries to pick the best martingale to optimize a certain objective. This connection is more transparent if the capacity of a resource in the network revenue management problem is thought of as an option that could be exercised in each period. This relationship is further analyzed in the subsequent section.

Intuitively, if the amount of information flowing into the system during short period is small, then the performance of adapted and predictable bid-price controls should be close to each other. This point is illustrated in the next section by constructing a near optimal bid-price control where the bid-price and capacity usage limit processes depend only on the information at a point strictly before decision making.

1.3. Predictable Generalized Bid-Price Controls

In an adapted bid-price control, the bid-price and the capacity usage limit processes are functions of the information available at the time of decision making. However, anyone using bid-price controls in practice will update bid prices less often than required for adaptedness. Thus, it is important from a practical standpoint to study bid prices that are predictable, that is, they depend only on the information available strictly before booking decisions.

Recall that the controls in the network revenue management problem (P) are adapted and use all the information available at the time of decision making. Hence, if we restrict bid prices to be predictable and use information only up to a point strictly before the booking decisions are being made, there will be an optimality gap in general. That is, the booking controls resulting from a predictable bid-price control may not be optimal for the network revenue management problem (P) in general. Therefore, a natural direction is to look for the "best" predictable bid-price control. Nevertheless, this a very broad question as one could come up with a staggeringly complex array of predictable bid-price controls. Therefore, we take a somewhat different approach and construct a predictable bid-price control which is near optimal, cf. Theorem 5.

To facilitate our analysis in this section, we introduce the parameter[3] $h > 0$ as the time gap between the last time the bid-price and capacity usage limit vectors are updated and the time at which the booking decisions are made. To be specific, the bid-price and capacity usage limit vectors to be used at decision time t_n can depend only on the information $\mathcal{F}_{t_n - h}$ available at time $t_n - h$ for $n = 1, \ldots, N$. As a preliminary to the construction of a near optimal predictable bid-price control, we impose further structure on the information structure $\{\mathcal{F}_t : 0 \leq t \leq T\}$ by assuming that it is semi-continuous. The following definition of a semi-continuous information structure is due to [**36**].

Definition 3. *An information structure $\{\mathcal{F}_t,\, 0 \leq t \leq T\}$ is semi-continuous if $\mathcal{F}_t = \mathcal{F}_{t-}$ for $t \in (0, T]$, where $\mathcal{F}_{t-}$ is the information available just before t. Formally, $\mathcal{F}_{t-}$ is the information generated by the collection of information sets $\{\mathcal{F}_s : s < t\}$.*

To add concreteness to the definition of semi-continuous information structures, consider the following familiar demand processes, cf. [**36**]. First of all, the information structure generated by the Poisson process is semi-continuous. Likewise, the information structure generated by renewal processes, functionals of Brownian motion and any rate based model of demand is semi-continuous. Hence, the demand process in our model can be any of the above or any other demand process whose natural information structure is semi-continuous[4].

[3]Our results carry over to the setting where one introduces a collection of parameters $\{h_n : n = 1, \ldots, N\}$; one for each decision time.

[4]The role played by the semi-continuous information structures in the construction of the near optimal predictable bid-price scheme is to ensure that all continuous-time martingales adapted to the information structure are continuous almost surely for any fixed time $t \in [0, T]$, cf. [**36**] and [**23**]. Notice, however, that the continuous-time martingales adapted to semi-continuous information structures need not have continuous sample paths.

To facilitate our construction of a predictable bid-price control, fix an optimal adapted bid-price control (π, λ), where $\{\pi(t_n) : n = 1, \ldots, N\}$ is a martingale, cf. Theorems 1 & 2. A natural approach for constructing a predictable bid-price control is to take the conditional expectation of (π, λ). Thus, for $h > 0$ and $n = 1, \ldots, N$ define

$$\widetilde{\pi}_h(t_n) = \mathbb{E}[\pi(t_n) \mid \mathcal{F}_{t_n-h}], \tag{1.2}$$

$$\widetilde{\lambda}_h(t_n) = \min\{\mathbb{E}[\lambda(t_n)|\mathcal{F}_{t_n-h}], C - \sum_{m=1}^{n-1} \widetilde{\lambda}_h(t_m)\}. \tag{1.3}$$

By construction, the bid-price process $\widetilde{\pi}_h(t_n)$ and the capacity usage limit process $\widetilde{\lambda}_h(t_n)$ are adapted to the information $\mathcal{F}_{t_n-h}$ available at time $t_n - h$. The following proposition establishes the martingale property of the predictable bid-price control $(\widetilde{\pi}_h, \widetilde{\lambda}_h)$ for each $h > 0$ and is proved in Appendix A.2.

Proposition 4. *For $h > 0$, the predictable bid-price process $\{\widetilde{\pi}_h(t_n) : n = 1, \ldots, N\}$ is a martingale adapted to $(\{\mathcal{F}_{t_n-h} : n = 1, \ldots, N\}, \mathbb{P})$.*

The execution of the proposed predictable bid-price control is similar to that of an adapted one, cf. Section 1.2. Namely, at each decision time t_n for $n = 1, \ldots, N$, the system manager solves the following linear program to determine the bookings.

$$\max_u \, (f(t_n) - \widetilde{\pi}_h(t_n)A) \cdot u + \eta(Au - \widetilde{\lambda}_h(t_n)) \cdot e$$

$$\text{subject to} \qquad (\mathrm{P}_h(t_n))$$

$$Au \leq \widetilde{\lambda}_h(t_n),$$

$$0 \leq u \leq D(t_n) - D(t_{n-1}).$$

As in the adapted case this linear program is lexicographic and the system manager prefers a maximal solution. However, the bid-price vector $\widetilde{\pi}_h(t_n)$ and the capacity usage limit vector $\widetilde{\lambda}_h(t_n)$ used in the linear program $\mathrm{P}_h(t_n)$ at decision time t_n are predictable since they use information only up to time $t_n - h$. The execution of the predictable bid-price control $(\widetilde{\pi}_h, \widetilde{\lambda}_h)$ using the set of linear programs $\mathrm{P}_h(t_n)$ results in a feasible control for the network revenue management problem. (The feasibility of these controls will be shown in Appendix A.3.) One would also hope that for small values of h, the performance of the proposed predictable bid-price control is close to that of the optimal adapted bid-price control. Indeed, the following theorem establishes the near optimality of the predictable bid-price control $(\widetilde{\pi}_h, \widetilde{\lambda}_h)$ constructed as in (1.2) and (1.3).

Theorem 5. *The predictable bid-price control $(\widetilde{\pi}_h, \widetilde{\lambda}_h)$ defined in (1.2)-(1.3) is near optimal. That is, as $h \searrow 0$, the expected revenue generated by the bookings resulting from $(\widetilde{\pi}_h, \widetilde{\lambda}_h)$ converges to the expected revenue generated by the optimal adapted bid-price control (π, λ), which, in turn, is equal to the optimal objective function value P^* of the network revenue management problem (P).*

The sequence of controls resulting from the predictable bid-price scheme are also close to an optimal solution to the network revenue management problem (P) for small values of h. That is, every cluster point of $(\widetilde{\pi}_h, \widetilde{\lambda}_h)$ is an optimal control for the network revenue management problem (P). Recall that $h > 0$ is the difference between the last time the bid-price and the permissible capacity vectors are updated and the time at which they are actually used. One would hope that for small values of h, the performance of the proposed predictable bid-price policy is close to that of the optimal adapted bid-price scheme. Indeed, the following theorem establishes the near optimality of the predictable bid-price scheme $(\widetilde{\pi}_h, \widetilde{\lambda}_h)$ constructed as in (1.2) and (1.3). It also shows that the sequence of controls resulting from the predictable bid-price scheme are close to an optimal solution to the network revenue management problem (P) for small values of h.

To be more precise, let $\mathcal{U}_n^h(\omega)$ denote the set of optimal solutions to the linear program $\mathrm{P}_h(\omega, t_n)$ for $n = 1, \ldots, N$ and a realization $\omega \in \Omega$; and let $\mathcal{U}^h(\omega) = \mathcal{U}_1^h(\omega) \times \cdots \times \mathcal{U}_N^h(\omega)$ for $\omega \in \Omega$ denote the set of solutions resulting from the predictable bid-price policy $(\widetilde{\pi}_h, \widetilde{\lambda}_h)$ for $h > 0$. In particular, denote a generic element of $\mathcal{U}^h(\omega)$ by $u^h(\omega)$. That is, for $\omega \in \Omega$, $h > 0$ and $u^h(\omega) \in \mathcal{U}^h(\omega)$, we have the following representation:

$$u^h(\omega) = (u^h(\omega, t_1), \ldots, u^h(\omega, t_N)),$$

where $u^h(\omega, t_n)$ is any optimal solution to $(\mathrm{P}_h(\omega, t_n))$ for $n = 1, \ldots, N$.

Then we consider the collection of the set of optimal controls $\{\mathcal{U}^h(\omega) : \omega \in \Omega\}_{h>0}$ resulting from the collection of predictable bid-price policies $\{(\widetilde{\pi}_h, \widetilde{\lambda}_h)\}_{h>0}$ and identify its cluster (limit) points. To this end, we next provide a precise definition of a cluster point of the predictable bid-price policies $\{(\widetilde{\pi}_h, \widetilde{\lambda}_h)\}_{h>0}$.

Definition 6. *$(u(\omega) : \omega \in \Omega)$ is a cluster point of the sequence of predictable bid-price controls $\{(\widetilde{\pi}_h, \widetilde{\lambda}_h)\}_{h>0}$, if for every $\omega \in \Omega$, there exists a sequence $h_m \searrow 0$ as $m \to \infty$ and $u^{h_m}(\omega) \in \mathcal{U}^{h_m}(\omega)$ for $m \geq 1$ such that*

$$u^{h_m}(\omega) \to u(\omega) \text{ as } m \to \infty.$$

For $\omega \in \Omega$, the set of cluster points $u(\omega)$ is denoted by $\mathcal{U}(\omega)$.

It is easy to see that $\mathcal{U}(\omega)$ is non-empty for each $\omega \in \Omega$; and the following theorem establishes the optimality of every cluster point $u(\omega) \in \mathcal{U}(\omega)$ of $\{(\widetilde{\pi}_h, \widetilde{\lambda}_h)\}_{h>0}$ and is proved in Appendix A.2.

Theorem 7. *The predictable bid-price scheme $(\widetilde{\pi}_h, \widetilde{\lambda}_h)$ defined in (1.2)-(1.3) is near optimal. That is, as $h \searrow 0$:*

a) The expected revenue generated by the bookings resulting from $(\widetilde{\pi}_h, \widetilde{\lambda}_h)$ converges to the expected revenue generated by the optimal adapted bid-price scheme (π, λ), which, in turn, is equal to the objective function value of the network revenue management problem (P).

b) Every cluster point of $(\widetilde{\pi}_h, \widetilde{\lambda}_h)$ is an optimal control for the network revenue management problem (P).

Theorem 5 sheds light on the significance of information for the performance of the bid-price controls. In particular, Theorem 5 suggests that in the presence of resource usage limits (capacity usage limit processes), the non-optimality of the predictable bid-price controls in general results from the information gap and not from the network effect or the large replacement of capacity. The near optimality of the proposed predictable bid-price control shows the robustness of the bid-price policies to small information distortions. In the same vein, martingale representation is also robust to informational distortions and still holds for predictable bid-prices.

1.4. A Class of Simple Predictable Bid-Price Controls

In this section, we introduce a class of simple bid-price controls, which only use bid-prices associated with various resources for decision making, hence, are easy implement. Moreover, bid-prices used in each period are updated in the previous period, i.e. they are predictable. To be more specific, the bid-price control we propose involves a bid-price process π, where $\pi(t_n)$ depends only on the information $\mathcal{F}_{t_{n-1}}$ available at time t_{n-1} for $n = 1, \ldots, N$, and a parameter $\varepsilon > 0$, and is executed as follows: At each decision time t_n, the system manager observes demand $D(t_n) - D(t_{n-1})$ for period n. Then, letting A^j denote the j^{th} column of the capacity consumption matrix A, she makes the booking decisions for various products sequentially with respect to the product index $j = 1, \ldots, J$ as follows:

$$
(1.4) \qquad u_j(t_n) = \begin{cases} 0 & \text{if } \; f_j(t_n) - \pi(t_n)A^j < 0, \\ D_j(t_n) - D_j(t_{n-1}) & \text{if } \; f_j(t_n) - \pi(t_n)A^j > \varepsilon, \\ \frac{f_j(t_n) - \pi(t_n)A^j}{\varepsilon}\left[D_j(t_n) - D_j(t_{n-1})\right] & \text{if } \; 0 \le f_j(t_n) - \pi(t_n)A^j \le \varepsilon, \end{cases}
$$

provided there is enough capacity, i.e. $u_j(t_n)A^j \le x(t_{n-1}) - \sum_{l=1}^{j-1} u_l(t_n) A^l$. Otherwise, $u_j(t_n)$ is scaled down as dictated by the remaining capacity. The reader may feel that the order in which the booking decisions for various products are made is arbitrary, which is

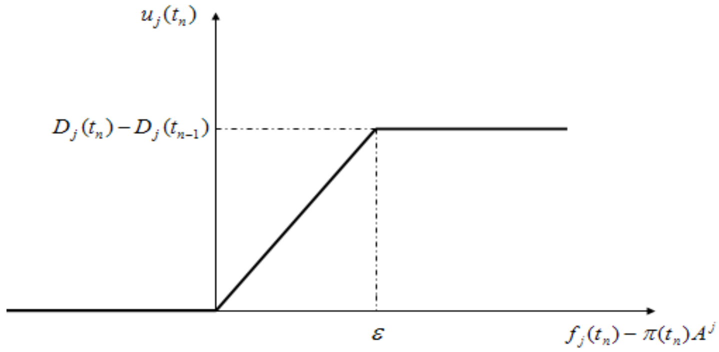

Figure 1.1. The bookings as a function of the difference $f_j(t_n) - \pi(t_n)A^j$.

indeed the case. As shall seen in the proof of Theorem 9, our results are independent of this ordering.

The bid-price control we propose can be seen as a perturbation of an ordinary bid-price control, cf. [**60**]. In particular, for each product $j = 1, \ldots, J$, the bookings in (1.4) are the same as those resulting from an ordinary bid-price control as long as $f_j(t_n) - \pi(t_n)A^j$ does not fall in the interval $(0, \varepsilon)$, in which case an ordinary bid-price control would dictate booking all the demand, i.e. $D_j(t_n) - D_j(t_{n-1})$. In contrast, the bid-price control we propose books

$$u_j(t_n) = \frac{f_j(t_n) - \pi(t_n)A^j}{\varepsilon}\left[D_j(t_n) - D_j(t_{n-1})\right] < D_j(t_n) - D_j(t_{n-1}).$$

Figure 1.1 displays $u_j(t_n)$ as a function of the difference $f_j(t_n) - \pi(t_n)A^j$. Indeed, the graph looks more and more like a step function as ε tends to zero, which would result from an ordinary bid-price control.

In what follows, we will view the set of decision points $\Gamma = \{t_1, \ldots, t_N\}$ as a decision to be made as well. One can view these decisions as being made hierarchically. The system manager first chooses the parameter $\varepsilon > 0$ and a partition Γ at time zero, followed by the bid-prices to be used, which are updated dynamically over time. To be more specific, we are interested in an asymptotic regime where the system manager makes decisions more and more frequently, i.e. the number of period N tends to infinity while the period lengths tend to zero. Nonetheless, the stochastic primitives of the problem and the planing horizon remain unchanged. We will denote the revenues resulting from the bookings in (1.4) by $\mathrm{Obj}(\pi, \varepsilon, \Gamma)$ to highlight its dependence on Γ, ε and π. The following definition is needed to state the main result of this section.

Definition 8. *Given a stochastic process* $\{Z(t) : 0 \leq t \leq T\}$ *and a partition* $\Gamma = \{t_0, t_1, \ldots, t_N\}$ *of* $[0, T]$ *with* $t_0 = 0$ *and* $t_N = T$*, the* p^{th} *variation of* Z *over the partition* Γ*, denoted by* $\mathcal{V}_p(Z, \Gamma)$*, is defined as follows*[5]*:*

$$\mathcal{V}_p(Z, \Gamma) = \sum_{n=1}^{N} |Z(t_n) - Z(t_{n-1})|^p, \quad p \geq 1.$$

The quadratic variation essentially captures the volatility of a stochastic process over time. Next we state the key result of this section, from which Corollaries 10 and 12 follow.

Theorem 9. *For* $\varepsilon > 0$ *and any partition* $\Gamma = \{t_0, t_1, \ldots, t_N\}$ *of* $[0, T]$*, there exists a bid-price process* π^ε *such that* $\{\pi^\varepsilon(t_n) : n = 1, \ldots, N\}$ *is a martingale adapted to* $(\{\mathcal{F}_{t_{n-1}} : n = 1, \ldots, N\}, \mathbb{P})$ *with*

$$0 \leq \pi^\varepsilon(t_n) \leq B \quad \text{for} \quad n = 1, \ldots, N.$$

[5] $|\cdot|$ is the sup norm in $\mathbb{R}^J$.

Moreover, we have the following bound on the optimality gap:

$$|Obj(\pi^{\varepsilon},\varepsilon,\Gamma) - P^*| \leq \kappa\varepsilon + \frac{C}{\varepsilon}\left[\mathbb{E}\mathcal{V}_p(\pi^{\varepsilon},\Gamma)\right]^{1/p}\left[\mathbb{E}\mathcal{V}_q(D,\Gamma)\right]^{1/q}, \tag{1.5}$$

for $p > 1$ and $q = p/(p-1)$, where B, C and κ are constants depending only on the capacity consumption matrix A, the upper bound F on the fare process and the expected cumulative demand over the planning horizon.

The bound (1.5) provided in Theorem 9 reflects two sources of error: The first term $\kappa\varepsilon$ is a perturbation error, cf. proof of Theorem 9, while the second term is due to the information gap between predictable versus adapted bid-price controls. Theorem 9 makes no assumptions on the primitives of the problem. That is, the various stochastic processes and the underlying information structure is very general. Next, we consider two important special cases. The first one assumes that demand over small periods is also small, which corresponds to the mathematical statement that sample paths of the demand process are continuous.

Corollary 10. *If the demand process has continuous sample paths, then*

$$|Obj(\pi^{\varepsilon},\varepsilon,\Gamma) - P^*| \leq \kappa_2\varepsilon + \frac{2\kappa_1\kappa_3}{\varepsilon}\sqrt{\mathbb{E}\mathcal{V}_2(D,\Gamma)}. \tag{1.6}$$

Moreover, for every $\varepsilon > 0$, one can choose a partition Γ^{ε} fine enough such that

$$|Obj(\pi^{\varepsilon},\varepsilon,\Gamma^{\varepsilon}) - P^*| \to 0 \quad as \quad \varepsilon \to 0.$$

Specializing the bound (1.5) of Theorem 9 to this case yields the upper bound (1.6) in terms of the quadratic variation $\mathcal{V}_2(D,\Gamma)$ of demand, which tends to zero as the partition

Γ gets finer. The upper bound in equation (1.6) (more generally the upper bound (1.5)) sheds light on how one should choose the decision points $\Gamma = \{t_0, t_1, \ldots, t_N\}$. To be more specific, given an absolute error tolerance, one picks an ε which determines the perturbation error $\kappa\varepsilon$. The smaller the perturbation error is, the finer the corresponding partition Γ^ε must be to respect the given error tolerance. Hence, the upper bound (1.6) helps the system manager trade-off the two sources of error by choices of ε and the corresponding partition Γ^ε. A similar trade-off can also be made in the context of Corollary 12, which concerns the case of continuous information. Roughly speaking, an information structure is continuous if the incremental information arriving over small periods is also small. The formal definition is provided next.

Definition 11. *An information structure $\{\mathcal{F}_t,\, t \in \mathbb{R}_+\}$ is said to be continuous if for every event E, the posterior probability assessment $\mathbb{P}(E \,|\, \mathcal{F}_t)$ is continuous.*

[**36**] proves that an information structure is continuos if and only if all stopping times are predictable, which in turn is a equivalent to the statement that every continuous-time martingale has continuous sample paths.

Corollary 12. *If the information structure $\{\mathcal{F}_t : 0 \leq t \leq T\}$ is continuous, then for every $\varepsilon > 0$, one can choose a partition Γ^ε fine enough such that*

$$|\,Obj\,(\pi^\varepsilon, \varepsilon, \Gamma^\varepsilon) - P^*| \to 0 \quad as \quad \varepsilon \to 0.$$

Combining the results of this section we conclude that in the usual case where periods are small compared to the planning horizon, there exists a simple predictable bid-price control which is near optimal if either the demand arriving during small periods is also small, or the incremental information arriving over small periods is small. These results also generate the insight that one must take additional care when the fare of a product exceeds the sum of the bid-prices of the resources it uses by a slight margin. In this case, one should book only a fraction of the demand, contrary to the ordinary bid-price controls which should dictate booking all demand.

1.5. A Special Case with Network Flow Structure

Although we have not imposed any integrality requirement on the bookings made in our model of network revenue management, cf. Section 1.1, in practice the booking vectors often should be integer valued. Moreover, the demand for products is typically not continuous but integral. Therefore, in this section, we consider a formulation where such integrality requirements are in place. In fact, the network revenue management problem (P) introduced in Section 1.1 corresponds to a continuous relaxation of the problem at hand. Next, imposing additional restrictions on the capacity consumption matrix A and assuming that the initial capacity vector C and the demand realizations the cumulative demand process D are integer valued, we show that in the continuous relaxation, bookings made by a generalized bid-price scheme are actually integral, and thus optimal for the formulation with integrality requirements.

To that end, first assume that the capacity consumption matrix has consecutive 1's in each column and 0's everywhere else (the first and the last row are assumed to be consecutive to each other). This special structure arises naturally in car rental and hotel revenue management applications, cf. [**15**]. We first show that A is totally unimodular, that is the determinant of each square submatrix of A is equal to 0, 1, or -1. The use of totally unimodular matrices in integer programming is typical; an integer program with a totally unimodular constraint matrix and integer right-hand side can be solved efficiently as a linear program since the LP relaxation provide integer solutions. In the same vein, total unimodularity of A will provide the key role in proving the integrality of bookings by making the extreme points of the feasible region in the linear programs ($\mathrm{P}(t_n)$) integer valued. Under the consecutive 1's assumption, there exists a unimodular transformation[6] that transforms A into a matrix with at most one 1 and one -1 in each column, cf. [**64**]. The totally unimodularity of A follows from the fact a matrix consisting of only 1, 0 and -1 is totally unimodular if it has no more than two nonzero entries in each column and the sum of the entries in a column that has two nonzero coefficients is zero. Assume further that the initial capacity vector C and the demand process D, are integer valued. This implies that, the feasible region

$$F_n(x) = \{u \in \mathbb{R}^J_+ : Au \leq x,\ u \leq D\,(t_n)\} \text{ for } n = 1, \ldots, N$$

is an integral polyhedron for integral $x \in \mathbb{R}^K_+$, provided that it is non-empty, cf. [**46**]. A nonempty polyhedron $F_n(x) \subseteq \mathbb{R}^J_+$ is integral if and only if all of its extreme points are integral. Particularly, since C is integer valued, $F_1(C)$ is an integral polyhedron and the bookings made in period 1 are integral as ($\mathrm{P}(t_1)$) is a linear program with feasible region $F_1(C)$. Then, the initial capacity vector at the beginning of period 2 would also be integral since A consists only of 0's and 1's. Therefore, it can be shown inductively that the bookings made by a generalized bid-price scheme (π, λ) are actually integral, thus optimal for the formulation with integrality requirements.

[6]A linear transformation $y = Bx$ is unimodular if the determinant of the matrix B is 1 or -1.

Moreover, since matrices with one 1 and one -1 in each column are node-arc incidence matrices, optimization problem $(\mathrm{P}(t_n))$ can be solved via network flow techniques for each $i \in \mathcal{I}$, which in turn makes the solution to $(\mathrm{P}(t_n))$ quite efficient. Indeed, the network flow structure arises naturally for many airlines as pointed out by [**34**].

CHAPTER 2

Bid-Price Controls for Network Revenue Management: Martingale Characterization of Optimal Bid Prices (joint with Barış Ata)

The network revenue management problems arise naturally in airline, railway, cruise-line and hotel revenue management, more generally, whenever, customers buy bundles of resources under various terms and conditions. In such settings, bid-price controls represent an intuitively appealing and powerful approach to quantity-based revenue management. To be specific, given a network of resources, a bid-price control assigns a threshold price, that is, a bid price, for each resource dynamically over time. Then, the decision to fulfill a booking request is made based on the availability of various resources and whether or not the revenue associated with the request exceeds the sum of the bid-prices of the resources it uses. Bid-price controls simplify decision making in network revenue management by reducing the number of parameters required for implementation (one bid price has to be specified for each resource) since evaluating a booking request requires only a simple comparison of the fare to the sum of the bid prices for the requested resources.

Bid-price controls are introduced by [**58**] and further analyzed by [**65**]. Williamson computes the bid prices of various resources by means of mathematical programming formulations and interprets the bid-price of a resource as the opportunity cost of using one additional unit of the resource. This reflects the intuitive notion that a booking request should be accepted only if its fare exceeds the opportunity cost of the reduction in resource capacities required to satisfy that request. Thus, bid prices retrieved from a revenue management system may facilitate decision making in other areas of management such as capacity planning or pricing.

[**59**] considers bid price controls in a discrete time model of network revenue management, where the discretion is fine enough such that in each period at most one request arrives. In this context, a bid-price control is implemented by specifying one bid price for each resource (leg) for each time period and capacity vector, and the request is accepted if the fare of the request is higher than the sum of the bid prices it uses. The authors show that the optimal policy need not correspond to a bid-price control, and provide a two-period counter example which shows that the bid-price policy as defined immediately above may not result in optimal accept/deny decisions. The insight they provide for why bid-price controls may not be optimal is that the bid price for a resource may not correspond to the opportunity cost of using one additional unit of that resource due to two reasons: First, selling one unit of capacity might be a large change in the capacity of several resources at the same time if the remaining capacity is low and hence the interpretation of the bid prices as the marginal value of one unit of additional capacity may not be correct. Second, the revenues may depend on the remaining capacity in a nonlinear way.

Despite this counter example, bid-price controls are widely used in practice, cf. [**49**], as they provide a simple, yet powerful approach to quantity-based network revenue management. In this vein, researchers have worked on various practical heuristic methods to derive bid-price controls. [**10**] proposed a new method based on approximate dynamic programming. Their method computes adaptive and nonadditive bid prices based on a linear programming approximation to the value function of a dynamic programming formulation. The authors provide a comparison of their method and the bid-price control based on deterministic linear programming approach and show that their algorithm results in higher revenues and more robust performance.

[**61**] revisits the network revenue management problem studied in [**59**] and proposes a new method to compute bid prices. Topaloglu explicitly considers the temporal dynamics of the customer arrivals and generates bid prices that depend on the remaining leg capacities. His method is based on relaxing certain capacity constraints that link decisions for different flight legs by associating Lagrange multipliers with them. Then the problem is decomposed by flight legs and one can concentrate on one flight leg at a time, which simplifies the problem tremendously. Topaloglu also shows through a numerical study that his method outperforms the standard heuristics significantly.

[**63**] follows a similar approach but uses a different relaxation of the capacity constraints which yields time-dependent prices. Their approach provides an upper bound on the optimal objective value of the problem, which is tighter than the one obtained from the so-called deterministic linear program. The authors also show that the bid prices they propose are asymptotically optimal as leg capacities and demand grow proportionally to infinity. Moreover, the authors discuss how to adopt their method to incorporate cancellations. Finally, the authors demonstrate through numerical examples that their method can improve on the existing methods. Another related paper is [**1**]. Adelman considers the dynamic programming formulation of the network revenue management problem.

Then assuming an affine functional form for the value function and using the linear programming representation of the dynamic programming formulation, the author computes time-dependent (deterministic) bid prices. He also shows that his approach yields an upper bound tighter than the one obtained from the deterministic linear program. Both [**1**] and [**63**] observe that their (approximate) method yields (deterministic) bid prices which are decreasing over time.

Another paper related to ours is [**38**], where the author considers a stylized (deterministic) fluid model of a general dynamic pricing problem for selling a network of resources. In Kleywegt's model prices are chosen dynamically to sell products (or itineraries) to multiple customer classes over time. Kleywegt's model is very general in terms of problem primitives and allows order cancellations. Moreover, Kleywegt observes that his model readily extends to incorporate probabilistic customer choice behavior. The author also develops a solution method and tests it with some numerical examples. Among other things, Kleywegt shows through an exact analysis that in his setting the opportunity cost of capacity under an optimal policy remains constant, which is in line with the martingale property of optimal bid prices in our setting. Indeed, looking more carefully at the numerical examples of [**1**] and [**63**] reveals that the bid prices seem to be constant except toward the end of planning horizon, which may be due to their approximate mode of analysis.

[**62**] presents a stochastic approximation method to compute bid prices in network revenue management problems by viewing the total expected revenue as a function of bid prices and using sample path derivatives to identify a good set of bid prices. The author demonstrates through numerical examples that the bid prices obtained by his

method outperforms the ones by the standard methods especially when bid prices are not computed frequently. [**31**] applies approximate dynamic programming ideas to revenue management problems. [**50**] develops a novel diffusion approximation to the network revenue management and advances a policy which is asymptotically optimal under the so-called diffusion scaling.

In this chapter, we analyze a continuous-time, rate-based model of network revenue management. Our main contribution is to prove ε-optimality of a simple bid-price control. The proposed bid-price control only uses bid-prices associated with various resources, hence, it is easy to implement. In what follows, we also construct an optimal generalized bid-price control which consists of a bid-price process and a capacity usage limit process, where the bid-price process forms a martingale. Although the generalized bid-price control we introduce here resembles the bid-price control of the previous chapter, it does give rise to new insights in this setting.

We provide further insights and implications of the martingale property of the (near) optimal bid prices, which become more transparent in our continuous-time model. Although the martingale property is primarily a theoretical contribution, it has surprising implications. For instance, exploiting the martingale property one can connect the optimal bid prices to Forward-Backward Stochastic Differential Equations (FBSDE). Given that there are readily available numerical methods for computing solutions FBSDEs, one may borrow that machinery to compute bid prices. Thus, this connection sets the stage up for a novel and analytically sound computational approach and is explored in Chapter 3.

Our analysis also sheds light on the non-optimality of bid-price controls defined in the classical sense, providing the new insight that the reason for potential non-optimality is not only the discreteness in demand but also the degeneracy or multiplicity of solutions, which will be elaborated on in Section 2.4. Although the bid prices defined in the classical sense may not be optimal in general, we provide some hypotheses guaranteeing their optimality (and ε-optimality). Unfortunately, identifying simple conditions on the problem primitives under which these hypotheses can be verified does not seem easy. Nevertheless, we feel that these hypotheses bring out the key step in proving such optimality results and help elucidate potential issues with the classical bid-price controls and why they fail to be optimal. In other words, our results provide further understanding of bid-price controls, their characterization and limitations, and the role they play in designing capacity allocation schemes in network revenue management problems.

From a methodological perspective, our analysis builds on the convex analysis framework of [**55**] and the duality results of [**11**]. [**11**] develops a new approach to problems of stochastic optimal control using convex duality. In particular, [**11**] defines the dual problems in stochastic optimal control and the coextremality conditions associated with the dual optima by applying general methods of convex analysis introduced by [**51**], [**52**], [**53**] and [**54**]. Bismut also provides results on the existence of optimal solutions for a general class of convex stochastic control problems, which include the stochastic control problems studied in this paper. This paper also illustrates the utility of stochastic duality techniques and their applicability in the revenue management context.

The rest of the chapter is structured as follows: Section 2.1 presents the model. Precise definition of several bid-price controls are introduced in Section 2.2. A dual formulation to the network revenue management problem and the associated coextremality conditions are provided in Section 2.3. An optimal generalized bid-price mechanism is defined in Section 2.4. There, we also provide sufficient conditions for the existence of (near) optimal bid-price controls in the classical sense. In Section 2.5, we discuss a perturbed network revenue management and its dual, based on which we also define an ε-optimal bid-price control. In Section 2.6, some concluding remarks are provided along with future research directions. The proofs, derivations and auxiliary results are relegated to Appendices B.2 through B.5 throughout the chapter. A summary of [**11**] is provided in Appendix B.1.

2.1. The Model

We analyze a continuous-time, rate-based model of network revenue management. There are K resources and J products. In an airline setting a resource is a flight leg and a product is a specific itinerary. A primitive of our model is a $K \times J$ non-negative capacity consumption matrix A, where A_{kj} denotes the amount of resource k capacity consumed by one unit of product j. The j^{th} column of A is denoted by A^j. The definition of a product contains all terms and conditions associated with the purchase. Thus, there may be more than one product that use the same amount of each resource but differ in price, purchase restrictions etc. Therefore, in practice, the number of products will be large compared to the number of resources. In our model, at each point in time the system manager observes the demand rate and chooses the corresponding booking rate for each product. The booking rates for the products translate into consumption rates for the resources through the capacity consumption matrix A. The objective is to maximize expected revenues over the time horizon $[0, T]$ subject to capacity and demand constraints.

Uncertainty is modeled by a given complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and the evolution of information is modeled through the increasing collection $\{\mathcal{F}_t,\, t \in \mathbb{R}_+\}$ of complete sub-σ-fields of $\mathcal{F}$. In particular, $\mathcal{F}_t$ represents the information available to the system manager at time t. All stochastic processes to appear will be adapted to the filtration $\{\mathcal{F}_t,\, t \in \mathbb{R}_+\}$. We assume that $\{\mathcal{F}_t,\, t \in \mathbb{R}_+\}$ is right-continuous and has no time discontinuity as in [**11**]. An information structure that has no time discontinuity is also referred to as a quasi-continuous information structure in [**36**], which also proves that the natural filtrations of most of the commonly encountered processes are quasi-continuous, including the natural filtrations generated by the Poisson process and Brownian motion. As a matter of fact, [**12**] extends the framework and results in [**11**] to the more general setting of the control of semi-martingales where the quasi-continuity assumption is also dropped.

The demand for the various products is generated by the J-dimensional demand rate process $\{d(\omega, t) : (\omega, t) \in \Omega \times [0, T]\}$. In particular, $d_j(\omega, t)$ is the rate at which demand for product j arrives at the system at time t along the sample path ω. Then, the cumulative demand observed by the system manager for product j over the interval $[t_1, t_2]$ is

$$\int_{t_1}^{t_2} d_j(\omega, s) ds,$$

if sample path $\omega \in \Omega$ is realized. The following are the only two assumptions we make on the demand rate process: We assume that the demand rate process $\{d(\omega, t) : (\omega, t) \in \Omega \times [0, T]\}$ is bounded and adapted to the filtration $\{\mathcal{F}_t : 0 \leq t \leq T\}$. In particular, we allow for non-stationary demand with an arbitrary dependence structure, including both inter-temporal and cross-product dependencies.

As the system evolves, the system manager exerts control on the system by selecting a nonnegative vector of booking rates at each point in time. That is, for each $\omega \in \Omega$ and $0 \leq t \leq T$, the system manager chooses the J-dimensional vector of booking rates, denoted by $u(\omega, t)$. In particular, $u_j(\omega, t)$ denotes the booking rate for product j at time t along the sample path ω for $j = 1, \ldots, J$. Then, under the control $\{u(\omega, t) : (\omega, t) \in \Omega \times [0, T]\}$, the cumulative bookings for product j up to time t along sample path ω is given by

$$U_j(\omega, t) = \int_0^t u_j(\omega, s) ds. \tag{2.1}$$

The system state at time $t \in [0, T]$ for the realization ω is the K-dimensional vector of remaining capacities denoted by $x(\omega, t)$. The component $x_k(\omega, t)$ denotes the remaining capacity for resource $k = 1, \ldots, K$ at time t. Given a control $\{u(\omega, t) : (\omega, t) \in \Omega \times [0, T]\}$, the system state evolves according to the following system dynamics equation

$$x(\omega, t) = C - AU(\omega, t) \text{ for } (\omega, t) \in \Omega \times [0, T], \tag{2.2}$$

where C is the initial capacity vector and $U(\omega, t)$ is the vector of cumulative bookings up to time t whose j^{th} component is given by (2.1). We use the shorthand notation x to denote the stochastic process $\{x(\omega, t) : (\omega, t) \in \Omega \times [0, T]\}$. Similarly, u denotes the booking rate process $\{u(\omega, t) : (\omega, t) \in \Omega \times [0, T]\}$.

A booking rate process u is feasible only if it satisfies demand and capacity restrictions. The demand restriction on bookings is that for each $(\omega, t) \in \Omega \times [0, T]$, the booking rate for each product should be less than or equal to the demand rate for that product. The capacity restriction on bookings is that the remaining capacity for each resource at the terminal time T should be nonnegative almost surely.

A booking rate vector $u(\omega, t)$ at time t given a sample path ω results in an instantaneous revenue rate of $f(\omega, t) \cdot u(\omega, t)$, where $f(\omega, t)$ is the exogenously set vector of fares. The process of fares $\{f(\omega, t) : (\omega, t) \in \Omega \times [0, T]\}$ is bounded, non-negative and adapted to the information structure $\{\mathcal{F}_t,\ t \in \mathbb{R}_+\}$. The fare process can be non-stationary and arbitrarily correlated with the demand process, which, in turn, allows us to model dependencies between the fare and demand process and potentially capture demand substitution across products and over time.

The objective is to choose a booking rate process u so as to maximize expected revenue subject to the demand and capacity restrictions. That is, choose booking rate vector $u(\omega, t)$ for each $(\omega, t) \in \Omega \times [0, T]$ so as to

$$
\begin{aligned}
&\text{maximize } \ \mathbb{E}\left[\int_0^T f(\omega, t) \cdot u(\omega, t)\, dt\right] \\
&\text{subject to} \\
&x(\omega, t) = C - AU(\omega, t), \quad (\omega, t) \in \Omega \times [0, T], \\
&U(\omega, t) = \int_0^t u(\omega, s) ds, \quad (\omega, t) \in \Omega \times [0, T], \\
&0 \leq u(\omega, t) \leq d(\omega, t), \quad (\omega, t) \in \Omega \times [0, T], \\
&x(\omega, T) \geq 0, \quad \omega \in \Omega,
\end{aligned}
\qquad (\mathrm{P}_{cont})
$$

where the first and second constraints describe how capacity evolves over time, and the third and fourth constraints are the demand and capacity restrictions, respectively. Throughout the rest of the paper we will refer to the formulation (P_{cont}) as the network revenue management problem.

As an aside, since the fare process can be an arbitrary adapted stochastic process, the formulation (P_{cont}) of the network revenue management problem subsumes possible discounted formulations.

2.2. Bid-Price Control Definitions and Summary of Results

In our setting, a bid-price process is a K-dimensional, non-negative stochastic process $\pi = \{\pi(\omega,t) : (\omega,t) \in \Omega \times [0,T]\}$, where $\pi_k(\omega,t)$ denotes the bid-price, or the shadow price, associated with resource k at time t along the sample path ω. Next, we introduce three closely related definitions of bid-price controls, which will be used in subsequent sections.

Definition 13. *(Bid-Price Control): Given a bid-price process* $\pi = \{\pi(\omega,t) : (\omega,t) \in \Omega \times [0,T]\}$ *and a booking function* $\phi : \mathbb{R}_+^3 \to \mathbb{R}_+$*, the pair* (π,ϕ) *is called a bid-price control, where the corresponding booking rates for each product* $j = 1,\ldots,J$ *are determined as follows:*

$$u_j(\omega,t) = \phi(\pi(\omega,t)A^j,\ f_j(\omega,t),\ d_j(\omega,t)) \quad \textit{for } (\omega,t) \in \Omega \times [0,T]. \tag{2.3}$$

Given a bid-price control (π,ϕ), for each product j and each (ω,t), the booking function ϕ compares the fare $f_j(\omega,t)$ with the sum of the bid prices for the resources used by product j, $\pi(\omega,t)A^j$, and dictates how much of the demand rate $d_j(\omega,t)$ to book. Our definition allows for non-linear booking functions. In particular, it may not result in a "bang-bang" booking process; u is called a bang-bang booking process if $u_j(\omega,t) \in \{0, d_j(\omega,t)\}$ for almost all $(\omega,t) \in \Omega \times [0,T]$ and $j = 1,\ldots,J$. In this sense, our definition of a bid-price control is more generous than the classical definition; and the latter can be viewed as a special case of ours. In particular, the booking decisions under a bid-price control in the classical sense necessarily result in a bang-bang booking process. The next definition introduces the bid-price control in the classical sense in our setting.

Definition 14. *(Classical Bid-Price Control) For each bid-price process π, there corresponds a classical bid-price control denoted by π, which dictates the following booking rates for each product $j = 1, \ldots, J$ and $(\omega, t) \in \Omega \times [0, T]$:*

$$u_j(\omega, t) = \begin{cases} d_j(\omega, t) & \textit{if } f_j(\omega, t) \geq \pi(\omega, t)A^j, \\ 0 & \textit{otherwise.} \end{cases}$$

One can view a classical bid-price control π as a specific bid-price control (π, ϕ) where $\phi(z) = z_3 \mathbf{1}_{\{z_2 \geq z_1\}}$ for $z \in \mathbb{R}^3_+$, from which it follows that

$$\phi(\pi(\omega, t)A^j,\ f_j(\omega, t),\ d_j(\omega, t)) = \begin{cases} d_j(\omega, t) & \text{if } f_j(\omega, t) \geq \pi(\omega, t)A^j, \\ 0 & \text{otherwise.} \end{cases} \tag{2.4}$$

A bid-price control (π, ϕ) is called optimal if the booking rates resulting form (π, ϕ), cf. (2.3), constitute an optimal solution to the network revenue management problem (P_{cont}). Given $\varepsilon > 0$, a bid-price control $(\pi^\varepsilon, \phi^\varepsilon)$ is called ε-optimal if the revenues associated with the resulting booking process u^ε is within ε of the optimal objective value of the network revenue management problem (P_{cont}).

An important virtue of bid-price controls is that they offer a tractable solution for a complex problem of allocating a network of resources to a large number of products. Bid-price controls simplify the control in network revenue management by reducing the number of parameters required for implementation (one bid price is specified for each resource.) In addition, a bid-price control decomposes the problem across time, sample paths and products. That is, given a bid-price control (π, ϕ), at each point in time and for every sample path, the booking rates are determined only as a function of the current bid-prices, fares and demand rates without having to account for the future impact of current decisions. Moreover, the booking decisions for each product can be made in isolation, independently of the booking decisions for other products.

Next, we consider a simple example to illustrate the classical bid-price controls. This example is indeed the continuous-time, rate-based version of the counter example provided by [**59**].

Example 1. There are two resources and three products with the associated capacity consumption matrix

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}.$$

The planning horizon is $[0, 2]$. Each resource has initial capacity of one, that is, $C = (1, 1)'$. There is no uncertainty or non-stationarity in product fares. In particular, the vector of product fares is given by

$$f = (250, 250, 500)'.$$

The only uncertainty is in the demand rate process. The evolution of uncertainty is suitably represented by an information tree in Figure 2.1. The terminal nodes of the tree correspond to specific sample paths. The intermediate set of nodes represent the resolved uncertainty by time $t = 1$. On each arc of the information tree displayed is the corresponding demand rate vector. There are six sample paths and the probability of each sample path is also displayed in Figure 2.1, from which one can deduce the probabilities of various events. In particular, during $[0,1]$ we will see demand for only one type of product (at rate 1). The probability of having demand for product 3 is 0.4; for product 1 it is 0.3 and for product 2 it is 0.3. On the other hand, during $[1,2]$, we see either demand for product 3 (at rate 1) with probability 0.8 or no demand with probability 0.2. To be more specific, the demand rate process displayed in Figure 2.1 is given as follows:

$$d(\omega_1,t) = (1,1)', \ t \leq 2 \qquad \text{and} \qquad d(\omega_2,t) = \begin{cases} (1,1)', & t \leq 1, \\ (0,0)', & t > 1, \end{cases}$$

$$d(\omega_3,t) = \begin{cases} (1,0)', & t \leq 1, \\ (1,1)', & t > 1, \end{cases} \qquad \text{and} \qquad d(\omega_4,t) = \begin{cases} (1,0)', & t \leq 1, \\ (0,0)', & t > 1, \end{cases}$$

$$d(\omega_5,t) = \begin{cases} (0,1)', & t \leq 1, \\ (1,1)', & t > 1, \end{cases} \qquad \text{and} \qquad d(\omega_6,t) = \begin{cases} (0,1)', & t \leq 1, \\ (0,0)', & t > 1. \end{cases}$$

It is easy to see that the optimal solution is to book only product 3, while denying all other requests. This results in the expected revenue of 440. Formally, the solution is given as follows:

$$u(\omega_1,t) = u(\omega_2,t) = \begin{cases} (0,0,1)', & t \leq 1, \\ (0,0,0)', & t > 1, \end{cases} \tag{2.5}$$

$$u(\omega_3,t) = u(\omega_5,t) = \begin{cases} (0,0,0)', & t \leq 1, \\ (0,0,1)', & t > 1, \end{cases} \tag{2.6}$$

$$u(\omega_4,t) = u(\omega_6,t) = \begin{cases} (0,0,0)', & t \leq 1, \\ (0,0,0)', & t > 1. \end{cases} \tag{2.7}$$

As an aside, this solution corresponds to the optimal solution of the Talluri-van Ryzin example, which also books only product 3 and results in expected revenue of 440.

Consider the following classical bid-price control: $\pi(\omega_1, t) = \pi(\omega_2, t) = (250, 250)'$, and

$$\pi(\omega_3, t) = \begin{cases} (251, 150)', & t \le 1, \\ (312.5, 187.5)', & t > 1, \end{cases} \quad \text{and} \quad \pi(\omega_4, t) = \begin{cases} (251, 150)', & t \le 1, \\ (5, 0)', & t > 1, \end{cases}$$

$$\pi(\omega_5, t) = \begin{cases} (150, 251)', & t \le 1, \\ (187.5, 312.5)', & t > 1, \end{cases} \quad \text{and} \quad \pi(\omega_6, t) = \begin{cases} (150, 251)', & t \le 1, \\ (0, 5)', & t > 1. \end{cases}$$

Observe that the classical bid-price control π results in the optimal bookings given in (2.5)-(2.7), and hence yields expected revenue of 440. It is also easy to check that the bid-price process π forms a martingale. Also note the choice of the optimal bid-price control is not unique. Once can easily come up with other optimal classical bid prices.

The bid-price control given immediately above shows that the classical bid-price controls are indeed optimal for this particular example. Thus, it sheds light onto reasons for non-optimality in the Talluri-van Ryzin example. As pointed out earlier in the literature, the non-optimality stems from discreteness, and, in particular, from the fact that each booking consumes a large fraction of remaining capacity. In contrast, in our setting the bookings at each point in time consumes only an infinitesimal amount of capacity. We also allow frequent (indeed continuous) updating of bid prices. These allow the bid-price controls to perform optimally. Therefore, in addition to discreteness of the problem, infrequent updating of bid prices may be another reason for non-optimality of classical bid-price controls.

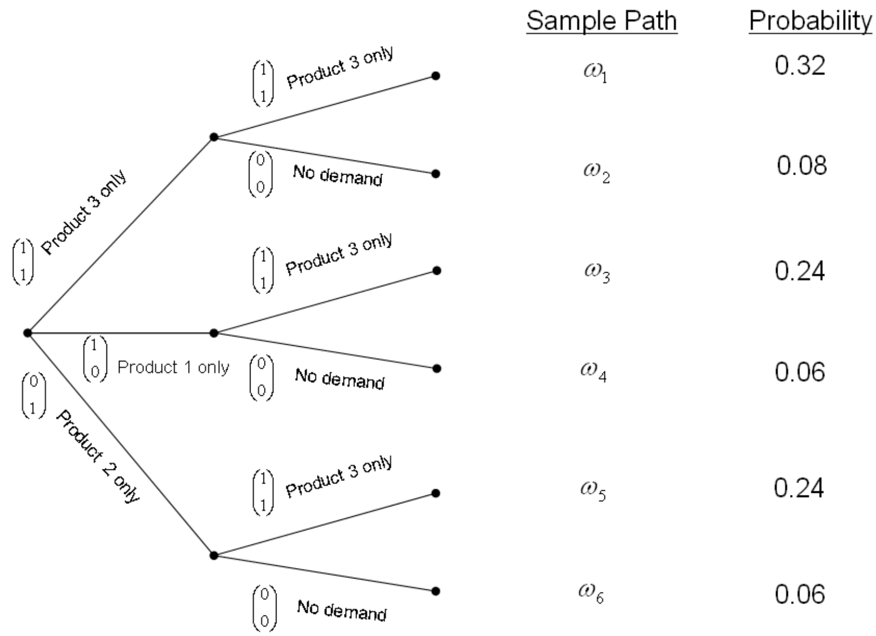

Figure 2.1. Evolution of uncertainty in Example 1.

Although the classical bid-price controls are optimal for the example immediately above, we next present an example where no classical bid price control can be optimal. Indeed, we show a stronger result that no bang-bang control can be optimal.

Example 2. We have a single resource and two products. As in the earlier example the fares are constant. In particular, we have

$$C = 1, \quad A = [1, 1] \quad \text{and} \quad f = (100, 200)'.$$

The planning horizon is $[0, 1]$. The underlying probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is defined as follows: $\Omega = [0, 1]$ and $\mathcal{F}$ is the collection of Borel subsets of $[0, 1]$ (suitably completed). Let τ be a random variable defined on $(\Omega, \mathcal{F})$ with strictly positive density g on $[0, 1]$. Then $\mathbb{P}$ is the probability measure induced by g; and each sample path can be thought of as a particular realization of τ. Demand for the two products is given as follows:

$$d_1(t) = \begin{cases} 2 & \text{if } t \leq \tau, \\ 0 & \text{if } t > \tau, \end{cases} \quad \text{and} \quad d_2(t) = \begin{cases} 0 & \text{if } t \leq \tau, \\ 1 & \text{if } t > \tau. \end{cases}$$

The information $\mathcal{F}_t$ available at time t is the σ-algebra generated by $\{d(s) : 0 \leq s \leq t\}$, suitably completed with the null sets of $\mathcal{F}$.

In this setting, we prove that no classical bid-price control π adapted to $\{\mathcal{F}_t, t \geq 0\}$ can be optimal. First, observe that the revenue management problem (P_{cont}) has a pathwise solution in this example, which is given as follows:

$$u_1(t) = \begin{cases} 1 & \text{if } t \leq \tau, \\ 0 & \text{if } t > \tau, \end{cases} \quad \text{and} \quad u_2(t) = \begin{cases} 0 & \text{if } t \leq \tau, \\ 1 & \text{if } t > \tau. \end{cases}$$

In particular, the optimal revenue is $200 - 100\tau$ along each sample path which results in the expected revenue of

$$200 - 100\mathbb{E}[\tau] = 200 - 100 \int_0^1 g(s) ds.$$

We argue by contradiction to conclude that no classical bid-price control can be optimal. Suppose that there exists an optimal bid-price control π. First, note that under π we must have

$$\int_0^\tau 2\,\mathbf{1}_{\{\pi(t) \leq 1\}} dt = \tau \quad \text{almost surely.} \tag{2.8}$$

That is, we must book exactly half of the requests for product 1. Suppose (2.8) does not hold. Then, with positive probability we have at least one of the following:

$$\int_0^\tau 2\,\mathbf{1}_{\{\pi(t)\leq 1\}}dt \quad < \quad \tau, \tag{2.9}$$

$$\int_0^\tau 2\,\mathbf{1}_{\{\pi(t)\leq 1\}dt} \quad > \quad \tau. \tag{2.10}$$

If we have (2.9) with positive probability, then for those sample paths the total revenue is given by

$$\int_0^\tau 200\,\mathbf{1}_{\{\pi(t)\leq 1\}}dt + 200(1-\tau) < 200 - 100\tau.$$

Thus, if (2.9) arises with positive probability, then the expected revenues under π will be strictly less than $200 - 100\mathbb{E}[\tau]$, the optimal objective. Similarly, if (2.10) happens with positive probability, then the revenue along such a sample path is

$$\int_0^\tau 200\,\mathbf{1}_{\{\pi(t)\leq 1\}}dt + 200\left[1 - \int_0^\tau 2\,\mathbf{1}_{\{\pi(t)\leq 1\}}dt\right] < 200 - 100\tau.$$

Thus, the expected revenue is strictly less than $200-100\mathbb{E}[\tau]$ in this case too, contradicting optimality. Therefore, we must have (2.8). That is, almost surely

$$\int_0^\tau 2\,\mathbf{1}_{\{\pi(t)\leq 1\}}dt = \tau. \tag{2.11}$$

Then since τ has a strictly positive density on $[0,1]$ and both sides are absolutely continuous functions of τ, we conclude by differentiating both sides of (2.11) with respect to τ

that for almost every t and sample path

$$\mathbf{1}_{\{\pi(t) \leq 1\}} = \frac{1}{2},$$

which clearly is a contradiction. Thus, no classical bid-price control can be optimal.

As this example shows no classical bid-price control can achieve optimal bookings in general. In this specific example, if we impose the upper bound of 1 on the capacity consumption rate and use the bid price of $\pi = 1$ at all times, then the resulting bookings will be optimal. Next, we show that this idea works in great generality. That is, using bid prices in conjunction with limits on capacity consumption rates results in optimal bookings. To this end, the next definition introduces a generalized bid-price control along the lines of Chapter 1. In our setting, a generalized bid-price control involves a pair of stochastic processes (π, λ), where π is a bid-price process and $\lambda = \{\lambda(\omega, t) : (\omega, t) \in \Omega \times [0, T]\}$ is a K-dimensional, nonnegative stochastic process; we will refer to λ as the capacity usage limit process.

Definition 15. *(Generalized Bid-Price Control) Given a bid-price process π and a capacity usage limit process λ, the pair (π, λ) is called a generalized bid-price control. For each $(\omega, t) \in \Omega \times [0, T]$, the booking rate vector $u(\omega, t)$ corresponding to (π, λ) is given by* *the solution to the following linear program: Choose booking rate vector u so as to*

$$(P(\omega,t)) \qquad \begin{aligned} &\textit{maximize} \ \ (f(\omega,t) - A^{'}\pi(\omega,t))\cdot u + \eta(Au - \lambda(\omega,t))\cdot \mathbf{e} \\ &\textit{subject to} \\ &Au \leq \lambda(\omega,t), \\ &0 \leq u \leq d(\omega,t), \end{aligned}$$

where $\eta > 0$ is arbitrarily small and $\mathbf{e}$ is the K-dimensional vector of ones.

As before, $\pi_k(\omega,t)$ is the bid-price or the shadow price for resource k at time t along the sample path ω. Similarly, $\lambda_k(\omega,t)$ is associated with resource k, and will be used as an upper bound on the consumption rate of resource k at time t along the sample path ω. The linear program (P(ω,t)) is lexicographic in the following sense. The system manager first solves (P(ω,t)) by setting $\eta = 0$. In the case of multiple optimal solutions, she selects the one that maximizes $(Au - \lambda(\omega,t))\cdot\mathbf{e}$. For concreteness, tie breaking is done as follows in case of multiple such solutions. The potential choices of basis matrices are numbered up-front. For each (ω,t), the optimal solutions are characterized by the extreme points, each of which corresponds to a basis matrix. In case of multiple optimal solutions, the system manager picks the (extreme) optimal solution which corresponds to the basis matrix with the lowest index. Ideally, the system manager tries to choose a "maximal" booking rate u that has $Au = \lambda(\omega,t)$. A generalized bid-price control (π,λ) is said to be optimal if the booking rate process u resulting from its execution is optimal for the network revenue management problem (P$_{cont}$). As will be seen below (cf. Proof of Theorem 18), if π and λ are chosen optimally, then the bookings under (π,λ) satisfy

$Au = \lambda(\omega, t)$ for a.e. (ω, t). In particular, the capacity usage limit process λ ensures that the system state x follows an optimal trajectory, which is crucial for optimality because of potential issues due to degeneracy or multiplicity of solutions, cf. Section 2.4.

As pointed out earlier, it is easy to see that setting

$$\pi(\omega, t) = 1 \quad \text{and} \quad \lambda(\omega, t) = 1 \quad \text{for } (\omega, t) \in \Omega \times [0, 2]$$

results in optimal bookings in Example 2.

In what follows, we first show that there exists an optimal generalized bid-price control (π, λ) for the network revenue management problem. We also show that the optimal bid-price process π forms a martingale. Second, we identify some sufficient conditions under which an optimal bid-price control in the classical sense exists. Although, identifying simpler conditions on the problem primitives under which these sufficient conditions can be verified does not seem easy, they bring out the key step in proving such results. Third, by the help of a perturbed version of the network revenue management problem, we construct an ε-optimal bid-price control $(\pi^{\varepsilon}, \phi^{\varepsilon})$ for each $\varepsilon > 0$, where the associated bid-price process π^{ε} forms a martingale. The bid-price control $(\pi^{\varepsilon}, \phi^{\varepsilon})$ can be viewed as a perturbation of the classical bid-price control corresponding to the bid-price process π^{ε}. In particular, it does not involve any capacity usage limits, and hence, is easier to implement. Finally, we show that for small values of ε, the booking process u^{ε} corresponding to the bid-price control $(\pi^{\varepsilon}, \phi^{\varepsilon})$ is close to an optimal booking process for (P_{cont}). These results are proved without making *any* assumptions on the stochastic structure of demand and fare processes, allowing non-stationary demand and fare processes with an arbitrary dependence structure, including both inter-temporal and cross-product dependencies.

To facilitate our analysis of bid-price controls, in the next section we present a stochastic control problem that is dual to the network revenue management problem (P_{cont}) in the sense of [**11**].

2.3. Dual Network Revenue Management Problem

In this section we present the dual problem formulation (D_{cont}) of the network revenue management problem (P_{cont}) laid out in Section 2.1, and the coextremality results between the two formulations. The dual problem associated with the network revenue management problem (P_{cont}) is obtained using the stochastic duality theory of [**11**]. [**11**] develops a new approach to problems of stochastic optimal control using convex duality, which enables us to express the network revenue management problem in an equivalent way but in a completely different context. In particular, [**11**] defines the dual problems in stochastic optimal control and the coextremality conditions associated with the dual optima by applying general methods of convex analysis introduced by [**51**], [**52**], [**53**] and [**54**]. A summary of [**11**] is provided in Appendix B.1.

Following [**11**] the dual problem of control associated with the network revenue management problem (P_{cont}) can be stated as follows (see Appendix B.3 for its derivation): Choose a K-dimensional, square integrable, random vector $y_0 \in \mathcal{F}_0$ and a K-dimensional, square-integrable martingale M, which is null at zero, stopped at time T and adapted to the filtration $\{\mathcal{F}_t,\, t \in \mathbb{R}_+\}$, so as to

$$\begin{aligned}
&\text{minimize } \ \mathbb{E}\left[\int_0^T d(\omega,t)\cdot[f(\omega,t)-y(\omega,t)A]^+dt + C\cdot y_0(\omega)\right]\\
&\text{subject to} \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (\mathrm{D}_{cont})\\
&y(\omega,t) = y_0(\omega) + M(\omega,t), \quad (\omega,t)\in\Omega\times[0,T],\\
&y(\omega,T)\geq 0,\ \omega\in\Omega,
\end{aligned}$$

where $[z]^+ = (\max\{0,z_1\},\ldots,\max\{0,z_J\})'$ for $z\in\mathbb{R}^J$.

In the dual problem formulation, y is the dual state variable. The value of the state variable y at time zero is given by $y_0 \in \mathcal{F}_0$, which is constant if there is no randomness at time zero. The dual state vector $y(\omega,t)$ can be interpreted as the shadow price for or the value assigned to the resources at time t along the sample path ω. Then the objective of the dual problem formulation can be interpreted as the value attributed to the network of resources by a given choice of the shadow price process. Thus, the dual problem formulation has the following interpretation: The system manager chooses the shadow price process y, which is a non-negative martingale, so as to minimize the expected value she attributes to her network of resources.

In dual problem formulation (D_{cont}), M is a predictive term, null at zero, or the best estimate at time t of the unresolved uncertainty. M is a term that integrates the information on environmental factors. That is, it contains the relevant information from the future and serves the purpose of integrating into the dual variable y this necessary information.

A remarkable feature of the dual problem is that the dual state variable y can have jumps, corresponding to the jumps of M. To elaborate on this, consider the setting where the information is revealed continuously over time. In particular, consider a stopping time τ which is defined as the first time an event happens. Under a continuous information structure this event is foretellable by a sequence of events. Intuitively, no event takes us by surprise under a continuous information structure. As put by [**23**], "We are forewarned by a succession of precursory signs, of the exact time the phenomenon will occur"; see [**36**] for a precise definition of a continuous information structure.

An equivalent characterization of continuous information structures is that all martingales have continuous sample paths, cf. [**36**]. Moreover, the martingale term M of the dual problem formulation (D_{cont}) will have jumps only if the information arrives discontinuously, say, because of unpredictable changes in the business environment, political situation etc. in which case the value of the resources reflected by the shadow prices has to be adjusted abruptly, in a discontinuous manner. That is, the new information can significantly increase or decrease the bid prices. Hence, continuity of M under a continuous information structure is the reflection of the continuos flow of information into the system.

To elaborate further, suppose that any martingale, and hence the optimal bid-prices can be represented as a stochastic integral with respect to a given Brownian motion w plus a martingale M that is orthogonal to w. Then, loosely speaking w contains the short term uncertainties and M is a prediction of the long-term uncertainties. The interpretation of the martingale term M is intuitively appealing since it enables us to express formally the distinction in the decision making process between continuous information processing which is done on a routine basis and discontinuous information processing done by reassessing the predictions.

The dual problem (D_{cont}) and the primal problem (P_{cont}) are closely linked to each other. Above all, the objective function values of (P_{cont}) and (D_{cont}) are equal. Moreover, any optimal primal solution and any optimal dual solution satisfy a set of coextremality conditions, which are necessary and sufficient conditions for optimality. The following proposition summarizes the duality results between the two formulations that are relevant for our purposes; its proof is given in Appendix B.3.

Proposition 16. *The network revenue management problem (P_{cont}), that is, the primal problem, and the dual problem (D_{cont}) have the same optimal objective value. Moreover, letting u be a feasible control for (P_{cont}) with the corresponding state trajectory x, and (y_0, M) be a feasible control for (D_{cont}) with the corresponding state trajectory y, the controls u and (y_0, M) are optimal for (P_{cont}) and (D_{cont}), respectively, if and only if they satisfy the coextremality conditions (D.1) and (2.13) given below:*

$$y(\omega, T) \cdot x(\omega, T) = 0, \ \ a.e.\ \omega \in \Omega, \tag{2.12}$$

and for $j = 1, \ldots, J$ and almost all $(\omega, t) \in \Omega \times [0, T]$ with $d_j(\omega, t) > 0$,

$$\begin{aligned}
&\text{if } \ u_j(\omega, t) = 0, && \text{then } \ y(\omega, t)A^j - f_j(\omega, t) \geq 0,\\
&\text{if } \ 0 < u_j(\omega, t) < d_j(\omega, t), && \text{then } \ y(\omega, t)A^j - f_j(\omega, t) = 0,\\
&\text{if } u_j(\omega, t) = d_j(\omega, t), && \text{then } \ y(\omega, t)A^j - f_j(\omega, t) \leq 0.
\end{aligned} \tag{2.13}$$

The following corollary is immediate from Proposition 16 and it provides an upper bound on the objective function value of the network revenue management problem (P_{cont}).

Corollary 17. *For any non-negative, square integrable martingale y adapted to the filtration $\{\mathcal{F}_t,\, t \in \mathbb{R}_+\}$,*

$$\mathbb{E}\left[\int_0^T d(\omega,t)\cdot[f(\omega,t) - y(\omega,t)A]^+ dt + C\cdot y_0(\omega)\right]$$

provides an upper bound on the objective function value of the network revenue management problem (P_{cont}).

2.4. An Optimal Generalized Bid-Price Control

In this section we show the existence of an optimal generalized bid-price control (π, λ) defined as in Section 2.2 such that the optimal bid-price process $\{\pi(\omega,t) : (\omega,t) \in \Omega \times [0,T]\}$ forms a martingale. Recall that a generalized bid-price policy (π, λ) is said to be optimal if the booking rate process $\{u(\omega,t) : (\omega,t) \in \Omega \times [0,T]\}$ resulting from the execution of (π, λ) is optimal for the network revenue management problem (P_{cont}), cf. Section 2.2. The following theorem is the main result of this section and is proved in Appendix B.4.

Theorem 18. *There exists an optimal generalized bid-price control (π, λ) such that the optimal bid-price process $\{\pi(\omega,t) : (\omega,t) \in \Omega \times [0,T]\}$ is a martingale adapted to $\{\mathcal{F}_t : t \in [0,T]\}$.*

As can be seen from the proof of Theorem 18, we construct an optimal generalized bid-price control (π, λ) by using an optimal state trajectory y for the dual problem (D_{cont}) as the bid-price process π. To elaborate on its connection to the classical bid-price controls, fix $(\omega, t) \in \Omega \times [0, T]$ and suppose that

$$f_j(\omega, t) \neq \pi(\omega, t)A^j \text{ for } j = 1, \dots, J.$$

Then it can be seen from the coextremality conditions (D.1) and (2.13) that the booking rate vector $u(\omega, t)$ is uniquely determined by the bid-price vector $\pi(\omega, t)$ as follows: For $j = 1, \dots, J$,

$$u_j(\omega, t) = \begin{cases} d_j(\omega, t) & \text{if } \; f_j(\omega, t) > \pi(\omega, t)A^j, \\ 0 & \text{if } \; f_j(\omega, t) < \pi(\omega, t)A^j. \end{cases} \tag{2.14}$$

Our construction of the generalized bid-price control (π, λ) will choose the capacity usage limit process $\lambda(\omega, t) = Au(\omega, t)$ in this case, and the optimal generalized bid-price control results in the same booking decisions as a classical bid-price control would, cf. (2.14). Moreover, arguing heuristically, one can simply set $\eta = 0$ and $\lambda(\omega, t) = \infty$, in which case the execution of the generalized bid-price control reduces to a classical bid-price control, because the problem ($\mathrm{P}(\omega, t)$) decomposes across products.

Differences may arise, though, when $f_j(\omega, t) = \pi(\omega, t)A^j$ for some products and the capacity usage limit vector $\lambda(\omega, t)$ can be crucial in separating an optimal booking rate from a non-optimal one. In this sense, the need for a capacity usage limit process λ for optimality is linked to the multiplicity of optimal solutions u to (P_{cont}) associated with a given optimal dual solution y to (D_{cont}). Recall that one reason cited for the non-optimality of the classical bid-price controls is the discreteness of demand and bookings, cf. [**59**]. In contrast, our analysis suggests that discreteness may not be the only reason for non-optimality as we avoid it by adapting a rate-based model, or a stochastic fluid model. Rather, the reason for potential non-optimality of the classical bid-price controls in our setting seems to be the multiplicity or degeneracy which arises when $f_j(\omega, t) = \pi(\omega, t)A^j$ for some product $j = 1, \ldots, J$ over a non-negligible set of (ω, t).

In the next subsection, we explore conditions under which an optimal primal solution u is uniquely characterized by a given bid-price process π and propose some hypotheses that guarantee optimality or near-optimality of the classical bid-price controls.

2.4.1. Sufficient Conditions for Existence of Optimal Classical Bid-Price Controls

In this subsection we identify conditions under which classical bid-prices are optimal. In particular, we introduce a series of hypotheses which we will relate to the existence of optimal classical bid-price controls. The strict complementary slackness condition introduced in the next definition is key to the optimality of the classical bid-price controls.

Definition 19. *A solution u to the network revenue management problem (P_{cont}) and a K-dimensional, non-negative process $z = \{z(\omega, t) : (\omega, t) \in \Omega \times [0, T]\}$ are said to satisfy strict complementary slackness condition for $(\omega, t) \in \Omega \times [0, T]$ if exactly one of the following is true for (ω, t) and $j = 1, \ldots, J$:*

(1) $u_j(\omega, t) = 0$ and $z(\omega, t)A^j > f_j(\omega, t)$,

(2) $u_j(\omega, t) > 0$ and $z(\omega, t)A^j \leq f_j(\omega, t)$.

Then the following hypotheses provide sufficient conditions for the optimality and ε-optimality of the classical bid-price controls; and Proposition 20 states the rather obvious consequences of the hypotheses.

Hypothesis 1 *There exists a bang-bang solution u to the network revenue management problem (P_{cont}) and a dual solution y to (D_{cont}) that satisfy the strict complementary slackness condition for a.e. $(\omega, t) \in \Omega \times [0, T]$.*

Hypothesis 2 *For every $\varepsilon > 0$, there exists a bang-bang solution u^{ε} to the network revenue management problem (P_{cont}) and a dual solution y^{ε} to (D_{cont}) such that u^{ε} and y^{ε} satisfy the strict complementary slackness condition except for a set of $dP \otimes dt$ measure $\varepsilon > 0$.*

Proposition 20. *The following hold:*

a) If Hypothesis 1 is true, then there exists an optimal classical bid-price control such that the associated bid-price process forms a martingale adapted to the filtration $\{\mathcal{F}_t, t \in \mathbb{R}_+\}$.

b) If Hypothesis 2 is true, then for every $\varepsilon > 0$, there exists an ε-optimal classical bid-price control such that the associated bid-price process forms a martingale adapted to the filtration $\{\mathcal{F}_t, t \in \mathbb{R}_+\}$.

Example 3. We have a single resource and two products. The fares are constant. In particular, we have

$$C = 3, \quad A = [1, 1], \quad \text{and} \quad f = (100, 200)'.$$

The planning horizon is $[0, 4]$. There is no uncertainty in the environment. Demands for the two products are given as follows:

$$d_1(t) = \begin{cases} 1 & \text{if } t \leq 2, \\ 0 & \text{otherwise,} \end{cases} \quad \text{and} \quad d_1(t) = \begin{cases} 0 & \text{if } t \leq 2, \\ 1 & \text{otherwise.} \end{cases}$$

It is clear by inspection that the optimal solution books half of class 1 requests and all class 2 requests, resulting in optimal revenue of 500. Neither hypothesis 1 nor hypothesis 2 are satisfied in this example. Note that the unique solution to the dual problem is $\pi(t) = 100$ for $0 \leq t \leq 4$, which is easy to see since additional capacity results in booking more demand for product 1. That is, the shadow price of the resource is 1. Restricting attention to bang-bang solutions of the primal problem (as required by the hypothesis), it is clear that the Lebesgue measure of the set $F = \{t \in [0, 1] : u_1(t) = 0\}$ under any optimal solution must be 0.5. Thus, for $t \in F$, we have $u_1(t) = 0$ and $\pi(t) \leq f_1$ so that the strict complementary slackness conditions are violated with measure 0.5. Of course, an interesting open question is that whether one can impose additional assumptions on problem primitives to rule out such examples, which is left as a topic for future research.

A sufficient condition for Hypothesis 1 to be true is that there exists an optimal state trajectory y for the dual network revenue management problem (D_{cont}) such that the measure of the set

$$E = \{(\omega, t) : y(\omega, t)A^j - f_j(\omega, t) = 0 \text{ for some } j = 1, \ldots, J\}$$

is zero $d\mathbb{P} \otimes dt$. In that case, every optimal solution u to (P_{cont}) is necessarily bang-bang since it should satisfy the coextremality conditions with y. This, in turn, would require the solution to the network revenue management problem (P_{cont}) to be unique since otherwise taking convex combinations of distinct bang-bang optimal solutions would produce optimal solutions that are not bang-bang. However, calling for a unique bang-bang solution to (P_{cont}) might be a strong assumption that would be hard to satisfy.

Admittedly, verifying the hypotheses stated immediately above for a general class of problems is not easy, and it remains an open question for future research to identify conditions on the problem primitives that will make these hypotheses true. Nonetheless, these hypotheses bring out multiplicity of solutions and strict complementary slackness as the key issues in proving the existence of optimal bid-price controls in the classical sense.

In the next section, we pursue an alternative path and construct an ε-optimal bid-price control for any given $\varepsilon > 0$. To this end, we introduce a perturbed version of the network revenue management problem. The perturbed problem for $\varepsilon > 0$, in turn, gives rise to an ε-optimal bid price control, which can be viewed as a perturbation of a classical bid-price control. The perturbation results in a strictly concave problem which has a unique solution. This is a form of regularization that gives the strict complementarity needed to ensure that the optimal primal solution can be derived directly from the dual problem.

2.5. An ε-Optimal Bid-Price Control

In this section, we introduce a perturbed version of the network revenue management problem (P_{cont}) and its dual. Then we derive the coextremality conditions between the two formulations, which eventually gives rise to an ε-optimal bid-price control. For each $\varepsilon > 0$, the perturbed problem (P^{ε}) can be stated as follows: Choose a booking rate vector

$u(\omega, t)$ for each $(\omega, t) \in \Omega \times [0, T]$ so as to

$$\begin{aligned}
&\text{maximize} \;\; \mathbb{E}\left[\int_0^T [f(\omega,t)\cdot u(\omega,t) - \frac{1}{2}\sum_{j=1}^{J} \varepsilon_j(\omega,t)\, u_j^2(\omega,t)]\, dt\right] \\
&\text{subject to} \\
&x(\omega,t) = C - AU(\omega,t), \quad (\omega,t)\in\Omega\times[0,T], \qquad\qquad (\mathrm{P}^{\varepsilon}) \\
&U(\omega,t) = \int_0^t u(\omega,s)ds, \quad (\omega,t)\in\Omega\times[0,T], \\
&0 \le u(\omega,t) \le d(\omega,t), \quad (\omega,t)\in\Omega\times[0,T], \\
&x(\omega,T) \ge 0, \quad \omega\in\Omega,
\end{aligned}$$

where $\varepsilon_j(\omega, t)$ is defined as follows: For $j = 1, \ldots, J$ and $(\omega, t) \in \Omega \times [0, T]$, let

$$\varepsilon_j(\omega,t) = \begin{cases} \frac{\varepsilon}{d_j(\omega,t)} & \text{if } d_j(\omega,t) > 0, \\ \varepsilon & \text{otherwise.} \end{cases} \tag{2.15}$$

The perturbed problem (P^{ε}) is the same as the network revenue management problem (P_{cont}), except for the quadratic term in its objective, which makes it a strictly concave problem. Thus, the perturbed problem (P^{ε}) has a unique solution. Consequently, by this simple perturbation we avoid the issues of multiplicity of the solution or degeneracy issues encountered earlier, which in turn leads to an easy characterization of the solution to the perturbed problem (P^{ε}) in terms of the optimal shadow price process derived from its dual formulation.

The dual problem (D^{ε}) of the perturbed network revenue management problem (P^{ε}) can be stated as follows (see Appendix B.5 for its derivation): Choose a K-dimensional, square integrable, random vector $y_0 \in \mathcal{F}_0$ and a K-dimensional, square-integrable martingale M, which is null at zero, stopped at time T, and adapted to the filtration $\{\mathcal{F}_t, t \in \mathbb{R}_+\}$, so as to

$$\begin{aligned} &\text{minimize } \ \mathbb{E}\left[\int_0^T g_\varepsilon(f(\omega,t) - y(\omega,t)A, d(\omega,t))dt + C \cdot y_0(\omega)\right] \\ &\text{subject to} \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (\mathrm{D}^\varepsilon) \\ &y(\omega,t) = y_0(\omega) + M(\omega,t), \quad (\omega,t) \in \Omega \times [0,T], \\ &y(\omega,T) \geq 0, \ \omega \in \Omega, \end{aligned}$$

where $g_\varepsilon(z,d) = \sum_{j=1}^J h_\varepsilon(z_j,d_j)$ and for $j = 1,\ldots,J$, h_ε is given by

$$h_\varepsilon(z_j,d_j) = \begin{cases} 0 & \text{if} \quad z_j \leq 0, \\ z_jd_j - \frac{\varepsilon}{2}d_j & \text{if} \quad z_j \geq \varepsilon, \\ \frac{z_j^2 d_j^2}{2\varepsilon} & \text{if} \quad 0 < z_j < \varepsilon. \end{cases} \tag{2.16}$$

As pointed out earlier, the difference between (P^ε) and (P_{cont}) is that (P^ε) has the strictly concave term $-\frac{1}{2}\sum_{j=1}^J \varepsilon_j(\omega,t)u_j^2(\omega,t)$ in its objective function in addition to the revenue term $f(\omega,t) \cdot u(\omega,t)$, which makes ($\mathrm{P}^\varepsilon$) a strictly concave problem. As a result, there is a unique optimal solution u^ε for (P^ε), cf. Proposition 22, which can be determined through the coextremality conditions between (P^ε) and (D^ε), cf. Proposition 21.

The following proposition summarizes the duality results between the two formulations that are relevant for our purposes; its proof is given in Appendix B.5.

Proposition 21. *The perturbed primal problem (P^ε) and its dual (D^ε) have the same optimal objective value. Moreover, letting u^ε be a feasible control for (P^ε) with the corresponding state trajectory x^ε, and $(y_0^\varepsilon, M^\varepsilon)$ be a feasible control for (D^ε) with the corresponding state trajectory y^ε, the controls u^ε and $(y_0^\varepsilon, M^\varepsilon)$ are optimal for (P^ε) and (D^ε), respectively, if and only if they satisfy the following coextremality conditions (2.17) and (2.18) given below:*

$$y^\varepsilon(\omega, T) \cdot x^\varepsilon(\omega, T) = 0, \quad a.e.\ \omega \in \Omega, \tag{2.17}$$

and for $j = 1, \ldots, J$ and almost all $(\omega, t) \in \Omega \times [0, T]$ with $d_j(\omega, t) > 0$,

$$\begin{array}{lll} \text{if } u_j^\varepsilon(\omega,t) = 0, & \text{then } f_j(\omega,t) - y^\varepsilon(\omega,t)A^j \leq 0, \\ \text{if } u_j^\varepsilon(\omega,t) = d_j(\omega,t), & \text{then } f_j(\omega,t) - y^\varepsilon(\omega,t)A^j \geq \varepsilon, \\ \text{if } 0 < u_j^\varepsilon(\omega,t) < d_j(\omega,t), & \text{then } \frac{f_j(\omega,t) - y^\varepsilon(\omega,t)A^j}{\varepsilon} d_j(\omega,t) = u_j^\varepsilon(\omega,t). \end{array} \tag{2.18}$$

One would expect that for small values of ε the objective value of the perturbed problem (P^ε) is close to that of the network revenue management problem (P_{cont}). Building on this intuition, we next construct a bid-price control $(\pi^\varepsilon, \phi^\varepsilon)$ which is ε-optimal for the network revenue management problem (P_{cont}) for $\varepsilon > 0$. To facilitate our construction, fix an optimal state trajectory y^ε for the perturbed dual problem (D^ε) for each $\varepsilon > 0$. Then, for $\varepsilon > 0$, let $\pi^\varepsilon = y^\varepsilon$ and

$$\phi^\varepsilon(z_j, f_j, d_j) = \begin{cases} 0 & \text{if } f_j < z_j, \\ d_j & \text{if } f_j > z_j + \varepsilon, \\ \frac{f_j - z_j}{\varepsilon} d_j & \text{if } z_j \leq f_j \leq z_j + \varepsilon. \end{cases} \tag{2.19}$$

It follows from Definition 13, cf. (2.3), that for each $\varepsilon > 0$, the bookings under the bid-price control $(\pi^\varepsilon, \phi^\varepsilon)$ are given as follows.

$$u_j^\varepsilon(\omega, t) = \begin{cases} 0 & \text{if } f_j(\omega, t) - \pi^\varepsilon(\omega, t)A^j < 0, \\ d_j(\omega, t) & \text{if } f_j(\omega, t) - \pi^\varepsilon(\omega, t)A^j > \varepsilon, \\ \frac{f_j(\omega,t) - \pi(\omega,t)A^j}{\varepsilon} d_j(\omega, t) & \text{if } 0 \le f_j(\omega, t) - \pi^\varepsilon(\omega, t)A^j \le \varepsilon \end{cases} \tag{2.20}$$

for $(\omega, t) \in \Omega \times [0, T]$ and $j = 1, \ldots, J$. The following proposition states the optimality of the control u^ε given in (2.20) for the perturbed problem (P^ε) and is proved in Appendix B.5.

Proposition 22. *For each $\varepsilon > 0$, the booking control $\{u^\varepsilon(\omega, t) : (\omega, t) \in \Omega \times [0, T]\}$ given in (2.20) is the unique optimal control for the perturbed problem (P^ε).*

One would hope that for small values of $\varepsilon > 0$, the performance of the bid-price control $(\pi^\varepsilon, \phi^\varepsilon)$ is close to the optimal objective value of (P_{cont}). Indeed, the following theorem establishes the ε-optimality of the bid-price control $(\pi^\varepsilon, \phi^\varepsilon)$. It also shows that the booking controls u^ε resulting from the bid-price controls $\{(\pi^\varepsilon, \phi^\varepsilon) : \varepsilon > 0\}$ are close to an optimal solution to the network revenue management problem (P_{cont}) for small values of ε. Viewing the booking controls u^ε for $\varepsilon > 0$ as an element of L^2, the space of square integrable functions on $\Omega \times [0, T]$, it is easy to see that the controls u^ε for $\varepsilon > 0$ are uniformly bounded in L^2. Thus, it follows from Alaoglu's Theorem, cf. [**28**], that the collection of booking controls $\{u^\varepsilon : \varepsilon > 0\}$ is weak* compact. Defining $\mathcal{U}$ as the collection of weak limit points of the sequences of booking controls $\{u^{\varepsilon_n} : n \ge 1\}$ where $\varepsilon_n \searrow 0$ as $n \to \infty$, the following theorem establishes the optimality of every weak limit $u \in \mathcal{U}$, and its proof is given in Appendix B.5.

Theorem 23. *The collection of bid-price controls* $\{(\pi^\varepsilon, \phi^\varepsilon) : \varepsilon > 0\}$ *and the associated booking controls* $\{u^\varepsilon : \varepsilon > 0\}$ *satisfy the following:*

a) For each $\varepsilon > 0$*, the bid-price process* π^ε *is a non-negative martingale.*

b) For each $\varepsilon > 0$*, the bid-price control* $(\pi^\varepsilon, \phi^\varepsilon)$ *is* $\kappa\varepsilon$*-optimal, where*

$$\kappa = \sum_{j=1}^{J} \int_0^T \mathbb{E}[d_j(\omega, t)] dt. \tag{2.21}$$

c) Every weak limit $u \in \mathcal{U}$ *of the booking controls* $\{u^\varepsilon : \varepsilon > 0\}$ *as* $\varepsilon \searrow 0$ *is an optimal booking control for the network revenue management problem* (P_{cont}).

For $\varepsilon > 0$ and product $j = 1, \ldots, J$, the bid-price control $(\pi^\varepsilon, \phi^\varepsilon)$ behaves in the same way as a classical bid-price control as long as $f_j(\omega, t) - \pi^\varepsilon(\omega, t)A^j$ does not fall in the interval $(0, \varepsilon)$, in which case a classical bid-price control would dictate booking all of the demand. In contrast, the bid-price control $(\pi^\varepsilon, \phi^\varepsilon)$ results in a booking rate of

$$\frac{f_j(\omega, t) - \pi^\varepsilon(\omega, t)A^j}{\varepsilon} d_j(\omega, t) < d_j(\omega, t).$$

Figure 2.2 displays $u_j^\varepsilon(\omega, t)$ as a function of the difference $f_j(\omega, t) - \pi^\varepsilon(\omega, t)A^j$. The slope of the line segment in the middle of the figure is $d_j(\omega, t)/\varepsilon$, and as $\varepsilon \searrow 0$, the graph looks more and more like a step function, corresponding to a bang-bang control, which would result from a classical bid-price control.

2.6. Discussion

We consider a continuous-time model of network revenue management. First, we prove that there exists an optimal generalized bid-price control, where the bid-price process

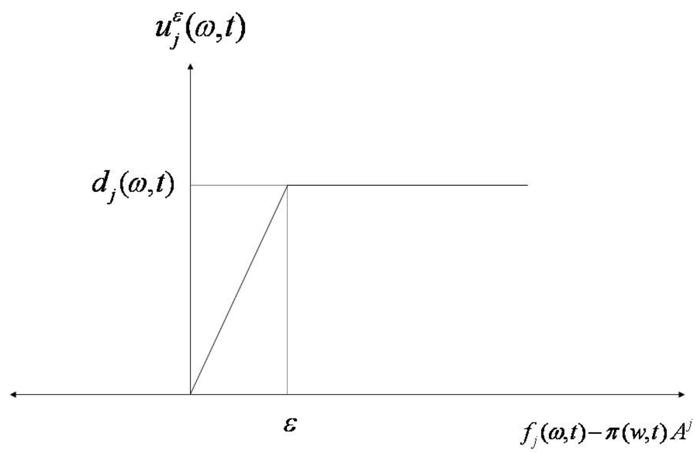

Figure 2.2. The bookings corresponding to the ε-optimal bid-price control $(\pi^{\varepsilon}, \phi^{\varepsilon})$

forms a martingale. A generalized bid-price control consists of a bid-price process and a capacity usage limit process, which creates limits on the instantaneous capacity usage rate of the resources. We also identify sufficient conditions under which an optimal bid-price control in the classical sense exists. Although identifying simpler conditions on the problem primitives under which these sufficient conditions can be verified does not seem easy, they still bring out the key step in proving such results. Next, we analyze a perturbed version of the network revenue management problem and its dual, using which we construct an ε-optimal bid-price control. The bid-price process associated with the ε-optimal bid-price control forms a martingale, too. Finally, we show that every weak* limit of the sequence of booking processes resulting from the ε-optimal bid-price controls as $\varepsilon \searrow 0$ is an optimal solution to the network revenue management problem.

Connection to Forward-Backward Stochastic Differential Equations (FBSDEs). In Chapter 3, we will show that through another perturbation, the optimal primal and dual state trajectories can be expressed as a solution to a FBSDE; see [**29**] and the references therein for an overview of FBSDEs. This intriguing connection is useful in two regards: First, the numerical methods for solving FBSDEs can be adapted to our setting to compute (near) optimal bid prices; see, for example, [**41**] and [**27**] for PDE methods and [**13**] and [**9**] for Monte-Carlo methods for solving FBSDEs. Indeed, the latter studies report encouraging results for solving FBSDEs in high dimensions. Second, the question of whether there exists an ε-optimal bid-price control in the classical sense can equivalently be stated as a question of the existence of a solution to a FBSDE; see Chapter 3 for further discussion.

Connection to Dynamic Programming. Recall that the stochastic primitives of our model are very general, allowing an arbitrary dependence structure both across time and across products. In particular, the underlying demand process need not be a Markovian process. [**22**] notes that "... Bellman equation approach is essentially limited to Markovian systems." Therefore, analyzing the network revenue management problem (P_{cont}) under general probabilistic assumptions by dynamic programming does not seem to be a viable approach. Thus, we adopt the convex analysis approach of [**11**]. Nonetheless, [**11**] formally derives a connection between his approach and the dynamic programming approach for controlled Ito processes, and shows formally (under various technical assumptions) that the optimal dual state variable equals the gradient of the value function obtained by dynamic programming along the optimal trajectory. Even if we look at the restrictive version of the network revenue management problem, which is Markovian, the state space constraints prevents us from making such a connection. Nonetheless, we intuitively expect in a Markovian setting that the optimal bid prices (or shadow prices) derived from the dual problem correspond to (generalized) gradients of the value function obtained from the dynamic programming formulation.

Practical insights and implementation issues. Although we study a stylized rate-based model, or a stochastic fluid model, the insights we provide carry over to more practical settings. For instance, one can model the demand for various products as a multidimensional, doubly-stochastic Poisson process, where the intensity of the Poisson process is given by a constant multiple of the demand rate process $\{d(\omega, t) : (\omega, t) \in \Omega \times [0, T]\}$ of Section 2.1. For such systems with high initial capacity and high demand, the model introduced in Section 2.1 is a good approximation. Indeed, it is the so-called associated fluid model. It is important, however, to point out that our fluid model is a stochastic fluid model, and it captures all key trade-offs faced by the system manager unlike most fluid models considered in the literature which are deterministic models. Moreover, viewing our rate-based network revenue management model as a fluid model of a more practical system and using its solution, one can propose near optimal policies. To elaborate further, one can implement the ε-optimal bid-price controls proposed in the preceding sections in a practical setting as follows. At every point in time, given the ε-optimal bid prices π^ε for our rate based model, a request for product j is accepted if $f_j \geq \pi^\varepsilon A^j + \varepsilon$, and it is rejected if $f_j \leq \pi^\varepsilon A^j$, while the system manager flips a coin with success probability $(f_j - \pi^\varepsilon A^j)/\varepsilon$ to decide when $0 < f_j - \pi^\varepsilon A^j < \varepsilon$. That is, she accepts the request with probability $(f_j - \pi^\varepsilon A^j)/\varepsilon$ and rejects it with probability $1 - (f_j - \pi^\varepsilon A^j)/\varepsilon$ when $0 < f_j - \pi^\varepsilon A^j < \varepsilon$. Moreover, as updating the bid-prices continuously may not be practical, one could use a discrete review policy with sufficiently small review periods. We conjecture that such policies can be shown to be near optimal for systems with large capacity and high demand. Although we provide no proof of these assertions, the literature on large call centers and their analysis via fluid models make these claims plausible; see for example [**7**] and [**8**].

The martingale representation of optimal bid-prices also leads to a connection to the literature on the pricing of American options where one tries to pick the best martingale to optimize a certain objective; see for example [**56**], [**3**], and [**35**]. Indeed, the dual network revenue management problem (D_{cont}) corresponds to picking the best non-negative martingale to minimize a certain objective.

Finally, answering the question of when a (near) optimal bid-price control in the classical sense exists remains a future research direction. Hypotheses 1-3 give sufficient conditions under which an optimal bid-price control in the classical sense exists. Although Proposition 20 sheds some light on the existence of optimal classical bid-price controls, determining conditions on problem primitives that render Hypotheses 1-3 true remains an open problem.

CHAPTER 3

Bid-Price Controls for Network Revenue Management: Connection to the Forward-Backward Stochastic Differential Equations (joint with Barış Ata)

In this chapter, we discuss the connection of the continuous-time network revenue management problem to the theory of (generalized) forward-backward stochastic differential equations (FBSDE); see [**29**] and the references therein. Our ultimate objective in this chapter is to relate the optimal primal and dual state trajectories to the solution of a forward-backward stochastic differential equation. To be more specific, we show how an ε-optimal bid-price control and the corresponding solution to the continuous-time network revenue management problem can be characterized as a solution to a (generalized) FBSDE. The connection of ε-optimal bid prices to FBSDEs is insightful in two regards: First, the numerical methods for solving FBSDEs can be adapted in our setting to compute bid prices. Second, the question of whether there exists an ε-optimal TvR bid-price control can equivalently be stated as a question of existence of a solution to a FBSDE.

To facilitate this connection, we next introduce yet another perturbation of the continuous time network revenue management problem and the corresponding dual problem. As a preliminary, first define $R(\delta) = 6TJFD/\delta^3$, where F and D are upper bounds on the fare and demand rate processes, respectively. Next, for $\delta > 0$ and $z \in \mathbb{R}$, define $\psi_\delta(z)$

as

$$\psi_\delta(z) = \begin{cases} 0 & \text{if } \; z \geq \delta, \\ R(\delta)\frac{(z-\delta)^2}{2} & \text{if } \; z < \delta, \end{cases} \tag{3.1}$$

Then, for $\varepsilon, \delta > 0$, the perturbed problem ($\mathrm{P}^{\varepsilon,\delta}$) can be stated as follows: Choose the booking rate vector $u(\omega, t)$ for each $(\omega, t) \in \Omega \times [0, T]$ so as to

$$\text{maximize} \;\; \mathbb{E}\left[\int_0^T \left(f(\omega,t) \cdot u(\omega,t) - \frac{1}{2}\sum_{j=1}^{J} \varepsilon_j(\omega,t)\, u_j^2(\omega,t) \right) dt \right] - \;\mathbb{E}\left[\int_0^T \sum_{k=1}^{K} \psi_\delta(x_k(\omega,t))\, dt \right]$$

subject to

$$x(\omega,t) = C - AU(\omega,t), \quad (\omega,t) \in \Omega \times [0,T], \tag{$\mathrm{P}^{\varepsilon,\delta}$}$$

$$U(\omega,t) = \int_0^t u(\omega,s)ds, \quad (\omega,t) \in \Omega \times [0,T],$$

$$0 \leq u(\omega,t) \leq d(\omega,t), \quad (\omega,t) \in \Omega \times [0,T],$$

where $\varepsilon_j(\omega, t)$ is defined by (2.15) for $(\omega, t) \in \Omega \times [0, T]$ and $j = 1, \ldots, J$.

In the perturbed problem ($\mathrm{P}^{\varepsilon,\delta}$), the term ψ_δ in the objective is a strictly convex cost function on the state variable x_k for $k = 1, \ldots, K$. For capacity levels below δ, ψ_δ results in a quadratic cost term, whereas no cost is incurred when the capacity of a resource is greater than or equal to δ, cf. (3.1). Although we do not have a hard constraint imposing the non-negativity of the remaining capacity at any time in the planning horizon, notice that $\psi_\delta(z) \nearrow \infty$ for $z < 0$ as $\delta \searrow 0$. Our definition of $R(\delta)$ simply ensures that the function $\psi_\delta(\cdot)$ results in sufficiently large penalty so as to ensure that the resulting state trajectory is non-negative. Indeed, it will be shown that for given $\varepsilon > 0$ and $\delta > 0$, any optimal booking rate control that solves ($\mathrm{P}^{\varepsilon,\delta}$) would almost surely have nonnegative capacity for each resource at the end of planning horizon, cf. Proposition 26.

The following definitions are needed to introduce the dual problem associated with ($\mathrm{P}^{\varepsilon,\delta}$): For $\delta > 0$ and $z \in \mathbb{R}^K$ let $\xi^\delta(z) = \sum_{k=1}^K \xi_k^\delta(z_k)$, where for $k = 1, \ldots, K$,

$$\xi_k^\delta(z_k) = \begin{cases} 0 & \text{if} \quad z_k > 0, \\ \frac{1}{R(\delta)} \frac{z_k^2}{2} + \delta z_k & \text{if} \quad z_k \le 0. \end{cases} \tag{3.2}$$

Then, the dual formulation ($\mathrm{D}^{\varepsilon,\delta}$) associated with ($\mathrm{P}^{\varepsilon,\delta}$) is stated as follows (see Appendix C for its derivation): Choose a K-dimensional, square integrable, random vector $y_0 \in \mathcal{F}_0$, a K-dimensional, square integrable stochastic process $\{\dot{y}(\omega, t) : (\omega, t) \in \Omega \times [0, T]\}$ and a K-dimensional, square integrable martingale M, which is null at zero and stopped at time T, and adapted to the filtration $\{\mathcal{F}_t,\, t \in \mathbb{R}_+\}$, so as to

$$\text{minimize} \;\; \mathbb{E}\left[\int_0^T [g_\varepsilon(f(\omega,t) - y(\omega,t)A, d(\omega,t)) + \xi^\delta(\dot{y}(\omega,t))]dt + C \cdot y_0(\omega)\right]$$

subject to

$$y(\omega,t) = y_0(\omega) + \int_0^t \dot{y}(\omega,s)ds + M(\omega,t), \quad (\omega,t) \in \Omega \times [0,T], \tag{$\mathrm{D}^{\varepsilon,\delta}$}$$

$$\dot{y}(\omega,t) \le 0, \quad (\omega,t) \in \Omega \times [0,T],$$

$$y(\omega,T) = 0, \;\; \omega \in \Omega,$$

where $g_\varepsilon(z,d) = \sum_{j=1}^J h_\varepsilon(z_j, d_j)$ and $h_\varepsilon(\cdot)$ is given by (2.16). The following proposition establishes the non-negativity of the dual state trajectories and is proved in Appendix C.

Proposition 24. *Every feasible dual state trajectory y for the perturbed problem ($D^{\varepsilon,\delta}$) is non-negative for a.e. $(\omega, t) \in \Omega \times [0, T]$.*

One difference between the perturbed problem ($\mathrm{D}^{\varepsilon,\delta}$) and the perturbed dual problem (D^{ε}) of Chapter 2 is that we choose $\dot{y}$, the rate of change or drift of the dual state variable y, and incur the infinitesimal cost rate of $\xi^{\delta}(\dot{y}(\omega, t))$ at $(\omega, t) \in \Omega \times [0, T]$. Another important difference is that, any feasible dual state trajectory y for ($\mathrm{D}^{\varepsilon,\delta}$) is equal to zero at time T for a.e. $\omega \in \Omega$, whereas the optimal dual state trajectory y^{ε} for (D^{ε}) satisfies

$$y^{\varepsilon}(\omega, T) \cdot x^{\varepsilon}(\omega, T) = 0, \ \omega \in \Omega,$$

where x^{ε} is the primal state trajectory associated with an optimal solution u^{ε} for (P^{ε}).

The following proposition summarizes the duality results between the formulations ($\mathrm{P}^{\varepsilon,\delta}$) and ($\mathrm{D}^{\varepsilon,\delta}$) that are relevant for our purposes, and its proof is given in Appendix C.

Proposition 25. *The perturbed primal problem ($P^{\varepsilon,\delta}$) and its dual ($D^{\varepsilon,\delta}$) have the same optimal objective value. Moreover, letting $u^{\varepsilon,\delta}$ be a feasible control for ($P^{\varepsilon,\delta}$) with the corresponding state trajectory $x^{\varepsilon,\delta}$, and $(y_0^{\varepsilon,\delta}, \dot{y}^{\varepsilon,\delta}, M^{\varepsilon,\delta})$ be a feasible control for ($D^{\varepsilon,\delta}$) with the corresponding state trajectory $y^{\varepsilon,\delta}$, the controls $u^{\varepsilon,\delta}$ and $(y_0^{\varepsilon,\delta}, \dot{y}^{\varepsilon,\delta}, M^{\varepsilon,\delta})$ are optimal for ($P^{\varepsilon,\delta}$) and ($D^{\varepsilon,\delta}$), respectively, if and only if they satisfy the following coextremality conditions (3.3) and (3.4): For $k = 1, \ldots, K$ and almost all $(\omega, t) \in \Omega \times [0, T]$,*

(3.3)

$$\dot{y}_k^{\varepsilon,\delta}(\omega, t) = 0 \quad \textit{if } x_k^{\varepsilon,\delta}(\omega, t) \geq \delta \quad \textit{and} \quad \dot{y}_k^{\varepsilon,\delta}(\omega, t) = R(\delta)(x_k^{\varepsilon,\delta}(\omega, t) - \delta) \quad \textit{if } x_k^{\varepsilon,\delta}(\omega, t) < \delta,$$

and for $j = 1, \dots, J$ and almost all $(\omega, t) \in \Omega \times [0, T]$ with $d_j(\omega, t) > 0$,

$$
\begin{array}{lll}
 & \textit{if } \; u_j^{\varepsilon,\delta}(\omega,t) = 0, & \textit{then } \; f_j(\omega,t) - y^{\varepsilon,\delta}(\omega,t)A^j \le 0, \\
(3.4) & \textit{if } \; u_j^{\varepsilon,\delta}(\omega,t) = d_j(\omega,t), & \textit{then } \; f_j(\omega,t) - y^{\varepsilon,\delta}(\omega,t)A^j \ge \varepsilon, \\
 & \textit{if } \; 0 < u_j^{\varepsilon,\delta}(\omega,t) < d_j(\omega,t), & \textit{then } \; \frac{f_j(\omega,t) - y^{\varepsilon,\delta}(\omega,t)A^j}{\varepsilon} d_j(\omega,t) = u_j^{\varepsilon,\delta}(\omega,t).
\end{array}
$$

Next, we define the bid-price control $(\pi^{\varepsilon,\delta}, \phi^{\varepsilon,\delta})$ for the continuous-time network revenue management problem for $\varepsilon, \delta > 0$. To facilitate our construction, fix an optimal state trajectory $y^{\varepsilon,\delta}$ for the perturbed dual problem $(\mathrm{D}^{\varepsilon,\delta})$ for each $\varepsilon, \delta > 0$. Then, for $\varepsilon, \delta > 0$, let $\pi^{\varepsilon,\delta} = y^{\varepsilon,\delta}$, and $\phi^{\varepsilon,\delta} = \phi^{\varepsilon}$ as defined in (A.23). Note that $\pi^{\varepsilon,\delta} = y^{\varepsilon,\delta}$ is a valid bid-price process, cf. Proposition 24. Then the bookings under the bid-price control $(\pi^{\varepsilon,\delta}, \phi^{\varepsilon,\delta})$ are given as follows, cf. (2.3).

$$
(3.5) \qquad u_j^{\varepsilon,\delta}(\omega,t) = \begin{cases} 0 & \text{if } \; f_j(\omega,t) - \pi^{\varepsilon,\delta}(\omega,t)A^j < 0, \\ d_j(\omega,t) & \text{if } \; f_j(\omega,t) - \pi^{\varepsilon,\delta}(\omega,t)A^j > \varepsilon, \\ \frac{f_j(\omega,t) - \pi^{\varepsilon,\delta}(\omega,t)A^j}{\varepsilon} d_j(\omega,t) & \text{if } \; 0 \le f_j(\omega,t) - \pi^{\varepsilon,\delta}(\omega,t)A^j \le \varepsilon, \end{cases}
$$

for $(\omega, t) \in \Omega \times [0, T]$ and $j = 1, \dots, J$. The following proposition states the optimality of the control $u^{\varepsilon,\delta}$ for the perturbed problem $(\mathrm{P}^{\varepsilon,\delta})$, and it is proved in Appendix C.

Proposition 26. *For each $\varepsilon, \delta > 0$, the control $\{u^{\varepsilon,\delta}(\omega, t) : (\omega, t) \in \Omega \times [0, T]\}$ given in (3.5) is the unique optimal control for the perturbed problem $(P^{\varepsilon,\delta})$. Moreover, for $\varepsilon, \delta > 0$, $u^{\varepsilon,\delta}$ is feasible for (P_{cont}).*

For small values of $\varepsilon, \delta > 0$, one would hope that expected revenue generated by $u^{\varepsilon,\delta}$ is close to optimal objective value of (P_{cont}). Indeed, the following theorem establishes the ε-optimality of the bid-price control $(\pi^{\varepsilon,\delta}, \phi^{\varepsilon,\delta})$. We define $\widetilde{\mathcal{U}}$ as the collection of weak limits of the booking controls $\{u^{\varepsilon,\delta} : \varepsilon, \delta > 0\}$ as $\varepsilon, \delta \searrow 0$, that is, $\widetilde{\mathcal{U}}$ is the collection of weak limit points of the sequences of booking controls $\{u^{\varepsilon_n,\delta_n} : n \geq 1\}$ where $\varepsilon_n, \delta_n \searrow 0$ as $n \to \infty$. Then, the next theorem also establishes the optimality of every weak limit $u \in \widetilde{\mathcal{U}}$ for the continuous-time network revenue management problem (P_{cont}), and its proof is given in Appendix C.

Theorem 27. *The collection of bid-price processes $\{(\pi^{\varepsilon,\delta}, \phi^{\varepsilon,\delta}) : \varepsilon, \delta > 0\}$ and the associated booking controls $\{u^{\varepsilon,\delta} : \varepsilon, \delta > 0\}$ satisfy the following:*

a) For each resource $k = 1, \ldots, K$ and $\varepsilon, \delta > 0$, the bid-price process $\pi_k^{\varepsilon,\delta}$ is a non-negative supermartingale adapted to the filtration $\{\mathcal{F}_t : t \in [0,T]\}$ with $\pi_k^{\varepsilon,\delta}(\omega, T) = 0$ for a.e. $\omega \in \Omega$, whereas $\pi_k^{\varepsilon,\delta}$ stopped at the first time the capacity of resource k drops below δ is a martingale.

b) For $\varepsilon, \delta > 0$, the bid-price control $(\pi^{\varepsilon,\delta}, \phi^{\varepsilon,\delta})$ is $(\kappa\varepsilon + \rho\delta)$-optimal, where κ is given by (2.21) and

$$\rho = \frac{KJF}{\min_{k,j}\{A_{kj} : A_{kj} > 0\}}. \tag{3.6}$$

c) Every weak limit $u \in \widetilde{\mathcal{U}}$ of the booking controls $\{u^{\varepsilon,\delta} : \varepsilon, \delta > 0\}$ as $\varepsilon, \delta \searrow 0$ is an optimal control for the continuous-time network revenue management problem (P_{cont}).

The next proposition shows how the dual state trajectory $y^{\varepsilon,\delta}$ associated with an ε-optimal bid-price control $(\pi^{\varepsilon,\delta}, \phi^{\varepsilon,\delta})$ and the corresponding solution to the continuous-time network revenue management problem can be characterized as the solution to a (generalized) Forward-Backward Stochastic Differential Equation. The generalized Forward-Backward Differential Equation defined in (3.7)-(3.9) is obtained by coupling the system dynamics equations governing primal-dual state trajectories of solutions to ($\mathrm{P}^{\varepsilon,\delta}$) and ($\mathrm{D}^{\varepsilon,\delta}$) with the coextremality conditions stated in Proposition 25 that characterize an optimal primal-dual solution pair. The proof of Proposition 28 is given in Appendix C.

Proposition 28. *For $\varepsilon, \delta > 0$, stochastic processes $x^{\varepsilon,\delta}$ and $y^{\varepsilon,\delta}$ correspond to optimal state trajectories for the primal problem ($P^{\varepsilon,\delta}$) and the dual problem ($D^{\varepsilon,\delta}$) if and only if $(x^{\varepsilon,\delta}, y^{\varepsilon,\delta}, M)$ jointly solve the following (generalized) FBSDE: For $(\omega, t) \in \Omega \times [0,T]$, and $k = 1, \ldots, K$,*

$$dx_k = -\left[\sum_{j=1}^{J} A_{kj} \phi^{\varepsilon,\delta}(y(\omega,t)A^j, f_j(\omega,t), d_j(\omega,t))\right] dt, \tag{3.7}$$

$$dy_k = \psi_\delta'(x_k(\omega,t))\, dt + dM_k, \tag{3.8}$$

$$x_k(0) = C_k \quad \textit{and} \quad y_k(\omega,T) = 0, \tag{3.9}$$

where M is a K-dimensional martingale, null at zero, stopped at time T and adapted to the filtration $\{\mathcal{F}_t,\ t \in \mathbb{R}_+\}$, and

$$\psi_\delta'(x_k) = \begin{cases} R(\delta)(x_k - \delta) & \textit{if} \;\; x_k < \delta, \\ 0 & \textit{if} \;\; x_k \geq \delta. \end{cases} \tag{3.10}$$

Moreover, given a solution $(x^{\varepsilon,\delta}, y^{\varepsilon,\delta}, M)$ to the (generalized) FBSDE, the corresponding optimal booking control $u^{\varepsilon,\delta}$ for ($P^{\varepsilon,\delta}$) is uniquely determined by the coextremality conditions (3.3)-(3.4).

The backward stochastic differential equations were first introduced by [**47**] and [**26**]; and is by now well studied. Forward-backward differential equations arise naturally in contingent claim valuation problems in mathematical finance. Solving the FBSDE as in

(3.7)-(3.9) requires the coupling of the forward equation (3.7) with the backward equation (3.9), which leads to a circular dependence.

The FBSDEs received considerable attention in recent years; see for example [**48**], [**29**] and the references therein. There are two main methods for the study of FBSDEs: The first one is purely probabilistic, and the main idea is to use Ito's formula and contraction mapping arguments to obtain local existence and uniqueness. The second method concerns a "four-steps scheme" (see [**41**] and [**27**]). The latter is a sort of combination of the methods of semi-linear partial differential equations and probability theory. In general, numerical solutions to FBSDE's can be obtained via several approximation schemes, see [**25**], [**4**], [**5**], [**17**] and [**42**] and the references therein. Clearly, one can adopt these techniques to compute ε-optimal bid prices for the continuous-time network revenue management problem. The connection to FBSDEs also sheds light on the question of the optimality of TvR bid-price controls.

Existence of ε-optimal bid-price controls. Replacing the booking function $\phi^{\varepsilon,\delta}$ in the generalized FBSDE in (3.7)-(3.9) by the booking function ϕ, where $\phi(z) = z_3\mathbf{1}_{\{z_2 \geq z_1\}}$, gives rise to the following FBSDE: For $(\omega, t) \in \Omega \times [0, T]$, and $k = 1, \ldots, K$,

$$dx_k = -\left[\sum_{j=1}^{J} A_{kj} d_j(\omega, t) \mathbf{1}_{\{f_j(\omega,t) \geq y(\omega,t) A^j\}}\right] dt, \tag{3.11}$$

$$dy_k = \psi_\delta'(x_k(\omega, t))\, dt + dM_k, \tag{3.12}$$

$$x_k(0) = C_k \quad \text{and} \quad y_k(\omega, T) = 0, \tag{3.13}$$

where M is a K-dimensional martingale, null at zero, stopped at time T and adapted to the filtration $\{\mathcal{F}_t,\, t \in \mathbb{R}_+\}$. The next proposition shows that we can state the question of existence of an ε-optimal TvR bid-price control as a question of existence to a FBSDE. The proof of Proposition 29 is given in Appendix C.

Proposition 29. *For $\delta > 0$, if (x, y, M) jointly solve the FBSDE defined in (3.11)-(3.13), then the TvR bid-price control with bid-price process $\pi = y$ is $\rho\delta$-optimal.*

3.1. The Continuous Information Case

In this subsection, we introduce an additional assumption on the underlying information structure, which enables us to reach sharper conclusions. Namely, we assume that the information structure $\{\mathcal{F}_t,\, t \in \mathbb{R}_+\}$ is continuous. An information structure is continuous if the posterior probability of any event is updated in a continuous manner. To be more precise, we next present the definition of a continuous information structure due to [**36**].

Definition 30. *The information structure $\{\mathcal{F}_t,\, t \in \mathbb{R}_+\}$ (already assumed to be right continuous) is said to be continuous if for every event $E \in \mathcal{F}$, the posterior probability assessment $\mathbb{P}(E \,|\, \mathcal{F}_t)$ has a continuous modification.*

Examples of continuous information structures include the filtration generated by a continuous process having the strong Markov property. Thus, the information structure generated by a diffusion process is continuous. The canonical example of a continuous information structure is the one generated by a Brownian motion. The next proposition states the equivalent characterizations of continuous information structures and is proved in [**36**].

Proposition 31. *The following statements are equivalent:*

(i) The information structure $\{\mathcal{F}_t,\, t \in \mathbb{R}_+\}$ is continuous.

(ii) Every martingale on $(\Omega, \mathcal{F}, \mathbb{P})$ *adapted to* $\{\mathcal{F}_t,\ t \in \mathbb{R}_+\}$ *has a continuous modification.*

(iii) All stopping times are predictable.

Consider a stopping time τ which is defined as the first time an event happens. Then this event is foretellable by a sequence of events except possibly on a set of probability zero, if τ is a predictable stopping time. Therefore it follows from Proposition 31 that under a continuous information structure no event can take us by surprise. That is, as put by [**23**], "we are forewarned by a succession of precursory signs, of the exact time the phenomenon will occur". Also, recall that the martingale term M in the dual problem of control contains the relevant information from future and serves the purpose of integrating into the dual variable y this necessary information. Principally, the information may arrive discontinuously because of unpredictable changes in the business environment, political situation etc. In these circumstances, the system manager should revise his predictions so as to take into account the new information that became available. This new information is incorporated exactly through the term dM. Then, the new information can significantly increase or decrease the bid prices which reflect the marginal value of resources. Hence, the continuity of M under a continuous information structure is the reflection of the continuous flow of information into the system.

To elaborate further, suppose that any martingale, and hence the optimal bid-prices, can be represented as a stochastic integral with respect to a given Brownian motion w plus a martingale M that is orthogonal to w in the sense of [**11**]. Then, loosely speaking, w contains the short term uncertainties and M is a prediction of the long-term uncertainties. The interpretation of the martingale term M is intuitively appealing since it enables us to express formally the distinction in the decision making process between continuous information processing which is done on a routine basis and discontinuous information processing which is done by reassessing the predictions.

3.1.1. FBSDE's Under Continuous Information

Taking the canonical continuous information structure generated by a Brownian motion as the underlying information structure, every martingale on this space is an Ito integral and the generalized FBSDE in (3.7)-(3.9) reduces to

$$dx_k = -\left[\sum_{j=1}^{J} A_{kj}\phi^{\varepsilon,\delta}(y(\omega,t)A^j, f_j(\omega,t), d_j(\omega,t))\right] dt, \tag{3.14}$$

$$dy_k = \psi'_\delta(x_k(\omega,t))\, dt + Z_k \cdot dw, \tag{3.15}$$

$$x_k(0) = C_k \quad \text{and} \quad y_k(\omega,T) = 0, \tag{3.16}$$

where w represents an m-dimensional Brownian motion which generates the underlying information structure, and Z_k is an m-dimensional adapted stochastic process.

3.2. FBSDEs under general information structures

Under general information structures, the martingale multiplicity of the information structure leads to a representation similar to the one in continuous information case, cf. Section 3.1. To that end, let $\mathcal{M}^2$ denote the space of square integrable martingales on $(\Omega, \mathcal{F}, \mathbb{P})$ which are null at zero. A collection of orthogonal martingales $M = (M_1, \ldots, M_N)$ adapted to $\{\mathcal{F}_t,\ t \in \mathbb{R}_+\}$ is an orthogonal basis with multiplicity $N(\mathcal{M}^2) \leq \infty$ for $\mathcal{M}^2$ if any square integrable martingale in $\mathcal{M}^2$ can be expressed as a stochastic integral with respect to M, cf. Chapter 4 of [**37**].

The path breaking work of [**39**] shows the existence of an orthogonal basis of martingales $M = (M_1, \ldots, M_N)$ for $\mathcal{M}^2$, where the multiplicity $N(\mathcal{M}^2)$ of $\mathcal{M}^2$ represents the dimension of uncertainty which could be resolved at any one time. A direct consequence of this martingale representation technique is that the (general) FBSDE (3.7)-(3.9) can now be written as

$$dx_k = -\left[\sum_{j=1}^{J} A_{kj}\phi^{\varepsilon,\delta}(y(\omega,t)A^j, f_j(\omega,t), d_j(\omega,t))\right] dt, \tag{3.17}$$

$$dy_k = \psi_\delta'(x_k(\omega,t))\, dt + \theta_k \cdot dM, \tag{3.18}$$

$$x_k(0) = C_k \quad \text{and} \quad y_k(\omega,T) = 0, \tag{3.19}$$

where M is the orthogonal martingale basis and θ_k is an $N(\mathcal{M}^2)$-dimensional predictable process. Then, the problem of solving for an ε-optimal bid-price control reduces to the one of looking for the process θ_k for $k = 1, \ldots, K$.

3.3. Discussion

By the help of a perturbation of the continuous-time network revenue management problem (P_{cont}), we write a (generalized) FBSDE whose solution gives us an optimal primal trajectory and an ε-optimal bid-price process. The ε-optimal bid-price process associated with resource $k = 1, \ldots, K$ forms a supermartingale and is equal to zero at the terminal time, whereas the bid-price process stopped at the first time the capacity of resource k drops below δ is a martingale. The connection between the network revenue management problem and the FBSDE's may lead to new insights and a novel computational approach.

We also discuss in Section 3.1, the special case of continuous information. Under continuous information structures, no event is a "surprise" and the optimal bid-prices are continuous martingales. An important continuous information structure is the one generated by the Brownian motion, under which all continuous martingales can be expressed as Ito integrals. This, in turn enables us to employ Ito calculus and the machinery of FBSDE's. References for several numerical studies are also provided in Section 3.

CHAPTER 4

Revenue Management by Sequential Screening (joint with Barış Ata and James D. Dana, Jr.)

Most models of revenue management consider dynamic demand – consumers arrive and sequentially choose whether or not to purchase. While this is a reasonable assumption for many non-durable consumer goods, for durable goods, or services that are purchased in advance, such as travel and lodging, consumers often choose both when and whether or not to purchase. This paper considers a revenue management model with strategic consumers, that is, consumers who are forward-looking and choose when to buy. We assume all consumers 'arrive' at the very beginning, but demand is nevertheless dynamic because we assume that consumers learn their valuations at different times. While consumers learn their valuations sequentially, the firm cannot observe when consumers learn their demands or what they are willing to pay when they learn those demands and so consumers are free to purchase at any time, including both before and after they learn their valuations.

Revenue management has two important elements. The first is dynamic capacity control, which is valuable when the total demand is uncertain. The second is price discrimination, which is valuable when consumers are heterogeneous and have private information about their demands. This chapter considers only this second element.

Our firm chooses its pricing policy to maximize its expected profit given knowledge of the distribution of consumer demands, but without knowing which consumer is which. We formulate the firm's optimal pricing policy as a mechanism design problem. In particular, we look for the profit maximizing direct-revelation pricing policy. While direct revelation mechanisms are not used in practice, because direct-revelation mechanisms are always optimal, they represent the theoretical benchmark against which all other pricing policies should be compared. After analyzing the direct revelation mechanism we demonstrate that it can be implemented using a menu of simple contingent contracts.

We find that the optimal pricing policy induces all consumers to make their purchases at the very beginning, before they learn their valuations, but allows consumers to cancel and claim a refund after they learn their valuations. Partially refundable sales are a commonly used pricing practice by airlines, hotels, theaters, and restaurants. Airlines, for example, often offer fully refundable, partially refundable, and non-refundable tickets. Moreover, the availability of these refund options changes over time. Lower priced, non-refundable tickets may be available only in advance, while higher-priced, refundable tickets are often available until the day of departure. Partially refundable, or equivalently, options contracts are also used by large manufacturers to purchase electric power. These contracts reduce the risk faced by manufacturers, but also allow electricity suppliers to extract more of the surplus from the manufacturers because the exercise prices in the options are typically less distortionary than electricity spot prices.

Our model generalizes the work of [**19**] who considered the optimal pricing policy when heterogeneous consumers learn about their demand at the same time. As in their work, the optimal pricing policy induces all consumers to make their purchases at the very beginning, before they learn their valuations, but allows consumers to cancel and claim a refund after they learn their valuations and the firm varies the size of the refund that it offers consumers in order to extract more surplus. However, we show that the firm does even better when it varies both the length of time that they are refundable as well as the size of the refund.

Offering refundable purchases, or equivalent option contracts, allows the firm to extract more total surplus from consumers and to better discriminate among them. When consumers are ex ante homogeneous, the firm charges each consumer a premium equal to marginal cost if they take the good, and a flat fee equal to their expected consumer surplus whether or not they take the good. This contract extracts all the surplus from consumers and is more profitable than the optimal spot price. When consumers are ex ante heterogeneous the firm can do better by offering a menu of contracts. Consumers vary in the ex ante distribution of their valuations and the point in time when they learn their valuation. By varying when the cost of taking the good (or the return price) and the point in time when consumers must make their final consumption decision, the firm can extract more surplus than if it offers a single contract. Indeed, under some assumptions on the demand distribution the firm can extract the entire consumer surplus even when consumers are heterogeneous.

We consider a variety of distributional assumptions for consumers' valuations. When consumers who learn their valuations later have higher valuations, in the sense of first order stochastic dominance, then the optimal pricing policy gives the highest value consumer the longest time to exercise the option and a refund price equal to marginal cost, or equivalently the longest time to claim a refund and a refund price equal to marginal cost. This induces the efficient consumption for this consumer. If the first best is feasible, lower valuation consumers purchase tickets that become non-refundable sooner, and the price paid is increasing in consumers' expected valuations, but the refund price is always equal to marginal cost. If the first best is not feasible, then lower valuation consumers still purchase tickets which become non-refundable sooner and the refund price is higher than marginal cost, at least for some consumers. Intuitively, offering a higher refund is more attractive to low valuation consumers than high valuation consumers because they exercise the option more often. So the firm can increase the refund price, and increase the original purchase price by a nearly offsetting amount, and make the contract less attractive to the highest valuation consumer, who will never be able to exercise that option because they will not learn their valuations in time.

When consumers who learn later have more dispersed valuations, that is, their distribution is a mean preserving spread of the distribution of valuations for all consumers who learn earlier, we show that the first best is always achievable. That is, consumers with less dispersed valuations purchase tickets that become non-refundable sooner, and the price paid is increasing in the variance of consumers' valuations, but the refund price is always equal to marginal cost.

Other papers in the economics literature have looked at pricing when consumers learn their valuations over time. [**18**] considers a monopolist with commitment power who chooses whether to sell to consumers at a uniform price before or after they have learned their demand. In a related paper, [**24**] considers a monopolist without commitment who intentionally creates a shortage of capacity to create a buying frenzy in the spot market and to induce buyers to purchase early, before they know their valuations. [**21**] considers a competitive model in which heterogeneous consumers divide their purchases between advanced sales, when they may not yet know their demands, and spot market sales, after their valuations have been realized. [**33**] consider a monopolist who sells to heterogeneous consumers, some of whom purchase early before they know their departure time preferences and some of whom purchase late after they learn their preferences. Like [**19**], [**21**] and [**33**] assume consumers learn their valuations at the same time. (See also [**43**], [**32**]).

Some papers in the operations management literature have considered optimal pricing and capacity controls when heterogeneous consumers purchase in an exogenously given, sequential order. [**40**] considers a setting where the consumers have either high or low valuations, which are known and observable. Consumers with low valuations arrive in the first period, while the consumers with high valuations arrive in the second period. There is aggregate uncertainty, and a system manager chooses how much capacity to reserve for the consumers with high valuations. [**40**] characterizes the optimal policy as a booking limit policy. [**14**] and [**20**] provide extensions of Littlewood's result to n customer classes, characterizing the optimal capacity control policy by nested booking limits. There are extensions of this capacity control models in various different directions in the operations literature. [**60**] provides an extensive review of that literature. Although most papers in the operations management literature ignore the strategic consumer behavior, modeling consumer behavior received some attention in recent years; see [**57**] for a review.

The rest of the chapter is structured as follows: Section 4.1 presents the model. The mechanism design problem is introduced in Section 4.1.1, where we also present the necessary and sufficient conditions for implementing the first-best solution. We consider a variety of distributional assumptions for consumers' valuations in Sections 4.2-4.4. We first consider the case where the consumer valuations increase over time in the sense of

First Order Stochastic Dominance in Section 4.2. Section 4.3 investigates the case with the consumer valuations exhibiting "reversed" First Order Stochastic Dominance. We analyze the situation where the valuations become more dispersed over time, i.e. the case of mean preserving spread, in Section 4.4. Some concluding remarks are provided in Section 4.5. The technical and lengthy proofs are relegated to Appendices. Chapter 5 establishes a revelation principle for our setting in continuous time. Appendix D.1 establishes some technical results regarding the mechanism design problem. Appendix D.2 consists of the lengthy and technical proofs in the First Order Stochastic Dominance case. Finally, the technical Appendix D.3 is dedicated to the derivation of the dual problems associated with convex dynamic control problems that arise in the analysis.

4.1. The Model

We assume that consumers are heterogeneous and that their types are continuously distributed on $[0, \bar{t}]$ with a strictly positive density function $h(t)$ and cumulative distribution function $H(t)$. That is, $h(t)$ represents the relative frequency of type t consumers in the population. Consumers learn their type at time zero. The type t determines the probability distribution of their valuations as well as the time at which they learn their valuations. Without loss of generality, we assume that type t consumer privately learns her realized valuation at time t. The valuation of a type t consumer is distributed according to the probability density function $g(v, t)$ on the interval $[\underline{v}, \overline{v}]$. Let $f(v, t)$ be the joint distribution of types and valuations. Clearly, it follows that

$$\int_{v,t} f(v,t)\, dvdt = 1, \quad h(t) = \int_{v} f(v,t)\, dv, \text{ and } g(v,t) = \frac{f(v,t)}{h(t)}.$$

We also assume that each unit of the good costs the monopolist c and that both the consumers and the monopolist are risk neutral, and the consumption takes place at time $\bar{t}$.

4.1.1. Mechanism Design Problem

In this section, we consider the profit maximizing incentive-compatible direct-revelation mechanisms for the monopolist. The revelation principle established in Chapter 5 proves that there is no loss of generality in assuming that the monopolist should structure his incentive system so that all consumers will be willing to reveal all of their information to him honestly. This result generalizes the revelation principle of [**45**] to continuous-time communication games, and is of interest on its own right. It also implies that in the optimal incentive-compatible direct-revelation mechanism, contracting takes place after the consumer privately learn their type but before they know their valuations perfectly, i.e. consumers are asked to report their type at time zero. Since the monopolist can do no better than the maximally centralized communication system in which, at every stage, each individual confidentially reports all his private information to a central mediator, a type t consumer is asked to report her realized valuation at time t, the exact moment she learns her valuation. Then, for each pair of reports of valuation v and type t, let $y(v,t)$ be the probability that the monopolist delivers the good, and let $x(v,t)$ denote the payment to the monopolist.

The monopolist's mechanism design problem can be stated as follows: Choose payments

$$\left\{x(v,t) : t \in [0,\overline{t}],\ v \in [\underline{v},\overline{v}]\right\}$$

and delivery probabilities

$$\left\{y(v,t) : t \in [0,\overline{t}],\ v \in [\underline{v},\overline{v}]\right\}$$

so as to

$$\text{maximize} \int_{v,t} f(v,t)\left[x(v,t) - cy(v,t)\right] dv dt$$

subject to

$$\int_{\underline{v}}^{\overline{v}} \left[vy(v,t) - x(v,t)\right] g(v,t)\, dv \ \geq 0 \text{ for } t \in [0,\overline{t}], \tag{IR}$$

$$vy(v,t) - x(v,t) \geq vy(v',t) - x(v',t) \ \text{ for } v, v' \in [\underline{v},\overline{v}] \text{ and } t \in [0,\overline{t}] \tag{IC$_t$}$$

$$\int_{\underline{v}}^{\overline{v}} \left[vy(v,t) - x(v,t)\right] g(v,t)\, dv$$
$$\geq \int_{\underline{v}}^{\overline{v}} \left[\max_{v'} \left\{vy(v',t') - x(v',t')\right\}\right] g(v,t)\, dv \ \text{ for } t' > t, \tag{$\overline{\text{IC}}_0$}$$

$$\int_{\underline{v}}^{\overline{v}} \left[vy(v,t) - x(v,t)\right] g(v,t)\, dv$$
$$\geq \max_{v'} \int_{\underline{v}}^{\overline{v}} \left[vy(v',t') - x(v',t')\right] g(v,t)\, dv \ \text{ for } t' < t, \tag{$\underline{\text{IC}}_0$}$$

$$0 \leq y(v,t) \leq 1 \text{ for } t \in [0,\overline{t}] \ \text{ and } v \in [\underline{v},\overline{v}]. \tag{F}$$

The first set of constraints are the individual rationality, or participation, constraints. These constraints are imposed to guarantee that the firm gives every consumer nonnegative *expected* surplus. Note that there is no ex-post individual rationality constraint, i.e. the ex-post utility $vy(v,t) - x(v,t)$ of type t with a realized valuation v could be negative. For example, the consumer might purchase a ticket to attend a meeting but not be eligible for a full refund if she later realizes that she will not be able to attend the meeting.

The second set of constraints, (IC_t), are the incentive compatibility constraints with respect to the consumers' realized valuations. These are imposed to guarantee that every consumer, conditional on reporting their type at time zero truthfully, finds it optimal to report her realized valuation truthfully at time t.

The third set of constraints, ($\overline{\mathrm{IC}}_0$) and ($\underline{\mathrm{IC}}_0$), are the global incentive compatibility constraints with respect to the reports of consumers' types. Note that there is a distinction between "upward" deviations and "downward" deviations because when a type t consumer reports a lower type, i.e., $t' < t$, she will be asked to report her valuation before see knows her true valuation, while if the consumer reports a higher type, i.e., $t' > t$, she will be asked to report her valuation after she knows her true valuation. ($\overline{\mathrm{IC}}_0$) guarantees that no consumer finds it profitable to deviate "upwards". If a type t consumer pretends to be type $t' > t$, she will be asked to report her valuation at time t' and she will report the valuation that maximizes her ex-post surplus. Thus, the expected surplus that a type t consumer gets by pretending to be type $t' > t$ is given by the right hand side of ($\overline{\mathrm{IC}}_0$), which should be less than or equal to the ex-ante surplus that a type t consumer gets by reporting her type and subsequently her valuation truthfully.

Similarly, ($\underline{\text{IC}}_0$) is imposed to guarantee that no consumer finds it profitable to deviate "downwards". In this case, type t consumer reports the valuation that maximizes her expected surplus given that she already reported her type as $t' < t$. Hence, ($\underline{\text{IC}}_0$) imposes the restriction that ex-ante surplus that a type t consumer gets by reporting her type and subsequently her valuation truthfully is larger than the expected surplus she will obtain by pretending to be type $t' < t$ and subsequently choosing her best report for valuation.

The final set of constraints, denoted by (F), require the delivery rule y to be feasible. The following lemma characterizes how the consumers report their valuation if they do not report their types truthfully at time zero and its proof is provided in the Appendix.

Lemma 32. *(i) If a type t consumer reports her type as $t' > t$ at time zero, then for any mechanism that satisfies the constraint (IC_t), when she reports her valuation at time t', it is optimal for her to report her true valuation, that is,*

$$v \in \arg\max_{v'} \left\{ vy\left(v', t'\right) - x\left(v', t'\right) \right\}.$$

(ii) If a type t consumer reports her type as $t' < t$ at time zero, then for any mechanism that satisfies (IC_t), when she reports her valuation, it is optimal for her to report her expected valuation at time t', that is,

$$\mathbb{E}_t\left[v\right] \in \arg\max_{v'} \int_{\underline{v}}^{\overline{v}} \left[vy\left(v', t'\right) - x\left(v', t'\right)\right] g\left(v, t\right) dv,$$

where

$$\mathbb{E}_t\left[v\right] = \int_{\underline{v}}^{\overline{v}} vg\left(v, t\right) dv \quad \textit{for} \quad t \in \left[0, \bar{t}\right].$$

Lemma 32 follows from (IC_t) constraints and proves that if type t consumer deviates upward and pretends to be a higher type $t' > t$, she will report her true valuation at time t', which she already learned at time t. On the other hand, if type t consumer pretends to be a lower type $t' < t$, she will report her best prediction of her valuation at time t', namely her expected valuation $\mathbb{E}_t[v]$, since she does not know her true valuation yet.

Using Lemma 32, we can simplify ($\underline{IC}_0$) and ($\overline{IC}_0$) and the monopolist's problem becomes: Choose the payments $\{x(v,t) : t \in [0,\bar{t}],\ v \in [\underline{v},\overline{v}]\}$ and the delivery probabilities $\{y(v,t) : t \in [0,\bar{t}],\ v \in [\underline{v},\overline{v}]\}$ so as to

$$\text{maximize} \int_{v,t} f(v,t)\left[x(v,t) - cy(v,t)\right] dvdt$$

subject to

$$\int_{\underline{v}}^{\overline{v}} \left[vy(v,t) - x(v,t)\right] g(v,t)\, dv \ \geq 0 \text{ for } t \in [0,\bar{t}], \tag{IR}$$

$$vy(v,t) - x(v,t) \geq vy(v',t) - x(v',t) \ \text{ for } v,v' \in [\underline{v},\overline{v}] \text{ and } t \in [0,\bar{t}] \tag{IC_t}$$

$$\int_{\underline{v}}^{\overline{v}} \left[vy(v,t) - x(v,t)\right] g(v,t)\, dv$$
$$\geq \int_{\underline{v}}^{\overline{v}} \left[vy(v,t') - x(v,t')\right] g(v,t)\, dv \ \text{ for } t' > t, \tag{$\overline{IC}_0$}$$

$$\int_{\underline{v}}^{\overline{v}} \left[vy(v,t) - x(v,t)\right] g(v,t)\, dv$$
$$\geq \int_{\underline{v}}^{\overline{v}} \left[vy(\mathbb{E}_t[v],t') - x(\mathbb{E}_t[v],t')\right] g(v,t)\, dv \ \text{ for } t' < t, \tag{$\underline{IC}_0$}$$

$$0 \leq y(v,t) \leq 1 \text{ for } t \in [0,\bar{t}] \ \text{ and } v \in [\underline{v},\overline{v}]. \tag{F}$$

Ignoring the incentive compatibility constraints, (IC_t), ($\underline{IC}_0$), and ($\overline{IC}_0$), it is easy to see that the optimal solution to the monopolist's problem is to set for all types $y(v,t) = 1$

if $v \geq c$ and $y(v,t) = 0$ if $v < c$ while extracting all of the *expected* surplus from the consumers, that is, the monopolist chooses the payments $x(v,t)$ such that

$$\int_{\underline{v}}^{\overline{v}} [vy(v,t) - x(v,t)]\, g(v,t)\, dv \ = 0 \text{ for all } t.$$

In other words, ignoring the incentive compatibility constraints, the monopolist allocates the good efficiently (i.e., $y(v,t) = 1$ if $v \geq c$ and $y(v,t) = 0$ if $v < c$) and extracts all of the *expected* surplus from the consumers.

A solution to the mechanism design problem $\left\{x(v,t), y(v,t) : t \in [0, \overline{t}],\ v \in [\underline{v}, \overline{v}]\right\}$ implements the first-best solution if it

(i) has efficient allocation of the good ($y(v,t) = 1$ if $v \geq c$ and $y(v,t) = 0$ if $v < c$),

(ii) extracts all the *expected* surplus from the consumers, i.e.

$$\int_{\underline{v}}^{\overline{v}} [vy(v,t) - x(v,t)]\, g(v,t)\, dv \ = 0 \text{ for all } t,$$

(iii) satisfies the incentive compatibility constraints (IC_t), ($\underline{\text{IC}}_0$) and ($\overline{\text{IC}}_0$).

The next proposition characterizes the conditions under which a monopolist can implement the first-best solution; and its proof is given in the Appendix. We introduce the following notation as an aid in the statement of the next proposition: Define $\mathbb{E}_t [v - c; v \geq c]$ as the expected surplus above the marginal cost of a type t consumer, i.e.

$$\mathbb{E}_t [v - c; v \geq c] = \int_c^{\overline{v}} (v - c)\, g(v,t)\, dv \quad \text{for} \quad t \in [0, \overline{t}] .$$

Proposition 33. *The monopolist can implement the first-best solution if and only if the following two conditions hold: For* $t \in [0, \bar{t}]$,

$$\text{(Condition 1)} \qquad \mathbb{E}_{t'}[v - c; v \geq c] \geq \mathbb{E}_t[v - c; v \geq c] \quad \textit{for } t' > t,$$

and for types t *such that* $\mathbb{E}_t[v] \geq c$,

$$\text{(Condition 2)} \qquad \mathbb{E}_{t'}[v - c; v \geq c] \geq \mathbb{E}_t[v - c] \;\; \textit{for} \;\; t' < t.$$

Proposition 33 makes it clear that the monopolist is better off when consumers learn their preferences sequentially. Specifically, when Conditions 1 and 2 hold, the monopolist is able to implement the unconstrained first-best. On the other hand, when the consumers learn their valuations at the same time, the monopolist cannot exploit the differences in learning times to screen consumers. In particular, the only setting when the monopolist can implement the first-best in [**19**] is the degenerate case that all the consumers have the same expected surplus from obtaining the good, i.e. $\mathbb{E}_t[v - c; v \geq c]$ does not depend on t.

Condition 1 of Proposition 33 is satisfied whenever the valuations of consumers "increase" as t increases. In that case, the higher types are more willing to pay for the good and they get the good with a higher probability while the deviation of the lower types to higher ones are deferred by the fact that their willingness to pay for the good is less. Condition 2 is satisfied when the expected valuations of different types are not too different or if the loss in the expected surplus due to reporting her valuation before the true realization is relatively large. For instance, if $\underline{v} > c$ (this is the case when $c = 0$ and $\underline{v} > 0$), then the expected surplus of a consumer is not affected by the timing of her report of her valuation and in this case Conditions 1 and 2 cannot both hold except in the degenerate case when $\mathbb{E}_t[v]$ is independent of t.

If Conditions 1 and 2 of Proposition 33 are satisfied, then the first-best solution can be implemented using the following menu of expiring refund contracts: For a report of type $t \in [0, \bar{t}]$ at time zero, charge $\mathbb{E}_t[v - c; v \geq c] + c$ for the ticket with a refund of c if the ticket is returned until time t. Since the refund is c for the returned tickets, only consumers with valuations higher than the cost will fly, which makes the allocation efficient. Moreover, the expected utility of all the consumers is equal to zero since

$$\begin{aligned}
& \int_{\underline{v}}^{\overline{v}} [vy(v,t) - x(v,t)]\, g(v,t)\, dv \\
= & \int_{c}^{\overline{v}} vg(v,t)\, dv - (1 - G(c,t))\,(\mathbb{E}_t[v - c; v \geq c] + c) \\
& -G(c,t)\,\mathbb{E}_t[v - c; v \geq c] \\
= & \ 0.
\end{aligned}$$

If Condition 1 of Proposition 33 is satisfied, no type t would want to purchase the ticket designed for a higher type $t' > t$. Similarly, if Condition 2 of Proposition 33 is satisfied, no type t would want to purchase the ticket designed for a lower type $t' < t$ since the refund of the ticket for type t' expires at time t', while type t consumer is still uncertain about her valuation for the ticket.

Next, we consider the optimal mechanism when Conditions 1 and 2 of Proposition 33 are not satisfied. Define $u(v,t) = vy(v,t) - x(v,t)$ to be the consumer's ex post surplus after she truthfully reports his type t and then his realized valuation v. Also define

$$U(t) = \int_{\underline{v}}^{\overline{v}} u(v,t)\, g(v,t)\, dv$$

to be the expected surplus of a consumer of type t at time zero. We make the technical assumption that U is of bounded variation, which is true for a very broad class of functions. Letting $G(\cdot,t)$ denote the cumulative probability distribution of the valuations of type t, we assume that it changes smoothly across types. That is, the partial derivative $\partial G(v,t)/\partial t$ exists for all v and t.

The next lemma establishes that for any optimal mechanism, when the consumer draws a greater valuation, he receives the good with a greater probability and has a greater consumer surplus. The proof of the lemma is standard in the mechanism design literature and therefore is skipped.

Lemma 34. *The incentive compatibility constraint (IC_t) is satisfied if and only if*

(i) $\partial u(v,t)/\partial v = y(v,t)$ *,*

(ii) $y(v,t)$ *is non-decreasing in* v*.*

In the next section, we analyze the case in which the types' valuations can be ordered using first order stochastic dominance, where Condition 1 of Proposition 33 is readily satisfied.

4.2. First order stochastic dominance

In this section, we focus attention on the case when the consumers' valuation increase with their type in the sense of first order stochastic dominance (FSD), which is the natural assumption in many settings. For example, in an airline context, business travellers often have higher valuations than the leisure travellers and they typically learn their realized valuation after the leisure types. The formal definition of first order stochastic dominance in the context of our model is as follows.

Definition 35. *Type t is "higher" than type t' if $G(v,t) \leq G(v,t')$ for all v. The type space $[0,\bar{t}]$ is ordered by first order stochastic dominance if $t > t'$ implies that t is higher than t' for all $t,t' \in [0,\bar{t}]$.*

Under first order stochastic dominance, Condition 1 of Proposition 33 is satisfied. To see this, first notice that, for a given t, we can write

$$\mathbb{E}_t[v-c; v \geq c] = \int_c^{\bar{v}} (v-c)\, g(v,t)\, dv = (\bar{v}-c) - \int_c^{\bar{v}} G(v,t)\, dv,$$

where the second equality is obtained using integration by parts. Then, for any $t' > t$, we have

$$\mathbb{E}_{t'}[v-c; v \geq c] - \mathbb{E}_t[v-c; v \geq c] = \int_c^{\bar{v}} [G(v,t) - G(v,t')]\, dv \geq 0,$$

since $G(v,t') \leq G(v,t)$ for all v by FSD.

If Condition 2 of Proposition 33 is also satisfied, then it is easy to see that the optimal solution to the mechanism design problem is given by $U(t) = 0$ for all t and $y(v,t) = 1$ if $v \geq c$ and 0 otherwise; and appropriate payment schemes are as described earlier.

However, if Condition 2 of Proposition 33 is violated, that is, if

$$\mathbb{E}_{\bar{t}}[v-c] > \mathbb{E}_0[v-c; v \geq c],$$

then for those types t such that $\mathbb{E}_t[v] \geq c$ and

$$\mathbb{E}_t[v-c] > \mathbb{E}_0[v-c; v \geq c],$$

it is profitable to deviate at time zero and pretend that they are type zero.

This implies that the firm cannot extract all the surplus and implement the efficient allocation without violating the incentive compatibility constraints. Therefore, we next consider the case in which the type space is ordered with respect to first order stochastic dominance and Condition 2 of Proposition 33 is violated.

In what follows, we first consider a relaxed version of the monopolist's problem and show that its optimal solution indeed satisfies all the constraints of the original problem. Hence, it is optimal for the original problem as well. As an intermediate step, we first prove that the expected surplus $U(t)$ of type t is nondecreasing under FSD. Subsequently, a key step in our approach is to establish that without loss of optimality, we can restrict attention to solutions of the form $y(v,t) \in \{0,1\}$, cf. Proposition 37, which in turn corresponds to refund contracts. We then characterize the best refund contract, cf. Proposition 39. Finally, we show that the refund contract characterized in Proposition 39 is indeed optimal for the monopolist's screening problem.

Building on Lemma 34, the following proposition characterizes the expected surplus function U, which facilitates our analysis. The proof of Proposition 36 is given in Appendix D.2.

Proposition 36. *Under FSD, without loss of generality, the objective function can be written as:*

$$\int_{v,t} f(v,t)\left[x(v,t) - cy(v,t)\right] dvdt =$$

$$\int_{v,t} f(v,t)(v-c)\, y(v,t)\, dvdt - U(0) - \int_0^{\bar{t}} (1 - H(t))\, dU(t)\,.$$

Moreover, the payments $\left\{x(v,t) : t \in [0,\bar{t}],\ v \in [\underline{v},\overline{v}]\right\}$ *are non-negative and if the incentive compatibility constraint* (IC_t) *is satisfied, then* U *is nondecreasing and satisfies*

$$0 \le U'(t) \le -\int_{\underline{v}}^{\overline{v}} y(v,t)\, \frac{\partial G(v,t)}{\partial t} dv. \tag{4.1}$$

Using Proposition 36, we next consider the following relaxed version of the monopolist's problem ($\mathrm{P}^1_{relaxed}$): Choose the delivery probabilities

$$\left\{y(v,t) : t \in [0,\bar{t}],\ v \in [\underline{v},\overline{v}]\right\}$$

so as to

$$\text{maximize} \int_{v,t} f(v,t)\left[(v-c)\,y(v,t)\right] dvdt - U(0) - \int_0^{\bar{t}} (1-H(t))\,dU(t)$$

subject to

$$\int_{\underline{v}}^{\overline{v}} \left[vy(v,t) - x(v,t)\right] g(v,t)\,dv \;\geq 0 \text{ for } t \in [0,\bar{t}], \tag{IR}$$

$$y(v,t) = \frac{\partial u(v,t)}{\partial v} \;\text{ and } y(v,t) \text{ is non-decreasing in } v, \tag{$\widetilde{\text{IC}}_t$}$$

$$0 \leq U'(t) \leq -\int_{\underline{v}}^{\overline{v}} y(v,t)\,\frac{\partial G(v,t)}{\partial t} dv, \tag{$\widetilde{\text{IC}}_0$}$$

$$U(t) \geq \max_{t'<t} \left\{\mathbb{E}_t[v]\,y(\mathbb{E}_t[v],t') - x(\mathbb{E}_t[v],t')\right\}. \tag{IC$_{global}$}$$

$$0 \leq y(v,t) \leq 1 \text{ for } t \in [0,\bar{t}] \;\text{ and } v \in [\underline{v},\overline{v}]. \tag{F}$$

The relaxed problem ($P^1_{relaxed}$) differs from the original mechanism design problem in several ways. First of all, the incentive compatibility constraints (IC_t) are replaced by their local counterparts ($\widetilde{IC}_t$), which are equivalent by Lemma 34. Moreover, the incentive compatibility constraints regarding "upward" deviations ($\overline{IC}_0$) are ignored, except for the local constraints ($\widetilde{IC}_0$), which were derived from Proposition 36, cf. equation (4.1). The incentive compatibility constraints ($\underline{IC}_0$) regarding "downward" deviations are rewritten as (IC_{global}). The individual rationality (IR) and feasibility (F) constraints are the same as those in the original mechanism design problem.

For the remainder of this section, we make the standard assumption that the monotone hazard rate condition holds on the valuation space, that is, $(1-G(v,t))/g(v,t)$ is non-increasing in v.

A key quantity for our analysis is the virtual surplus function $\phi(v,t)$ defined as

$$\phi(v,t) = v - c + \frac{(1-H(t))}{h(t)} \frac{\partial G(v,t)/\partial t}{g(v,t)}.$$

The virtual surplus function $\phi(v,t)$ is similar to the one defined by [**44**] for one-dimensional non-linear pricing problems. The first part of $\phi(v,t)$ corresponds to the social surplus of a type t consumer with valuation v from consuming one unit of the good, whereas the second part is the distortion due to inducing truth-telling for a type t consumer with valuation v. In addition to the hazard rate $(1-H(t))/h(t)$ as in the standard one-dimensional non-linear pricing problems, the second term also includes an "informativeness measure" $(\partial G(v,t)/\partial t)/g(v,t)$, which represents how informative the type of the consumer is regarding his valuation. The distortions increase with the informativeness measure since it is more difficult for the monopolist (i.e. more information rent has to be given to the consumer) in order to prevent marginally different consumers from pretending to be a type t consumer with valuation v. Throughout the remainder of this section, we assume that $\phi(v,t)$ is increasing in v for a given t and increasing in t for a given v. Intuitively, this implies that the revenue contribution of a consumer taking into account the incentive constraints is increasing in valuation and type.

The following proposition partially characterizes the optimal solution to the relaxed problem, and its proof is given in Appendix D.2.

Proposition 37. *Under FSD, there exists an optimal solution to the relaxed problem* $(P^1_{relaxed})$ *such that* $y(v,t) \in \{0,1\}$*. Moreover, for* $t \in [0,\bar{t}]$*, there exists a cut off* $k(t)$ *such that* $y(v,t) = 1$ *if and only if* $v \geq k(t)$*.*

The following corollary of Proposition 37 will be instrumental in characterizing the optimal contract and is proved in Appendix D.2.

Corollary 38. *Given an optimal solution to the relaxed problem* $(P^1_{relaxed})$ *as in Proposition 37 such that* $y(v,t) \in \{0,1\}$ *with corresponding cutoffs* $\{k(t) : 0 \leq t \leq \bar{t}\}$, *we have for all* $t \in [0,\bar{t}]$ *that*

$$\max_{t'<t} \{\mathbb{E}_t[v]\, y(\mathbb{E}_t[v], t') - x(\mathbb{E}_t[v], t')\}$$
$$= \mathbb{E}_t[v]$$
$$+ \max_{t' \in \{\tau : y(\mathbb{E}_t[v],\tau)=1\}} \left\{ U(t') - \int_{k(t')}^{\overline{v}} (v - k(t'))\, g(v,t')\, dv - k(t') \right\}.$$

Corollary 38 tells that if a type t were to pretend to be a lower type, than it is in his best interest to choose a type t' such that reporting his expected valuation of $\mathbb{E}_t[v]$ at time t', type t will be able to obtain the good with probability one, i.e. $\mathbb{E}_t[v] \geq k(t')$ cf. Proposition 37, in which case the payment to the monopolist would be

$$x(\mathbb{E}_t[v], t') = U(t') - \int_{k(t')}^{\overline{v}} (v - k(t'))\, g(v,t')\, dv - k(t').$$

Using Corollary 38, we rewrite the constraint (IC_{global}) of the formulation ($\text{P}^1_{relaxed}$) and define the relaxed problem formulation ($\text{P}^2_{relaxed}$) as follows: Choose the delivery probabilities $\left\{y(v,t) : t \in [0,\bar{t}],\ v \in [\underline{v},\overline{v}]\right\}$ so as to

$$\text{maximize} \int_{v,t} f(v,t)\left[(v-c)\,y(v,t)\right] dvdt - U(0) - \int_0^{\bar{t}} (1 - H(t))\, dU(t)$$

subject to

$$\int_{\underline{v}}^{\overline{v}} \left[vy(v,t) - x(v,t)\right] g(v,t)\, dv \ \geq 0 \text{ for } t \in [0,\bar{t}], \tag{IR}$$

$$y(v,t) = \frac{\partial u(v,t)}{\partial v} \ \text{ and } y(v,t) \text{ is non-decreasing in } v, \tag{$\widetilde{\text{IC}}_t$}$$

$$0 \leq U'(t) \leq -\int_{\underline{v}}^{\overline{v}} y(v,t)\, \frac{\partial G(v,t)}{\partial t} dv, \tag{$\widetilde{\text{IC}}_0$}$$

$$U(t) - \mathbb{E}_t[v] \geq$$

$$\max_{t' \in \{\tau : y(\mathbb{E}_t[v],\tau)=1,\ \tau < t\}} \left\{ U(t') - \int_{k(t')}^{\overline{v}} (v - k(t'))\, g(v,t')\, dv - k(t') \right\} \tag{IC_{global}}$$

$$0 \leq y(v,t) \leq 1 \text{ for } t \in [0,\bar{t}] \ \text{ and } v \in [\underline{v},\overline{v}]. \tag{R}$$

The next proposition characterizes the optimal solution to the problem ($\text{P}^2_{relaxed}$) and is proved in Appendix D.2.

Proposition 39. *There exists an optimal solution to relaxed problem ($P^2_{relaxed}$), which is characterized by the cut-off points $\{k(t) : 0 \leq t \leq \bar{t}\}$ such that the allocation y satisfies $y(v,t) = 1$ for $v \geq k(t)$ and $y(v,t) = 0$ otherwise. Moreover, for that optimal solution, the global incentive compatibility constraint (IC_{global}) binds for the highest type $\bar{t}$ only, who is indifferent between her contract and the contract choice of the lowest type, i.e.*

$$U(\bar{t}) = \mathbb{E}_{\bar{t}}[v] - \mathbb{E}_0\left[\max\{v, k(0)\}\right].$$

The optimal solution is further characterized by two thresholds t_1, t_2 *with* $0 < t_2 < \bar{t}$ *and* $0 \leq t_1 \leq t_2$ *such that* $U(t) = 0$ *for* $t \leq t_2$ *and* $U(t) > 0$ *otherwise. To be more specific,*

$$U'(t) = -\int_{k(t)}^{\overline{v}} \frac{\partial G(v,t)}{\partial t} dv \quad \textit{for} \quad t \geq t_2,$$

while $U'(t) = 0$ *for* $t \leq t_2$*. Finally, the optimal cutoff points* $\{k(t) : 0 \leq t \leq \bar{t}\}$ *are characterized as follows: For* $t \geq t_2$*,* $k(t) \geq c$ *and* $k(t)$ *is nonincreasing and is the unique solution of* $\phi(k(t), t) = 0$*, where*

$$\phi(v,t) = (v - c) + \frac{(1 - H(t))}{h(t)} \frac{\partial G(v,t) / \partial t}{g(v,t)}.$$

Similarly, $k(t)$ *is nonincreasing with* $k(t) \geq c$ *for* $t \leq t_2$*. In particular,* $k(0) > c$ *with* $k(t)$ *(strictly) decreasing for* $t \leq t_1$*, while* $k(t) = c$ *for* $t \in (t_1, t_2)$*.*

Given the cutoff points $\{k(t) : 0 \leq t \leq \bar{t}\}$ characterized in Proposition 39, the transfer payments can be written as follows:

$$x(v,t) = \begin{cases} \underline{x}(t) & \text{if} \quad v < k(t), \\ \overline{x}(t) & \text{if} \quad v \geq k(t), \end{cases}$$

where $\overline{x}(t) = \underline{x}(t) + k(t)$. Note that the expected surplus of type t consumer can be written as

$$U(t) = -\underline{x}(t) + \int_{k(t)}^{\bar{v}} (1 - G(v,t)) \, dv. \tag{4.2}$$

Then, since $U(t) = 0$ for $t \in [0, t_2]$, we write

$$\underline{x}(t) = \int_{k(t)}^{\bar{v}} (1 - G(v,t))\, dv \quad \text{for } t \leq t_2.$$

For $t \geq t_2$, we write the following by taking the derivatives of both sides of (4.2).

$$\text{(4.3)} \qquad \underline{x}'(t) = -U'(t) - \frac{dk(t)}{dt}(1 - G(k(t), t)) - \int_{k(t)}^{\bar{v}} \frac{\partial G(v,t)}{\partial t} dv,$$

Therefore, we can calculate $\underline{x}(t)$ for $t \geq t_2$ from (4.3) and the boundary condition that

$$\underline{x}(t_2) = \int_{k(t_2)}^{\bar{v}} (1 - G(v, t_2))\, dv.$$

Having characterized $\underline{x}(t)$, we write $\overline{x}(t) = \underline{x}(t) + k(t)$ for $t \in [0, \bar{t}]$, and interpret $\{(\overline{x}(t), k(t)) : 0 \leq t \leq \bar{t}\}$ as a menu of expiring refund contracts where type t is charged the initial price $\overline{x}(t)$ and is offered a refund of $k(t)$ if he chooses not to consume the good or use the service before time t. In other words, the refund $k(t)$ is only good before time t.

The following proposition establishes the optimality of this refund contract for the monopolist's original screening problem introduced in Section 4.1.

Proposition 40. *The menu of refund contracts $\{(\overline{x}(t), k(t)) : 0 \leq t \leq \bar{t}\}$ characterized immediately above is an optimal mechanism for the screening problem of the monopolist presented in Section 4.1.*

Proof of Proposition 40. Since the cutoff points $\{k(t) : 0 \leq t \leq \bar{t}\}$ are optimal for $(\mathrm{P}^2_{relaxed})$ and $(\mathrm{P}^2_{relaxed})$ is more relaxed than the original screening problem of the monopolist, if the menu of refund contracts $\{(\overline{x}(t), k(t)) : 0 \leq t \leq \bar{t}\}$ satisfy the constraints of the original problem, then it is an optimal mechanism for the screening problem of the monopolist.

We check whether the constraints of the screening problem of the monopolist is satisfied by the menu of refund contracts $\{(\overline{x}(t), k(t)) : 0 \leq t \leq \bar{t}\}$. Since $U(t) \geq 0$ for all t, cf. Proposition 39, individual rationality (IR) constraints are satisfied. The feasibility (F) constraints are readily satisfied. Moreover, from Lemma 34 ($\widetilde{\text{IC}}_t$) constraint of ($\text{P}^2_{relaxed}$) and the incentive compatibility constraint (IC_t) of the original problem are equivalent and hence (IC_t) of the original problem are satisfied. Since the incentive compatibility constraints ($\underline{\text{IC}}_0$) regarding "downward" deviations were rewritten as (IC_{global}) in ($\text{P}^2_{relaxed}$) and the cutoff points $\{k(t) : 0 \leq t \leq \bar{t}\}$ are optimal for ($\text{P}^2_{relaxed}$), the refund contracts

$$\{(\overline{x}(t), k(t)) : 0 \leq t \leq \bar{t}\}$$

satisfy ($\underline{\text{IC}}_0$) of the original screening problem.

What remains is to show that the refund contracts $\{(\overline{x}(t), k(t)) : 0 \leq t \leq \bar{t}\}$ satisfy ($\overline{\text{IC}}_0$). We first prove that any type $t \in [t_2, \bar{t}]$ does not find it profitable to pretend to be some higher type $t' > t$. To that end, let $U(t, t')$ denote the expected utility of type t by pretending to be type $t' > t$. We have

$$\begin{aligned} U(t,t') &= \int_{\underline{v}}^{\overline{v}} [vy(v,t') - x(v,t')]\, g(v,t)\, dv \\ &= \int_{k(t')}^{\overline{v}} [v - \overline{x}(t')]\, g(v,t)\, dv + \int_{\underline{v}}^{k(t')} [-\underline{x}(t')]\, g(v,t)\, dv \\ &= \int_{k(t')}^{\overline{v}} [v - (\underline{x}(t') + k(t'))]\, g(v,t)\, dv + \int_{\underline{v}}^{k(t')} [-\underline{x}(t')]\, g(v,t)\, dv \\ &= -\underline{x}(t') + \int_{k(t')}^{\overline{v}} [v - k(t')]\, g(v,t)\, dv. \qquad (4.4) \end{aligned}$$

Hence, taking the derivative of both sides of equation (4.2), we obtain

$$\frac{\partial U(t,t')}{\partial t'} = -\frac{d\underline{x}(t')}{dt'} - \frac{dk(t')}{dt'}\left(1 - G(k(t'),t)\right). \qquad (4.5)$$

From Proposition 39, $k'(t) \leq 0$ for $t \geq t_2$. Then,

$$\begin{aligned} \frac{\partial U(t,t')}{\partial t'} &\leq -\frac{d\underline{x}(t')}{dt'} - \frac{dk(t')}{dt'}\left(1 - G(k(t'),t')\right), \\ &= U'(t') + \int_{k(t')}^{\bar{v}} \frac{\partial G(v,t')}{\partial t'} dv, \\ &\leq 0, \end{aligned}$$

where the first inequality is obtained using equation (4.5), $k'(t) \leq 0$ and the fact that

$$\left(1 - G(k(t'),t')\right) \geq \left(1 - G(k(t'),t')\right),$$

which is due to FSD. The equality in the second step follows from taking the derivatives of both sides of the equation

$$U(t) = -\underline{x}(t) + \int_{k(t)}^{\bar{v}} (1 - G(v,t))\, dv. \tag{4.6}$$

Finally, the third line is true since from Proposition 36, we have

$$U'(t') \leq -\int_{k(t')}^{\bar{v}} \frac{\partial G(v,t')}{\partial t'} dv.$$

Then, by integrating both sides of the inequality,

$$\frac{\partial U(t,t')}{\partial t'} \leq 0,$$

and hence, we obtain $U(t,t') \leq U(t)$. This proves that any type $t \in [t_2, \bar{t}]$ does not find it profitable to pretend to be type some type $t' > t$.

Since t_2 does not find the deviation to some type $t' \in [t_2, \bar{t}]$ profitable and since $U(t_2) = 0$, FSD implies that any type $t \in [0, t_2]$ does not want to pretend to be type $t' \in [t_2, \bar{t}]$ as well. To see this, note that

$$\begin{aligned}
\frac{\partial U(t,t')}{\partial t} &= \int_{k(t')}^{\bar{v}} (v - k(t')) \frac{\partial g(v,t)}{\partial t} dv, \\
&= (\bar{v} - k(t')) \frac{\partial G(\bar{v},t)}{\partial t} - \int_{k(t')}^{\bar{v}} \frac{\partial G(v,t)}{\partial t} dv, \\
&= -\int_{k(t')}^{\bar{v}} \frac{\partial G(v,t)}{\partial t} dv, \\
&> 0,
\end{aligned}$$

and

$$U(t,t') = -\underline{x}(t') + \int_{k(t')}^{\overline{v}} [v - k(t')]\, g(v,t)\, dv$$

is increasing in t for a given t'. Hence, to conclude the argument that $(\overline{\text{IC}}_0)$ is satisfied by the refund contract $\{(\overline{x}(t), k(t)) : 0 \leq t \leq \overline{t}\}$, we show that any type $t \in [0, t_2]$ does not want to pretend to be type $t' \in (t, t_2)$. To be more specific,

$$\begin{aligned} U(t,t') &= -\underline{x}(t') + \int_{k(t')}^{\overline{v}} [v - k(t')]\, g(v,t)\, dv, \\ &\leq -\underline{x}(t') + \int_{k(t')}^{\overline{v}} [v - k(t')]\, g(v,t')\, dv, \\ &= U(t') \\ &= 0, \end{aligned}$$

which completes the proof that $(\overline{\text{IC}}_0)$ is satisfied.

Hence, the menu of refund contracts $\{(\overline{x}(t), k(t)) : 0 \leq t \leq \overline{t}\}$ is a feasible mechanism for the screening problem of the monopolist presented in Section 4.1, which also proves its optimality. ■

The next proposition investigates how the optimal initial price $\{\overline{x}(t) : 0 \leq t \leq \overline{t}\}$ changes over time and is proved in Appendix D.2.

Proposition 41. *The optimal menu of refund contracts* $\{(\overline{x}(t), k(t)) : 0 \leq t \leq \overline{t}\}$ *has the following properties: The optimal price path* $\{\overline{x}(t) : 0 \leq t \leq \overline{t}\}$ *is continuous except for an upward jump at time* t_2*. Moreover, the optimal initial price* $\overline{x}(t)$ *is constant for* $t < t_1$ *and is strictly increasing with rate* $\overline{x}'(t) = -\int_c^{\overline{v}} v \frac{\partial G(v,t)}{\partial t} dv$ *for* $t \in (t_1, t_2)$*. Then, the* *price has an upward jump at* t_2 *and is strictly decreasing with rate* $\overline{x}'(t) = k'(t)\, G(k(t), t)$ *for* $t \in [t_2, \bar{t}]$ *while* $\overline{x}(\bar{t}) > \overline{x}(t)$ *for all* $t < t_2$.

Proposition 41 characterizes the optimal menu of expiring refund contracts. Illustrative optimal price and refund paths are given in Figures 4.1 and 4.2, where the specific shape of the curve would depend on the distribution functions $G(v, t)$ and $H(t)$.

Although the optimal initial price is decreasing for $t > t_2$ as can be seen from Proposition 41 and Figure 4.2, the "effective" price $\overline{x}(t) - G(k(t), t)\, k(t)$, defined as the expected transfer from the consumer to the monopolist, is increasing since the rate of change of the effective price is

$$\begin{aligned} & \overline{x}'(t) - G(k(t), t)\, k'(t) - \left[g(k(t), t)\, k'(t) + \frac{\partial G(k(t), t)}{\partial t} \right] k(t) \\ = & -\left[g(k(t), t)\, k'(t) + \frac{\partial G(k(t), t)}{\partial t} \right] k(t) \\ > & \ 0. \end{aligned}$$

The effective price is clearly increasing for $t \leq t_1$, since the refund size and the likelihood of exercising the refund is decreasing while the initial price remains constant. Similarly, for $t \in [t_1, t_2]$, the initial price is increasing whereas the refund size is constant and the likelihood if exercising the refund is decreasing.

4.3. First order stochastic dominance reversed

In this section, we focus on the case that the types who learn their valuation early have a higher valuation in the sense of first order stochastic dominance (FSD). For instance, in an airline context, business travellers often have higher valuations than some tourists

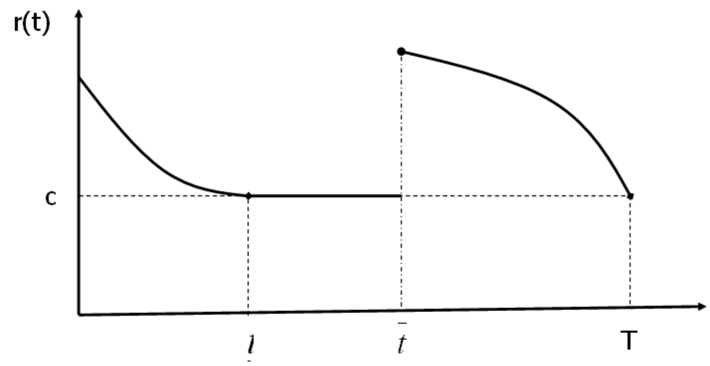

Figure 4.1. Optimal refund size as a function of time.

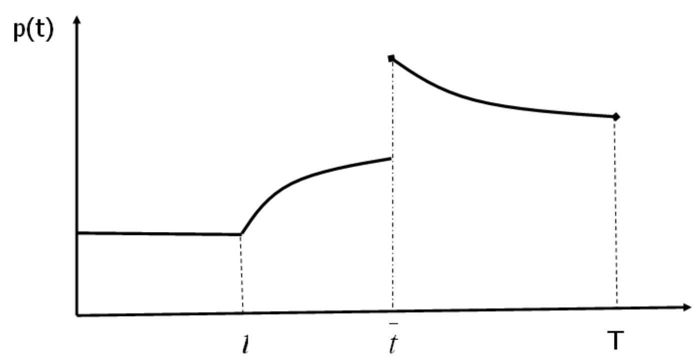

Figure 4.2. Optimal initial price as a function of time.

who might try to purchase the ticket at the last minute. The formal definition of the "reversed" first order stochastic dominance in the context of our model is as follows.

Definition 42. *The type space* $[0, \bar{t}]$ *is ordered by "reversed" FSD if for any* $t > t'$, $G(v, t) \geq G(v, t')$ *for all* v.

Condition 1 of Proposition 33 is violated in this case, and hence, first-best solution cannot be achieved. Nonetheless, it is easy to characterize the optimal mechanism, which can again be implemented by a menu of expiring refund contracts. The following proposition characterizes the optimal mechanism in this case.

Proposition 43. *Under "reversed" FSD, there exists cut-off points* $\{k(t) : 0 \leq t \leq \bar{t}\}$ *such that the optimal allocation* y *satisfies* $y(v,t) = 1$ *for* $v \geq k(t)$ *and* $y(v,t) = 0$ *otherwise. The expected utility function* U *is nonincreasing and satisfies*

$$U'(t) = -\int_{k(t)}^{\bar{v}} \frac{\partial G(v,t)}{\partial t} dv \quad \text{for } t \in [0, \bar{t}].$$

Finally, the optimal cutoff points $\{k(t) : 0 \leq t \leq \bar{t}\}$ *are characterized as follows:* $k(t)$ *is the unique solution of* $\psi(k(t), t) = 0$*, where*

$$\psi(v,t) = (v - c) - \frac{H(t)}{h(t)} \frac{\partial G(v,t)/\partial t}{g(v,t)}.$$

The payments are then given as follows: For $t \in [0, \bar{t}]$*,*

$$x(v,t) = \begin{cases} \underline{x}(t) & \text{if} \quad v < k(t), \\ \overline{x}(t) & \text{if} \quad v \geq k(t), \end{cases}$$

with $\overline{x}(t) = \underline{x}(t) + k(t)$ *and*

$$\underline{x}(t) = \int_{k(t)}^{\bar{v}} (v - k(t)) g(v,t) dv - U(t).$$

Proof. By definition, we have

$$\partial G(v,t)/\partial t = \lim_{h \searrow 0} \frac{G(v, t+h) - G(v,t)}{h} \geq 0.$$

As in Proposition 36, we can prove that without loss of optimality, the objective of the monopolist can be written as

$$\int_{v,t} f(v,t)(v-c)\,y(v,t)\,dvdt - U(\bar{t}) + \int_{[0,\bar{t}]} H(t)\,dU(t)\,. \tag{4.7}$$

To see this, note that using integration by parts

$$\int_0^{\bar{t}} h(t)\,U(t)\,dt = \{U(t)\,H(t)\}\,|_{t=0}^{t=\bar{t}} - \int_{[0,\bar{t}]} H(t)\,dU(t) = U(\bar{t}) - \int_{[0,\bar{t}]} H(t)\,dU(t)\,.$$

Then consider the problem:

$$\begin{aligned}
&\text{maximize} \int_{v,t} f(v,t)(v-c)\,y(v,t)\,dvdt - U(\bar{t}) + \int_{[0,\bar{t}]} H(t)\,dU(t) \\
&\text{subject to} \\
&U'(t) \leq -\int_{\underline{v}}^{\overline{v}} y(v,t)\,\frac{\partial G(v,t)}{\partial t} dv \;\; \text{for } t \in [0,\bar{t}]\,, \\
&0 \leq y(v,t) \leq 1 \text{ for } t \in [0,\bar{t}] \;\text{ and } v \in [\underline{v},\bar{v}]\,,
\end{aligned}$$

ignoring the global IC constraints.

We prove that in an optimal solution, it should be that

$$U'(t) = -\int_{\underline{v}}^{\overline{v}} y(v,t)\,\frac{\partial G(v,t)}{\partial t} dv.$$

We argue by contradiction. Suppose not. Let U, y denote an optimal solution to the relaxed problem stated immediately above with the property that

$$U'(t) < -\int_{\underline{v}}^{\overline{v}} y(v,t)\,\frac{\partial G(v,t)}{\partial t} dv$$

for some interval $[\tau_1, \tau_2]$. Suppose we decrease $U(0)$ by the amount

$$\int_{\tau_1}^{\tau_2} \left[-U'(t) + \int_{\underline{v}}^{\overline{v}} y(v,t) \frac{\partial G(v,t)}{\partial t} dv \right] dt.$$

At the same time, we keep the allocation y and $U'(t)$ for $t \in [0, \bar{t}] \setminus [\tau_1, \tau_2]$ the same, while setting

$$U'(t) = -\int_{\underline{v}}^{\overline{v}} y(v,t) \frac{\partial G(v,t)}{\partial t} dv$$

for $t \in [\tau_1, \tau_2]$. Let $\widetilde{U}$ be the modified expected utility function. That is, the modified expected utility function $\widetilde{U}$ is defined as follows using the original utility function U: Let

$$\begin{aligned} \widetilde{U}(0) &= U(0) - \int_{\tau_1}^{\tau_2} \left[-U'(t) + \int_{\underline{v}}^{\overline{v}} y(v,t) \frac{\partial G(v,t)}{\partial t} dv \right] dt, \\ \widetilde{U}(t) &= U'(t) \text{ for } t \in [0, \bar{t}] \setminus [\tau_1, \tau_2], \\ \widetilde{U}'(t) &= -\int_{\underline{v}}^{\overline{v}} y(v,t) \frac{\partial G(v,t)}{\partial t} dv \text{ for } t \in [\tau_1, \tau_2]. \end{aligned}$$

Then, $\widetilde{U}(t) > U(t)$ for $t < \tau_2$ and $\widetilde{U}(t) = U(t)$ for $t \geq \tau_2$. The change in the objective function resulting from the modified expected utility function $\widetilde{U}$ is

$$\int_{\tau_1}^{\tau_2} H(t) \left[-U'(t) + \int_{\underline{v}}^{\overline{v}} y(v,t) \frac{\partial G(v,t)}{\partial t} dv \right] dt$$

which is strictly positive. Thus, the objective improves. Hence, we have

$$U'(t) = -\int_{\underline{v}}^{\overline{v}} y(v,t) \frac{\partial G(v,t)}{\partial t} dv \quad \text{for all } t.$$

Since

$$U'(t) = -\int_{\underline{v}}^{\overline{v}} y(v,t) \frac{\partial G(v,t)}{\partial t} dv \leq 0 \quad \text{for all } t,$$

it is optimal to have $U(\bar{t}) = 0$, in which case using (4.7) the optimal y can be found by solving the following problem: Choose the allocation y so as to

$$\text{maximize} \int_{v,t} f(v,t)\,\psi(v,t)\,y(v,t)\,dvdt$$

subject to

$$0 \leq y(v,t) \leq 1,$$

where

$$\psi(v,t) = (v - c) - \frac{H(t)}{h(t)} \frac{\partial G(v,t)/\partial t}{g(v,t)}.$$

Then, y satisfies $y(v,t) = 1$ for $v \geq k(t)$ and $y(v,t) = 0$ otherwise where $k(t)$ is the unique solution of $\psi(k(t), t) = 0$. Then, it is easy to check that (IC_t) constraints are satisfied using the payments x characterized in the statement of Proposition 43. Moreover, we can show that the proposed solution satisfies the constraints ($\underline{\text{IC}}_0$) and ($\overline{\text{IC}}_0$) in a way similar to the analysis in the proof of Proposition 40, which concludes the proof of Proposition 43. ■

When the types who learn their valuation early have a higher valuation, the firm cannot use the learning dynamics to separate the types. Indeed, in the reversed first order stochastic dominance case, the firm can never implement the first-best solution. The reason is that the types who learn early can always choose to wait to make their purchase and the monopolist cannot deter them from doing so since they make consumption decisions when they know their demand if they wait to purchase.

Since $\partial G(v,t)/\partial t \geq 0$ due to reversed first order stochastic dominance, $k(t) \geq c$ for all t, and hence all types are rationed except type zero who has the highest valuation distribution. The expected utility of the consumers decrease over and the latest type to learn his valuation gets zero expected surplus. The first component of the virtual surplus function $\psi(v,t)$ is again the social surplus of a type t consumer with valuation v from consuming one unit of the good, while the second part is the distortion due to inducing truth-telling for a type t consumer with valuation v. The rate $H(t)/h(t)$ in the second component of $\psi(v,t)$ differs from the hazard rate $(1-H(t))/h(t)$ of the standard one-dimensional non-linear pricing problems since one unit of the informational rent supplied to type t, has to be supplied to all the lower types as opposed to all the higher types as in the virtual surplus function $\phi(v,t)$ of Section 4.2.

The following corollary shows that the optimal mechanism can be implemented via a menu of expiring refund contracts.

Corollary 44. *Under reversed first order stochastic dominance, the menu of expiring refund contracts* $\{(\overline{x}(t), k(t)) : 0 \leq t \leq \overline{t}\}$ *with* $\overline{x}(t)$ *and* $k(t)$ *as in Proposition 43 is an optimal mechanism for the screening problem of the monopolist presented in Section 4.1. Moreover, since* $\partial G(v,t)/\partial t \geq 0$, *it is easy to see that*

$$k(t) = c + \frac{H(t)}{h(t)} \frac{\partial G(k(t),t)/\partial t}{g(v,t)} \geq c,$$

that is, all types are rationed again and the allocation of type zero is efficient, i.e. $k(0) = c$.

4.4. Mean preserving spread

In this section, we consider the case when the consumers' valuations are ordered by second order stochastic dominance. In particular, the consumers who learn their valuations later also face greater uncertainty regarding their valuations. To be more specific, we assume that all consumers have the same expected valuation for the good, but for any $t > t'$, the valuation distribution of type t is a mean preserving spread of that of type t'. The formal definition is as follows:

Definition 45. *Let $z = \mathbb{E}_t [v]$ for all t. If $t > t'$ then $G(v,t) \geq G(v,t')$ for all $v < z$ and $G(v,t) \leq G(v,t')$ for all $v > z$.*

Throughout this section, we assume that the types are ordered in the sense of mean preserving spread. It is immediate from Definition 45 that $\partial G(v,t)/\partial t \leq 0$ for $v > z$ and $\partial G(v,t)/\partial t \geq 0$ for $v < z$. The next proposition characterizes the optimal mechanism under MPS.

Proposition 46. *Under MPS the first-best solution can be implemented.*

Proof. The next identity follows from integration by parts and will facilitate the proof.

$$\mathbb{E}_t [v - c; v \geq c] = (\bar{v} - c) - \int_c^{\bar{v}} G(v,t)\, dv \text{ for all } t. \tag{4.8}$$

Consider the case $z \leq c$, where Condition 2 of Proposition 33 is readily satisfied since $z = \mathbb{E}_t [v] \leq c$ for all t. Moreover, it follows from (4.8) that for $t' > t$,

$$\mathbb{E}_{t'} [v - c; v \geq c] - \mathbb{E}_t [v - c; v \geq c] = \int_c^{\bar{v}} [G(v,t) - G(v,t')]\, dv \geq 0,$$

where the last inequality follows since $\partial G(v,t)/\partial t \leq 0$ for $v \geq z$. Condition 1 of Proposition 33 also holds and the result follows.

Next consider the case $z > c$, where Condition 2 holds because for $t' < t$,

$$\mathbb{E}_{t'}\left[v - c; v \geq c\right] - \mathbb{E}_{t}\left[v - c\right] = \mathbb{E}_{t'}\left[v - c; v \geq c\right] - \mathbb{E}_{t'}\left[v - c\right] = \mathbb{E}_{t'}\left[v - c; v < c\right] < 0.$$

To check Condition 1, note the following identity, which follows from integration by parts:

$$\int_{c}^{\bar{v}} G(v,t)\, dv = \bar{v} - z - \int_{\underline{v}}^{c} G(v,t)\, dv \text{ for all } t,$$

where the integral on the right hand side is increasing in t by Definition 45 since $z > c$. Thus, the integral on the left is decreasing in t. Combining that with (4.8) shows that Condition 1 holds. ■

From Proposition 46, the menu of refund contracts $\{(\overline{x}(t), k(t)) : 0 \leq t \leq \bar{t}\}$ where the refund size is equal to the marginal cost, i.e. $k(t) = c$ for $t \in [0, \bar{t}]$ and the initial price is is equal to $\overline{x}(t) = \mathbb{E}_t\left[v - c; v \geq c\right] + c$ is the optimal mechanism for the screening problem of the monopolist presented in Section 4.1.

The next corollary investigates how the optimal initial price $\{\overline{x}(t) : 0 \leq t \leq \bar{t}\}$ changes over time and follows from Proposition 47 and Condition 1 of Proposition 33.

Corollary 47. *Under MPS, the optimal price path $\{\overline{x}(t) : 0 \leq t \leq \bar{t}\}$ is increasing over time.*

Proposition 46 and Corollary 47 prove that consumers with less dispersed valuations purchase tickets that become non-refundable sooner, and the price paid is increasing in the variance of consumers' valuations, but the refund price is always equal to marginal cost.

4.5. Discussion

We consider a revenue management model with strategic, i.e. forward looking consumers. Consumers vary in the ex ante distribution of their valuations and the time at which they learn their valuation. Consumers privately learn their valuations at different times and can purchase the good at any instance, both before and after they learn their valuations. The firm observes neither the time at which consumers learn their valuations nor their valuations and chooses a selling policy to maximize total expected profits.

We analyze the firm's optimal pricing policy in a mechanism design framework. Without loss of generality, the firm looks for the profit maximizing direct-revelation pricing policies and induces all consumers to buy the good at time zero. In other words, it is optimal for the firm to sell to the consumer before they learn their valuations while permits consumers to return the good sold for a certain refund after they learn their valuations.

Looking for the profit maximizing direct-revelation pricing policies, we establish the necessary and sufficient conditions for the firm to implement the first-best solution. Specifically, when Conditions 1 and 2 of Proposition 33 hold, the monopolist is able to implement the unconstrained first-best. In particular, Condition 1 of Proposition 33 is satisfied when the consumers who learn their demand late in the horizon have "higher" valuations. In that case, consumers who learn late are more willing to pay more for the good and they get the good with a higher probability. Condition 2 of Proposition 33 is satisfied when the expected valuations of different types of consumers are not too different or if the loss in the expected surplus due to making the consumption decisions before knowing one's true valuation is relatively large. If Conditions 1 and 2 of Proposition 33 are satisfied, then the first-best solution can be implemented using a menu of expiring refund contracts where the refund size is equal to the marginal cost and the prices increase over time. The monopolist is better off if the consumers learn their valuations sequentially since this enables her to exploit the differences in learning times to screen consumers.

We consider a variety of distributional assumptions for consumers' valuations. First, we analyze the case under which consumers who learn their valuations later have higher valuations, in the sense of first order stochastic dominance. Then the optimal pricing policy always provides the highest value consumer with the longest time to exercise the return option and a refund price equal to marginal cost. If the first-best is feasible, then a menu of expiring refund contracts with a refund size equal to the marginal cost and increasing prices is optimal. If the first best is not feasible, a menu of expiring refund contracts is still optimal for the firm and lower valuation consumers still purchase tickets which become non-refundable sooner while the refund price is higher than marginal cost. The allocation of the highest type is efficient whereas the refund size of some other types are distorted with respect to the first best. While the initial price and the refund size are distorted with respect to the first best, nevertheless, the effective price paid by the consumers is increasing over time.

The second case of interest is the one where consumers who learn their valuations early have higher valuations, in the sense of first order stochastic dominance, which we refer as reversed first order stochastic dominance. In this case, the firm can never achieve the first-best solution and a menu of expiring refund contracts is again the optimal mechanism and all consumer types are rationed again while the allocation of the type who learns the earliest is efficient.

Finally, we investigate the case when consumers who learn later have more dispersed valuations in the sense that their distribution is a mean preserving spread of the distribution of valuations for all consumers who learn earlier. Then, we show that the first best is always achievable. A menu of expiring refund contracts with the refunds equal to marginal cost is optimal and consumers with less variable demand purchase tickets that become non-refundable sooner while the price is increasing in the dispersion of consumers' valuations.

For any distributional assumption on consumers' valuations, exploiting the sequential learning of the consumers gives the firm an additional screening instrument. In certain cases, the monopolist is even able to extract all the expected surplus from the consumers. In particular, if the valuations of the consumers who learn later are "higher" and the expected valuations of the consumers are not to far away from each other (or the loss in surplus due to making consumption decisions before fully knowing the demand is high), then the monopolist can implement the first best. Offering refundable purchases allows the firm to extract more surplus from consumers compared to spot market sales. Moreover, when consumers vary in the ex ante distribution of their valuations and the time at which they learn their valuations, optimal contract varies not only the price of the ticket and the size of the refund but also the time at which the refunds are expiring.

CHAPTER 5

Revelation Principle for Continuous-Time Communication Games (joint with Barış Ata and James D. Dana, Jr.)

In this chapter, we establish a revelation principle that there is no loss of generality in assuming that the monopolist should structure his incentive system so that all consumers will be willing to reveal all of their information to him honestly. This result generalizes the revelation principle of [**45**] to continuous-time communication games, and is of interest on its own right.

Consider a setting with one principal and one agent. Let $\Theta = [0, \bar{t}] \times \{[\underline{v}, \bar{v}] \cup \xi\}$ denote the set of all possible states of the agent's private information. To be more specific, $\theta = (t, v)$ implies that the agent is of type t with valuation v whereas (t, ξ) for $t \in [0, \bar{t}]$ means that the agent knows that he is of type t but his true valuation has not yet realized. Let D denote the principal's decision domain. That is, D denotes the set of decisions that the principal can take at each point in time. Similarly, let M denote the set of messages that the principal can send to the agent and R denote the report that the agent can send to the principal at each point in time. The generality of these spaces allows us to model mixed, i.e. randomized, actions and messages.

Uncertainty is modeled by the complete probability space $(\Omega, \mathcal{H}, \mathbb{P})$, where Ω denotes the sample space and $\mathcal{H}$ is a σ-field of subsets of Ω. The evolution of information is modeled through the increasing sequences $\{\mathcal{F}_t,\ t \in \mathbb{R}_+\}$ and $\{\mathcal{G}_t,\ t \in \mathbb{R}_+\}$ of complete sub-σ-fields of $\mathcal{H}$. In particular, $\mathcal{F}_t$ represents the private information available to the agent at time t and $\mathcal{G}_t$ denotes the information observable to the principal and the agent at time t. Then, define

$$\mathcal{H}_t = \mathcal{F}_t \vee \mathcal{G}_t \ \text{ for } t \geq 0$$

to be the information available to the agent at time t and let $\mathcal{R}$ denote the space of R-valued, $\mathcal{H}_t-$predictable stochastic processes and let $\{\mathcal{A}_t^{\mathbf{r}}, t \in \mathbb{R}_+\}$ be the natural filtration generated by the stochastic process $\mathbf{r} \in \mathcal{R}$, satisfying the usual conditions, and

$$\mathcal{B}_t^{\mathbf{r}} = \mathcal{A}_t^{\mathbf{r}} \vee \mathcal{G}_t \ \text{ for } \ t \geq 0,$$

which denotes the information available to the principal at time t upon seeing the agent's reports until then.

Following [**45**], to describe a typical coordination mechanism which could be established by the principal, let $\mathcal{M}$ be the set of all possible processes of messages which the agent might receive from the principal. That is, $\mathcal{M}$ consists of M-valued, $\mathcal{B}_t^{\mathbf{r}}$-adapted stochastic processes for $\mathbf{r} \in \mathcal{R}$. Then, $\mathcal{D}$ is the set of all possible processes of actions that can be taken by the principal. More precisely, $\mathcal{D}$ consists of D-valued, $\mathcal{B}_t^{\mathbf{r}}$-adapted stochastic processes for $\mathbf{r} \in \mathcal{R}$. In our setting, $\mathbf{d} \in \mathcal{D}$ may represent a description of how the principal might plan to sell to the consumers and this includes any actions on the side of the consumers that can be observed by the principal as these are considered as part of the principal's decision domain.

Let $U_p : \mathcal{D} \times \Theta \rightarrow \mathbb{R}$ denote the utility function of the principal and $u : \mathcal{D} \times \Theta \rightarrow \mathbb{R}$ denote the utility function of the agent. That is, $u\left(\mathbf{d}, v, t\right)$ denotes the utility of an agent of type t and valuation v if the principal follows the decision rule $\mathbf{d} \in \mathcal{D}$. Notice that $u(\mathbf{d}, v, t)$ denotes the ex-post utility of the consumer since there are no random variables that she cannot observe at the terminal time $\bar{t}$. Given these structures, the principal's problem is to coordinate his decisions and those of his agents so as to maximize his expected utility. We assume that the principal has complete control over all communication, that he can request any information which the agent is willing to send, and that he can send messages and recommendations to the agent. However, the principal cannot directly observe an agent's type in Θ.

A coordination mechanism for the principal consists of the message space M, report space R as defined above, where the agent can choose among the reporting strategies $\mathbf{r} \in \mathcal{R}$ and the principal can choose the message strategy $\mathbf{m} \in \mathcal{M}$. Notice that the messages that the agent receives should depend on the reports she has sent. Similarly, the decision strategy $\mathbf{d}$ of the principal in $\mathcal{D}$ can also depend on these reports and

$$\pi(\mathbf{d}, \mathbf{m}; \mathbf{r})$$

denotes the probability measure on the decision strategy $\mathbf{d}$ and the messages $\mathbf{m}$ that the principal will choose given that the agent is planning to send the reports according to the strategy $\mathbf{r} \in \mathcal{R}$. Whenever we write $\pi(\mathbf{d}, \mathbf{m}; \mathbf{r})$, it should be understood that for all $t \in [0, \bar{t}]$, the agent reports $\{r(s) : 0 \leq s \leq t\}$ to the principal at time t. Then, $(\pi, \mathcal{D}, \mathcal{M}, \mathcal{R})$ completely describes the coordination mechanism established by the principal.

In the context of this coordination mechanism $(\pi, \mathcal{D}, \mathcal{M}, \mathcal{R})$ agent controls his choice of reporting strategy in $\mathcal{R}$ as a function of his type. Recall that there are no private decisions or actions by the agent in our setting. Hence, the agent selects a reporting strategy $\boldsymbol{\rho}$ as a function of her private information and each such reporting strategy $\boldsymbol{\rho}$ generates reports in the set $\mathcal{R}$. Then, $\pi(\mathbf{d}, \mathbf{m}; \boldsymbol{\rho})$ denotes the probability measure on the decision strategy $\mathbf{d}$ and the messages $\mathbf{m}$ that the principal will choose given that the agent is planning to send the reports according to the reporting strategy $\boldsymbol{\rho}$. The expected utility of the agent of type τ at time t is given by

$$U_t(\boldsymbol{\rho}|\tau, \mathbf{d}, \mathbf{m}) = \int\limits_{\mathcal{D}\times\mathcal{M}\times[\underline{v},\overline{v}]} u(\mathbf{d}, v, \tau)\, d\pi(\mathbf{d}, \mathbf{m}; \boldsymbol{\rho})\, d\mathbb{P}_t(v)$$

where the reporting strategy of the agent is dictated by $\boldsymbol{\rho}$ and $\mathbb{P}_t(\cdot) = \mathbb{P}(\cdot\,|\mathcal{H}_t)$ is the distribution over the valuations[1] for the agent given her information $\mathcal{H}_t$ up to time t. Notice that since $u(\mathbf{d}, v, \tau)$ is the ex-post utility of the agent, $U_t(\boldsymbol{\rho}|\tau, \mathbf{d}, \mathbf{m})$ represents the expected utility of the agent looking into the future standing at time t.

The reporting strategy $\boldsymbol{\rho}$ of the agent forms an equilibrium of this communication game if and only if $\boldsymbol{\rho}$ is superior to any other reporting strategy at each point in time. That is,

$$U_t(\boldsymbol{\rho}|\tau, \mathbf{d}, \mathbf{m}) \geq U_t(\hat{\boldsymbol{\rho}}|\tau, \mathbf{d}, \mathbf{m}) \text{ for all } t \in [0, \bar{t}]. \tag{5.1}$$

Even though $U_t(\boldsymbol{\rho}|\tau, \mathbf{d}, \mathbf{m})$ denotes the expected utility of the agent looking into the future standing at time t, the constraint (5.1) is equivalent to requiring that the continuation payoff of the agent under the reporting strategy $\boldsymbol{\rho}$ is higher than any other $\hat{\boldsymbol{\rho}}$ since any cost or benefit incurred until time t by the agent is sunk and does not affect the decisions thereafter.

[1] $\mathbb{P}_t(\cdot)$ takes values in $\{0, 1\}$ if the agent has already learned her true valuation at time t and it is equivalent to $g(\cdot, \tau)$ if the agent knows that her true type is τ but has not learned her true valuation yet.

The principal's problem is to find a coordination mechanism $(\pi, \mathcal{D}, \mathcal{M}, \mathcal{R})$ such that there is an equilibrium reporting strategy which gives the principal highest expected profits.

We say that a coordination mechanism is direct if the reporting strategy $\boldsymbol{\rho}$ takes values in the type space Θ, i.e. $R = \Theta$. More precisely, the agent reports her private information at each time point denoted by $\omega = \{\omega_t : t \in [0, \bar{t}]\}$ and recalling that the reporting strategy $\boldsymbol{\rho}$ is a function of the private information of the agent, with an abuse of notation, we have $\boldsymbol{\rho}(t, \omega) = \omega(t, \omega)$. Then, ω_0 corresponds to the type of the agent and subsequently ω_t includes the information regarding the valuation of the agent if the agent learns it before time t. Under a direct mechanism, the probability function $\pi(\mathbf{d}, \mathbf{m}; \hat{\omega})$ is the probability measure of the principal choosing the decisions $\mathbf{d} \in \mathcal{D}$ and the messages $\mathbf{m} \in \mathcal{M}$ given the report of the private information $\hat{\omega} = \{\hat{\omega}_t : t \in [0, \bar{t}]\}$ on the side of the consumer. The direct mechanism is honest if $\hat{\omega}_t = \omega_t$ for all t where ω_t denotes the true information of the consumer at time t.

Proposition 48. *Given any equilibrium of reporting strategy $\boldsymbol{\rho}$ and coordination mechanism $(\pi, \mathcal{D}, \mathcal{M}, \mathcal{R})$, there exists an incentive-compatible direct mechanism in which the principal gets the same expected utility (when the agents are honest) as in the given equilibrium of the given mechanism. Thus, the optimal incentive-compatible direct coordination mechanism is also optimal in the class of all coordination mechanisms.*

Proof of Proposition 48. Given the equilibrium of reporting strategy $\boldsymbol{\rho}$ and coordination mechanism $(\pi, \mathcal{D}, \mathcal{M}, \mathcal{R})$, consider the following direct mechanism: The reporting strategy $\boldsymbol{\rho}^*$ takes values in the type space Θ and the agent reports her private information at each time point denoted by $\{\omega_t : t \in [0, \bar{t}]\}$. The principal uses the probability measure π^* on $\mathcal{D} \times \mathcal{M} \times \mathcal{R}$ such that

$$\pi^*(\cdot, \cdot; \hat{\omega}) = \pi(\cdot, \cdot; \boldsymbol{\rho}(\hat{\omega})).$$

That is, under the direct mechanism, if the agent chooses to report the private information $\hat{\omega} = \{\hat{\omega}_t : t \in [0, \bar{t}]\}$, the principal assigns the same probability measure to all the decision and message strategies $(\mathbf{d}, \mathbf{m})$ as in the original coordination mechanism where the agent uses the reporting strategy $\boldsymbol{\rho}(\hat{\omega}) \in \mathcal{R}$ since $\boldsymbol{\rho}$ is a function of agent's private information.

We prove that the direct mechanism consisting of the reporting strategy $\boldsymbol{\rho}^*$ and coordination mechanism $(\pi^*, \mathcal{D}, \mathcal{M}, \mathcal{R})$ is incentive compatible. We argue by contradiction. Suppose that the agent finds it more profitable to report her information untruthfully at some point. That is, suppose that the agent finds reporting her information as $\hat{\omega}$ instead of ω. That is,

$$\int\limits_{\mathcal{D}\times\mathcal{M}\times[\underline{v},\overline{v}]} u(\mathbf{d}, v, \tau)\, d\pi^*(\mathbf{d}, \mathbf{m};\, \hat{\omega})\, d\mathbb{P}_t(v) > \int\limits_{\mathcal{D}\times\mathcal{M}\times[\underline{v},\overline{v}]} u(\mathbf{d}, v, \tau)\, d\pi^*(\mathbf{d}, \mathbf{m};\omega)\, d\mathbb{P}_t(v)$$

for some $\hat{\omega} = \{\hat{\omega}_t : t \in [0, \bar{t}]\}$ while the true information of the agent evolves according to $\omega = \{\omega_t : t \in [0, \bar{t}]\}$. Then, we also have

$$\int\limits_{\mathcal{D}\times\mathcal{M}\times[\underline{v},\overline{v}]} u(\mathbf{d}, v, \tau)\, d\pi(\mathbf{d}, \mathbf{m};\widetilde{\boldsymbol{\rho}})\, d\mathbb{P}_t(v) > \int\limits_{\mathcal{D}\times\mathcal{M}\times[\underline{v},\overline{v}]} u(\mathbf{d}, v, \tau)\, d\pi(\mathbf{d}, \mathbf{m}; \boldsymbol{\rho})\, d\mathbb{P}_t(v),$$

where $\widetilde{\boldsymbol{\rho}}(\omega) = \boldsymbol{\rho}(\hat{\omega})$. In other words, in the original coordination mechanism, the agent would also find it more profitable to choose the reporting strategy $\boldsymbol{\rho}(\hat{\omega}) \in \mathcal{R}$ instead of

$\boldsymbol{\rho}(\omega)$ if ω were her true information, which is a contradiction to the fact that $\boldsymbol{\rho}$ is an equilibrium reporting strategy. ■

Proposition 48 establishes the revelation principle that in order to maximize her total expected profits, the monopolist can consider only incentive-compatible direct-revelation mechanisms in our setting.

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APPENDIX A

Proofs of the Results in Chapter 1

A.1. Proof of the Results Regarding Adapted Bid-Price Controls

In this section, we prove Theorems 1 and 2 along with some auxiliary results regarding a Lagrangian representation of the network revenue management problem (P). Our proof can be broken down to four major steps. First, we prove the existence of an optimal solution to the network revenue management problem. Second, we show the validity of a Lagrangian representation for the network revenue management problem. To this end, we introduce a discretized version of the network revenue management problem in Section A.1.1 where at each decision time the demand and fare processes may have only finitely many realizations. Third, we interpret the Lagrangian representation as an adapted bid-price control. In particular, we construct an optimal adapted bid-price control (π, l), which, of course, provides a constructive proof of Theorem 1. Finally, the complementary slackness conditions associated with the Lagrange multipliers gives rise to the martingale property of optimal bid prices, proving Theorem 2.

We first establish the existence of an optimal solution to the network revenue management problem. The proof of the next proposition is straightforward and it is skipped.

Proposition 49. *There exists an optimal control for the network revenue management problem.*

Proof of Proposition 49. The proof is based on induction where the induction will be on the number of periods until the terminal time. First, we establish the induction basis by proving that there exists an optimal solution to (P) if $N = 1$. That is, consider the last decision time t_N. For a given capacity process $\overline{x}(\omega, t_{N-1}) \geq 0$, $\omega \in \Omega$, we solve the following problem to determine the bookings

$$\begin{aligned}
&\max \ \mathbb{E}[f(\omega, t_N) \cdot u(\omega, t_N)] \\
&\text{subject to} \\
&Au(\omega, t_N) \leq \overline{x}(\omega, t_{N-1}), \quad \omega \in \Omega, \\
&0 \leq u(\omega, t_N) \leq D(\omega, t_N) - D(\omega, t_{N-1}), \quad \omega \in \Omega.
\end{aligned}$$

Notice that, we can maximize this problem along each sample path ω. That is, for each $\omega \in \Omega$, we solve

$$\begin{aligned}
&\max_u \ f(\omega, t_N) \cdot u \\
&\text{subject to} \\
&Au(\omega, t_N) \leq \overline{x}(\omega, t_{N-1}), \\
&0 \leq u(\omega, t_N) \leq D(\omega, t_N) - D(\omega, t_{N-1}).
\end{aligned}$$

Let $u^*(\omega, t_N)$ denote an optimal solution to this problem which exists since the objective function is continuous in u and the feasible region is compact. Then, the resulting control $\{u^*(\omega, t_N) : \omega \in \Omega\}$ would maximize the expected revenue. Moreover, the objective function value of the linear program we solve for each $\omega \in \Omega$ is continuous in $\overline{x}(\omega, t_{N-1})$ since we have a linear program.

To facilitate our analysis, let $V_{n+1}(\omega, \overline{x})$ denote the optimal revenue that can be generated in decision times $n+1$ through N if we have capacity $\overline{x}$ at the end of period n and ω is realized. As the induction hypothesis, assume that an optimal solution exists for decision times $n+1$ through N and $V_{n+1}(\omega, \overline{x}(\omega, t_n))$ is a continuous function of $\overline{x}(\omega, t_n)$, where $\overline{x}(\omega, t_n)$ is the process of remaining capacities at decision time t_n. At decision time t_n we solve the following program for the given capacity process $\overline{x}(\omega, t_{n-1})$:

$$\max \mathbb{E}[f(\omega, t_n) \cdot u(\omega, t_n)] + \mathbb{E}[V_{n+1}(\omega, \overline{x}(\omega, t_{n-1}) - Au(\omega, t_n))]$$

subject to

$$Au(\omega, t_n) \leq \overline{x}(\omega, t_{n-1}), \quad \omega \in \Omega,$$

$$0 \leq u(\omega, t_n) \leq D(\omega, t_n) - D(\omega, t_{n-1}), \quad \omega \in \Omega.$$

We can optimize again for each sample path $\omega \in \Omega$. Then, for $\omega \in \Omega$, we solve

$$\max_u f(\omega, t_n) \cdot u + V_{n+1}(\omega, \overline{x}(\omega, t_{n-1}) - Au)$$

subject to

$$Au(\omega, t_n) \leq \overline{x}(\omega, t_{n-1}),$$

$$0 \leq u(\omega, t_n) \leq D(\omega, t_n) - D(\omega, t_{n-1}).$$

Note that the objective function is continuous in u while the feasible region is compact and hence, an optimal solution $u^*(\omega, t_n)$ exists for $\omega \in \Omega$ and $n = 1, \ldots, N$. Thus, an optimal solution exists for decision times n through N. Moreover, $V_n(\omega, \overline{x}(\omega, t_{n-1}))$, the objective function value of the linear program solved at decision time t_n along ω, is continuous in $\overline{x}(\omega, t_{n-1})$. ■

A.1.1. A Discrete Approximation of the Network Revenue Management Problem

In this section, we introduce a discrete approximation to the network revenue management problem (P). To be specific, we introduce a sequence of problems indexed by $m = 1, 2, \ldots$ such that in each problem the distributions of the demand during each period and the vector of fares have finite support so that these problems reduce to finite linear programs, allowing us to use the machinery of linear programming. As the readers will see, this sequence of problems will be helpful in proving statements about the network revenue management problem (P), cf. Theorem 1 and Propositions 50 & 51.

Since for each product the cumulative demand has finite mean and the fare process is bounded, there exist constants F and K^m for $m = 1, 2, \ldots$ and $n = 1, \ldots, N$ such that

$$\text{(A.1)} \quad \mathbb{P}[D(t_n) - D(t_{n-1}) \geq IK^m]\mathbb{E}[D(t_n) - D(t_{n-1}) \mid D(t_n) - D(t_{n-1}) \geq IK^m] \leq 1/3^m,$$

$$\text{(A.2)} \qquad \sup_{\omega \in \Omega,\ t \in \Gamma} f(\omega, t) \leq IF,$$

where I denotes the J-dimensional vector of ones. Without loss of generality assume K^m and F to be multiples of $1/2^m$. Fixing m, for each product j we truncate the demand during a period by K^m. Then we discretize demand during each period by taking a J-dimensional dyadic partition of the cube $[0, K^m]^J$ comprised of equal size grids, each of which is a J-dimensional cube with a side length of $1/2^m$. Similarly, we discretize the vector of fares in each period by taking a J-dimensional dyadic partition of the cube $[0,F]^J$ comprised also of equal size grids, each of which is a J-dimensional cube with a side length of $1/2^m$. Combining these two partitions, we construct a dyadic partition of the $2J$-dimensional set $[0,K^m]^J \times [0,F]^J$ comprised of $2J$-dimensional cubes with side lengths of $1/2^m$. Hereafter, we will refer to these $2J$-dimensional cubes as grids.

In formulating the discrete approximation, we pretend that the system manager cannot distinguish the demand and fare realizations in a grid and regards them as a single realization of demand and fare. When the system manager cannot distinguish the demand and fare realizations in a grid, she acts as if the demand and fare realizations were at their lowest possible level in that grid. Then, using this discretization one can represent the evolution of information as a finite information tree. That is, we get a finite number of "information nodes" for each decision time and the problem can be formulated as a linear program. To be more precise, an information node corresponds to a subset of Ω and two sample paths ω and ω' belong to the same information node for the m^{th} partition at decision time t_n only if the demand and fare realizations $D(\omega,t_r) - D(\omega,t_{r-1})$ and $f(\omega,t_r)$, and $D(\omega',t_r) - D(\omega',t_{r-1})$ and $f(\omega',t_r)$ are in the same grid for each $r = 1,\ldots,n$. Let $\mathcal{I}^m$ denote the set of information nodes resulting from the discretization of demand and fare processes. Associated with each information node is the probability of visiting that information node which is the probability measure of the sample paths corresponding to that information node. Let p^i denote the probability of visiting information node i for $i \in \mathcal{I}^m$, where $p^i = \mathbb{P}(i)$ viewing information node i as a measurable subset. Every information node $i \in \mathcal{I}^m$ has a unique predecessor in the information tree denoted by $i-$

(associated with the previous period). Similarly, let $i+$ denote a generic element of the set of nodes that can be subsequent to information node i. The set of information nodes at decision time t_n are denoted by $\mathcal{I}_n^m$, where $\mathcal{I}_0^m$ contains a single information node for all m. To be precise, $\mathcal{I}_n^m$ corresponds to all combinations of the different discrete realizations of demand and fare at decision times $t_1, \ldots, t_n$.

To repeat, the system manager behaves as if the sample paths in each grid results in a single demand realization and a single fare realization. Since the number of possible such realizations are finite, we get a finite linear program. Consider the finite linear program resulting from the m^{th} discretization. Upon entering information node $i \in \mathcal{I}^m$, the system manager behaves as if the demand realization is equal to the lowest possible demand realization and the fare realization is equal to the lowest possible fare in information node i. That is, if information node i corresponds to the case that for product j, we have at decision time t_n

$$s/2^m \leq D_j(\omega, t_n) - D_j(\omega, t_{n-1}) < (s+1)/2^m,$$

$$r/2^m \leq f_j(\omega, t_n) < (r+1)/2^m,$$

then the system manager acts as if the actual demand and fare realizations are $s/2^m$ and $r/2^m$ respectively.

In essence, we approximate (P) from "below" by a sequence of finite linear programs. At each decision time the system manager decides on the bookings for each product so that the capacity and demand restrictions are not violated. (Recall that for each product the demand realizations are governed by the lowest value of the demand for that product in a grid.) The objective is again to maximize expected revenue subject to the feasibility constraints.

The following notation is needed to proceed with the analysis. Let $d^{i,m}$ and $f^{i,m}$denote the discretized demand and fare at information node $i \in \mathcal{I}^m$, respectively. Let $u^{i,m}$ denote the booking vector at information node $i \in \mathcal{I}^m$. Similarly, denote vector of remaining capacities *upon entering* information node $i \in \mathcal{I}^m$ by $x^{i,m}$. Then, the finite linear program resulting from the m^{th} discretization (denoted by (P^m)) is given by

$$
(\text{P}^m) \qquad
\begin{aligned}
&\text{Maximize} \sum_{i \in \mathcal{I}^m} p^i f^{i,m} \cdot u^{i,m} \\
&\text{subject to} \\
&x^{i,m} = C \ , \ \ i \in \mathcal{I}_1^m, \\
&x^{i,m} = x^{i-,m} - Au^{i-,m}, \ \ i \in \mathcal{I}_n^m, \ \ n = 2, \ldots, N, \\
&Au^{i,m} \leq x^{i,m}, \ \ i \in \mathcal{I}_n^m, \ \ n = 1, \ldots, N, \\
&0 \leq u^{i,m} \leq d^{i,m}, \ i \in \mathcal{I}_n^m, \ \ n = 1, \ldots, N.
\end{aligned}
$$

For a realization $\omega \in \Omega$, we visit a sequence of information nodes (one for each decision time). Given an optimal control $\{u^{i,m}\}_{i \in \mathcal{I}^m}$ to (P^m), we can rewrite controls $u^{i,m}$, $i \in \mathcal{I}^m$, as a function of the sample paths ω by tracking which information nodes we visit at each decision time. Formally, for $\omega \in \Omega$, let

$$u^m(\omega, t_n) = u^{i,m} \ \text{ if } \ \omega \in i \text{ and } i \in \mathcal{I}_n^m.$$

Note that as the approximation of (P) through the discretized problems gets finer, that is, as $m \nearrow \infty$, we get a sequence of booking controls. The following proposition shows that the optimal controls for the discretized problems converge to an optimal control for the network revenue management problem (P).

Proposition 50. *There exists an optimal control* $\widetilde{u}(\cdot,\cdot) : \Omega \times \Gamma \to \mathbb{R}^J_+$ *for the network revenue management problem (P) constructed from the sequence of controls* u^m *for* $m = 1, 2, \ldots$. *More specifically, for every* $\omega \in \Omega$ *and* $n = 1, \ldots, N$ *there exists a subsequence* $m_{n,r}$ *such that*

$$\widetilde{u}(\omega, t_n) = \lim_{r\to\infty} u^{m_{n,r}}(\omega, t_n),$$

and $m_{n,r}$ *is a further subsequence of* $m_{n-1,r}$ *for* $n \geq 2$.

Proof of Proposition 50. The proof can be divided into three steps. First, we prove the existence of a limiting process $\widetilde{u}$. Second, we prove the feasibility of $\widetilde{u}$ for (P). Finally, we prove the optimality of $\widetilde{u}$ for (P). As our first step, we use the formal setup introduced above to show that for every $\omega \in \Omega$ and $n = 1, \ldots, N$ there exists a subsequence $m_{n,r}(\omega)$ such that for $n \geq 2$, $m_{n,r}(\omega)$ is a further subsequence of $m_{n-1,r}(\omega)$ and

$$\widetilde{u}(\omega, t_n) = \lim_{r\to\infty} u^{m_{n,r}(\omega)}(\omega, t_n) \quad \text{for } n = 1, \ldots, N.$$

We will next explicitly construct such a subsequence. To that end, first consider the decision time t_1. For each $\omega \in \Omega$, the sequence $\{u^m(\omega, t_1)\}$ is nonnegative and bounded from above. In particular, there exists a finite number M such that $\sup_{\omega\in\Omega,\ t\in\Gamma} u^m_j(\omega, t) \leq M$ for $j = 1, \ldots, J$. (Note that such constant M exists since the capacity C is finite.) We apply the following procedure to construct a convergent subsequence of $\{u^m(\omega, t_1)\}$.

Recall that the elements of $\{u^m(\omega, t_1)\}$ are J-dimensional vectors. We illustrate the construction for the case $J = 2$ for simplicity. The generalization for the cases $J \geq 3$ is straightforward. Let $J = 2$. Then, the sequence $\{u^m(\omega, t_1)\}$ resides in the square $[0, M] \times [0, M]$. The construction proceeds by repeated division to produce a sequence of nested squares whose common point will be shown to be a limit point of $\{u^m(\omega, t_1)\}$. First, divide the square $[0, M] \times [0, M]$ into four squares of side length $M/2$. Index these squares from 1 to 4 clockwise from top. That is, the square $[0, M/2] \times [M/2, M]$ has index 1, the square $[M/2, M] \times [M/2, M]$ has index 2, the square $[M/2, M] \times [0, M/2]$ has index 3 and so on. At least one of these squares should contain an infinite subsequence of $\{u^m(\omega, t_1)\}$. Pick the square with the smallest index that contains an infinite subsequence of $\{u^m(\omega, t_1)\}$. The indexing rule generates an unambiguous choice of the square to pick. We now divide the square of length $M/2$ that we picked again into four squares of side length $M/4$. Index these squares clockwise from top and select the square with the smallest index that contains an infinite subsequence of $\{u^m(\omega, t_1)\}$. If we continue in this manner, we obtain a sequence of nested squares that contain infinite subsequences of $\{u^m(\omega, t_1)\}$ and the side length of the square resulting from the r^{th} division is $M/2^r$. Let $m_{1,r}(\omega)$ denote the subsequence that is constructed by the application of this procedure to the sequence $\{u^m(\omega, t_1)\}$. That is, we construct $m_{1,r}(\omega)$ by repeatedly selecting a subsequence of $\{u^m(\omega, t_1)\}$ that is contained in the square we selected as a result of the division process. A common element of the sequence of squares resulting from the division procedure exists and is unique from the Nested Intervals Property, cf. [**6**]. Denote this point by $\widetilde{u}(\omega, t_1)$. Then, we have that $\widetilde{u}(\omega, t_1) = \lim_{r\to\infty} u^{m_{1,r}(\omega)}(\omega, t_1)$. Since for each $\omega \in \Omega$, the choice of the nested squares resulting from the division process in unambiguous, we have that $m_{1,r}(\omega) \in \mathcal{F}_{t_1}$ for $r = 1, 2, \ldots$ by construction.

Next, we consider the second decision time t_2. We consider the sequence

$$\{u^{m_{1,r}(\omega)}(\omega, t_1)\},$$

where $m_{1,r}(\omega)$ is the subsequence constructed for decision time t_1 using the division method described above. We apply the same procedure of repeated division to the sequence

$$\{u^{m_{1,r}(\omega)}(\omega, t_1)\}$$

and construct a subsequence $m_{2,r}(\omega)$ such that the limit

$$\widetilde{u}(\omega, t_2) = \lim_{r\to\infty} u^{m_{2,r}(\omega)}(\omega, t_2)$$

exists. We apply the procedure inductively for decision times t_n for $n = 1, \ldots, N$. Then, for every $\omega \in \Omega$ and $n = 1, \ldots, N$ there exists a subsequence $m_{n,r}(\omega)$ such that for $n \geq 2$, $m_{n,r}(\omega)$ is a further subsequence of $m_{n-1,r}(\omega)$ and

$$\widetilde{u}(\omega, t_n) = \lim_{r\to\infty} u^{m_{n,r}(\omega)}(\omega, t_n) \quad \text{for } n = 1, \ldots, N.$$

Moreover, $m_{n,r}(\omega) \in \mathcal{F}_{t_n}$ for $r = 1, 2, \ldots$ and $n = 1, \ldots, N$ by construction and this concludes the existence step of the proof of Proposition 50.

Our second step is to prove the feasibility of $\widetilde{u}$ for (P). Note that the demand restrictions of (P) are satisfied by $\widetilde{u}$ since for every $\omega \in \Omega$, along the convergent subsequence

$m_{n,r}(\omega)$, we have

$$0 \le u^{m_{n,r}(\omega)}(\omega, t_n) \le D(\omega, t_n) - D(\omega, t_{n-1}), \quad n = 1, \dots, N,$$

and passing to the limit as $r \to \infty$ we get

$$0 \le \widetilde{u}(\omega, t_n) \le D(\omega, t_n) - D(\omega, t_{n-1}), \ \ n = 1, \dots, N.$$

Since for every $\omega \in \Omega$ and for all m along the convergent subsequence $m_{N,r}(\omega)$, we have $\sum_{n=1}^{N} Au^{m_{N,r}(\omega)}(\omega, t_n) \le C$, and in the limit we obtain $\sum_{n=1}^{N} A\widetilde{u}(\omega, t_n) \le C$ due to Dominated Convergence Theorem. Thus, $\widetilde{u}$ satisfies the capacity constraints.

We now check the measurability of $\widetilde{u}$, i.e. $\widetilde{u}(\omega, t_n) \in \mathcal{F}_{t_n}$. For $a \in \mathbb{R}$ and $n = 1, \dots, N$, we have

$$(u^m(\omega, t_n))^{-1}(a, \infty) \in \mathcal{F}_{t_n},$$

since $u^m(\omega, t_n)$ takes finitely many values for all m due to discretization of demand ($u^m(\omega, t_n)$ takes a single value for each information node in the set $\mathcal{I}_n^m$) and every information node in $\mathcal{I}_n^m$ is a subset of $\mathcal{F}_{t_n}$. We can rewrite $u^{m_{n,r}(\omega)}(\omega, t_n)$ as

$$u^{m_{n,r}(\omega)}(\omega, t_n) = \sum_{k=1}^{\infty} \mathbf{1}_{\{m_{n,r}(\omega)=k\}} u^k(\omega, t_n),$$

and, hence $u^{m_{n,r}(\omega)}(\omega, t_n) \in \mathcal{F}_{t_n}$ for all r which follows from the fact that

$$\mathbf{1}_{\{m_{n,r}(\omega)=k\}} \in \mathcal{F}_{t_n}$$

and $u^k(\omega, t_n) \in \mathcal{F}_{t_n}$ for all k. Since $u^{m_{n,r}(\omega)}(\omega, t_n) \to \widetilde{u}(\omega, t_n)$ almost surely as $r \to \infty$, we get $\widetilde{u}(\omega, t_n) \in \mathcal{F}_{t_n}$.

As a preliminary to establishing the optimality of $\widetilde{u}$, we show the convergence of the objective function values of the discretized problems. To be specific, we show that the objective function values of the discretized problems converge to the expected revenue generated by the feasible control $\widetilde{u}$ for the network revenue management problem (P). That is,

$$\lim_{m\to\infty} \sum_{n=1}^{N} \mathbb{E}[f^m(\omega, t_n) \cdot u^m(\omega, t_n)] = \sum_{n=1}^{N} \mathbb{E}[f(\omega, t_n) \cdot \widetilde{u}(\omega, t_n)], \tag{A.3}$$

where $f^m(\omega, t_n) = f^{i,m}$ if $\omega \in i$ and $i \in \mathcal{I}_n^m$.

We first show that the revenue generated along a sample path ω,

$$\sum_{n=1}^{N} f^m(\omega, t_n) \cdot u^m(\omega, t_n),$$

is non-decreasing in m almost surely. That is, $\mathbb{P}$ almost surely,

$$\sum_{n=1}^{N} f^m(\omega, t_n) \cdot u^m(\omega, t_n)$$

increases weakly as the partition gets finer. To see this, first note that u^m is feasible for the $(m+1)^{st}$ discretization. Moreover, the fare and demand that is assumed by the system manager to realize weakly increases on each grid as the partition gets finer. Hence, applying the control u^m for the $(m+1)^{st}$ discretized problem generates at least as much revenue as u^m on each grid of the m^{th} discretized problem. However, u^{m+1} does at least as good as this control and the revenue generated along a sample path ω weakly increases.

Moreover, total revenue generated along each sample path is also bounded by above as the initial capacity C is bounded and so is the revenue that can be generated along each sample path. Then, objective function values of the discretized problems form a monotone sequence of real numbers which should have a limit. For every $\omega \in \Omega$, this limit also coincides with the revenue generated by $\widetilde{u}$ along ω. Then,

$$\sum_{n=1}^{N} f^m(\omega, t_n) \cdot u^m(\omega, t_n)$$

converges to

$$\sum_{n=1}^{N} f(\omega, t_n) \cdot \widetilde{u}(\omega, t_n)$$

for every ω, and the result follows from the Dominated Convergence Theorem.

To conclude our last step, we prove that $\widetilde{u}$ is optimal for the network revenue management problem (P). We start with an optimal solution to (P), which was shown to exist in Proposition 49. Then, we construct a feasible solution to each discretized problem using that optimal solution. We show that the objective function values of these feasible solutions converge to the objective function value of (P), which proves the optimality of $\widetilde{u}$ for (P). Let u^* be an optimal solution to (P). Construct a feasible solution $\overline{u}^m$ to the m^{th} discretized problem as follows. For decision time t_n and information node $i \in \mathcal{I}_n^m$, the booking for product j is given by

$$\text{(A.4)} \qquad \overline{u}_j^m(\omega, t_n) = \begin{cases} \max\{0, \mathbb{E}[u_j^*(\omega, t_n)|\ i] - 1/2^m\} & \text{if} \quad \omega \in i,\ d_j^{i,m} < K^m, \\ \mathbb{E}[\min\{u_j^*(\omega, t_n), K^m\}|\ i] & \text{if} \quad \omega \in i,\ d_j^{i,m} = K^m, \end{cases}$$

where $d_j^{i,m}$ is the discretized demand for product j at information node i and K^m is the level at which we truncate the demand. This control satisfies the demand restrictions in the case of $d_j^{i,m} < K^m$ because of the feasibility of u^* for (P). The case of $d_j^{i,m} = K^m$ clearly satisfy the demand restrictions.

The proposed bookings $\overline{u}^m$ also satisfy the capacity restrictions since

$$\sum_{n=1}^{N} u_j^*(\omega, t_n) \leq C \text{ for all } \omega$$

and the cumulative bookings under the proposed control is less than or equal to a conditional expectation of $\sum_{n=1}^{N} u_j^*(\omega, t_n)$ and hence is in turn less than or to the capacity C. Moreover, $\overline{u}^m$ is clearly adapted. The expected revenue under $\overline{u}^m$ is greater than or equal to

$$\text{(A.5)} \qquad \sum_{n=1}^{N} \sum_{i \in \mathcal{I}_n^m} p^i \mathbb{E}[f(\omega, t_n) \cdot u^*(\omega, t_n) \mid i\,] - JNF/2^m - NF^2 2^J 2^m/3^m - JNM/2^m,$$

where F is the bound on the fare process and M is a finite number such that

$$\sup_{\omega \in \Omega,\; t \in \Gamma} u_j^*(\omega, t) \leq M$$

for $j = 1, \ldots, J$. (Note that such constant M exists since the capacity C is finite.) The second and third terms in (A.5) give upper bounds on the loss in revenue due to approximation of the demand and the last term gives an upper bound for the loss in fare due to approximation of fare from below.

To be more specific, at an information node $i \in \mathcal{I}_n^m$ such that $d_j^{i,m} < K^m$, the feasible control $\overline{u}_j^m(\omega, t_n)$ as defined in (A.4) books $\mathbb{E}[u_j^*(\omega, t_n)|\; i] - 1/2^m$ for product j provided that it is positive. On the other hand, at an information node $i \in \mathcal{I}_n^m$ such that $d_j^{i,m} \geq K^m$, we have $\overline{u}_j^m(\omega, t_n) = \mathbb{E}[\min\{u_j^*(\omega, t_n), K^m\}|i]$ by (A.4). In the latter case, multiplying

$\overline{u}_j^m(\omega, t_n)$ with the probability of being in information node $i \in \mathcal{I}_n^m$ we get

$$\begin{aligned}
\mathbb{P}(i)\overline{u}_j^m(\omega, t_n) &= \mathbb{P}(i)\mathbb{E}[\min\{u_j^*(\omega, t_n), K^m\}|i] \\
&= \mathbb{P}(i)\mathbb{E}[u_j^*(\omega, t_n)|i] - \mathbb{P}(i)\mathbb{E}[\max\{u_j^*(\omega, t_n) - K^m, 0\}|i] \\
&\geq \mathbb{P}(i)\mathbb{E}[u_j^*(\omega, t_n)|i] \\
&\quad -\mathbb{P}(i)\mathbb{E}[\max\{D_j(\omega, t_n) - D_j(\omega, t_{n-1}) - K^m, 0\}|i] \\
&\geq \mathbb{P}(i)\mathbb{E}[u_j^*(\omega, t_n)|i] - 1/3^m, \qquad \text{(A.6)}
\end{aligned}$$

where the second line is obtained by rearranging terms. The first inequality is a result of replacing $\max\{u_j^*(\omega, t_n) - K^m, 0\}$ by $\max\{D_j(\omega, t_n) - D_j(\omega, t_{n-1}) - K^m, 0\}$ and the fact that $u_j^*(\omega, t_n) \leq D_j(\omega, t_n) - D_j(\omega, t_{n-1})$ by feasibility of u^*. Finally, to get (A.6) we use the fact that cumulative demand process has finite mean and the inequality (A.1) holds. The expected revenue under $\overline{u}_j^m$ for the m^{th} discretized problem is

$$\begin{aligned}
\sum_{n=1}^{N} p^i f^{i,m} \cdot \mathbb{E}[\overline{u}^m(\omega, t_n) \mid i\ ] &\geq \sum_{n=1}^{N} \sum_{i \in \mathcal{I}_n^m} p^i f^{i,m} \cdot \mathbb{E}[u^*(\omega, t_n)\ |i\ ] \\
&\quad - \sum_{n=1}^{N} \sum_{i \in \mathcal{I}_n^m} \frac{p^i f^{i,m} J}{2^m} - \sum_{n=1}^{N} \frac{F^2 2^m 2^J}{3^m} \\
&\geq \sum_{n=1}^{N} \sum_{i \in \mathcal{I}_n^m} p^i \mathbb{E}[f(\omega, t_n) \cdot u^*(\omega, t_n)\ |i\ ] \\
&\quad - \frac{JNF}{2^m} - \frac{NF^2 2^J 2^m}{3^m} - \frac{JNM}{2^m}, \qquad \text{(A.7)}
\end{aligned}$$

where the first inequality is obtained by replacing $\overline{u}^m$ with u^* and accounting for the fact that at each information node $i \in \mathcal{I}_n^m$ such that $d_j^{i,m} < K^m$, the difference between

$\mathbb{E}[\overline{u}^m(\omega, t_n) \mid i\,]$ and $\mathbb{E}[u^*(\omega, t_n) \mid i\,]$ is bounded by $1/2^m$, whereas at an information node $i \in \mathcal{I}_n^m$ such that $d_j^{i,m} \geq K^m$ (which exists at most $F\,2^J\,2^m$ times at each period) we have

$$\mathbb{P}(i)\overline{u}_j^m(\omega, t_n) \geq \mathbb{P}(i)\mathbb{E}[u_j^*(\omega, t_n)|i] - 1/3^m$$

by (A.6). Finally, the second inequality is obtained by replacing the term $f^{i,m} \cdot \mathbb{E}[u^*(\omega, t_n) \mid i\,]$ with $\mathbb{E}[f(\omega, t_n) \cdot u^*(\omega, t_n) \mid i\,]$. Then, the last term in (A.7) is an upper bound in the loss in expected revenue due to approximating the fare of each product at each information node.

Therefore, the expected revenue under the optimal solution to the m^{th} discretized problem is greater than or equal to (A.5). Since (A.5) converges to the objective function value of the network revenue management problem (P) as $m \to \infty$, it follows from (A.3) that $\widetilde{u}$ is optimal for (P). ■

Having proved the optimality of $\widetilde{u}$ for (P), we next show the existence of the Lagrange multipliers for the network revenue management problem (P) as the last step towards the proofs of Theorems 1 and 2. The Lagrange multiplier for a resource also forms a martingale until the capacity of that resource is exhausted due to complementary slackness conditions.

Proposition 51. *There exists $\widetilde{y}(\omega, t_n) \in \mathcal{F}_{t_n}$ for $\omega \in \Omega$ and $n = 1, \ldots, N$ such that $u^* = \{u^*(\omega, t)\}_{(\omega,t)\in\Omega\times\Gamma}$ is an optimal solution to the network revenue management problem (P) if and only if it maximizes the Lagrangian*

$$\begin{aligned} L(u; \widetilde{y}) \;\; = \;\; & \sum_{n=1}^{N} \mathbb{E}[f(\omega, t_n) \cdot u(\omega, t_n)] - \mathbb{E}[\widetilde{y}(\omega, t_1) \cdot C] \\ & - \sum_{n=2}^{N} \mathbb{E}[\widetilde{y}(\omega, t_n) \cdot (x(\omega, t_{n-1}) - x(\omega, t_{n-2}) + Au(\omega, t_{n-1}))] \end{aligned}$$

among all feasible solutions to (P). Moreover, there exists an optimal solution $\widetilde{u}$ to (P) such that its corresponding state trajectory $\widetilde{x}$ satisfies the following: For $n = 1, \ldots, N-1$ and $\omega \in \Omega$,

$$(\widetilde{y}(\omega, t_n) - \mathbb{E}[\widetilde{y}(\omega, t_{n+1})|\ \mathcal{F}_{t_n}]) \cdot \widetilde{x}(\omega, t_{n-1}) = 0. \tag{A.8}$$

Proof of Proposition 51. Let $\{\widetilde{u}^{i,m}\}_{i\in\mathcal{I}^m}$ denote an optimal control for the m^{th} discretized problem (P^m), and $\{\widetilde{x}^{i,m}\}_{i\in\mathcal{I}^m}$ denote the corresponding state trajectory. Recall that $\widetilde{x}^{i,m}$ denotes the vector of remaining capacities upon entering information node $i \in \mathcal{I}^m$. The dual linear program to the m^{th} discretized problem (P^m) is given by

$$\begin{aligned}
&\text{Min}_{\{v^{i,m}\}_{i\in\mathcal{I}^m}} \sum_{i\in\mathcal{I}^m} p^i[(0 \vee (f^{i,m} - A'y^{i,m})) \cdot d^{i,m}] + C \cdot y^{0,m} \\
&\text{subject to} \\
&y^{i,m} = \mathbb{E}[y^{i+,m} \mid i] - v^{i,m},\ i \in \mathcal{I}^m, \\
&v^{i,m} \le 0,\ i \in \mathcal{I}^m,
\end{aligned} \tag{D^m}$$

where $y^{i+,m} = 0$ for $i \in \mathcal{I}_N^m$ by convention and $d^{i,m}$ denote the discretized demand at information node $i \in \mathcal{I}^m$. From linear programming duality, the optimal dual variables $\{\widetilde{y}^{i,m}\}_{i\in\mathcal{I}^m}$ satisfy the complementary slackness conditions, which in turn imply the following.

$$(\widetilde{y}^{i,m} - \mathbb{E}[\widetilde{y}^{i+,m} \mid i]) \cdot \widetilde{x}^{i,m} = 0 \text{ for } i \in \mathcal{I}^m. \tag{A.9}$$

Recall that $i+$ denotes a generic information node that is preceded by the information node i and given i, it can be thought of as a discrete random variable. The primal optimal state trajectory $\{\widetilde{x}^{i,m}\}_{i\in\mathcal{I}^m}$ is nonnegative and uniformly bounded by the initial capacity C. We next prove that the dual variables $\{\widetilde{y}^{i,m}\}_{i\in\mathcal{I}^m}$ are nonnegative and uniformly bounded, too. The non-negativity of an optimal dual solution $\widetilde{y}^{i,m}$ then follows from the facts that $v^{i,m} \leq 0$ for $i \in \mathcal{I}^m$ and $\widetilde{y}^{i+,m} = 0$ for $i \in \mathcal{I}^m_N$. The objective function value of the discretized network revenue management problems are less than or equal to $\sum_{n=1}^N \mathbb{E}[f(\omega, t_n) \cdot (D(\omega, t_n) - D(\omega, t_{n-1}))]]$, which is uniformly bounded since the demand process has finite mean and fare process is bounded. Then, the objective function value of (D^m) are also uniformly bounded due to strong duality. For an optimal solution $\{\widetilde{y}^{i,m}\}_{i\in\mathcal{I}^m}$ to (D^m) we have that its objective function value is in the interval $[C\cdot\widetilde{y}^{0,m}, \sum_{n=1}^N \mathbb{E}[f(\omega, t_n)\cdot (D(\omega, t_n) - D(\omega, t_{n-1}))]]$ and $\widetilde{y}^{0,m}$ is uniformly bounded. The same is true, by a similar argument, for all $\widetilde{y}^{i,m}$ if we consider the smaller primal and dual problems that starts at information node i, and hence, $\{\widetilde{y}^{i,m}\}_{i\in\mathcal{I}^m}$ are uniformly bounded.

To facilitate our analysis, fix an optimal dual solution $\widetilde{y}$ and define the following: If $\omega \in i$, $i \in \mathcal{I}^m_n$ and $n = 1, \ldots, N$,

$$u^m(\omega, t_n) = \widetilde{u}^{i,m}, \;\; y^m(\omega, t_n) = \widetilde{y}^{i,m}, \;\;\; x^m(\omega, t_{n-1}) = \widetilde{x}^{i,m}.$$

We will show that the limit of $y^m(\omega, t_n)$ gives us the processes $y(\omega, t_n)$ as in the statement of Proposition 51. Recall that, for each $\omega \in \Omega$, we have constructed in Proposition 50 the subsequences $m_{n,r}(\omega)$, $n = 1, \ldots, N$ such that $m_{n,r}(\omega)$ is a further subsequence of $m_{n-1,r}(\omega)$ for $n \geq 2$ and $\{u^{m_{n,r}(\omega)}(\omega, t_n)\}$ has a limit as $r \to \infty$. Here, we use the same subsequences. Then, $\{x^{m_{n,r}(\omega)}(\omega, t_n)\}$ also has a limit as $r \to \infty$ since

$$x^{m_{n,r}(\omega)}(\omega, t_n) = C - A \sum_{k=1}^{n} u^{m_{n,r}(\omega)}(\omega, t_k) \text{ for } n = 1, \ldots, N.$$

For each $\omega \in \Omega$, the procedure described in the proof of Proposition 50 can be repeated if necessary to construct further subsequences of $m_{n,r}(\omega)$ for $n = 1, \ldots, N$ so that $\{y^{m_{n,r}(\omega)}(\omega, t_n)\}$ have a limit as well. ($\{y^{m_{n,r}(\omega)}(\omega, t_n)\}$ forms a nonnegative and uniformly bounded sequence.) Thus, without loss of generality assume for every $\omega \in \Omega$ and $n = 1, \ldots, N$ that $\{y^{m_{n,r}(\omega)}(\omega, t_n)\}$ converges as $r \to \infty$. Denote for $\omega \in \Omega$ and $n = 1, \ldots, N$

$$\begin{aligned} \widetilde{u}(\omega, t_n) &= \lim_{r\to\infty} u^{m_{n,r}(\omega)}(\omega, t_n), \\ \widetilde{x}(\omega, t_n) &= \lim_{r\to\infty} x^{m_{n,r}(\omega)}(\omega, t_n), \\ \widetilde{y}(\omega, t_n) &= y^{m_{n,r}(\omega)}(\omega, t_n). \end{aligned}$$

By construction of the subsequences $m_{n,r}(\omega)$, we have that $\widetilde{u}(\omega, t_n), \widetilde{y}(\omega, t_n) \in \mathcal{F}_{t_n}$ for $n = 1, \ldots, N$. We showed in Proposition 50 that $\widetilde{u}$ is an optimal solution to (P). We can rewrite (A.9) as

$$(y^{m_{n,r}(\omega)}(\omega, t_n) - \mathbb{E}[y^{m_{n+1,r}(\omega)}(\omega, t_{n+1})|\, \mathcal{F}_{t_n}]) \cdot x^{m_{n,r}(\omega)}(\omega, t_{n-1}) = 0,$$

for $n = 1, \ldots, N-1$ and $\omega \in \Omega$. As $r \to \infty$, $y^{m_{n+1,r}(\omega)}(\omega, t_{n+1})$ converges to $\widetilde{y}(\omega, t_{n+1})$. From Dominated Convergence Theorem, we get as $r \to \infty$

$$\text{(A.10)} \quad (\widetilde{y}(\omega, t_n) - \mathbb{E}[\widetilde{y}(\omega, t_{n+1})|\, \mathcal{F}_{t_n}]) \cdot \widetilde{x}(\omega, t_{n-1}) = 0, \;\; n = 1, \ldots, N-1 \;\text{ and }\; \omega \in \Omega,$$

proving (A.8).

Now consider a feasible solution u^* to (P) which maximizes $L(u;\widetilde{y})$ among all feasible solutions to (P), and let x^* be its corresponding state trajectory. Since $\widetilde{y}$ is nonnegative and u^* is a feasible solution to (P),

$$\text{(A.11)}\quad \sum_{n=2}^{N} \mathbb{E}[\widetilde{y}(\omega,t_n)\cdot(x^*(\omega,t_{n-1}) - x^*(\omega,t_{n-2}) + Au^*(\omega,t_{n-1}))] + \mathbb{E}[\widetilde{y}(\omega,t_1)\cdot C] \geq 0.$$

whereas (A.11) with $\widetilde{u}$ instead of u^* is zero from (A.10). Then, we should have

$$\text{(A.12)}\qquad \sum_{n=1}^{N} \mathbb{E}[f(\omega,t_n)\cdot\widetilde{u}(\omega,t_n)] \leq \sum_{n=1}^{N} \mathbb{E}[f(\omega,t_n)\cdot u^*(\omega,t_n)],$$

since u^* maximizes $L(u;\widetilde{y})$. The optimality of $\widetilde{u}$ for (P) implies that u^* is optimal for (P) and, (A.12) holds with equality. Combining this with (A.10) and the fact that u^* maximizes $L(u;\widetilde{y})$ we prove that (A.11) holds with equality.

Next, consider any optimal solution u^{**} to (P) and its corresponding state trajectory x^{**}. We have

$$\sum_{n=2}^{N} \mathbb{E}[\widetilde{y}(\omega,t_n)\cdot(x^{**}(\omega,t_{n-1}) - x^{**}(\omega,t_{n-2}) + Au^{**}(\omega,t_{n-1}))] + \mathbb{E}[\widetilde{y}(\omega,t_1)\cdot C] = 0.$$

Hence, $L(u^{**};\widetilde{y}) = \sum_{n=1}^{N} \mathbb{E}[f(\omega,t_n)\cdot\widetilde{u}(\omega,t_n)] = L(\widetilde{u};\widetilde{y})$, and u^{**} maximizes $L(u;\widetilde{y})$ among all feasible solutions to (P). ■

Proof of Theorem 1. We interpret the Lagrange multipliers in Proposition 51 as the opportunity cost of resources and construct optimal adapted bid-price and permissible capacity processes for the network revenue management problem (P). To this end, let $\widetilde{y}$ denote Lagrange multipliers as in Proposition 51. Let $\widetilde{u}$ be an optimal solution to (P) such that its corresponding state trajectory $\widetilde{x}$ satisfies (A.8). The existence of such an optimal solution is established in Proposition 51. An optimal adapted bid-price control will be constructed using $\widetilde{y}$ and $\widetilde{u}$. First, consider the Lagrangian as in Proposition 51 (we suppress the dependence of the stochastic process ω for brevity):

$$\begin{aligned}
L(u;\widetilde{y}) &= \sum_{n=1}^{N} \mathbb{E}[f(t_n)\cdot u(t_n)] - \sum_{n=2}^{N} \mathbb{E}[\widetilde{y}(t_n)\cdot(x(t_{n-1}) - x(t_{n-2}) + Au(t_{n-1}))] \\
&\quad -\mathbb{E}[C\cdot \widetilde{y}(t_1)] \\
&= \sum_{n=1}^{N} \mathbb{E}[f(t_n)\cdot u(t_n)] - \sum_{n=2}^{N} \mathbb{E}[(\widetilde{y}(t_n) - \widetilde{y}(t_{n+1}))\cdot x(t_{n-1})] \\
&\quad -\sum_{n=1}^{N} \mathbb{E}[\widetilde{y}(\omega, t_{n+1})Au(t_n)] - \mathbb{E}[C\cdot \widetilde{y}(t_1)] \\
&= \sum_{n=1}^{N} \mathbb{E}[(f(t_n) - \mathbb{E}[\widetilde{y}(t_{n+1})|\, \mathcal{F}_{t_n}]A)\cdot u(t_n)] \\
&\quad -\sum_{n=2}^{N} \mathbb{E}[(\widetilde{y}(t_n) - \mathbb{E}[\widetilde{y}(t_{n+1})|\, \mathcal{F}_{t_n}])\cdot x(t_{n-1})] - C\mathbb{E}\widetilde{y}(t_1),
\end{aligned}$$

where second equality is a rearrangement of terms and the last equality follows from the definition of conditional expectation. By Proposition 51, if a feasible solution maximizes $L(u;\widetilde{y})$ among all feasible solutions to (P), then it is an optimal solution to (P). The problem of maximizing $L(u;\widetilde{y})$ among the feasible solutions to (P) decomposes by each decision time t_n and all $\omega \in \Omega$. Thus, a feasible solution $\overline{u}$ is optimal for (P) if and only if, for each t_n and almost every $\omega \in \Omega$, $\overline{u}(\omega, t_n)$ solves the following problem:

$$\max_u \left(f(\omega, t_n) - \mathbb{E}[\widetilde{y}(\omega, t_{n+1})|\ \mathcal{F}_{t_n}]A\right) \cdot u$$

subject to

$$Au \leq \overline{x}(\omega, t_{n-1}),$$

$$0 \leq u \leq D(\omega, t_n) - D(\omega, t_{n-1}),$$

where $\overline{x}$ is the state trajectory corresponding to $\overline{u}$.

Given the Lagrange multipliers $\widetilde{y}$ and an optimal solution $\widetilde{u}$ to the network revenue management problem (P) and its corresponding state trajectory $\widetilde{x}$ as in Proposition 51, we define the adapted bid-price and capacity usage limit processes as follows: $\widetilde{\pi}(\omega, t_n) = \mathbb{E}[\widetilde{y}(\omega, t_{n+1})|\ \mathcal{F}_{t_n}]$ for $\omega \in \Omega$ and $n = 1, \ldots, N$,

$$\widetilde{\pi}(\omega, t_n) = \mathbb{E}[\widetilde{y}(\omega, t_{n+1})|\ \mathcal{F}_{t_n}] \text{ for } \omega \in \Omega \text{ and } n = 1, \ldots, N, \tag{A.13}$$

$$\widetilde{\lambda}(\omega, t_n) = \widetilde{x}(\omega, t_{n-1}) - \widetilde{x}(\omega, t_n) \text{ for } \omega \in \Omega \text{ and } n = 1, \ldots, N. \tag{A.14}$$

First note that the proposed bid price process $\widetilde{\pi}(\omega, t_n)$ is measurable with respect to $\mathcal{F}_{t_n}$ since it is defined as a conditional expectation with respect to $\mathcal{F}_{t_n}$. The proposed capacity usage limit vector $\widetilde{\lambda}(\omega, t_n)$ is also measurable with respect to $\mathcal{F}_{t_n}$ since at decision time t_n along the sample path ω, the decision maker knows the optimal bookings $\widetilde{u}(\omega, t_n)$.

Next we show that the proposed adapted bid-price control $(\widetilde{\pi}, \widetilde{\lambda})$ defined as in (A.13) and (A.14) is an optimal adapted bid-price control. That is, the booking controls resulting from $(\widetilde{\pi}, \widetilde{\lambda})$ constitute an optimal solution to the network revenue management problem

(P). Recall that the proposed bid-price control $(\widetilde{\pi}, \widetilde{\lambda})$ is executed as follows: At each decision time t_n for $n = 1, \ldots, N$ the system manager first observes the demand realization $D(\omega, t_n) - D(\omega, t_{n-1})$ for period n along the sample path ω. Then, she solves the linear program $(P(\omega, t_n))$ to determine the booking levels:

$$
\begin{aligned}
& \text{Max}_u (f(\omega, t_n) - A^{'}\widetilde{\pi}(\omega, t_n)) \cdot u + \eta(Au - \widetilde{\lambda}(\omega, t_n)) \cdot \mathbf{e} \\
(\text{P}(\omega, t_n)) \qquad & \text{subject to} \\
& Au \leq \widetilde{\lambda}(\omega, t_n), \\
& 0 \leq u \leq D(\omega, t_n) - D(\omega, t_{n-1}),
\end{aligned}
$$

where $\varepsilon > 0$ is arbitrarily small and $\mathbf{e}$ is the K-dimensional vector of ones. We define $(u^{(\widetilde{\pi},\widetilde{\lambda})}(\omega, t))_{(\omega,t)\in\Omega\times\Gamma}$ as the controls associated with the generalized bid-price control $(\widetilde{\pi}, \widetilde{\lambda})$, i.e. $u^{(\widetilde{\pi},\widetilde{\lambda})}(\omega, t_n)$ is an arbitrary optimal solution to $(\text{P}(\omega, t_n))$ for $\omega \in \Omega$ and $n = 1, \ldots, N$. Let $(x^{(\widetilde{\pi},\widetilde{\lambda})}(\omega, t))_{(\omega,t)\in\Omega\times\Gamma}$ denote the state trajectory associated with $(u^{(\widetilde{\pi},\widetilde{\lambda})}(\omega, t))_{(\omega,t)\in\Omega\times\Gamma}$. We will show that $(u^{(\widetilde{\pi},\widetilde{\lambda})}(\omega, t))_{(\omega,t)\in\Omega\times\Gamma}$ forms an optimal solution to the network revenue management problem (P) of Section 1.1.

To establish the optimality of $u^{(\widetilde{\pi},\widetilde{\lambda})}$, we will show that $u^{(\widetilde{\pi},\widetilde{\lambda})}$ maximizes $L(u, \widetilde{y})$. From Proposition 51, the problem of maximizing $L(u, \widetilde{y})$ among feasible solutions to (P) decomposes by each $\omega \in \Omega$ and decision time t_n for $n = 1, \ldots, N$. Then, we need to show that $u^{(\widetilde{\pi},\widetilde{\lambda})}(\omega, t_n)$ solves

$$\begin{aligned} &\max_u \left(f(\omega,t_n) - \mathbb{E}[\widetilde{y}(\omega,t_{n+1})|\ \mathcal{F}_{t_n}]A\right)\cdot u \\ &\text{subject to} \qquad\qquad (\mathrm{P}^{(\widetilde{\pi},\widetilde{\lambda})}(\omega,t_n)) \\ &Au \le x^{(\widetilde{\pi},\widetilde{\lambda})}(\omega,t_{n-1}), \\ &0 \le u \le D(\omega,t_n) - D(\omega,t_{n-1}) \end{aligned}$$

at each $\omega \in \Omega$ and decision time t_n for $n = 1, \ldots, N$. Since $\widetilde{u}$ is an optimal solution to the network revenue management problem (P), it maximizes $L(u, \widetilde{y})$ among all feasible solutions to (P). Thus, $\widetilde{u}(\omega, t_n)$ solves the following problem at each $\omega \in \Omega$ and decision time t_n for $n = 1, \ldots, N$

$$\begin{aligned} &\max_u \left(f(\omega,t_n) - \mathbb{E}[\widetilde{y}(\omega,t_{n+1})|\ \mathcal{F}_{t_n}]A\right)\cdot u \\ &\text{subject to} \qquad\qquad (\widetilde{\mathrm{P}}(\omega,t_n)) \\ &Au \le \widetilde{x}(\omega,t_{n-1}), \\ &0 \le u \le D(\omega,t_n) - D(\omega,t_{n-1}). \end{aligned}$$

To show that $u^{(\widetilde{\pi},\widetilde{\lambda})}(\omega, t_n)$ solves $(\mathrm{P}^{(\widetilde{\pi},\widetilde{\lambda})}(\omega, t_n))$, we first prove that

$$x^{(\widetilde{\pi},\widetilde{\lambda})}(\omega,t_{n-1}) = \widetilde{x}(\omega,t_{n-1}) \text{ for all } \omega \in \Omega \text{ and for } n = 1,\ldots,N.$$

For all $\omega \in \Omega$ and for $n = 1, \ldots, N$ we know that $\widetilde{u}(\omega, t_n)$ solves $(\widetilde{P}(\omega, t_n))$ by the decomposition result, and that $A\widetilde{u}(\omega, t_n) = \widetilde{\lambda}(\omega, t_n)$ by (A.14). Therefore, $\widetilde{u}(\omega, t_n)$ is an optimal solution to $(\mathrm{P}(\omega, t_n))$ as well, which in particular exhausts the permissible capacity. Then, since the linear program $(\mathrm{P}(\omega, t_n))$ is lexicographic, all optimal solutions to $(\mathrm{P}(\omega, t_n))$ exhaust the permissible capacity, that is, the capacity constraint necessarily binds. Therefore, since $u^{(\widetilde{\pi},\widetilde{\lambda})}(\omega, t_n)$ is also an optimal solution to $(\mathrm{P}(\omega, t_n))$, it must be that

$$Au^{(\widetilde{\pi},\widetilde{\lambda})}(\omega, t_n) = \widetilde{\lambda}(\omega, t_n) = \widetilde{x}(\omega, t_{n-1}) - \widetilde{x}(\omega, t_n),$$

by (A.14). This, in turn, ensures that

$$\begin{aligned} x^{(\widetilde{\pi},\widetilde{\lambda})}(\omega, t_{n-1}) &= C - \sum_{m=1}^{n-1} Au^{(\widetilde{\pi},\widetilde{\lambda})}(\omega, t_m), \\ &= C - \sum_{m=1}^{n-1} (\widetilde{x}(\omega, t_{m-1}) - \widetilde{x}(\omega, t_m)) = \widetilde{x}(\omega, t_{n-1}), \end{aligned}$$

where $x^{(\widetilde{\pi},\widetilde{\lambda})}(\omega, t_{n-1})$ is the capacity vector at the end of period $n-1$ under the control $u^{(\widetilde{\pi},\widetilde{\lambda})}$. Essentially, we are following the "optimal trajectory", that is, the trajectory of $\widetilde{u}$. An immediate implication of this result is that the problems $(\mathrm{P}^{(\widetilde{\pi},\widetilde{\lambda})}(\omega, t_n))$ and $(\widetilde{\mathrm{P}}(\omega, t_n))$ are equivalent for all $\omega \in \Omega$ and for $n = 1, \ldots, N$.

What remains to be shown is that $u^{(\widetilde{\pi},\widetilde{\lambda})}(\omega, t_n)$ solves $(\mathrm{P}^{(\widetilde{\pi},\widetilde{\lambda})}(\omega, t_n))$ at each $\omega \in \Omega$ and for $n = 1, \ldots, N$. To see this, recall that $\widetilde{u}(\omega, t_n)$ is an optimal solution to $(\mathrm{P}(\omega, t_n))$ for all $\omega \in \Omega$ and for $n = 1, \ldots, N$. Therefore, the objective function values of $\widetilde{u}(\omega, t_n)$ and $u^{(\widetilde{\pi},\widetilde{\lambda})}(\omega, t_n)$ for $(\mathrm{P}(\omega, t_n))$ are equal. That is,

$$\begin{aligned} & (f(\omega, t_n) - A'\mathbb{E}[\widetilde{y}(\omega, t_{n+1}) |\, \mathcal{F}_{t_n}]) \cdot \widetilde{u}(\omega, t_n) \\ = \; & (f(\omega, t_n) - A'\mathbb{E}[\widetilde{y}(\omega, t_{n+1}) |\, \mathcal{F}_{t_n}]) \cdot u^{(\widetilde{\pi},\widetilde{\lambda})}(\omega, t_n), \end{aligned} \tag{A.15}$$

since $\widetilde{\pi}(\omega, t_n) = \mathbb{E}[\widetilde{y}(\omega, t_{n+1})|\, \mathcal{F}_{t_n}]$ by (A.13) and

$$A\widetilde{u}(\omega, t_n) - \widetilde{\lambda}(\omega, t_n) = \ Au^{(\widetilde{\pi},\widetilde{\lambda})}(\omega, t_n) - \widetilde{\lambda}(\omega, t_n) = 0.$$

Then, since $\widetilde{u}(\omega, t_n)$ solves $(\widetilde{\mathrm{P}}(\omega, t_n))$ so does $u^{(\widetilde{\pi},\widetilde{\lambda})}(\omega, t_n)$ by (A.15). Moreover, as

$$(\mathrm{P}^{(\widetilde{\pi},\widetilde{\lambda})}(\omega, t_n))$$

and $(\widetilde{\mathrm{P}}(\omega, t_n))$ are equivalent for all $\omega \in \Omega$ and for $n = 1, \dots, N$ this implies that $u^{(\widetilde{\pi},\widetilde{\lambda})}(\omega, t_n)$ solves $(\mathrm{P}^{(\widetilde{\pi},\widetilde{\lambda})}(\omega, t_n))$ for $n = 1, \dots, N$ and $\omega \in \Omega$. This in turn implies that $u^{(\widetilde{\pi},\widetilde{\lambda})}$ maximizes $L(u; \widetilde{y})$ among all feasible solutions to (P) and hence by Proposition 51, $u^{(\widetilde{\pi},\widetilde{\lambda})}$ is an optimal solution for the network revenue management problem (P), proving the optimality of the proposed adapted bid-price control $(\widetilde{\pi}, \widetilde{\lambda})$. ■

Proof of Theorem 2. Proposition 51 shows the existence of an optimal solution $\widetilde{u}$ to the network revenue management problem (P) such that the associated state trajectory $\widetilde{x}$ satisfies (A.8). Fix such an optimal solution. Then, using the Lagrange multiplier $\widetilde{y}$ as in Proposition 51, construct the adapted bid-price process π and the permissible capacity processes λ as follows: For $\omega \in \Omega$ and $n = 1, \dots, N$,

$$\pi(\omega, t_n) = \mathbb{E}[\widetilde{y}(\omega, t_{n+1})|\, \mathcal{F}_{t_n}] \ \text{ and } \ \lambda(\omega, t_n) = \widetilde{x}(\omega, t_{n-1}) - \widetilde{x}(\omega, t_n). \tag{A.16}$$

The bid-price control (π, λ) constructed as such is shown to form an optimal adapted bid-price control for the network revenue management problem (P) in Theorem 1. Then, the bid-price process π_k for $k = 1, \dots, K$ forms a martingale until the last period at the end of which the capacity of resource k is exhausted. Formally, define the stopping time

σ_k for resource k as

$$\sigma_k(\omega) = \inf\{n : \widetilde{x}_k(\omega, t_n) = 0\},$$

which is the first period at the end of which the capacity resource k is exhausted, where the infimum of the empty set is ∞ by convention. For the Lagrange multiplier $\widetilde{y}$ as in Proposition 51, we have that

$$(\widetilde{y}(\omega, t_n) - \mathbb{E}[\widetilde{y}(\omega, t_{n+1})|\ \mathcal{F}_{t_n}]) \cdot \widetilde{x}(\omega, t_{n-1}) = 0, \quad n = 1, \ldots, N-1 \text{ and } \omega \in \Omega,$$

cf. (A.8). This implies that the stopped process $\{\widetilde{y}_k(\omega, t_n \wedge \sigma_k(\omega)) : n = 1, \ldots, N\}$ is a martingale. Let $\widetilde{y}(\omega, t_n \wedge \sigma(\omega))$ denote the stochastic process whose k^{th} component is the stopped process $\widetilde{y}_k(\omega, t_n \wedge \sigma_k(\omega))$. Then, the adapted bid-price process constructed from $\widetilde{y}(\omega, t_n \wedge \sigma(\omega))$ will form a martingale.

We next prove that the bookings resulting from the execution of the bid-price control, whose adapted bid-price process is constructed using $\widetilde{y}(\omega, t_n \wedge \sigma(\omega))$, is optimal for the network revenue management problem. Since the bid-price control constructed using $\widetilde{y}$ as in (A.16) is optimal, it suffices to show that for each decision time t_n where $n = 1, \ldots, N$ and $\omega \in \Omega$, if $u^* \in \mathbb{R}_+^J$ solves the maximization problem

$$\begin{aligned}
&\max_u \,(f(\omega, t_n) - \mathbb{E}[\widetilde{y}(\omega, t_{n+1} \wedge \sigma(\omega))|\ \mathcal{F}_{t_n}]A) \cdot u \\
&\text{subject to} \\
&Au \leq \lambda(\omega, t_n), \qquad\qquad (\mathrm{P}_{\text{stopped}}(\omega, t_n)) \\
&0 \leq u \leq D(\omega, t_n) - D(\omega, t_{n-1}),
\end{aligned}$$

then, it is also optimal for the problem

$$\begin{aligned}
&\max_u \left(f(\omega, t_n) - \mathbb{E}[\widetilde{y}(\omega, t_{n+1})|\ \mathcal{F}_{t_n}]A\right) \cdot u \\
&\text{subject to} \\
&Au \leq \lambda(\omega, t_n), \qquad\qquad (\mathrm{P}_{\text{unstopped}}(\omega, t_n)) \\
&0 \leq u \leq D(\omega, t_n) - D(\omega, t_{n-1}).
\end{aligned}$$

Note first that if $t_n < \sigma(\omega)$, then $t_{n+1} \leq \sigma(\omega)$ and $y(\omega, t_{n+1} \wedge \sigma(\omega)) = y(\omega, t_{n+1})$ which makes the statement true. Now consider the case when $t_n \geq \sigma(\omega)$. Then, the capacity of at least one resource is zero at decision time t_n. Consider a generic resource k whose capacity has already exhausted at decision time t_n, i.e. $\widetilde{x}_k(\omega, t_{n-1}) = 0$. We have for resource k

$$\lambda_k(\omega, t_n) = \widetilde{x}_k(\omega, t_{n-1}) - \widetilde{x}_k(\omega, t_n) = 0,$$

and for all products that use resource k we should have zero bookings. That is, if $A_{kj} > 0$, we have $u_j = 0$. This implies,

$$\begin{aligned}
&(\mathbb{E}[\widetilde{y}(\omega, t_{n+1} \wedge \sigma(\omega))|\ \mathcal{F}_{t_n}]A) \cdot u \\
= &\sum_{j=1}^{J} \sum_{k=1}^{K} \mathbb{E}[\widetilde{y}_k(\omega, t_{n+1} \wedge \sigma(\omega))|\ \mathcal{F}_{t_n}] A_{kj} u_j, \\
= &\sum_{j=1}^{J} \sum_{k=1}^{K} \mathbb{E}[\widetilde{y}_k(\omega, t_{n+1})|\ \mathcal{F}_{t_n}] A_{kj} u_j, \\
= &(\mathbb{E}[\widetilde{y}(\omega, t_{n+1})|\ \mathcal{F}_{t_n}]A) \cdot u.
\end{aligned}$$

Now, let u^* solve $\mathrm{P}_{\mathrm{stopped}}(\omega, t_n)$ and u^{**} solve $\mathrm{P}_{\mathrm{unstopped}}(\omega, t_n)$. Clearly, u^* is feasible for $\mathrm{P}_{\mathrm{unstopped}}(\omega, t_n)$. Moreover,

$$\begin{aligned}(f(\omega,t_n)-\mathbb{E}[\widetilde{y}(\omega,t_{n+1}|\ \mathcal{F}_{t_n}]A)\cdot u^{**} &= (f(\omega,t_n)-\mathbb{E}[\widetilde{y}(\omega,t_{n+1}\wedge\sigma(\omega))|\ \mathcal{F}_{t_n}]A)\cdot u^{**},\\ &\leq (f(\omega,t_n)-\mathbb{E}[\widetilde{y}(\omega,t_{n+1}\wedge\sigma(\omega))|\ \mathcal{F}_{t_n}]A)\cdot u^{*},\\ &= (f(\omega,t_n)-\mathbb{E}[\widetilde{y}(\omega,t_{n+1}|\ \mathcal{F}_{t_n}]A)\cdot u^{*},\end{aligned}$$

and u^* is optimal for $\mathrm{P}_{\mathrm{unstopped}}(\omega, t_n)$ as well. ■

A.2. Proof of the Results Regarding Predictable Bid-Price Controls

In this section we prove Theorem 5 which establishes the near optimality of the predictable bid-price control constructed as in (1.2)-(1.3). Recall that an optimal adapted bid-price control (π, λ) is constructed in the proofs of Theorems 1 and 2 so that the adapted bid-price process $\{\pi_k(\omega, t_n) : n = 1, \ldots, N\}$ for resource $k = 1, \ldots, K$ is a martingale adapted to $(\{\mathcal{F}_{t_n} : n = 1, \ldots, N\}, \mathbb{P})$. For the rest of the argument we fix such an optimal adapted bid-price control (π, λ).

As a preliminary to the proof of Theorem 5, we show that the predictable bid-price and permissible capacity processes converge to their adapted counterparts as the time gap disappears.

Lemma 52. $\lim_{h\searrow 0} \widetilde{\pi}_h(\omega, t_n) = \pi(\omega, t_n)$ *and* $\lim_{h\searrow 0} \widetilde{\lambda}_h(\omega, t_n) = \lambda(\omega, t_n)$ *for a.e.* $\omega \in \Omega$ *and* $n = 1, \ldots, N$.

Proof of Lemma 52. Proposition 6.1 of [**36**] states that if Z is a martingale adapted to the semi-continuous filtration $\{\mathcal{F}_t,\ 0 \leq t \leq T\}$, then $Z(\omega, t-) = \mathbb{E}[Z(\omega, t) \mid \mathcal{F}_{t-}]$ for a.e. ω,where $\mathcal{F}_{t-} = \bigvee_{s<t} \mathcal{F}_s$. That is,

$$\lim_{s \nearrow t} Z(\omega, s) = \mathbb{E}[Z(\omega, t) | \mathcal{F}_{t-}] \text{ for a.e. } \omega \in \Omega.$$

Notice that $\{\mathbb{E}[\lambda(\omega, t_n) \mid \mathcal{F}_{s+t_{n-1}}] : s \in (0, t_n - t_{n-1}]\}$ is a martingale adapted to

$$\{\mathcal{F}_t, t_{n-1} \leq t \leq t_n\}.$$

Then, for $\omega \in \Omega$ and $n = 1, \ldots, N$

$$\begin{aligned} \lim_{h \searrow 0} \mathbb{E}[\lambda(\omega, t_n) \mid \mathcal{F}_{t_n - h}] &= \mathbb{E}[\, \mathbb{E}[\lambda(\omega, t_n) \mid \mathcal{F}_{t_n}] \mid \mathcal{F}_{t_n -}], \\ &= \mathbb{E}[\lambda(\omega, t_n) \mid \mathcal{F}_{t_n}] = \lambda(\omega, t_n), \end{aligned}$$

from the semi-continuity of the information structure. Similarly, $\lim_{h \searrow 0} \mathbb{E}[\pi(\omega, t_n) \mid \mathcal{F}_{t_n - h}] = \pi(\omega, t_n)$ and hence $\lim_{h \searrow 0} \widetilde{\pi}_h(\omega, t_n) = \pi(\omega, t_n)$ for $\omega \in \Omega$ and $n = 1, \ldots, N$.

We next argue that $\lim_{h \searrow 0} \widetilde{\lambda}_h(\omega, t_n) = \lambda(\omega, t_n)$ for $n = 1, \ldots, N$ and $\omega \in \Omega$, by induction. First recall that given the Lagrange multipliers $\widetilde{y}$ and an optimal solution $\widetilde{u}$ to the network revenue management problem (P) and its corresponding state trajectory $\widetilde{x}$ as in Proposition 51, the adapted permissible capacity process λ is constructed as follows:

$$\lambda(\omega, t_n) = \widetilde{x}(\omega, t_{n-1}) - \widetilde{x}(\omega, t_n) \text{ for } \omega \in \Omega \text{ and } n = 1, \ldots, N.$$

As the induction basis consider the first decision time t_1. We have

$$\begin{aligned}\widetilde{\lambda}_h(\omega,t_1) &= \min\{\mathbb{E}[\lambda(\omega,t_1) \mid \mathcal{F}_{t_1-h}], C\},\\ &= \min\{\mathbb{E}[C-\widetilde{x}(\omega,t_1) \mid \mathcal{F}_{t_1-h}], C\},\\ &= \mathbb{E}[C-\widetilde{x}(\omega,t_1) \mid \mathcal{F}_{t_1-h}],\\ &= \mathbb{E}[\lambda(\omega,t_1) \mid \mathcal{F}_{t_1-h}],\end{aligned}$$

and the convergence of $\widetilde{\lambda}_h(\omega,t_1)$ to $\lambda(\omega,t_1)$ follows from the argument above regarding the martingales adapted to semi-continuous information structures. As the induction hypothesis assume that $\lim_{h\searrow 0}\widetilde{\lambda}_h(\omega,t_m) = \lambda(\omega,t_m)$ for $m = 1,\ldots,n-1$ where $n \geq 2$. Then

$$\begin{aligned}\lim_{h\searrow 0}\widetilde{\lambda}_h(\omega,t_n) &= \lim_{h\searrow 0}\min\{\mathbb{E}[\lambda(\omega,t_n) \mid \mathcal{F}_{t_n-h}], C-\sum_{m=1}^{n-1}\widetilde{\lambda}_h(\omega,t_m)\},\\ &= \min\{\lambda(\omega,t_n)\ , C-\sum_{m=1}^{n-1}\lambda(\omega,t_m)\},\\ &= \lambda(\omega,t_n).\end{aligned}$$

The second equality follows from the induction hypothesis, whereas the last equality is true since $\lambda(\omega,t_n) \leq \widetilde{x}(\omega,t_{n-1})$ for $n = 1,\ldots,N$. ■

Proof of Proposition 4. Recall that the optimal adapted bid-price process π is a martingale. Then, for $n = 1,\ldots,N$

$$\widetilde{\pi}_h(\omega,t_n) = \mathbb{E}[\pi(\omega,t_n) \mid \mathcal{F}_{t_n-h}] = \mathbb{E}[\pi(\omega,T) \mid \mathcal{F}_{t_n-h}],$$

and $\{\widetilde{\pi}_h(\omega, t_n) : n = 1, \ldots, N\}$ is a martingale by construction. The second equality is obtained via replacing $\pi(\omega, t_n)$ by its value at the terminal time T, and this this replacement is valid due to the fact that optimal adapted bid-price process π is a martingale and it is described completely by its terminal value and the associated filtration. ■

Proof of Theorem 5. The execution of the proposed adapted and predictable bid-price policies involve the maximization of the linear programs $(\mathrm{P}(\omega, t_n))$ and $(\mathrm{P}_h(\omega, t_n))$ for $n = 1, \ldots, N$ and $\omega \in \Omega$. We prove that as $h \searrow 0$, the expected revenue generated by the bookings resulting from $(\widetilde{\pi}_h, \widetilde{\lambda}_h)$ for $h > 0$ converges to the expected revenue generated by the optimal adapted bid-price control (π, λ), which, in turn, is equal to the objective function value of the network revenue management problem (P). To that end, we prove that the objective function values of the linear program $(\mathrm{P}_h(\omega, t_n))$ converges to that of $(\mathrm{P}(\omega, t_n))$ as $h \searrow 0$. Recall that from Lemma 52, $\lim_{h \searrow 0} \widetilde{\pi}_h(\omega, t_n) = \pi(\omega, t_n)$ and $\lim_{h \searrow 0} \widetilde{\lambda}_h(\omega, t_n) = \lambda(\omega, t_n)$ for $n = 1, \ldots, N$. This implies that the objective function value of the linear program $(\mathrm{P}_h(\omega, t_n))$ converges to that of $(\mathrm{P}(\omega, t_n))$ as $h \searrow 0$. Next we prove that for every $\omega \in \Omega$ and $n = 1, \ldots, N$, the revenue generated by an optimal solution to $(\mathrm{P}_h(\omega, t_n))$ converges as $h \searrow 0$ to the revenue generated by an optimal solution to $(\mathrm{P}(\omega, t_n))$. To that end, let $u^h(\omega, t_n)$ be an optimal solution to $(\mathrm{P}_h(\omega, t_n))$ for $n = 1, \ldots, N$ and $\omega \in \Omega$. Similarly, let $u(\omega, t_n)$ be an optimal solution to $(\mathrm{P}(\omega, t_n))$ for $n = 1, \ldots, N$ and $\omega \in \Omega$. We already know that

$$\begin{aligned} & \lim_{h \searrow 0} (f(\omega, t_n) - \widetilde{\pi}_h(\omega, t_n)A) \cdot u^h(\omega, t_n) + \eta(Au^h(\omega, t_n) - \widetilde{\lambda}_h(\omega, t_n)) \cdot e \\ \text{(A.17)} \quad = \; & (f(\omega, t_n) - \pi(\omega, t_n)A) \cdot u + \eta(Au(\omega, t_n) - \lambda(\omega, t_n)) \cdot e. \end{aligned}$$

By definition of $\lambda(\omega, t_n)$, cf. (A.14), we have $Au(\omega, t_n) - \lambda(\omega, t_n)$. Notice that

$$\lim_{h \searrow 0}(Au^h(\omega, t_n) - \widetilde{\lambda}_h(\omega, t_n)) \cdot e = 0.$$

We prove this by contradiction. Suppose

$$\lim_{h \searrow 0}(Au^h(\omega, t_n) - \widetilde{\lambda}_h(\omega, t_n)) \cdot e < 0.$$

(Recall that $Au^h(\omega, t_n) \leq \widetilde{\lambda}_h(\omega, t_n)$ by feasibility of $u^h(\omega, t_n)$ for $(\mathrm{P}_h(\omega, t_n))$.) Then, appropriately trimming $u(\omega, t_n)$ so as to satisfy the capacity usage limit imposed by $\widetilde{\lambda}_h(\omega, t_n)$ and passing to the limit as $h \searrow 0$, we get a contradiction to the fact that the objective function value of the linear program $(\mathrm{P}_h(\omega, t_n))$ converges to that of $(\mathrm{P}(\omega, t_n))$ as $h \searrow 0$. Hence,

$$\lim_{h \searrow 0}(Au^h(\omega, t_n) - \widetilde{\lambda}_h(\omega, t_n)) \cdot e = 0$$

and we can write

$$\begin{aligned}
& \lim_{h \searrow 0} (f(\omega, t_n) - \widetilde{\pi}_h(\omega, t_n)A) \cdot u^h(\omega, t_n) + \eta(Au^h(\omega, t_n) - \widetilde{\lambda}_h(\omega, t_n)) \cdot e \\
= & \lim_{h \searrow 0} (f(\omega, t_n) - \widetilde{\pi}_h(\omega, t_n)A) \cdot u^h(\omega, t_n), \\
= & \lim_{h \searrow 0} f(\omega, t_n) \cdot u^h(\omega, t_n) - \pi(\omega, t_n) \cdot \lambda(\omega, t_n),
\end{aligned}$$

which implies together with (A.17) that $\lim_{h \searrow 0} f(\omega, t_n) \cdot u^h(\omega, t_n) = f(\omega, t_n) \cdot u(\omega, t_n)$. Hence, the function $\sum_{n=1}^{N} f(\omega, t_n) \cdot u^h(\omega, t_n)$ converges almost surely. From Dominated Convergence Theorem, the expected revenue generated by the controls resulting from the predictable bid-price control converge to the expected revenue generated by an optimal solution to (P), proving Theorem 5. ■

The execution of the proposed adapted and predictable bid-price policies involve the maximization of the linear programs $(\mathrm{P}(\omega, t_n))$ and $(\mathrm{P}_h(\omega, t_n))$ for $n = 1, \ldots, N$ and $\omega \in \Omega$. (For clarity, I will emphasize the dependence of the programs $(\mathrm{P}(\omega, t_n))$ and $(\mathrm{P}_h(\omega, t_n))$ on ω for the remainder of the proof.) To facilitate our analysis of the convergence of the problems $(\mathrm{P}_h(\omega, t_n))$ to $(\mathrm{P}(\omega, t_n))$ as $h \searrow 0$, we eliminate the hard constraints in linear problems $(\mathrm{P}(\omega, t_n))$ and $(\mathrm{P}_h(\omega, t_n))$ by incorporating penalty expressions in the objective function. The formulation with penalty expressions is more flexible because it allows us to study the convergence of maximization problems in terms of convergence of extended real valued functions. To that end, define the indicator function $\chi_F(\cdot)$

$$\chi_F(x) = \begin{cases} 0 & \text{if } x \in F, \\ \infty & \text{otherwise.} \end{cases} \tag{A.18}$$

Let h_m be any sequence such that $h_m \searrow 0$ as $m \to \infty$. Then, for $\omega \in \Omega$ and $n = 1, \ldots, N$ consider the minimization of the extended real valued function $g^m(\cdot, \omega, t_n)$ on $\mathbb{R}^J$ where $g^m(u, \omega, t_n)$ is defined as

$$\begin{aligned} g^m(u, \omega, t_n) \quad &= \quad (\widetilde{\pi}_{h_m}(\omega, t_n)A - f(\omega, t_n)) \cdot u + \chi_{\{v:\, Av \leq \widetilde{\lambda}_{h_m}(\omega, t_n)\}}(u) \\ &\quad + \chi_{\{v:\, 0 \leq v \leq D(\omega, t_n) - D(\omega, t_{n-1})\}}(u). \end{aligned}$$

The problem of minimizing $g^m(\cdot,\omega,t_n)$ is equivalent to maximizing $\mathrm{P}_{h_m}(\omega,t_n)$ for $\omega\in\Omega$ and $n=1,\ldots,N$. Also define the function $g(\cdot,\omega,t_n)$ to represent $(\mathrm{P}(\omega,t_n))$ as follows:

$$\begin{aligned} g(u,\omega,t_n) &= (\pi(\omega,t_n)A - f(\omega,t_n))\cdot u + \chi_{\{v:Av\leq\lambda(\omega,t_n)\}}(u) \\ &\quad + \chi_{\{v:0\leq v\leq D(\omega,t_n)-D(\omega,t_{n-1})\}}(u). \end{aligned}$$

For the definitions of the terms used in the subsequent statements and proofs, see [**55**].

Lemma 53. *The sequence $\{g^m(\cdot,\omega,t_n)\}_{m\in\mathbb{N}_+}$ is eventually level-bounded for $\omega\in\Omega$ and $n=1,\ldots,N$. At the same time $g(\cdot,\omega,t_n)$ and $g^m(\cdot,\omega,t_n)$ are lower semi-continuous and proper. Moreover, $g^m(\cdot,\omega,t_n)$ converges epigraphically to $g(\cdot,\omega,t_n)$, i.e.*

$$g^m(\cdot,\omega,t_n) \to_e g(\cdot,\omega,t_n)$$

for $\omega\in\Omega$ and $n=1,\ldots,N$.

Proof of Lemma 53. For $\omega\in\Omega$ and $n=1,\ldots,N$, $g^m(\cdot,\omega,t_n)$ and $g(\cdot,\omega,t_n)$ are lower semi-continuous since their epigraphs are closed in $\mathbb{R}^J\times\mathbb{R}$. $g^m(\cdot,\omega,t_n)$ and $g(\cdot,\omega,t_n)$ are proper since $u=0$ is feasible for both. To show the epigraphical convergence of $g^m(\cdot,\omega,t_n)$ to $g(\cdot,\omega,t_n)$, we use Proposition 7.2 of [**55**] which proves that $f^m\to_e f$ if and only if at each point x one has

$$\begin{aligned} \liminf_{m\to\infty} f^m(x^m) &\geq f(x) \quad \text{for every sequence } \; x^m\to x, \\ \limsup_{m\to\infty} f^m(x^m) &\leq f(x) \quad \text{for some sequence } \; x^m\to x. \end{aligned}$$

First consider a point $u \in \mathbb{R}^J$ such that $g(u, \omega, t_n) < \infty$, i.e. u is a feasible point for the problem $(\mathrm{P}(\omega, t_n))$. Consider an arbitrary sequence such that $u^m \to u$ as $m \to \infty$. Then, $\liminf_{m\to\infty} g^m(u^m, \omega, t_n)$ is attained along a subsequence whose elements are feasible for the corresponding problems, i.e. $g^m(u^m, \omega, t_n) < \infty$. Without loss of generality assume that u^m are feasible for $\mathrm{P}^m(\omega, t_n)$. Then, for all m along the subsequence, we have

$$Au^m \le \widetilde{\lambda}_{h_m}(\omega, t_n),\ 0 \le u^m \le D(\omega, t_n) - D(\omega, t_{n-1})$$

and

$$g^m(u^m, \omega, t_n) = (\widetilde{\pi}_{h_m}(\omega, t_n)A - f(\omega, t_n)) \cdot u^m.$$

Moreover, we also have

$$\lim_{m\to\infty} g^m(u^m, \omega, t_n) = (\pi(\omega, t_n)A - f(\omega, t_n)) \cdot u = g(u, \omega, t_n).$$

Hence, along the same subsequence we have $\limsup_{m\to\infty} g^m(u^m, \omega, t_n) \le g(u, \omega, t_n)$ as well as $\liminf_{m\to\infty} g^m(u^m, \omega, t_n) \ge g(u, \omega, t_n)$.

Now consider some u such that $g(u, \omega, t_n) = \infty$, i.e. u is infeasible. Let $u^m \to u$. The inequality $\limsup_{m\to\infty} g^m(u^m, \omega, t_n) \le g(u, \omega, t_n) = \infty$ is already satisfied. Next, we prove that $\liminf_{m\to\infty} g^m(u^m, \omega, t_n) = g(u, \omega, t_n)$. This is true since

$$\lim_{m\to\infty} \widetilde{\lambda}_{h_m}(\omega, t_n) = \lambda(\omega, t_n), \quad \lim_{m\to\infty} \widetilde{\pi}_{h_m}(\omega, t_n) = \pi(\omega, t_n),$$

and u does not satisfy the constraints of the problem $g(\cdot, \omega, t_n)$ represents. Thus, if $u^m \to u$, then after some large enough index M, for all $m \ge M$, u^m would be infeasible for $g^m(\cdot, \omega, t_n)$, too. Hence, we will have $g^m(u^m, \omega, t_n) = \infty$ for all $m \geq M$ which imply that $\liminf_{m\to\infty} g^m(u^m, \omega, t_n) = g(u, \omega, t_n)$. ■

Proof of Theorem 7. To execute the proposed predictable bid-price policy, the system manager solves at each decision time t_n for $n = 1, \ldots, N$ and for each realization of $\omega \in \Omega$, the linear program $(\mathrm{P}_h(\omega, t_n))$ to determine the bookings. We first establish the feasibility of the controls (bookings) resulting from the proposed predictable bid-price policy for the network revenue management problem (P). Let $u^h(\omega)$ denote an arbitrary element of $\mathcal{U}^h(\omega)$. That is, for $\omega \in \Omega$, $h > 0$, we have

$$u^h(\omega) = (u^h(\omega, t_1), \ldots, u^h(\omega, t_N)),$$

where $u^h(\omega, t_n)$ is any optimal solution to $(\mathrm{P}_h(\omega, t_n))$ for $n = 1, \ldots, N$. The control $\{u^h(\omega, t)\}_{(\omega,t)\in(\Omega,\Gamma)}$ clearly satisfies the demand restrictions of the network revenue management problem (P). To check whether $\{u^h(\omega, t)\}_{(\omega,t)\in(\Omega,\Gamma)}$ satisfies the capacity restrictions, we prove that $\sum_{n=1}^N \widetilde{\lambda}_h(\omega, t_n) \leq C$ for all $\omega \in \Omega$. For $n = 1$, we have $\widetilde{\lambda}_h(\omega, t_1) \leq C$ due to the definition of $\widetilde{\lambda}_h(\omega, t_n)$. For $n \geq 2$,

$$\widetilde{\lambda}_h(\omega, t_n) = \min\{\mathbb{E}[\lambda(\omega, t_n) \mid \mathcal{F}_{t_n - h}], C - \sum_{i=1}^{n-1} \widetilde{\lambda}_h(\omega, t_i)\}, \tag{A.19}$$

$$\leq C - \sum_{i=1}^{n-1} \widetilde{\lambda}_h(\omega, t_i), \tag{A.20}$$

and, hence, $\sum_{i=1}^n \widetilde{\lambda}_h(\omega, t_i) \leq C$. Since the bookings $\{u^h(\omega, t)\}_{(\omega,t)\in(\Omega,\Gamma)}$ satisfy

$$\sum_{n=1}^N A u^h(\omega, t_n) \leq \sum_{n=1}^N \widetilde{\lambda}_h(\omega, t_n) \leq C,$$

they are feasible for the network revenue management problem (P).

Having proved the feasibility of the bookings resulting from $(\widetilde{\pi}_h, \widetilde{\lambda}_h)$ for $h > 0$, we next prove that as $h \searrow 0$, the expected revenue generated by those bookings converges to the expected revenue generated by the optimal adapted bid-price scheme (π, λ), which, in turn, is equal to the objective function value of the network revenue management problem (P). To study the limit as $h \searrow 0$, we analyze the limits along arbitrary sequences such that $h_m \searrow 0$ as $m \to \infty$. The properties proved in Lemma 53 enable us to use Theorem 7.33 of [**55**], from which it follows that $\inf\, g^m(\cdot, \omega, t_n) \to \inf\, g(\cdot, \omega, t_n)$ (finite) as $m \to \infty$ for $\omega \in \Omega$ and $n = 1, \ldots, N$. At the same time, for v in some index set $\mathcal{N} \in N_\infty$, where N_∞ is the set of all subsequences of natural numbers $\mathbb{N}$ containing all v beyond some $\overline{v} \in \mathbb{N}$, that is, $N_\infty = \{N \subset \mathbb{N}\colon\ \mathbb{N}\backslash N \text{ finite}\}$,the sets $\arg\min g^v(\cdot, \omega, t_n)$ are non-empty and form a bounded sequence with

$$\text{(A.21)} \qquad \limsup_{v}\ (\arg\min g^v(\cdot, \omega, t_n)) \subset \arg\min g(\cdot, \omega, t_n).$$

Recall that from Lemma 52, $\lim_{m\to\infty} \widetilde{\pi}_{h_m}(\omega, t_n) = \pi(\omega, t_n)$ and $\lim_{m\to\infty} \widetilde{l}_{h_m}(\omega, t_n) = l(\omega, t_n)$ for $n = 1, \ldots, N$. Combining this with (A.21), we see that for every $\omega \in \Omega$, the revenue generated along ω by the controls $u^{h_m}(\omega)$ converge as $m \to \infty$ to the revenue generated by an optimal solution to (P) along ω, where $u^{h_m}(\omega) \in \mathcal{U}^h(\omega)$ for all m. Hence, the function $\sum_{n=1}^{N} f(\omega, t_n) \cdot u^{h_m}(\omega, t_n)$ converges almost surely. From Dominated Convergence Theorem, the expected revenue generated by the controls resulting from the predictable bid-price policy converge to the expected revenue generated by an optimal solution to (P), proving the first part of Theorem 7.

Finally, note that for every $\omega \in \Omega$, by definition we have

$$\mathcal{U}(\omega) = \limsup_{r\to\infty} (\arg\min\, g^{h_r}(\cdot,\omega,t_1) \times \cdots \times \arg\min g^{h_r}(\cdot,\omega,t_N)).$$

From (A.21) it follows that for $\omega \in \Omega$ and $n = 1,\ldots,N$, $u(\omega,t_n) \in \arg\max(\mathrm{P}(\omega,t_n))$ for $u(\omega) \in \mathcal{U}(\omega)$. Then, the optimality of the adapted bid-price policy (π,λ) for (P) implies that $u(\omega)$ is an optimal solution to (P) along ω. Hence, every cluster point $(u(\omega) : \omega \in \Omega)$ of the sequence of predictable bid-price controls $\{(\widetilde{\pi}_h, \widetilde{\lambda}_h)\}_{h>0}$ is an optimal control for the network revenue management problem, proving second part of Theorem 7. ■

A.3. Proofs Regarding Section 1.4

As our first step, we introduce a perturbed version of the network revenue management problem (P). For each $\varepsilon > 0$, the perturbed problem (P^ε) can be stated as follows: Choose $u(t_n) \in \mathcal{F}_{t_n}$ for $n = 1,\ldots,N$ so as to

$$\text{Maximize } \sum_{n=1}^{N} \mathbb{E}\left[f(t_n)\cdot u(t_n) - \frac{1}{2}\sum_{j=1}^{J} \varepsilon_j(t_n)\, u_j^2(t_n) \right]$$

subject to

$$x(t_0) = C, \tag{P^ε}$$

$$x(t_n) = x(t_{n-1}) - Au(t_n), \quad n = 1,\ldots,N,$$

$$Au(t_n) \le x(t_{n-1}), \;\; n = 1,\ldots,N,$$

$$0 \le u(t_n) \le D(t_n) - D(t_{n-1}), \quad n = 1,\ldots,N,$$

where $\varepsilon_j(t_n)$ is defined as follows: For $j = 1, \ldots, J$ and $n = 1, \ldots, N$, let

$$\varepsilon_j(t_n) = \begin{cases} \frac{\varepsilon}{D_j(t_n) - D_j(t_{n-1})} & \text{if } D_j(t_n) - D_j(t_{n-1}) > 0, \\ \varepsilon & \text{otherwise.} \end{cases}$$

The difference between (P^ε) and (P) is that (P^ε) has the strictly concave term

$$-\frac{1}{2} \sum_{j=1}^{J} \varepsilon_j(t_n)\, u_j^2(t_n)$$

in its objective function in addition to the revenue term $f(t_n) \cdot u(t_n)$, which makes (P^ε) a strictly concave problem. Note that from (P^ε), we recover the network revenue management problem (P) for $\varepsilon = 0$. The existence of a unique optimal solution to (P^ε) can be shown using an argument similar to Proposition 49.

Lemma 54. *For each $\varepsilon > 0$, there exists a bid-price process π with $\pi(t_n) \in \mathcal{F}_{t_n}$ for $n = 1, \ldots, N$ such that the booking controls u defined as in (1.4) using π constitute an optimal solution to the perturbed network revenue management problem (P^ε) and for each resource $k = 1, \ldots, K$, the bid-price process $\{\pi_k(t_n) : n = 1, \ldots, N\}$ is a martingale. Moreover, $0 \le \pi(t_n) \le B$ for $n = 1, \ldots, N$ where $B = JF \max_j \mathbb{E}[D_j(\omega, T)]$.*

Proof of Lemma 54. The proof can be divided into four major steps. We first introduce a discrete approximation to the perturbed problem (P^ε). We derive the dual convex problem associated with the discretized problem (P^ε) and the resulting coextremality conditions. Second, we prove that the limit $\overline{u}$ of the optimal controls for the discretized problems is indeed an optimal control for (P^ε). Third, we define the bid-price process π using the limit of the dual variables to the discretized problems. We also show that u as defined in (1.4) using π constitute an optimal solution to the perturbed network revenue management problem (P^{ε}). Finally, we show that the bid-price process π we defined forms a martingale.

We first introduce a discrete approximation to the perturbed problem (P^{ε}). To be specific, we use the discretization introduced in Section A.1.1. Then, we have a sequence of problems indexed by $m = 1, 2, \ldots$ such that in each problem the distributions of the demand during each period and the vector of fares have finite support so that these problems reduce to finite convex programs, allowing us to use the machinery of convex programming.

Let $d^{i,m}$ and $f^{i,m}$denote the discretized demand and fare at information node $i \in \mathcal{I}^m$, respectively. Then, define $\varepsilon_j^{i,m}$ as follows: Let $\varepsilon_j^{i,m} = \varepsilon$ if $d_j^{i,m} = 0$ and $\varepsilon^{i,m} = \varepsilon / d_j^{i,m}$ if $d_j^{i,m} > 0$. Let $u^{i,m}$ denote the booking vector at information node $i \in \mathcal{I}^m$. Similarly, denote vector of remaining capacities *upon entering* information node $i \in \mathcal{I}^m$ by $x^{i,m}$. Then, the finite convex program resulting from the m^{th} discretization is given by the following finite convex problem (denoted by $(\mathrm{P}_\varepsilon^m)$):

$$(\mathrm{P}_\varepsilon^m) \qquad \begin{aligned}
&\text{Maximize} \sum_{i\in\mathcal{I}^m} \left[p^i f^{i,m}\cdot u^{i,m} - \frac{1}{2}\sum_{j=1}^{J} \varepsilon_j^{i,m} \left(u_j^{i,m}\right)^2 \right] \\
&\text{subject to} \\
&x^{i,m} = C\ ,\ \ i\in\mathcal{I}_1^m, \\
&x^{i,m} = x^{i-,m} - Au^{i-,m},\ \ i\in\mathcal{I}_n^m,\ \ n=2,\ldots,N, \\
&Au^{i,m} \le x^{i,m},\ \ i\in\mathcal{I}_N^m,\ \ n=1,\ldots,N, \\
&0\le u^{i,m}\le d^{i,m},\ \ i\in\mathcal{I}_n^m,\ \ n=1,\ldots,N,
\end{aligned}$$

where the capacity constraints $Au^{i,m} \le x^{i,m}$ are imposed only for information nodes $i\in\mathcal{I}_N^m$ in the last period. Imposing the capacity constraints only for the information nodes in the last period is equivalent to imposing them for every node $i\in\mathcal{I}^m$.

Let $\{\widetilde{u}^{i,m}\}_{i\in\mathcal{I}^m}$ denote an optimal control for the m^{th} discretized problem $(\mathrm{P}_\varepsilon^m)$, and $\{\widetilde{x}^{i,m}\}_{i\in\mathcal{I}^m}$ denote the corresponding state trajectory. Using the convex duality framework of [**55**], the dual convex program to the m^{th} discretized problem $(\mathrm{P}_\varepsilon^m)$ is given by

$$\begin{aligned}
&\mathrm{Min}_{\{v^{i,m}\}_{i\in\mathcal{I}^m}} \sum_{i\in\mathcal{I}^m} p^i[g_\varepsilon(f^{i,m} - y^{i,m}A, d^{i,m})] + C\cdot y^{0,m} \\
&\text{subject to} \\
&y^{i,m} = \mathbb{E}[y^{i+,m}\mid i] - v^{i,m},\ \ i\in\mathcal{I}^m, \\
&v^{i,m}\le 0,\ \ i\in\mathcal{I}^m,
\end{aligned} \qquad (\mathrm{D}_\varepsilon^m)$$

where $g_\varepsilon(z,d) = \sum_{j=1}^{J} h_\varepsilon(z_j,d_j)$ and for $j = 1,\ldots,J$, h_ε is defined as

$$\text{(A.22)} \qquad h_\varepsilon(z_j,d_j) = \begin{cases} 0 & \text{if} \quad z_j \le 0, \\ z_j d_j - \frac{\varepsilon}{2} d_j & \text{if} \quad z_j \ge \varepsilon, \\ \frac{z_j^2 d_j^2}{2\varepsilon} & \text{if} \quad 0 < z_j < \varepsilon. \end{cases}$$

We have $y^{i+,m} = 0$ for $i \in \mathcal{I}_N^m$ by convention and $d^{i,m}$ denote the discretized demand at information node $i \in \mathcal{I}^m$. The non-negativity of an optimal dual solution $\widetilde{y}^{i,m}$ then follows from the facts that $v^{i,m} \le 0$ for $i \in \mathcal{I}^m$ and $\widetilde{y}^{i+,m} = 0$ for $i \in \mathcal{I}_N^m$. Moreover, the objective function value of the discretized problems (P_ε^m) are less than or equal to $\sum_{n=1}^{N} \mathbb{E}[f(\omega,t_n)\cdot(D(\omega,t_n) - D(\omega,t_{n-1}))]$, which is uniformly bounded since the demand process has finite mean and fare process is bounded. Then, the objective function value of (D_ε^m) are also uniformly bounded due to strong duality. For an optimal solution $\{\widetilde{y}^{i,m}\}_{i\in\mathcal{I}^m}$ to (D_ε^m) we have that its objective function value is in the interval $[C\cdot\widetilde{y}^{0,m}, JF\mathbb{E}[D(\omega,T)]$ where F is the bound on the fare process and $\widetilde{y}^{0,m}$ is uniformly bounded. By a similar argument, we can prove that $\widetilde{y}^{i,m}$ is uniformly bounded by $JF\mathbb{E}[D(\omega,T)]$ if we consider the smaller primal and dual problems that starts at information node i.

From convex programming duality, the optimal primal-dual variables $\{\widetilde{u}^{i,m}\}_{i\in\mathcal{I}^m}$ and $\{\widetilde{y}^{i,m}\}_{i\in\mathcal{I}^m}$ satisfy a set of coextremality conditions which are necessary and sufficient for optimality. To facilitate our analysis, first define for $\varepsilon > 0$ the booking function ϕ^ε as follows:

$$\text{(A.23)} \qquad \phi^\varepsilon(z_j,f_j,d_j) = \begin{cases} 0 & \text{if} \quad f_j < z_j, \\ d_j & \text{if} \quad f_j > z_j + \varepsilon, \\ \frac{f_j - z_j}{\varepsilon} d_j & \text{if} \quad z_j \le f_j \le z_j + \varepsilon. \end{cases}$$

It is easy to see that ϕ^ε is continuous in all of its arguments. Then, the convex duality results of [**55**] imply that primal-dual variables $\{\widetilde{u}^{i,m}\}_{i\in\mathcal{I}^m}$ and $\{\widetilde{y}^{i,m}\}_{i\in\mathcal{I}^m}$ are optimal for (P^m_ε) and (D^m_ε), respectively if and only if they satisfy the following coextremality conditions:

$$(\widetilde{y}^{i,m} - \mathbb{E}[\widetilde{y}^{i+,m} \mid i]) \cdot \widetilde{x}^{i,m} = 0 \text{ for } i \in \mathcal{I}^m_N, \tag{A.24}$$

and for $j = 1, \dots, J$ and for all j with $d^{i,m}_j > 0$,

$$\widetilde{u}^{i,m}_j = \phi^\varepsilon(\mathbb{E}[\widetilde{y}^{i+,m}|i]A^j,\ f^{i,m}_j,\ d^{i,m}_j) \ \text{ for } \ i \in \mathcal{I}^m_N.$$

Having completed the first step of the proof of Lemma 54, we rewrite controls $\widetilde{u}^{i,m}$ and $\widetilde{y}^{i,m}$, $i \in \mathcal{I}^m$, as a function of the sample paths ω by tracking which information nodes we visit at each decision time. Formally, for $\omega \in \Omega$, let

$$\widetilde{u}^{i,m}_j(\omega, t_n) = u^{i,m} \text{ and } \widetilde{y}^m(\omega, t_n) = \widetilde{y}^{i,m} \ \text{ if } \ \omega \in i \text{ and } i \in \mathcal{I}^m_n.$$

Next, to embark on the second step of the proof of Lemma 54, we show the existence of a limiting process $\overline{u}$ such that for every $\omega \in \Omega$ and $n = 1, \dots, N$ there exists a subsequence $m_{n,r}(\omega)$ so that for $n \geq 2$, $m_{n,r}(\omega)$ is a further subsequence of $m_{n-1,r}(\omega)$ and

$$\overline{u}(\omega, t_n) = \lim_{r\to\infty} \widetilde{u}^{m_{n,r}(\omega)}(\omega, t_n) \ \text{ and } \ \overline{y}(\omega, t_n) = \lim_{r\to\infty} \widetilde{y}^{m_{n,r}(\omega)}(\omega, t_n) \ \text{ for } n = 1, \dots, N.$$

The subsequences $m_{n,r}(\omega) \in \mathcal{F}_{t_n}$ can be constructed as in the proof of Proposition 51. Moreover, we have $\overline{u}(\omega, t_n), \overline{y}(\omega, t_n) \in \mathcal{F}_{t_n}$ by construction. The feasibility of $\overline{u}$

for (P) has been established in the proof of Proposition 50. Hence, $\overline{u}$ is also feasible for the perturbed problem (P^{ε}). Since the booking function ϕ^{ε} is continuous and $\widetilde{u}_j^{i,m} = \phi^{\varepsilon}(\mathbb{E}[\widetilde{y}^{i+,m}|i]A^j,\ f_j^{i,m},d_j^{i,m})$ for $i \in \mathcal{I}_N^m$, we have

$$\overline{u}(\omega,t_n) = \phi^{\varepsilon}(\mathbb{E}[\overline{y}(\omega,t_{n+1})|\mathcal{F}_n]A^j,\ f_j(\omega,t_n), D_j(\omega,t_n) - D_j(\omega,t_{n-1})) \tag{A.25}$$

for $\omega \in \Omega$ and $n = 1,\ldots,N$.

Next we show that $\overline{u}$ is optimal for the perturbed problem (P^{ε}) and the booking controls u defined as in (1.4) using the bid-price process $\pi = \mathbb{E}[\overline{y}(\omega,t_{n+1})|\mathcal{F}_n]$ are equal to $\overline{u}$, proving the optimality of u for (P^{ε}). As a preliminary to establishing the optimality of $\overline{u}$, we show the convergence of the objective function values of the discretized problems. To be specific, we show that the objective function values of the discretized problems converge to the objective function value generated by the feasible control $\overline{u}$ for the perturbed network revenue management problem (P^{ε}).

We first show that the objective function value generated along a sample path ω is non-decreasing in m almost surely. That is, $\mathbb{P}$ almost surely,

$$\sum_{n=1}^{N}\left[f^m(\omega,t_n)\cdot\widetilde{u}^m(\omega,t_n) - \frac{1}{2}\sum_{j=1}^{J}\varepsilon_j^m(\omega,t_n)\ \left(\widetilde{u}_j^m(\omega,t_n)\right)^2\right]$$

increases weakly as the partition gets finer. To see this, first note that $\widetilde{u}^m$ is feasible for the $(m+1)^{st}$ discretization. Moreover, the fare and demand that is assumed by the system manager to realize weakly increases on each grid as the partition gets finer. The value of $\varepsilon_j^m(\omega,t_n)$ is either constant for all $m \geq 0$ (this is the case when $D_j(\omega,t_n) - D_j(\omega,t_{n-1}) = 0$) or it is increasing for all $m \geq Q$ for some constant $Q \geq 1$ (this is the case when

$D_j(\omega, t_n) - D_j(\omega, t_{n-1}) > 0$.) Hence, applying the control $\widetilde{u}^m$ for the $(m+1)^{st}$ discretized problem generates at least as much revenue as $\widetilde{u}^m$ on each grid of the m^{th} discretized problem for m large enough. However, $\widetilde{u}^{m+1}$ does at least as good as this control and the revenue generated along a sample path ω weakly increases. Moreover, total revenue generated along each sample path is also bounded by above as the initial capacity C is bounded and so is the revenue that can be generated along each sample path. Then, objective function values of the discretized problems form a monotone sequence of real numbers which should have a limit. For every $\omega \in \Omega$, this limit also coincides with the revenue generated by $\overline{u}$ along ω and the result follows from the Dominated Convergence Theorem.

To we prove that $\overline{u}$ is optimal for the perturbed network revenue management problem (P^ε), we start with an optimal solution to (P^ε) and construct a feasible solution to each discretized problem. We show that the objective function values of these feasible solutions converge to the objective function value of (P^ε), which proves the optimality of $\overline{u}$ for (P^ε). Let u^* be an optimal solution to (P^ε). Construct a feasible solution $\widehat{u}^m$ to the m^{th} discretized problem as follows. For decision time t_n and information node $i \in \mathcal{I}_n^m$, the booking for product j is given by

$$\widehat{u}_j^m(\omega, t_n) = \begin{cases} \max\{0, \mathbb{E}[u_j^*(\omega, t_n)|\, i] - 1/2^m\} & \text{if} \quad \omega \in i,\, d_j^{i,m} < K^m, \\ \mathbb{E}[\min\{u_j^*(\omega, t_n), K^m\}|\, i] & \text{if} \quad \omega \in i,\, d_j^{i,m} = K^m, \end{cases}$$

where K^m is the level at which we truncate the demand. The control $\widehat{u}^m$ satisfies the demand constraints and is clearly adapted. The expected revenue under $\widehat{u}^m$ is greater than or equal to

$$\sum_{n=1}^{N} \sum_{i \in \mathcal{I}_n^m} p^i \mathbb{E}\left[f(\omega, t_n) \cdot u^*(\omega, t_n) - \frac{1}{2} \sum_{j=1}^{J} \varepsilon_j(\omega, t_n) \, \left(u_j^*(\omega, t_n)\right)^2 \middle| \; i \; \right]$$

$$-JNF/2^m - NF^2 2^J 2^m / 3^m - JNM/2^m - \frac{JN\varepsilon}{2^m}, \tag{A.26}$$

where F is the bound on the fare process and M is a finite number such that

$$\sup_{\omega \in \Omega, \; t \in \Gamma} u_j^*(\omega, t) \leq M$$

for $j = 1, \ldots, J$. (Note that such constant M exists since the capacity C is finite.) Therefore, the expected revenue under the optimal solution to the m^{th} discretized problem is greater than or equal to (A.26). Since (A.26) converges to the objective function value of the perturbed network revenue management problem (P^ε) as $m \to \infty$, it follows that $\overline{u}$ is optimal for (P^ε), completing the second step of the proof of Lemma 54.

Having proved the optimality of $\overline{u}$ for (P^ε), we next prove that the booking controls u defined as in (1.4) using the bid-price process $\pi = \mathbb{E}[\overline{y}(\omega, t_{n+1})|\mathcal{F}_n]$ is indeed an optimal solution to (P^ε). Feasibility of u for (P^ε) follows from its definition. Moreover, by definition of u and the coextremality condition (A.25), we have $\overline{u} = u$ and u is optimal for (P^ε) because given the bid prices $\mathbb{E}[\overline{y}(\omega, t_{n+1})|\mathcal{F}_n]$, the booking function ϕ^ε uniquely defines a booking control.

To conclude the proof of Lemma 54, we prove that the bid-price process

$$\pi = \mathbb{E}[\overline{y}(\omega, t_{n+1})|\mathcal{F}_n]$$

forms a martingale which follows from the dual system dynamics. Moreover, since

$$\{\widetilde{y}^{i,m}\}_{i\in\mathcal{I}^m}$$

are uniformly bounded by

$$JF\mathbb{E}[D(\omega,T)],$$

so are $\overline{y}(\omega,t_{n+1})$ for $n=1,\ldots,N$ and $\omega\in\Omega$. Thus,

$$0\leq\pi\left(t_n\right)\leq B$$

for $B=JF\max_j\mathbb{E}[D_j(\omega,T)]$. ■

Proof of Theorem 9. Fix an $\varepsilon>0$ and a partition $\Gamma=\{t_0,t_1,\ldots,t_N\}$ of $[0,T]$. Let $\{\pi\left(t_n\right):n=1,\ldots,N\}$ with $\pi\left(t_n\right)\in\mathcal{F}_{t_n}$ for $n=1,\ldots,N$, be a bid-price process as in Lemma 54. Then, the booking controls u defined as in (1.4) using π constitute an optimal solution to the perturbed network revenue management problem (P^{ε}) and for each resource $k=1,\ldots,K$, the bid-price process $\{\pi_k(t_n):n=1,\ldots,N\}$ is a martingale.

Define the predictable bid-price process $\{\pi^{\varepsilon}\left(t_n\right):n=1,\ldots,N\}$ as follows:

$$\pi^{\varepsilon}\left(t_n\right)=\mathbb{E}\left[\pi\left(t_N\right)|\mathcal{F}_{n-1}\right]\quad\text{for } n=1,\ldots,N.$$

That is, the predictable bid-price process π^{ε} is constructed by taking the conditional expectation of the adapted bid-price process π as in Lemma 54. Then, by construction the predictable bid-price process $\{\pi^{\varepsilon}\left(t_n\right):n=1,\ldots,N\}$ is a martingale adapted to $(\{\mathcal{F}_{t_{n-1}}:n=1,\ldots,N\},\mathbb{P})$ as well. The fact that $0\leq\pi^{\varepsilon}\left(t_n\right)\leq B$ for $n=1,\ldots,N$

follows from Lemma 54, which states that $0 \leq \pi(t_n) \leq B$ for $n = 1, \ldots, N$ where $B = JF \max_j \mathbb{E}[D_j(\omega, T)]$.

Next we provide the optimality gap (1.5) in Theorem 9. Let u^ε denote the booking controls defined as in (1.4) using the adapted bid-price process π. Then, from Lemma 54, u^ε is the unique optimal solution to the perturbed network revenue management problem (P^ε). Let u be an optimal solution to the network revenue management problem (P). As our first step, we prove that

$$\left| \sum_{n=1}^{N} \mathbb{E}[f(t_n) \cdot u^\varepsilon(t_n)] - \mathrm{P}^* \right| \leq \kappa\varepsilon, \tag{A.27}$$

where P^* is the objective function value of (P) and is equal to $\sum_{n=1}^{N} \mathbb{E}[f(t_n) \cdot u(t_n)]$. To prove (A.27), notice that u is also feasible for the perturbed problem (P^ε) and we have

$$\begin{aligned} & \sum_{n=1}^{N} \mathbb{E}[f(t_n) \cdot u(t_n)] - \varepsilon \sum_{j=1}^{J} \mathbb{E}\left[D_j(T)\right] \\ \leq \; & \sum_{n=1}^{N} \mathbb{E}\left[f(t_n) \cdot u(t_n) - \frac{1}{2} \sum_{j=1}^{J} \varepsilon_j(t_n) \, (u_j(t_n))^2 \right] \\ \leq \; & \sum_{n=1}^{N} \mathbb{E}\left[f(t_n) \cdot u^\varepsilon(t_n) - \frac{1}{2} \sum_{j=1}^{J} \varepsilon_j(t_n) \, \left(u_j^\varepsilon(t_n)\right)^2 \right], \\ \leq \; & \sum_{n=1}^{N} \mathbb{E}\left[f(t_n) \cdot u^\varepsilon(t_n)\right]. \end{aligned}$$

The first inequality follows from the definition of $\varepsilon_j(t_n)$, and the fact that $u(t_n) \leq D_j(t_n) - D_j(t_{n-1})$ for $n = 1, \ldots, N$. The second inequality is given by feasibility of u for (P^ε) and optimality of u^ε. The last inequality proves (A.27) since u^ε is also feasible for (P).

Finally, to conclude the proof of Theorem 9, we prove that

$$\left|\mathrm{Obj}\left(\pi^{\varepsilon},\varepsilon,\Gamma\right)-\sum_{n=1}^{N}\mathbb{E}\left[f(t_n)\cdot u^{\varepsilon}(t_n)\right]\right|\leq\frac{C}{\varepsilon}\left[\mathbb{E}\mathcal{V}_p\left(\pi^{\varepsilon},\Gamma\right)\right]^{1/p}\left[\mathbb{E}\mathcal{V}_q\left(D,\Gamma\right)\right]^{1/q}.$$

First define the booking function ϕ^{ε} for $\varepsilon>0$ as follows:

$$\phi^{\varepsilon}(z_j,f_j,d_j)=\begin{cases}0 & \text{if} \quad f_j<z_j,\\ d_j & \text{if} \quad f_j>z_j+\varepsilon,\\ \frac{f_j-z_j}{\varepsilon}d_j & \text{if} \quad z_j\leq f_j\leq z_j+\varepsilon.\end{cases}$$

It is easy to see that ϕ^{ε} is continuous in all of its arguments. Then, we can write

$$\begin{aligned}&\left|\mathrm{Obj}\left(\pi^{\varepsilon},\varepsilon,\Gamma\right)-\sum_{n=1}^{N}\mathbb{E}\left[f(t_n)\cdot u^{\varepsilon}(t_n)\right]\right|\\ \leq\ &\left|\sum_{n=1}^{N}\sum_{j=1}^{J}\mathbb{E}\left[f_j(t_n)\phi^{\varepsilon}(\pi^{\varepsilon}A^j,\ f_j\left(t_n\right),D_j(t_n)-D_j(t_{n-1}))\right]-\sum_{n=1}^{N}\mathbb{E}\left[f(t_n)\cdot u^{\varepsilon}(t_n)\right]\right|\\ &+\left|\mathrm{Obj}\left(\pi^{\varepsilon},\varepsilon,\Gamma\right)-\sum_{n=1}^{N}\sum_{j=1}^{J}\mathbb{E}\left[f_j(t_n)\phi^{\varepsilon}(\pi^{\varepsilon}A^j,\ f_j\left(t_n\right),D_j(t_n)-D_j(t_{n-1}))\right]\right|\end{aligned}$$

We first consider the following term:

$$\left|\mathrm{Obj}\left(\pi^{\varepsilon},\varepsilon,\Gamma\right)-\sum_{n=1}^{N}\sum_{j=1}^{J}\mathbb{E}\left[f_j(t_n)\phi^{\varepsilon}(\pi^{\varepsilon}A^j,\ f_j\left(t_n\right),D_j(t_n)-D_j(t_{n-1}))\right]\right|,$$

quantifies the loss in revenue from ignoring the capacity constraints. For notational brevity, define $\Delta D_j(t_n) = D_j(t_n) - D_j(t_{n-1})$. We have

$$\begin{aligned}
& \mathrm{Obj}\,(\pi^{\varepsilon}, \varepsilon, \Gamma) \\
\geq\ & \sum_{n=1}^{N} \sum_{j=1}^{J} \mathbb{E}\left[f_j(t_n)\phi^{\varepsilon}(\pi^{\varepsilon} A^j,\ f_j(t_n), \Delta D_j(t_n)))\right] \\
& - \frac{F}{a}\mathbb{E}\left[\sum_{k=1}^{K} \sum_{n=1}^{N} \sum_{j=1}^{J} A_{kj}\phi^{\varepsilon}(\pi^{\varepsilon} A^j,\ f_j(t_n), \Delta D_j(t_n)) - C_k\right]^{+}.
\end{aligned}$$

Since

$$\begin{aligned}
& \frac{F}{a}\mathbb{E}\left[\sum_{k=1}^{K} \sum_{n=1}^{N} \sum_{j=1}^{J} A_{kj}\phi^{\varepsilon}(\pi^{\varepsilon} A^j,\ f_j(t_n), \Delta D_j(t_n)) - C_k\right]^{+} \\
\leq\ & \frac{F}{a}\mathbb{E}\left[\sum_{k,n,j} A_{kj}\left[\phi^{\varepsilon}(\pi^{\varepsilon} A^j,\ f_j(t_n), \Delta D_j(t_n)) - \phi^{\varepsilon}(\pi A^j,\ f_j(t_n), \Delta D_j(t_n))\right]\right]^{+} \\
\leq\ & \frac{FEK}{a}\mathbb{E}\left[\sum_{n,j} \left|\phi^{\varepsilon}(\pi^{\varepsilon} A^j,\ f_j(t_n), \Delta D_j(t_n)) - \phi^{\varepsilon}(\pi A^j,\ f_j(t_n), \Delta D_j(t_n))\right|\right], \\
\leq\ & \frac{FEK}{a\varepsilon}\left[\left(\mathbb{E}\left[\mathcal{V}_p(\pi^{\varepsilon}, \Gamma)\right]\right)^{1/p}\left(\mathbb{E}\left[\mathcal{V}_q(D, \Gamma)\right]\right)^{1/q}\right],
\end{aligned}$$

we get

$$\begin{aligned}
\mathrm{Obj}\,(\pi^{\varepsilon}, \varepsilon, \Gamma)\ \geq\ & \sum_{n=1}^{N} \sum_{j=1}^{J} \mathbb{E}\left[f_j(t_n)\phi^{\varepsilon}(\pi^{\varepsilon} A^j,\ f_j(t_n), D_j(t_n) - D_j(t_{n-1}))\right] \\
& - \frac{FEK}{a\varepsilon}\left[\left(\mathbb{E}\left[\mathcal{V}_p(\pi^{\varepsilon}, \Gamma)\right]\right)^{1/p}\left(\mathbb{E}\left[\mathcal{V}_q(D, \Gamma)\right]\right)^{1/q}\right].
\end{aligned}$$

That is, we have

$$\left|\text{Obj}\left(\pi^{\varepsilon},\varepsilon,\Gamma\right)-\sum_{n=1}^{N}\sum_{j=1}^{J}\mathbb{E}\left[f_j(t_n)\phi^{\varepsilon}(\pi^{\varepsilon}A^j,\ f_j\left(t_n\right),D_j(t_n)-D_j(t_{n-1}))\right]\right|$$
$$\leq \quad \frac{FEK}{a\varepsilon}\left[\left(\mathbb{E}\left[\mathcal{V}_p\left(\pi^{\varepsilon},\Gamma\right)\right]\right)^{1/p}\left(\mathbb{E}\left[\mathcal{V}_q\left(D,\Gamma\right)\right]\right)^{1/q}\right].$$

Then, we also have

$$\left|\text{Obj}\left(\pi^{\varepsilon},\varepsilon,\Gamma\right)-\sum_{n=1}^{N}\mathbb{E}\left[f(t_n)\cdot u^{\varepsilon}(t_n)\right]\right|$$
$$\leq \quad \left(\frac{FEK}{a\varepsilon}+\frac{KJFE}{\varepsilon}\right)\left[\left(\mathbb{E}\left[\mathcal{V}_p\left(\pi^{\varepsilon},\Gamma\right)\right]\right)^{1/p}\left(\mathbb{E}\left[\mathcal{V}_q\left(D,\Gamma\right)\right]\right)^{1/q}\right]$$

Now we focus on the term

$$\left|\sum_{n=1}^{N}\sum_{j=1}^{J}\mathbb{E}\left[f_j(t_n)\phi^{\varepsilon}(\pi^{\varepsilon}A^j,\ f_j\left(t_n\right),D_j(t_n)-D_j(t_{n-1}))\right]-\sum_{n=1}^{N}\mathbb{E}\left[f(t_n)\cdot u^{\varepsilon}(t_n)\right]\right|$$

Then, we have

$$\left|\sum_{n=1}^{N}\sum_{j=1}^{J}\mathbb{E}\left[f_j(t_n)\phi^{\varepsilon}(\pi^{\varepsilon}A^j,\ f_j\left(t_n\right),D_j(t_n)-D_j(t_{n-1}))\right]-\sum_{n=1}^{N}\mathbb{E}\left[f(t_n)\cdot u^{\varepsilon}(t_n)\right]\right|$$

is less than or equal to

$$\frac{FJ}{\varepsilon}\left|\sum_{n=1}^{N}\mathbb{E}\left[\left|\pi^{\varepsilon}\left(t_n\right)A-\pi\left(t_n\right)A\right|\,\left|D(t_n)-D(t_{n-1})\right|\right]\right|,$$

where F is the upper bound on the fare process, $E=\max_{k,j}\left\{A_{kj}\right\}$ and the inequality follows from the fact that the booking function ϕ^{ε} is Lipschitz continuous in its first argument with Lipschitz constant less than

$$\left|D_j(t_n) - D_j(t_{n-1})\right| / \varepsilon.$$

From this, we get

$$\begin{aligned} &\left|\sum_{n=1}^{N}\left[\sum_{j=1}^{J}\mathbb{E}\left[f_j(t_n)\phi^{\varepsilon}(\pi^{\varepsilon}A^j,\ f_j\left(t_n\right), D_j(t_n) - D_j(t_{n-1}))\right] - \mathbb{E}\left[f(t_n)\cdot u^{\varepsilon}(t_n)\right]\right]\right| \\ \text{(A.28)}\ \leq\ & \frac{KJFE}{\varepsilon}\left|\sum_{n=1}^{N}\mathbb{E}\left[\left|\pi^{\varepsilon}\left(t_n\right) - \pi\left(t_n\right)\right|\ \left(D_j(t_n) - D_j(t_{n-1})\right)\right]\right|. \end{aligned}$$

Treating $\sum_{n=1}^{N}\sum_{j=1}^{J}\mathbb{E}\left(\cdot\right)$ as a product measure and applying Holder's inequality to (A.28), we get

$$\begin{aligned} &\left|\mathrm{Obj}\left(\pi^{\varepsilon},\varepsilon,\Gamma\right) - \sum_{n=1}^{N}\mathbb{E}\left[f(t_n)\cdot u^{\varepsilon}(t_n)\right]\right| \\ \text{(A.29)}\qquad &\leq \left(\frac{FEK}{a\varepsilon} + \frac{KJFE}{\varepsilon}\right)\left[\left(\mathbb{E}\left[\mathcal{V}_p\left(\pi^{\varepsilon},\Gamma\right)\right]\right)^{1/p}\left(\mathbb{E}\left[\mathcal{V}_q\left(D,\Gamma\right)\right]\right)^{1/q}\right], \end{aligned}$$

where $q = p/\left(p-1\right)$. Then, (A.29) together with (A.27) completes the proof of Theorem 9. ■

Proof of Corollary 10. Suppose the demand process $\{D\left(t\right) : 0 \leq t \leq T\}$ has continuous sample paths. For each $\varepsilon > 0$ and for each partition Γ, from (1.5) of Theorem 9 we have

$$\begin{aligned} \left|\mathrm{Obj}\left(\pi^{\varepsilon},\varepsilon,\Gamma\right) - \mathrm{P}^*\right| &\leq \kappa\varepsilon + \frac{C}{\varepsilon}\left[\left(\mathbb{E}\left[\mathcal{V}_2\left(\pi^{\varepsilon},\Gamma\right)\right]\right)^{1/2}\left(\mathbb{E}\left[\mathcal{V}_2\left(D,\Gamma\right)\right]\right)^{1/2}\right], \\ &= \kappa\varepsilon + \frac{2BC}{\varepsilon}\sqrt{\mathbb{E}\left[\mathcal{V}_2\left(D,\Gamma\right)\right]}, \end{aligned}$$

since $\left(\mathbb{E}\left[\mathcal{V}_2\left(\pi^{\varepsilon},\Gamma\right)\right]\right)^{1/2} \leq 2B$. To see this, note that by Theorem 9, we have $\pi^{\varepsilon} \leq B$ where $B = JF \max_j \mathbb{E}[D(\omega, T)]$. Hence,

$$\mathbb{E}\left[\mathcal{V}_2\left(\pi^{\varepsilon},\Gamma\right)\right] = \mathbb{E}\left[\sum_{n=1}^{N}\left(\pi^{\varepsilon}\left(t_n\right) - \pi^{\varepsilon}\left(t_{n-1}\right)\right)^2\right] = \mathbb{E}\left[\left(\pi^{\varepsilon}\left(t_N\right) - \pi^{\varepsilon}\left(t_0\right)\right)^2\right] \leq 4B^2,$$

where the second equality follows from the fact that π^{ε} is a martingale and the last inequality is true since $\pi^{\varepsilon} \leq B$.

Moreover, since demand process is continuous, $\mathbb{E}\left[\mathcal{V}_2\left(D,\Gamma\right)\right] \rightarrow 0$ as the the partitions get finer. Then, for each ε, we can choose a partition Γ^{ε} fine enough such that $\sqrt{\mathbb{E}\left[\mathcal{V}_2\left(D,\Gamma\right)\right]} \leq \varepsilon^2$. Thus, we have

$$\left|\mathrm{Obj}\left(\pi^{\varepsilon},\varepsilon,\Gamma^{\varepsilon}\right) - \mathrm{P}^*\right| \rightarrow 0 \quad \text{as} \quad \varepsilon \rightarrow 0,$$

which concludes the proof of Corollary 10. ■

Proof of Corollary 12. Suppose that the information structure $\{\mathcal{F}_t : 0 \leq t \leq T\}$ is continuous. Then, the sample paths of π^{ε} are continuous. For each $\varepsilon > 0$ and for each partition Γ, from Theorem 9 we have

$$\begin{aligned}\left|\mathrm{Obj}\left(\pi^{\varepsilon},\varepsilon,\Gamma\right) - \mathrm{P}^*\right| &\leq \kappa\varepsilon + \frac{C}{\varepsilon}\left[\left(\mathbb{E}\left[\mathcal{V}_3\left(\pi^{\varepsilon},\Gamma\right)\right]\right)^{1/3}\left(\mathbb{E}\left[\mathcal{V}_{3/2}\left(D,\Gamma\right)\right]\right)^{2/3}\right], \\ &= \kappa\varepsilon + \frac{2C\sum_{j=1}^{J}\mathbb{E}\left[D_j\left(T\right)\right]}{\varepsilon}\left(\mathbb{E}\left[\mathcal{V}_3\left(\pi^{\varepsilon},\Gamma\right)\right]\right)^{1/3}.\end{aligned}$$

Now observe that we can write

$$\begin{aligned}\mathbb{E}\left[\mathcal{V}_3\left(\pi^{\varepsilon},\Gamma\right)\right] &= \mathbb{E}\left[\sum_{n=1}^{N}\left(\pi^{\varepsilon}\left(t_n\right)-\pi^{\varepsilon}\left(t_{n-1}\right)\right)^3\right], \\ &= \mathbb{E}\left[\sup_n\left(\pi^{\varepsilon}\left(t_n\right)-\pi^{\varepsilon}\left(t_{n-1}\right)\right)\sum_{n=1}^{N}\left(\pi^{\varepsilon}\left(t_n\right)-\pi^{\varepsilon}\left(t_{n-1}\right)\right)^2\right].\end{aligned}$$

As the partitions Γ^{ε} get finer, we have $\lim_{\varepsilon\to 0}\sup_n\left(\pi^{\varepsilon}\left(t_n\right)-\pi^{\varepsilon}\left(t_{n-1}\right)\right) = 0$ since the sample paths of π^{ε} are continuous. Moreover, since $\sum_{n=1}^{N}\left(\pi^{\varepsilon}\left(t_n\right)-\pi^{\varepsilon}\left(t_{n-1}\right)\right)^2$ is bounded by $JF\max_{j=1,\dots,J}\mathbb{E}[D(\omega,T)]$, we get $\lim_{\varepsilon\to 0}\mathbb{E}\left[\mathcal{V}_3\left(\pi^{\varepsilon},\Gamma\right)\right] = 0$ by Dominated Convergence Theorem. Thus, for each ε, we can choose a partition Γ^{ε} fine enough such that $\left(\mathbb{E}\left[\mathcal{V}_3\left(\pi^{\varepsilon},\Gamma\right)\right]\right)^{1/3} \leq \varepsilon^2$ and we get

$$\left|\mathrm{Obj}\left(\pi^{\varepsilon},\varepsilon,\Gamma^{\varepsilon}\right)-\mathrm{P}^*\right| \to 0 \quad \text{as} \quad \varepsilon \to 0,$$

which concludes the proof of Corollary 12. ■

APPENDIX B

Proofs of the Results in Chapter 2

B.1. Summary of Bismut (1973)

We first introduce the probability framework used in the paper. Let $(\Omega, \mathcal{F}, \mathbb{P})$ denote a complete probability space. $\{\mathcal{F}_t,\, t \in \mathbb{R}_+\}$ is an increasing sequence of complete sub-σ-fields of $\mathcal{F}$. We assume that the sequence of complete sub-σ-fields is right-continuous and has no time discontinuity, That is, for any increasing sequence $(S_n)_{n\in\mathbb{N}}$ of stopping times, we have

$$\mathcal{F}_{(\lim_n S_n)} = \bigvee_n \mathcal{F}_{S_n}.$$

An information structure that has no time discontinuity is also referred to as a quasi-continuous information structure in [**36**], which also proves that the natural filtrations of most of the commonly encountered processes are quasi-continuous, including the natural filtrations generated by the Poisson process and Brownian motion. As a matter of fact, [**12**] extends the framework and results in [**11**] to the more general setting of the control of semi-martingales where the quasi-continuity assumption is also dropped. Let $\mathcal{J}$ be the σ-field of $\Omega \times [0, \infty)$ of the well measurable or optional sets. That is, $\mathcal{J}$ is the σ-field generated by the adapted processes which are right-continuous with left limits. $\mathcal{J}^*$ is the completion for the measure $d\mathbb{P} \otimes dt$. w denotes an m-dimensional Brownian motion on $(\Omega, \mathcal{F}, \mathbb{P})$ adapted to $\{\mathcal{F}_t,\, t \in \mathbb{R}_+\}$. V is an n-dimensional vector space.

We next define the following spaces of functions. For any stopping time σ, L_2^σ is the space of square integrable $\mathcal{F}_\sigma$ measurable random variables, with values in V. Let L_{21} denote the space of $d\mathbb{P}\otimes dt$ classes u of $\mathcal{J}^*$ measurable functions with values in V such that

$$\mathbb{E}(\int_0^T |u(\omega,t)|\, dt)^2 < \infty.$$

Similarly, let L_{22} denote the space of $d\mathbb{P}\otimes dt$ classes H of $\mathcal{J}^*$ measurable functions that have values in V such that

$$\mathbb{E}\int_0^T |H(\omega,t)|^2\, dt < \infty.$$

$\underline{L}$ is the space of square integrable martingales which take values in V, are stopped at time T and null at 0. W is the subspace of $\underline{L}$ generated by the stochastic integrals relative to w of elements of L_{22}. Let $W^\perp$ be an orthogonal of W in $\underline{L}$ in the sense of [**23**]. $W^\perp$ can then be decomposed into the sum of two orthogonal subspaces of martingales W_1 and W_2, i.e. $W^\perp = W_1 \oplus W_2$. In practice, we would either have $W_1 = W^\perp$ or $W_1 = \{\mathbf{0}\}$ where $\mathbf{0}$ is the constant martingale equal to zero.

The problem of control can be defined as follows. We first describe the set of admissible controls for the primal and the dual problems. Define R_1 and R_2 by

$$\begin{aligned} R_1 &= L_2^0 \times L_{21} \times L_{22} \times W_1, \\ R_2 &= L_2^0 \times L_{21} \times L_{22} \times W_2. \end{aligned}$$

Each $x = (x_0, \dot{x}, H, M) \in R_1$ defines uniquely the stochastic process x_t by

(B.1)
$$x_t = x_0 + \int_0^t \dot{x}_s ds + \int_0^t H_s w_s + M_t.$$

Similarly, each $y = (y_0, \dot{y}, H^{'}, M^{'}) \in R_2$ defines uniquely the stochastic process y_t by

$$y_t = y_0 + \int_0^t \dot{y}_s ds + \int_0^t H_s^{'} dw_s + M_t^{'}.$$

The elements of R_1 and R_2 denote the spaces of stochastic processes that are admissible as primal and dual solutions to the control problems we will define shortly. In defining the control problems, further constraints may be included through objective functions taking values on extended real line $\mathbb{R} \cup \{\infty\}$ so that the feasible admissible controls is a proper subset of R_1 or R_2.

The primal problem of control is concerned with minimizing the functional $\Phi_{l,L}$ defined on R_1 as

$$\Phi_{l,L}(x) = \begin{cases} l(x_0, x_T) + \mathbb{E} \int_0^T L(\omega, t, x(\omega, t), \dot{x}(\omega, t), H(\omega, t))dt & \text{if } x \in R_1, \\ \infty & \text{otherwise,} \end{cases}$$

where $x = (x_0, \dot{x}, H, M)$ is a stochastic process defined as in (B.1). A control $x \in R_1$ is feasible only if $\Phi_{l,L}(x) < \infty$. L is a normal convex integrand in the sense of [**51**], defined on $\Omega \times [0, T] \times V \times V \times V^m$. The functional l helps us define the boundary conditions of the problem and consists of two convex, lower semi-continuous functionals l_0 and l_T as follows:

$$l(x_0, x_T) = l_0(x_0) + l_T(x_T).$$

To be more specific, the functional l_0 is defined on L_2^0 with values in $\mathbb{R} \cup \{\infty\}$ and will assist in setting the initial conditions of the problem. On the other hand, l_T is defined on

L_2^T with values in $\mathbb{R} \cup \{\infty\}$ and introduces the boundary conditions and penalties at the terminal time T.

In convex duality framework, every function f is coupled with its conjugate function f^* defined as

$$f^*(y) := \sup_x \{x \cdot y - f(x)\}. \tag{B.2}$$

We refer to f^* as the dual of f. The definition of the dual problem of control relies on the duals of the integrand L and the functional l. Let L^* be the dual integrand of L and define M on $\Omega \times [0, T] \times V \times V \times V^m$ as

$$M(\omega, t, p, s, H') = L^*(\omega, t, s, p, H').$$

Note that we have swapped the order of the terms s and p in defining M in terms of L^*. Similarly, define m on $L_2^0 \times L_2^T$ by

$$m(y_0, y_T) = l_0^*(y_0) + l_T^*(-y_T),$$

where l_0^* and l_T^* are the duals of l_0 and l_T, respectively. The dual problem of control, then, consists of the minimization of the functional $\Phi_{m,M}$ on R_2, where for $y = (y_0, \dot{y}, H', M')$, $\Phi_{m,M}(y)$ is given by

$$\Phi_{m,M}(y) = \begin{cases} m(y_0, y_T) + \mathbb{E} \int_0^T M(\omega, t, y(\omega, t), \dot{y}(\omega, t), H'(\omega, t))dt & \text{if } y \in R_2, \\ \infty & \text{otherwise.} \end{cases}$$

The following theorem establishes the duality between the two problems of stochastic control represented by $\Phi_{l,L}$ *and* $\Phi_{m,M}$.

Theorem IV-1 (Bismut) $\inf_{x\in R_1} \Phi_{l,L}(x) = -\inf_{y\in R_2} \Phi_{m,M}(y)$ provided $\Phi_{l,L}$ *or* $\Phi_{m,M}$ *are not identically* ∞*.*

Having derived the dual problem of control, we next analyze the necessary and sufficient conditions for primal-dual control pairs to be optimal. Let ∂f denote the subgradient of a function f, cf. [**55**].

Definition 55. $x \in R_1$ *and* $y \in R_2$ *are said to be coextremal if*

a) $d\mathbb{P} \otimes dt$ *a.s.* $(\dot{y}(\omega,t), y(\omega,t), H^{'}(\omega,t)) \in \partial L(\omega,t,x(\omega,t),\dot{x}(\omega,t),H(\omega,t))$

b) $y_0 \in \partial l_0(x_0)$, $-y_T \in \partial l_T(x_T)$.

Note that the definition of coextremality is symmetric between the primal and the dual problems.

Theorem IV-2 (Bismut) *The following assertions are equivalent:*

a) x *and* y *are coextremal;*

b) x *minimizes* $\Phi_{l,L}$ *on* R_1*,* y *minimizes* $\Phi_{m,M}$ *on* R_2 *and* $\Phi_{l,L}(x) = -\Phi_{m,M}(y)$*.*

Theorem IV-2 proves that x is optimal for the primal problem of control and y is optimal for the corresponding dual problem of control if and only if x and y are coextremal. Moreover, in that case the objective function value of the primal problem of control is equal to the negative of the objective function value of the dual control problem.

B.2. An Auxiliary Weak* Convergence Lemma

As a preliminary, first let $L^2(\Omega \times [0,T], \mathcal{F}_T \otimes \mathcal{B}[0,T], \mathbb{P})$ denote the set of functions $X : \Omega\times[0,T] \to \mathbb{R}^J$ that are measurable with respect to the product σ-algebra $\mathcal{F}_T\otimes\mathcal{B}[0,T]$

such that

$$\mathbb{E}\int_0^T |X(\omega,t)|^2\,dt < \infty,$$

where $\mathcal{B}[0,T]$ is the Borel σ-algebra on $[0,T]$. The space $L^2(\Omega \times [0,T],\, \mathcal{F}_T \otimes \mathcal{B}[0,T],\, \mathbb{P})$ is endowed with the usual inner product $\langle \cdot, \cdot \rangle$ given by

$$\langle X, Y \rangle = \mathbb{E}\left[\int_0^T X(\omega,t)\cdot Y(\omega,t)dt\right]$$

for $X, Y \in L^2(\Omega \times [0,T],\, \mathcal{F}_T \otimes \mathcal{B}[0,T],\, \mathbb{P})$ so that it is a Hilbert space.

We also view the stochastic processes as mappings from $\Omega \times [0,T]$ into $\mathbb{R}^J$. To be more specific, we view the adapted stochastic processes as the elements of $L^2(\Omega \times [0,T],\, \mathcal{J}_T^*,\, \mathbb{P})$, where $\mathcal{J}_T^*$ is the completion for the measure $d\mathbb{P} \otimes dt$ of the σ-field $\mathcal{J}_T$ generated by the adapted processes on $\Omega \times [0,T]$ which are right-continuous with left limits. Then, $L^2(\Omega \times [0,T],\, \mathcal{J}_T^*,\, \mathbb{P})$ is also a Hilbert space endowed with the inner product $\langle \cdot, \cdot \rangle$. In particular, $L^2(\Omega \times [0,T],\, \mathcal{J}_T^*,\, \mathbb{P})$ can be viewed as a closed subset of $L^2(\Omega \times [0,T],\, \mathcal{F}_T \otimes \mathcal{B}[0,T],\, \mathbb{P})$, For completeness, we next state the definition of weak* convergence, which is followed by the main result of this section.

Definition 56. *The sequence $\{X_n\}$ of elements of $L^2(\Omega \times [0,T],\, \mathcal{F}_T \otimes \mathcal{B}[0,T],\, \mathbb{P})$ is said to converge in the weak* topology to an element X of $L^2(\Omega \times [0,T],\, \mathcal{F}_T \otimes \mathcal{B}[0,T],\, \mathbb{P})$ if for all $Y \in L^2(\Omega \times [0,T],\, \mathcal{F}_T \otimes \mathcal{B}[0,T],\, \mathbb{P})$,*

$$\langle X^n, Y \rangle \to \langle X, Y \rangle \text{ as } n \to \infty.$$

Lemma 57. *Let $\{u^n : n \geq 1\}$ be a sequence of feasible controls for the network revenue management problem (P_{cont}) which converges to u in the weak* topology. Then, u is a* *feasible booking control for (P_{cont}). Moreover, the expected revenue under u^n converges to that under u as $n \to \infty$. That is,*

$$\lim_{n\to\infty} \mathbb{E}\left[\int_0^T f(\omega,t)\cdot u^n(\omega,t)dt\right] = \mathbb{E}\left[\int_0^T f(\omega,t)\cdot u(\omega,t)dt\right].$$

Proof. To prove that u is a feasible control for the network revenue management problem (P_{cont}), we need to check that u is adapted and satisfies demand and capacity restrictions. First, we prove that $u \in L^2(\Omega \times [0,T],\, \mathcal{J}_T^*,\, \mathbb{P})$, and hence u is adapted to the filtration $\{\mathcal{F}_t,\ t \in \mathbb{R}_+\}$. To this end, note that for all $n \geq 1$ and $v \in (L^2(\Omega \times [0,T],\, \mathcal{J}_T^*,\, \mathbb{P}))^\perp$, we have $\langle u^n, v\rangle = 0$. Then, we also have

$$\lim_{n\to\infty} \langle u^n, v\rangle = \langle u, v\rangle = 0,$$

which implies that $u \in L^2(\Omega \times [0,T],\, \mathcal{J}_T^*,\, \mathbb{P})$.

Second, we show that u satisfies the demand restrictions, i.e.

$$0 \leq u(\omega,t) \leq d(\omega,t) \text{ for a.e. } (\omega,t) \in \Omega \times [0,T].$$

Note that $\langle u^n, v\rangle \geq 0$ for all $n \geq 1$ and $v \in L^2(\Omega \times [0,T],\, \mathcal{F}_T \otimes \mathcal{B}[0,T],\, \mathbb{P})$ such that $v \geq 0$ for a.e. $(\omega,t) \in \Omega \times [0,T]$. Then, we have that $\langle u, v\rangle \geq 0$ as well for all $v \in L^2(\Omega \times [0,T],\, \mathcal{F}_T \otimes \mathcal{B}[0,T],\, \mathbb{P})$ such that $v \geq 0$, and hence $u(\omega,t)$ is non-negative for a.e. $(\omega,t) \in \Omega \times [0,T]$. To show that $u(\omega,t) \leq d(\omega,t)$ for a.e. $(\omega,t) \in \Omega \times [0,T]$, observe that for all $v \in L^2(\Omega \times [0,T],\, \mathcal{F}_T \otimes \mathcal{B}[0,T],\, \mathbb{P})$ such that $v \geq 0$ for a.e. $(\omega,t) \in \Omega \times [0,T]$, we have

$$\begin{aligned} 0 &\leq \langle d - u^n, v \rangle \, , \\ &= \langle d, v \rangle - \langle u^n, v \rangle \, . \end{aligned}$$

Then, we have $\langle d, v\rangle - \langle u, v\rangle = \langle d - u, v\rangle \geq 0$ and hence $d(\omega, t) - u(\omega, t) \geq 0$ for a.e. $(\omega, t) \in \Omega \times [0, T]$.

Third, we show that $A \int_0^T u(\omega, t) dt \leq C$ for a.e. $\omega \in \Omega$. It suffices to show that

$$\mathbb{E}\left[(C - A \int_0^T u(\omega, t)) \cdot \alpha(\omega) \right] \geq 0$$

for all square integrable α such that $\alpha \in \mathcal{F}_T$ and $\alpha(\omega) \geq 0$ for a.e. $\omega \in \Omega$. From feasibility of u^n, we have for all n,

$$\mathbb{E}\left[(C - A \int_0^T u^n(\omega, t)) \cdot \alpha(\omega) \right] \geq 0.$$

Assume without loss of generality that the capacity consumption matrix A has rank K. If not, we can simply consider a new capacity consumption matrix $\widetilde{A} = [I \ A]$ of dimension $K \times (K + J)$, where the demand for the first K products is equal to zero for all $(\omega, t) \in \Omega \times [0, T]$ and the demand for the rest of the products is given as before. Then, the definitions of $L^2(\Omega \times [0, T], \mathcal{F}_T \otimes \mathcal{B}[0, T], \mathbb{P})$ and the inner product are modified accordingly and the analysis to follow carry over to this problem.

Let $\beta \in \mathbb{R}^J$ such that $A\beta = C$. Such β exists since the column space of A is $\mathbb{R}^K$. Then, we have that $\left\langle \frac{\beta}{T} - u^n, A'\alpha \right\rangle = \mathbb{E}\left[(C - A \int_0^T u^n(\omega, t)) \cdot \alpha(\omega) \right]$. To see this, note that

$$\begin{aligned}
\left\langle \frac{\beta}{T} - u^n, A^{'}\alpha \right\rangle &= \mathbb{E}\left[\int_0^T \left(\frac{\beta}{T} - u^n(\omega,t)\right) \cdot A^{'}\alpha(\omega)dt\right], \\
&= \mathbb{E}\left[\int_0^T \left(\frac{A\beta}{T} - Au^n(\omega,t)\right) \cdot \alpha(\omega)dt\right], \\
&= \mathbb{E}\left[\left(C - A\int_0^T u^n(\omega,t)\right) \cdot \alpha(\omega)\right].
\end{aligned}$$

Then, $\left\langle \frac{\beta}{T} - u^n, A^{'}\alpha \right\rangle \geq 0$ for all n and square integrable $\alpha(\omega) \geq 0$ such that $\alpha \in \mathcal{F}_T$. Since

$$\lim_{n\to\infty} \left\langle \frac{\beta}{T} - u^n, A^{'}\alpha \right\rangle = \left\langle \frac{\beta}{T} - u, A^{'}\alpha \right\rangle,$$

this implies that for all square integrable $\alpha(\omega) \geq 0$ such that $\alpha \in \mathcal{F}_T$, we have

$$\mathbb{E}\left[\left(C - A\int_0^T u(\omega,t)\right) \cdot \alpha(\omega)\right] \geq 0.$$

Finally, we show that for all adapted fare processes $\{f(\omega,t) : (\omega,t) \in \Omega \times [0,T]\}$,

$$\mathbb{E}\left[\int_0^T f(\omega,t) \cdot u^n(\omega,t)dt\right] \to \mathbb{E}\left[\int_0^T f(\omega,t) \cdot u(\omega,t)dt\right] \text{ as } n \to \infty.$$

This follows simply from the definition of the weak limit and the fact that $f \in L^2(\Omega \times [0,T], \mathcal{F}_T \otimes \mathcal{B}[0,T], \mathbb{P})$. ■

B.3. Derivation of the dual network revenue management problem and the coextremality results

Derivation of the dual network revenue management problem ($\mathbf{D}_{cont}$). We will follow the road map provided by [**11**] to derive the dual problem of control associated with the network revenue management problem (P_{cont}). In particular, we first append the penalty expressions corresponding to the demand and capacity restrictions on bookings in the objective function by defining the convex, extended real valued integrand L and the convex functional l. We also formulate the problem towards minimization. Next, we compute the conjugate convex functions associated with L and l so as to define the dual integrand M and the dual functional m. The dual problem of control is defined using M and m.

The system dynamics equation for the network revenue management problem is given by

$$x(\omega,t) = C - \int_0^t Au(\omega,s)ds, \quad (\omega,t) \in \Omega \times [0,T]. \tag{B.3}$$

Comparing (B.3) with the set of admissible controls for the primal problem in the framework of [**11**], cf. Proposition I-1 of [**11**], first thing to note is that there is no stochastic integration term and martingale term in (B.3).

To facilitate the analysis to follow, define the indicator function $\chi_F(\cdot)$ for a given set F by

$$\chi_F(x) = \begin{cases} 0 & \text{if } x \in F \text{ a.s.,} \\ \infty & \text{otherwise.} \end{cases}$$

We express the network revenue management problem (P_{cont}) in terms of the convex integrand L and the convex lower semi-continuous functional l which are defined as follows. Define L on $\Omega \times [0,T] \times \mathbb{R}^K \times \mathbb{R}^K$ as

$$L(\omega, t, x, \dot{x}) = \begin{cases} -f(\omega,t) \cdot u + \chi_{\mathbb{R}^J_+}(u) + \chi_{\mathbb{R}^J_-}(u - d(\omega,t)) & \text{if } \dot{x} = -Au, \\ \infty & \text{otherwise.} \end{cases} \tag{B.4}$$

In (B.4), $\dot{x}$ denotes the rate of change of x, where x is the state variable denoting the vector of remaining capacities. The integrand L serves the purpose of eliminating the hard constraints of the network revenue management problem (P_{cont}) by appending them to the objective function as penalty expressions. In this sense, the penalty expression $\chi_{\mathbb{R}^J_+}(u)+\chi_{\mathbb{R}^J_-}(u-d(\omega,t))$ is the demand restriction on bookings and replaces the constraint

$$0 \leq u \leq d(\omega, t).$$

Notice also that we have reformulated the problem towards minimization and $-f(\omega,t) \cdot u$ is the negative of the rate at which revenue is generated. The system dynamics equation (B.3) is incorporated in L by the fact that we require $\dot{x}$ to be equal to $-Au$.

Next step is to define the functional l on $L_2^0 \times L_2^T$ with values on $\mathbb{R} \cup \{\infty\}$ so as to initiate the problem with capacity vector C and dictate non-negativity of remaining capacity at the terminal time T. The functional l is defined as

$$l(x_0, x_T) = l_0(x_0) + l_T(x_T), \tag{B.5}$$

where the convex, lower semi-continuous functionals l_0 and l_T are given by

$$l_0(x_0) = \chi_{\{C\}}(x_0), \quad l_T(x_T) = \chi_{\mathbb{R}_+^K}(x_T). \tag{B.6}$$

The functional l_0 replaces the constraint that $x(\omega, 0) = C$ for a.e. $\omega \in \Omega$ and l_T replaces the capacity constraint $x(\omega, T) \geq 0$ for a.e. $\omega \in \Omega$. Then, the network revenue management problem (P_{cont}) can equivalently be stated as a problem of minimizing

$$\mathbb{E}\left[\int_0^T L(\omega, t, x(\omega, t), \dot{x}(\omega, t))dt + l(x_0, x_T)\right].$$

As our second step in deriving the dual problem of control, we compute the conjugates to the functions L and l. Let L^* denote the conjugate to L. To be specific,

$$L^*(\omega, t, s, p) = \sup_{z \in \mathbb{R}^K,\, y \in \mathbb{R}^K} \{z \cdot s + y \cdot p - L(\omega, t, z, y)\} \text{ for } s, p \in \mathbb{R}^K. \tag{B.7}$$

We can express L^* more explicitly as follows. Note that $L(\omega, t, z, y) < \infty$ only if there exists some $u \in \mathbb{R}^J$ such that $y = -Au$ and $0 \leq u \leq d(\omega, t)$. Then, for $s, p \in \mathbb{R}^K$, we can write L^* as

$$\begin{aligned} L^*(\omega, t, s, p) &= \sup_{z \in \mathbb{R}^K,\, 0 \leq u \leq d(\omega, t)} \{z \cdot s - pAu - (-f(\omega, t) \cdot u)\}, \\ &= \sup_{z \in \mathbb{R}^K} \{z \cdot s\} + \sup_{0 \leq u \leq d(\omega, t)} \{(f(\omega, t) - pA) \cdot u\}, \\ &= \chi_{\{0\}}(s) + [f(\omega, t) - pA]^+ \cdot d(\omega, t). \end{aligned}$$

The first line is obtained by replacing y with $-Au$ for $0 \leq u \leq d(\omega, t)$ and noting that $L(\omega, t, z, y) = -f(\omega, t) \cdot u$. The second line follows from the observation that we can take the supremum in the first line separately for z and u. To get the third line, note that

$$\sup_{0 \leq u \leq d(\omega,t)} \{(f(\omega, t) - pA) \cdot u\} = [f(\omega, t) - pA]^{+} \cdot d(\omega, t),$$

by simple constrained maximization. Finally, we have $\sup_{z \in \mathbb{R}^K}\{z \cdot s\} = \chi_{\{0\}}(s)$, since $\sup_{z \in \mathbb{R}^K}\{z \cdot s\}$ takes the value ∞ if $s_k \neq 0$ for $k = 1, \ldots, K$.

Using the conjugate L^* of the primal integrand L, we calculate the dual integrand M. For $(\omega, t) \in \Omega \times [0, T]$ and $s, p \in \mathbb{R}^K$, the dual integrand M is given by

$$M(\omega, t, p, s) = L^*(\omega, t, s, p).$$

That is, for $(\omega, t) \in \Omega \times [0, T]$ we have

$$\begin{aligned} M(\omega, t, y(\omega, t), \dot{y}(\omega, t)) &= L^*(\omega, t, \dot{y}(\omega, t), y(\omega, t)), \\ &= \chi_{\{0\}}(\dot{y}(\omega, t)) + [0 \vee (f(\omega, t) - y(\omega, t)A)] \cdot d(\omega, t), \end{aligned}$$

where the expression $\chi_{\{0\}}(\dot{y}(\omega, t))$ in the second line forces $\dot{y}(\omega, t) = 0$ for a.e. $(\omega, t) \in \Omega \times [0, T]$. Then, the dynamics of the dual variable y is given by

$$\begin{aligned} y(\omega, t) &= y_0 + \int_0^t \dot{y}(\omega, s) ds + M(\omega, t), \\ &= y_0 + M(\omega, t), \end{aligned}$$

where M is a square integrable martingale null at zero.

What remains is to derive the terminal conditions associated with the dual problem. To that end, define the functional m on $L_2^0 \times L_2^T$ as follows:

$$m(y_0, y_T) = l_0^*(y_0) + l_T^*(-y_T),$$

where l_0^* and l_T^* are the conjugates of l_0 and l_T.. We calculate l_0^* as follows:

$$\begin{aligned} l_0^*(y) &= \sup_x \{y \cdot x - l_0(x)\}, \\ &= \sup_{x \in \{C\}} \{y \cdot x\}, \\ &= C \cdot y. \end{aligned}$$

A similar calculation yields $l_T^*(y) = \chi_{\mathbb{R}_-^K}(y)$. From [**11**], the functional m for the dual problem is given by

$$\begin{aligned} m(y_0, y_T) &= l_0^*(y_0) + l_T^*(-y_T), \\ &= C \cdot y_0 + \chi_{\mathbb{R}_-^K}(-y_T), \\ &= C \cdot y_0 + \chi_{\mathbb{R}_+^K}(y_T), \end{aligned} \tag{B.8}$$

where the expression $\chi_{\mathbb{R}_+^K}(y_T)$ imposes that $y(\omega, T) \geq 0$ for a.e. $\omega \in \Omega$.

The dual problem of control is then to minimize

$$\mathbb{E} \int_0^T M(\omega, t, y(\omega, t), \dot{y}(\omega, t))dt + m(y_0, y_T),$$

which is equivalent to minimizing

$$\mathbb{E}\left[\int_0^T d(\omega,t)\cdot[f(\omega,t)-y(\omega,t)A]^+dt + C\cdot y_0(\omega)\right]$$

subject to $\qquad$ (D$_{cont}$)

$$y(\omega,t) = y_0(\omega) + M(\omega,t), \quad (\omega,t)\in\Omega\times[0,T]$$

$$y(\omega,T)\geq 0,\ \omega\in\Omega,$$

where M is a square integrable martingale stopped at T, null at zero and adapted to the filtration $\{\mathcal{F}_t,\ t\in\mathbb{R}_+\}$. Since the network revenue management problem (P$_{cont}$) is trivially feasible (simply let $u(\omega,t)=0$ for all $(\omega,t)\in\Omega\times[0,T]$), the objective function values of (P$_{cont}$) and (D$_{cont}$) are equal to each other, cf. Theorem IV-1 of [**11**]. ■

Proof of Proposition 16. The network revenue management problem (P$_{cont}$) and the dual problem (D$_{cont}$) have the same optimal objective value by Theorem IV-1 of [**11**]. Moreover, by Theorem IV-2 of [**11**], letting u be a feasible control for (P$_{cont}$) with the corresponding state trajectory x, and (y_0,M) be a feasible control for (D$_{cont}$) with the corresponding state trajectory y, the controls u and (y_0,M) are optimal for (P$_{cont}$) and (D$_{cont}$), respectively, if and only if they satisfy the coextremality conditions stated in Definition IV-1 of [**11**]. To be more specific about the coextremality conditions for the network revenue management problem and its dual problem, we derive the subgradients of L, l_0 and l_T, where L is a convex integrand and l_0 and l_T are convex functionals as in the derivation of the dual network revenue management problem (D$_{cont}$).

First, we calculate the subgradient of L from its epigraphical normals. To that end, we use Theorem 8.9 of [**55**] which proves that for $h:\mathbb{R}^n\to[-\infty,+\infty]$ and any point $\overline{x}$

at which h is finite, one has

$$\partial h(\overline{x}) = \{v : (v, -1) \in N_{\text{epi } h}(\overline{x}, h(\overline{x}))\},$$

where, epi h denotes the epigraph of h defined as

$$\text{epi } h := \{(x, \alpha) \in \mathbb{R}^n \times \mathbb{R} : \alpha \geq h(x)\},$$

and $N_{\text{epi } h}(\overline{x}, h(\overline{x}))$ is the set of vectors normal to the set epi h at $(\overline{x}, h(\overline{x}))$ in the general sense as in Definition 6.3 of [**55**].

For $(\omega, t) \in \Omega \times [0, T]$, the epigraph of the integrand L at (ω, t) is given by

$$\text{epi } L(\omega, t) = \{(x, \dot{x}, \alpha) \in \mathbb{R}^{2K} \times \mathbb{R} : \ \dot{x} = -Au, \ 0 \leq u \leq d(\omega, t), \ \alpha \geq -f(\omega, t) \cdot u\},$$

since the points $(x, \dot{x}) \in \mathbb{R}^{2K}$ where $L(\omega, t, x, \dot{x}) = \infty$ are such that the vertical line $(x, \dot{x}) \times \mathbb{R}$ misses epi $L(\omega, t)$. Then, we can write

$$\text{(B.9)} \qquad \partial L(\omega, t, \overline{x}, \overline{\dot{x}}) = \{(v^1, v^2) \in \mathbb{R}^{2K} : (v^1, v^2, -1) \in N_{\text{epi } L(\omega,t)}(\overline{x}, \overline{\dot{x}}, L(\omega, t, \overline{x}, \overline{\dot{x}}))\}.$$

First, note that for $(\omega, t) \in \Omega \times [0, T]$, epi $L(\omega, t)$ is a convex set and the point

$$(\overline{x}, \overline{\dot{x}}, L(\omega, t, \overline{x}, \overline{\dot{x}})$$

is an element of epi $L(\omega, t)$ for $(\overline{x}, \overline{\dot{x}}) \in \mathbb{R}^{2K}$. Let $\mathbf{v}$ denote an arbitrary element of $\mathbb{R}^{2K+1}$, where the first K components of $\mathbf{v}$ is denoted as v^1, the subsequent K components by v^2 and the last component by v^α. That is, $\mathbf{v} = [v^1, v^2, v^\alpha]'$, where $v^1, v^2 \in \mathbb{R}^K$ and $v^\alpha \in \mathbb{R}$.

Then, Theorem 6.9 of [**55**], gives

$$\text{(B.10)}\quad N_{\text{epi } L(\omega,t)}(\overline{x}, \overline{\dot{x}}, L(\omega,t,\overline{x},\overline{\dot{x}}))$$

$$\text{(B.11)}\quad = \{\mathbf{v} \in \mathbb{R}^{2K+1} : \ [(x, \dot{x}, \alpha) - (\overline{x}, \overline{\dot{x}}, L(\omega,t,\overline{x},\overline{\dot{x}}))] \cdot \mathbf{v} \ \leq 0, \ \forall (x, \dot{x}, \alpha) \in \text{epi } L(\omega,t)\}.$$

We next establish the following properties of $N_{\text{epi } L(\omega,t)}(\overline{x}, \overline{\dot{x}}, L(\omega,t,\overline{x},\overline{\dot{x}}))$ for $(\omega,t) \in \Omega \times [0,T]$, which will assist us in finding the subgradients of L.

Property 1. *For* $(\omega,t) \in \Omega \times [0,T]$, *if*

$$v = (v^1, v^2, v_\alpha)' \in N_{\text{epi } L(\omega,t)}(\overline{x}, \overline{\dot{x}}, L(\omega,t,\overline{x},\overline{\dot{x}})),$$

then $v^1 = 0$.

To verify Property 1, first note that any $\mathbf{v} = (v^1, v^2, v_\alpha)'$ such that $v^1_k < 0$ cannot be in $N_{\text{epi } L(\omega,t)}(\overline{x}, \overline{\dot{x}}, L(\omega,t,\overline{x},\overline{\dot{x}}))$. Suppose not. Then, we could find an element $(\widetilde{x}, \widetilde{\dot{x}}, \widetilde{\alpha})$ of epi $L(\omega,t)$ such that it is equal to $(\overline{x}, \overline{\dot{x}}, L(\omega,t,\overline{x},\overline{\dot{x}}))$ except the k^{th} component of $\widetilde{x}$, where we have $\widetilde{x}_k < \overline{x}_k$. However, we have

$$[(\widetilde{x}, \widetilde{\dot{x}}, \widetilde{\alpha}) - (\overline{x}, \overline{\dot{x}}, L(\omega,t,\overline{x},\overline{\dot{x}})] \cdot \mathbf{v} = (\widetilde{x}_k - \overline{x}_k) v^1_k > 0,$$

contradicting the fact that $(v^1, v^2, v_\alpha) \in N_{\text{epi } L(\omega,t)}(\overline{x}, \overline{\dot{x}}, L(\omega,t,\overline{x},\overline{\dot{x}}))$, cf. (D.28). Similarly, any (v^1, v^2, v_α) such that $v^1_k > 0$ cannot be an element of

$$N_{\text{epi } L(\omega,t)}(\overline{x}, \overline{\dot{x}}, L(\omega,t,\overline{x},\overline{\dot{x}})),$$

which proves Property 1. Coupled with (B.9) and Theorem IV-1 of [**11**], Property 1 proves that $\dot{y}(\omega,t) = 0$ for a.e. $(\omega,t) \in \Omega \times [0,T]$.

Property 2. *For* $(\omega,t) \in \Omega \times [0,T]$ *and* $j = 1,\ldots,J$ *with* $d_j(\omega,t) > 0$*, if* $\overline{\dot{x}} = -A\overline{u}$ *for* $\overline{u}$ *such that* $0 \leq \overline{u} \leq d(\omega,t)$ *and* $v = (v^1, v^2, v_\alpha) \in N_{\text{epi } L}(\overline{x}, \overline{\dot{x}}, L(\omega,t,\overline{x},\overline{\dot{x}}))$*, then, the following conditions hold.*

$$
\begin{aligned}
(v^2 A + v^\alpha f(\omega,t))_j \geq & \ \ 0, \text{ if } \ \overline{u}_j = 0, \\
(v^2 A + v^\alpha f(\omega,t))_j = & \ \ 0, \text{ if } \ 0 < \overline{u}_j < d(\omega,t), \\
(v^2 A + v^\alpha f(\omega,t))_j \leq & \ \ 0, \text{ if } \ \overline{u}_j = d_j(\omega,t).
\end{aligned}
$$

To establish Property 2, first recall that for any $(x, \dot{x}, \alpha) \in \text{epi } L(\omega,t)$, there exists some $u \in \mathbb{R}^J$ such that $\dot{x} = -Au$, $0 \leq u \leq d(\omega,t)$ and $\alpha \geq -f(\omega,t)\cdot u$. Consider now an element $(\overline{x}, \dot{x}, -f(\omega,t)\cdot u)$ of epi $L(\omega,t)$, where $\dot{x} = -Au$. Then, we the following holds for $\mathbf{v} = (v^1, v^2, v_\alpha) \in N_{\text{epi } L}(\overline{x}, \overline{\dot{x}}, L(\omega,t,\overline{x},\overline{\dot{x}}))$.

$$
\begin{aligned}
& [(\overline{x}, \dot{x}, -f(\omega,t)\cdot u) - (\overline{x}, \overline{\dot{x}}, L(\omega,t,\overline{x},\overline{\dot{x}})] \cdot \mathbf{v} \\
= \ & v^1 \cdot (\overline{x} - \overline{x}) + v^2 \cdot (\dot{x} - \overline{\dot{x}}) + v^\alpha(-f(\omega,t)\cdot u - L(\omega,t,\overline{x},\overline{\dot{x}})), \\
= \ & v^2 \cdot (A\overline{u} - Au) + v^\alpha(-f(\omega,t)\cdot u + f(\omega,t)\cdot \overline{u}), \\
= \ & (v^2 A + v^\alpha f(\omega,t)) \cdot (\overline{u} - u).
\end{aligned}
$$

First, consider the case when $\overline{u}_j = 0$ for some $j = 1,\ldots,J$. If $u = \overline{u}$ except for the j^{th} component, we have $(v^2 A + v^\alpha f(\omega,t))\cdot(\overline{u} - u) \leq 0$, only if $(v^2 A + v^\alpha f(\omega,t))_j \geq 0$. From (D.28), since $(\overline{x}, \dot{x}, -f(\omega,t)\cdot u)$ is an element of epi $L(\omega,t)$, this proves the first part of Property 2, namely, if $\overline{u}_j = 0$, then $(v^2 A + v^\alpha f(\omega,t))_j \geq 0$. The argument is similar for the cases when $0 < \overline{u}_j < d(\omega,t)$ and $\overline{u}_j = d(\omega,t)$ and this completes the proof of Property 2.

To summarize, for $(\overline{x}, \overline{\dot{x}})$ such that $L(\omega,t,\overline{x},\overline{\dot{x}}) < \infty$ and $\overline{\dot{x}} = -A\overline{u}$, if $(v^1, v^2, v_\alpha) \in N_{\text{epi } L(\omega,t)}(\overline{x}, \overline{\dot{x}}, L(\omega,t,\overline{x},\overline{\dot{x}}))$, then from Property 2, the following conditions hold for $j = 1,\ldots,J$.

$$\begin{aligned}
(v^2 A + v^\alpha f(\omega,t))_j \geq & \quad 0, \;\text{ if } \;\overline{u}_j = 0, \\
(v^2 A + v^\alpha f(\omega,t))_j = & \quad 0, \;\text{ if } \;0 < \overline{u}_j < d_j(\omega,t), \\
(v^2 A + v^\alpha f(\omega,t))_j \leq & \quad 0, \;\text{ if } \;\overline{u}_j = d_j(\omega,t).
\end{aligned}$$

Recall that the subgradient of L is related to the normal cone of its epigraph as follows.

$$\partial L(\omega,t,\overline{x},\overline{\dot{x}}) = \{(v^1,v^2) : (v^1,v^2,-1) \in N_{\text{epi } L}(\overline{x},\overline{\dot{x}},L(\omega,t,\overline{x},\overline{\dot{x}}))\},$$

The coextremality conditions in Definition IV-1 of [**11**], state that $d\mathbb{P}\otimes dt$ a.s.

$$(\dot{y}(\omega,t), y(\omega,t)) \in \partial L(\omega,t,x(\omega,t),\dot{x}(\omega,t)).$$

That is, for a.e. $(\omega,t) \in \Omega\times[0,T]$,

$$(\dot{y}(\omega,t), y(\omega,t), -1) \in N_{\text{epi } L}(x(\omega,t),\dot{x}(\omega,t),L(\omega,t,x(\omega,t),\dot{x}(\omega,t)))\}.$$

This implies that for $j = 1, \ldots, J$ and a.e. $(\omega, t) \in \Omega \times [0, T]$, we have

$$(y(\omega,t)A - f(\omega,t))_j \geq 0, \text{ if } u_j(\omega,t) = 0, \tag{B.12}$$

$$(y(\omega,t)A - f(\omega,t))_j = 0, \text{ if } 0 < u_j(\omega,t) < d_j(\omega,t), \tag{B.13}$$

$$(y(\omega,t)A - f(\omega,t))_j \leq 0, \text{ if } u_j(\omega,t) = d_j(\omega,t), \tag{B.14}$$

which establishes the coextremality conditions stated in (2.13).

To complete the proof of Proposition 16, we calculate the subgradients $\partial l_0(x_0)$, and $\partial l_T(x_T)$, and derive the coextremality condition (D.1). First, consider l_0, which is defined as $l_0(x_0) = \chi_{\{C\}}(x_0)$.We will use Theorem 8.9 of [**55**] to calculate $\partial l_0(\overline{x})$. At any point $\overline{x}$ for which l_0 is finite, we have

$$\partial l_0(\overline{x}) = \{v : (v, -1) \in N_{\text{epi } l_0}(\overline{x}, l_0(\overline{x}))\},$$

where epi l_0 is given by

$$\begin{aligned} \text{epi } l_0 &= \{(x, \alpha) \in \mathbb{R}^K \times \mathbb{R} : x = C, \ \alpha \geq 0\}, \\ &= C_1 \times \ldots \times C_K \times \mathbb{R}_+. \end{aligned}$$

Notice that epi l_0 is a box, and hence, we can use Example 6.10 of [**55**] to calculate its normal cone. As a result,

$$N_{\text{epi } l_0}(\overline{x}, l_0(\overline{x})) = N_{C_1}(\overline{x}_1) \times \ldots \times N_{C_K}(\overline{x}_K) \times N_{\mathbb{R}_+}(l_0(\overline{x})),$$

where $N_{C_k}(\overline{x}_1) = (-\infty, \infty)$ for $k = 1, \ldots, K$ since C_k is a one-point interval. Finally, $N_{\mathbb{R}_+}(l_0(\overline{x})) = (-\infty, 0]$ because for a feasible $\overline{x}$, we have $l_0(\overline{x}) = 0$, and in that case we are at the left end point of the interval $\mathbb{R}_+$, which implies through Example 6.10 of [**55**] that $N_{\mathbb{R}_+}(l_0(\overline{x})) = (-\infty, 0]$. In consequence, the coextremality condition $y_0 \in \partial l_0(x_0)$, cf. Definition IV-1 of [**11**], for a feasible y_0 is equivalent to

$$(y_0, -1) \in N_{\text{epi } l_0}(y_0, l_0(y_0))$$

and places no further restrictions on y_0.

Finally, we calculate the subgradient of $l_T(x_T)$ where $l_T(x_T) = \chi_{\mathbb{R}_+^K}(x_T)$.Again, epi l_0 is a box. Indeed, epi $l_0 = \mathbb{R}_+^{J+1}$, and we can resort to Example 6.10 of [**55**]. We have

$$N_{\text{epi } l_T}(\overline{x}, l_T(\overline{x})) = N_{\mathbb{R}_+}(\overline{x}_1) \times \ldots \times N_{\mathbb{R}_+}(\overline{x}_K) \times N_{\mathbb{R}_+}(l_T(\overline{x})).$$

Consequently, $N_{C_k}(\overline{x}_1) = (-\infty, 0]$ for $\overline{x}_1 \geq 0$ and $N_{C_k}(\overline{x}_1) = \{0\}$ if $\overline{x}_1 > 0$ for $k = 1, \ldots, K$. Finally, $N_{\mathbb{R}_+}(l_T(\overline{x})) = (-\infty, 0]$ since $l_T(\overline{x}) = 0$ for a feasible $\overline{x}$. Thus, the coextremality condition $-y_T \in \partial l_T(x_T)$ in Definition IV-1 of [**11**], implies that $y(\omega, T) \geq 0$ and $y(\omega, T) \cdot x(\omega, T) = 0$ for a.e. $\omega \in \Omega$. This establishes the coextremality condition D.1 and completes the proof of Proposition 16. ■

B.4. Proofs in Section 2.4

Proof of Theorem 18. We interpret the dual variables in dual network revenue management problem (D$_{cont}$) as the opportunity cost of resources and construct optimal bid-price and capacity usage limit processes for the network revenue management problem (P$_{cont}$) using them. To this end, fix an optimal solution u to (P$_{cont}$) and an optimal solution

y to its dual (D_{cont}). Given the optimal solutions u and y, define bid-price process π and the capacity usage limit process λ as follows:

$$\text{(B.15)} \qquad \pi(\omega,t) := y(\omega,t) \;\text{ and }\; \lambda(\omega,t) := Au(\omega,t) \text{ for } (\omega,t) \in \Omega\times[0,T].$$

Then, the bid-price process π is a martingale adapted to $\{\mathcal{F}_t,\, t\in\mathbb{R}_+\}$ since y is so. Having defined π and λ, let $u^{(\pi,\lambda)}(\omega,t)$ denote the booking rate vector under the generalized bid-price control (π,λ) for $(\omega,t)\in\Omega\times[0,T]$. That is, $u^{(\pi,\lambda)}(\omega,t)$ solves ($\mathrm{P}(\omega,t)$) for $(\omega,t)\in\Omega\times[0,T]$. Observe that the booking rate process $\{u^{(\pi,\lambda)}\,(\omega,t) : (\omega,t)\in\Omega\times[0,T]\}$ is clearly feasible for the network revenue management problem (P_{cont}). To see this, note that since $u^{(\pi,\lambda)}(\omega,t)$ solves ($\mathrm{P}(\omega,t)$), it clearly satisfies the demand constraints. Moreover, we have

$$Au^{(\pi,\lambda)}(\omega,t) \le \lambda(\omega,t).$$

Then integrating both sides of this over $[0,T]$, using the definition of $\lambda(\omega,t)$, cf. (B.15), and the fact that u is feasible for (P_{cont}), we conclude that

$$AU^{(\pi,\lambda)\cdot}(\omega,t) \le \int_0^T \lambda(\omega,t)dt = A\int_0^T u(\omega,t) = AU(\omega,T) \le C,$$

where $U^{(\pi,\lambda)}(\omega,t)$ denotes the vector of cumulative bookings under (π,λ) up to time T. Thus, the booking policy $u^{(\pi,\lambda)}$ is feasible for (P_{cont}).

To establish the optimality of $u^{(\pi,\lambda)}$, we will show that $u^{(\pi,\lambda)}$ maximizes the expected revenues, that is,

$$\mathbb{E}\int_0^T f(\omega,t)\cdot u^{(\pi,\lambda)}(\omega,t)dt = \mathbb{E}\int_0^T f(\omega,t)\cdot u(\omega,t)dt.$$

To that end, first note that $u(\omega,t)$ is feasible for $(\mathrm{P}(\omega,t))$. To see this, note that $0 \leq u(\omega,t) \leq d(\omega,t)$, which follows because u solves (P_{cont}), and that $Au(\omega,t) = \lambda(\omega,t)$ by definition of λ, cf. (B.15). Then $u(\omega,t)$ solves $(\mathrm{P}(\omega,t))$ for a.e. $(\omega,t) \in \Omega \times [0,T]$ because $u_j(\omega,t) = d_j(\omega,t)$ whenever $y(\omega,t)A^j < f_j(\omega,t)$ by the coextremality conditions, cf. Proposition 16. That is, $u(\omega,t)$ is an optimal solution to $(\mathrm{P}(\omega,t))$, which in particular exhausts the capacity usage limit, *i.e.* $Au(\omega,t) = \lambda(\omega,t)$ by construction of λ. In other words, $u(\omega,t)$ not only maximizes the first term in the objective of $(\mathrm{P}(\omega,t))$ but also the second term by setting it to zero, which is the maximum it can be since we require $Au \leq \lambda(\omega,t)$ in $(\mathrm{P}(\omega,t))$. Then since $(\mathrm{P}(\omega,t))$ is lexicographic any bookings under (π,λ) must not only maximize the first term but also it must set the second term to zero. In particular, we must have

$$Au^{(\pi,\lambda)}(\omega,t) = \lambda(\omega,t) \ \text{ for a.e. } (\omega,t) \in \Omega \times [0,T].$$

Moreover, $Au(\omega,t) = \lambda(\omega,t)$ by construction of (π,λ), cf. (B.15). In other words, the primal controls u and $u^{(\pi,\lambda)}$ result in the same state trajectory.

Since both $u(\omega,t)$ and $u^{(\pi,\lambda)}(\omega,t)$ are optimal solutions for $(\mathrm{P}(\omega,t))$, for a.e. $(\omega,t) \in \Omega \times [0,T]$, we have

$$(f(\omega,t) - A'\pi(\omega,t))\cdot u^{(\pi,\lambda)}(\omega,t) = (f(\omega,t) - A'\pi(\omega,t))\cdot u(\omega,t). \tag{B.16}$$

Moreover, as argued immediately above we also have

$$Au^{(\pi,\lambda)} = \lambda(\omega, t) = Au(\omega, t). \tag{B.17}$$

Then combining (B.16)-(B.17), we conclude that

$$f(\omega, t) \cdot u^{(\pi,\lambda)}(\omega, t) = f(\omega, t) \cdot u(\omega, t) \quad \text{for a.e.} \quad (\omega, t) \in \Omega \times [0, T].$$

Thus, the expected revenue generated by $u^{(\pi,\lambda)}$ is equal to the expected revenue generated by u, proving the optimality of the generalized bid-price control (π, λ) for the continuous network revenue management problem (P_{cont}). ■

B.5. Proofs in Section 2.5

Derivation of the dual problem ($\mathbf{D}^{\varepsilon}$). We follow the same steps as in the derivation of (D_{cont}). That is, we first append the penalty expressions associated with the demand and capacity restrictions on bookings in the objective function by defining the convex, extended real valued integrand L_{ε} and the convex functional l_{ε}. The problem is also formulated towards minimization. Next, we compute the conjugate convex functions associated with L_{ε} and l_{ε} and define the dual integrand M_{ε} and the dual functional m_{ε}, by the help of which we define the dual problem of control.

Define L_{ε}, the normal convex integrand in the sense of [**51**], on $\Omega \times [0, \infty) \times \mathbb{R}^K \times \mathbb{R}^K$, as follows:

$$L_{\varepsilon}(\omega, t, x, \dot{x}) = -f(\omega, t) \cdot u + \sum_{j=1}^{J} \frac{\varepsilon_j(\omega, t) u_j^2}{2} \cdot u + \chi_{\mathbb{R}^J_+}(u) + \chi_{\mathbb{R}^J_-}(u - d(\omega, t)), \tag{B.18}$$

if $\dot{x} = -Au$ and $L_\varepsilon(\omega, t, x, \dot{x}) = \infty$ otherwise. Since the terminal conditions of (P^ε) are the same as the terminal conditions of (P_{cont}), the functional l_ε, which specifies the terminal conditions and the initial system parameters, is the same as l, cf. (D.22) and (D.23). Hence, all the terminal conditions in the dual formulation (D^ε) are the same as the terminal conditions of (D_{cont}). Then, (P^ε) is equivalent to minimizing

$$\mathbb{E}\int_0^T L_\varepsilon(\omega, t, x(\omega, t), \dot{x}(\omega, t))dt + l_\varepsilon(x_0, x_T).$$

In order to define the dual problem of control, we derive the conjugate to the function L_ε. Let L_ε^* be the dual integrand of L_ε. That is,

$$\begin{aligned}
L_\varepsilon^*(\omega, t, s, p) &= \sup_{z\in\mathbb{R}^K,\ y\in\mathbb{R}^K} \{z\cdot s + y\cdot p - L_\varepsilon(\omega, t, z, y)\} \quad \text{for} \quad s, p \in \mathbb{R}^K \\
&= \sup_{0\le u\le d(\omega,t)} \{z\cdot s - pAu - (-f(\omega,t)\cdot u + \sum_{j=1}^{J}\frac{\varepsilon_j(\omega,t)u_j^2}{2})\}, \\
&= \sup_{z\in\mathbb{R}^K}\{z\cdot s\} + \sup_{0\le u\le d(\omega,t)}\{(f(\omega,t) - pA)\cdot u - \sum_{j=1}^{J}\frac{\varepsilon_j(\omega,t)u_j^2}{2})\}, \\
&= \chi_{\{0\}}(s) + g_\varepsilon(f(\omega,t) - pA, d(\omega,t)),
\end{aligned}$$

where $g_\varepsilon(z,d) = \sum_{j=1}^{J} h_\varepsilon(z_j, d_j)$ and h_ε is given by (2.16). The second line is obtained by replacing y with $-Au$ for $0 \le u \le d(\omega,t)$ and noting that $L_\varepsilon(\omega,t,z,y) = -f(\omega,t)\cdot u + \sum_{j=1}^{J}\frac{\varepsilon_j(\omega,t)u_j^2}{2}$. The third line follows from the observation that we can take the supremum in the second line separately for z and u. To get the fourth line, note that $g_\varepsilon(z, d(\omega,t))$ is the value function of the following maximization problem.

$$\text{Maximize}_{0\le v\le d(\omega,t)}\ z\cdot v-\sum_{j=1}^{J}\frac{\varepsilon_j(\omega,t)}{2}v_j^2,$$

whose solution for $j=1,\dots,J$ is given by

$$v_\varepsilon^*(z,\omega,t)=(v_{1,\varepsilon}^*,\dots,v_{J,\varepsilon}^*),\ v_{j,\varepsilon}^*(z_j,\omega,t)=\begin{cases}0 & \text{if} \quad z_j\le 0,\\ d_j(\omega,t) & \text{if} \quad z_j\ge\varepsilon,\\ \frac{z_j}{\varepsilon}d_j(\omega,t) & \text{if} \quad 0<z_j<\varepsilon.\end{cases}$$

The terminal conditions associated with the dual problem to (P^ε) are derived as follows. As mentioned above, we have $l_\varepsilon=l$ and, hence $l_\varepsilon^*=l^*$. This, in turn implies that $m_\varepsilon=m$, where m is defined in (B.8). The dual problem of control is then to minimize

$$\mathbb{E}\int_0^T M_\varepsilon(\omega,t,y(\omega,t),\dot{y}(\omega,t))dt+m_\varepsilon(y_0,y_T),$$

which is equivalent to minimizing

$$\mathbb{E}\left[\int_0^T g_\varepsilon(f(\omega,t)-y(\omega,t)A,d(\omega,t))dt+C\cdot y_0(\omega)\right]$$

subject to (D^ε)

$$y(\omega,t)=y_0(\omega)+M(\omega,t),\quad (\omega,t)\in\Omega\times[0,T]$$

$$y(\omega,T)\ge 0,\ \omega\in\Omega,$$

where M is a square integrable martingale stopped at T, null at zero and adapted to the filtration $\{\mathcal{F}_t,\ t\in\mathbb{R}_+\}$. Since the primal problem (P^ε) is trivially feasible (simply let

$u(\omega,t)=0$ for all $(\omega,t)\in\Omega\times[0,T]$), the objective function values of (P^{ε}) and (D^{ε}) are equal to each other, cf. Theorem IV-2 of [**11**]. ■

Proof of Proposition 21. The perturbed problem (P^{ε}) and its dual problem (D^{ε}) have the same optimal objective value by Theorem IV-1 of [**11**]. From Theorem IV-2 of [**11**], letting u^{ε} be a feasible control for (P^{ε}) with the corresponding state trajectory x^{ε}, and $(y_0^{\varepsilon}, M^{\varepsilon})$ be a feasible control for (D^{ε}) with the corresponding state trajectory y^{ε}, the controls u^{ε} and $(y_0^{\varepsilon}, M^{\varepsilon})$ are optimal for (P^{ε}) and (D^{ε}), respectively, if and only if they satisfy the coextremality conditions stated in Definition IV-1 of [**11**]. Recall that the terminal conditions of the perturbed problem (P^{ε}) and the network revenue management problem (P_{cont}) are the same. Thus, the coextremality conditions for the problems (P^{ε}) and (D^{ε}) regarding the terminal conditions are the same as those for the problems (P_{cont}) and (D_{cont}). This, in turn, establishes the coextremality condition (2.17). To compute the coextremality conditions stated in Definition IV-1 of [**11**], the subgradient of the convex integrand L_{ε} defined in (B.18) needs to be calculated.

To calculate the subgradient of L_{ε}, first note that for $(\omega,t)\in\Omega\times[0,T]$, we can write the following:

$$\text{(B.19)} \qquad L_{\varepsilon}(\omega,t,x,\dot{x}) - L(\omega,t,x,\dot{x}) - \sum_{j=1}^{J}\frac{\varepsilon_j(\omega,t)u_j^2}{2} = 0 \ \text{ if } \dot{x} = -Au,$$

where L is the convex integrand for the network revenue management problem (P_{cont}) given in (B.4). Let $\partial_u L_{\varepsilon}(\omega,t,x,\dot{x})$ and $\partial_{\dot{x}} L_{\varepsilon}(\omega,t,x,\dot{x})$ denote the subgradients of L_{ε} with respect to u and $\dot{x}$, respectively. That is, $\partial_{\dot{x}} L_{\varepsilon}(\omega,t,x,\dot{x})$ is the projection of the set $\partial L_{\varepsilon}(\omega,t,x,\dot{x})$ on $\dot{x}$ axis. $\partial_u L_{\varepsilon}(\omega,t,x,\dot{x})$ is similarly defined by viewing $\dot{x}$ as a function of u. Notice that L and L_{ε} are convex and piecewise linear quadratic, cf. Definition 10.20

of [**55**]. Therefore, from Exercise 10.22 of [**55**], we can calculate the subgradients of each term in (B.19) separately. The subgradient of L with respect to $\dot{x}$ is already calculated in the proof of Proposition 16. To calculate $\partial_u L(\omega, t, x, \dot{x})$ and $\partial_u L_\varepsilon(\omega, t, x, \dot{x})$ at $(x, \dot{x})$ such that $L(\omega, t, x, \dot{x}), L_\varepsilon(\omega, t, x, \dot{x}) < \infty$, we will use the basic chain rule for subgradients as in Theorem 10.6 of [**55**], which implies that

$$\partial_{\dot{x}} L_\varepsilon(\omega, t, x, \dot{x})^*(-A') - \partial_{\dot{x}} L(\omega, t, x, \dot{x})^*(-A') - \Delta^\varepsilon(\omega, t, u) = \{0\},$$

where $\Delta_j^\varepsilon(\omega, t, u) = \varepsilon_j(\omega, t) u_j$. Then, $y(\omega, t) \in \partial_{\dot{x}} L_\varepsilon(\omega, t, x, \dot{x})$ if and only if

$$(-y(\omega, t)A - \Delta^\varepsilon(\omega, t, u)) \in -A^* \partial_{\dot{x}} L(\omega, t, x, \dot{x}). \tag{B.20}$$

Definition IV-1 of [**11**] and the coextremality conditions in Proposition 16 imply that if $v \in A^* \partial_{\dot{x}} L(\omega, t, x, \dot{x})$, then for $j = 1, \ldots, J$ with $d_j(\omega, t) > 0$,

$$v_j - f_j(\omega, t) \geq 0, \text{ if } u_j = 0, \tag{B.21}$$

$$v_j - f_j(\omega, t) = 0, \text{ if } 0 < u_j < d_j(\omega, t), \tag{B.22}$$

$$v_j - f_j(\omega, t) \leq 0, \text{ if } u_j = d_j(\omega, t). \tag{B.23}$$

Then, together with (B.20), the conditions (B.21)-(B.23) establish the coextremality conditions (2.18). ■

Proof of Proposition 22. For each $\varepsilon > 0$, the optimality of the control $\{u^\varepsilon(\omega, t) : (\omega, t) \in \Omega \times [0, T]\}$ given in (2.20) for the perturbed problem (P^ε) follows from Theorem IV-2 of [**11**], and the fact that u^ε and y^ε satisfy the coextremality conditions stated in Proposition 21 where y^ε is an optimal state trajectory for (D^ε). We next argue that for each $\varepsilon > 0$, (P^{ε}) has a unique solution. Suppose not. Then, there exists booking controls $\widetilde{u}$ and $\overline{u}$ that are optimal for (P^{ε}) and yet are not equal on a set of strictly positive $d\mathbb{P} \otimes dt$ measure. From Theorem IV-2 of [**11**], both $\widetilde{u}$ and $\overline{u}$ should satisfy the coextremality conditions (2.17)-(2.18) with y^{ε}. However, this implies that $\widetilde{u}(\omega, t)$ and $\overline{u}(\omega, t)$ are equal for a.e. $(\omega, t) \in \Omega \times [0, T]$. Contradiction. Hence, for each $\varepsilon > 0$, u^{ε} is the unique solution for (P^{ε}). ■

Proof of Theorem 23. The bid-price process π^{ε} is defined as $\pi^{\varepsilon} = y^{\varepsilon}$, where y^{ε} is an optimal state trajectory for the perturbed dual problem (D^{ε}). Since y^{ε} forms a martingale, so does π^{ε}. The terminal condition $y^{\varepsilon}(\omega, T) \geq 0$ for a.e. $\omega \in \Omega$ and the fact that y^{ε} is a martingale guarantees that $\pi^{\varepsilon}(\omega, t) \geq 0$ for a.e. $(\omega, t) \in \Omega \times [0, T]$, which proves the first part of Theorem 23**.**

Next, we prove that the bid-price policy $(\pi^{\varepsilon}, \phi^{\varepsilon})$ is κ-optimal, where κ is given by (2.21). To that end, let u be an optimal booking control for the network revenue management problem (P_{cont}). Notice that u is also feasible for the perturbed problem (P^{ε}). From Proposition 22, the control $\{u^{\varepsilon}(\omega, t) : (\omega, t) \in \Omega \times [0, T]\}$ resulting from the bid-price policy $(\pi^{\varepsilon}, \phi^{\varepsilon})$ is the unique optimal control for the perturbed problem (P^{ε}) for each $\varepsilon > 0$. Then, the objective value of u^{ε} is greater than or equal to the objective value of u

for the perturbed problem (P^{ε}). Thus, we have

$$
\begin{aligned}
&\mathbb{E}\left[\int_0^T [f(\omega,t)\cdot u(\omega,t)]\,dt\right] - \varepsilon \sum_{j=1}^{J} \int_0^T \mathbb{E}[d_j(\omega,t)]dt \\
\leq\ & \mathbb{E}\left[\int_0^T \Big[f(\omega,t)\cdot u(\omega,t) - \frac{1}{2}\sum_{j=1}^{J} \varepsilon_j(\omega,t)\, u_j^2(\omega,t)\Big]\,dt\right] \\
\leq\ & \mathbb{E}\left[\int_0^T \Big[f(\omega,t)\cdot u^{\varepsilon}(\omega,t) - \frac{1}{2}\sum_{j=1}^{J} \varepsilon_j(\omega,t)\, (u_j^{\varepsilon}(\omega,t))^2\Big]\,dt\right], \\
\leq\ & \mathbb{E}\left[\int_0^T [f(\omega,t)\cdot u^{\varepsilon}(\omega,t)]\,dt\right]. \qquad \text{(B.24)}
\end{aligned}
$$

The first inequality follows from the definition of $\varepsilon_j(\omega,t)$, cf. (2.15), and the fact that $u(\omega,t) \leq d(\omega,t)$ for a.e. $(\omega,t) \in \Omega\times[0,T]$. The second inequality is given by feasibility of u for (P^{ε}) and optimality of u^{ε}. The last inequality holds since $\sum_{j=1}^{J} \varepsilon_j(\omega,t)\,(u_j^{\varepsilon}(\omega,t))^2 \geq 0$, and it proves that u^{ε} is $\kappa\varepsilon$-optimal since u^{ε} is also feasible for (P_{cont}) and completes the proof of the second part of Theorem 23.

Finally, we show that every weak limit $\widetilde{u} \in \mathcal{U}$ of the booking controls $\{u^{\varepsilon} : \varepsilon > 0\}$ is an optimal booking control for the network revenue management problem (P_{cont}). Let $\{u^{\varepsilon_n} : n \geq 1\}$ be a sequence of feasible controls for the network revenue management problem (P) which converges to $\widetilde{u}$ in the weak* topology as $\varepsilon_n \searrow 0$. From Lemma 57, $\widetilde{u}$ is a feasible booking control for (P) and, we have

$$
\lim_{n\to\infty} \mathbb{E}\left[\int_0^T f(\omega,t)\cdot u^{\varepsilon_n}(\omega,t)dt\right] = \mathbb{E}\left[\int_0^T f(\omega,t)\cdot \widetilde{u}(\omega,t)dt\right].
$$

The optimality of $\widetilde{u}$, then follows from (B.24). ■

APPENDIX C

Proofs of the Results in Chapter 3

Derivation of the dual problem ($\mathbf{D}^{\varepsilon,\delta}$). As in the derivations of (D_{cont}) and (D^{ε}), we follow the road map given by [**11**]. We express the primal problem ($P^{\varepsilon,\delta}$) in terms of the convex integrand $L_{\varepsilon,\delta}$ and the convex lower semi-continuous functional $l^{\varepsilon,\delta}$ which are defined as follows. Define the normal convex integrand $L_{\varepsilon,\delta}$ on $\Omega \times [0,\infty) \times \mathbb{R}^K \times \mathbb{R}^K$ as:

(C.1)
$$L_{\varepsilon,\delta}(\omega,t,x,\dot{x}) = -f(\omega,t)\cdot u + \sum_{j=1}^{J} \frac{\varepsilon_j(\omega,t)u_j^2}{2}\cdot u + \sum_{k=1}^{K}\psi_\delta(x_k) + \chi_{\mathbb{R}_+^J}(u) + \chi_{\mathbb{R}_-^J}(u - d(\omega,t))$$

if if $\dot{x} = -Au$ and $L_{\varepsilon,\delta}(\omega,t,x,\dot{x}) = \infty$ otherwise. Define the functional $l^{\varepsilon,\delta}$ on $L_2^0 \times L_2^T$ with values on $\mathbb{R}\cup\{\infty\}$, which will initiate the problem with capacity vector C. The functional $l^{\varepsilon,\delta}$ is defined as

$$l^{\varepsilon,\delta}(x_0,x_T) = l_0^{\varepsilon,\delta}(x_0) + l_T^{\varepsilon,\delta}(x_T), \tag{C.2}$$

where the convex, lower semi-continuous functionals $l_0^{\varepsilon,\delta}$ and $l_T^{\varepsilon,\delta}$ are given by

$$l_0^{\varepsilon,\delta}(x_0) = \chi_{\{C\}}(x_0), \quad l_T^{\varepsilon,\delta}(x_T) = 0. \tag{C.3}$$

The functional $l_0^{\varepsilon,\delta}$ sets $x(\omega,0) = C$ for a.e. $\omega \in \Omega$, and $l_T^{\varepsilon,\delta}(x_T) = 0$ reflects the fact that there are no hard constraints on the remaining capacity at the terminal time. Then, the problem ($P^{\varepsilon,\delta}$) is equivalent to the problem of minimizing

$$\mathbb{E}\int_0^T L_{\varepsilon,\delta}(\omega,t,x(\omega,t),\dot{x}(\omega,t))dt + l^{\varepsilon,\delta}(x_0,x_T).$$

In order to define the dual problem of control, we derive the conjugate to the function $L_{\varepsilon,\delta}$. Let $L^*_{\varepsilon,\delta}$ denote the conjugate to $L_{\varepsilon,\delta}$. That is,

$$\begin{aligned}
& L^*_{\varepsilon,\delta}(\omega,t,s,p) \\
= \quad & \sup_{z\in\mathbb{R}^K,\ y\in\mathbb{R}^K} \{z\cdot s + y\cdot p - L_{\varepsilon,\delta}(\omega,t,z,y)\} \ \text{ for } \ s,p\in\mathbb{R}^K \\
= \quad & \sup_{z,\ 0\le u\le d(\omega,t)} \{z\cdot s - pAu - (-f(\omega,t)\cdot u + \sum_{j=1}^J \frac{\varepsilon_j(\omega,t)u_j^2}{2} + \sum_{k=1}^K \psi_\delta(z_k))\}, \\
= \quad & \sup_{z\in\mathbb{R}^K} \{z\cdot s - \sum_{k=1}^K \psi_\delta(z_k)\} \\
& + \sup_{0\le u\le d(\omega,t)} \{(f(\omega,t)-pA)\cdot u - \sum_{j=1}^J \frac{\varepsilon_j(\omega,t)u_j^2}{2})\}, \\
= \quad & \xi^\delta(s) + \chi_{\mathbb{R}^K_-}(s) + g_\varepsilon(f(\omega,t)-pA, d(\omega,t)),
\end{aligned}$$

We get the second line by replacing y with $-Au$ for $0\le u\le d(\omega,t)$ and noting that $L_{\varepsilon,\delta}(\omega,t,z,y) = -f(\omega,t)\cdot u + \sum_{j=1}^J \frac{\varepsilon_j(\omega,t)u_j^2}{2} + \sum_{k=1}^K \psi_\delta(z_k)$. The third line follows from the observation that we can take the supremum in the second line separately for z and u. To obtain the fourth line, note that $g_\varepsilon(f(\omega,t)-pA, d(\omega,t))$ is the value function of the following maximization problem:

$$\text{Maximize}_{0\le v\le d(\omega,t)}\ (f(\omega,t)-pA)\cdot v - \sum_{j=1}^J \frac{\varepsilon_j(\omega,t)}{2} v_j^2.$$

Finally, $\sup_{z\in\mathbb{R}^K}\{z\cdot s-\sum_{k=1}^K \psi_\delta(z_k)\}$ is equal to $\xi^\delta(s)+\chi_{\mathbb{R}^K_-}(s)$, where $\xi^\delta(z)=\sum_{k=1}^K \xi_k^\delta(z_k)$ and ξ_k^δ is given by (3.2) for $k=1,\dots,K$.

The dual integrand $M_{\varepsilon,\delta}$ is calculated from the conjugate $L^*_{\varepsilon,\delta}$ of the primal integrand $L_{\varepsilon,\delta}$ as follows. For $(\omega,t)\in\Omega\times[0,T]$ and $s,p\in\mathbb{R}^K$, the dual integrand $M_{\varepsilon,\delta}$ is given by

$$M_{\varepsilon,\delta}(\omega,t,p,s)=L^*_{\varepsilon,\delta}(\omega,t,s,p).$$

Then, for $(\omega,t)\in\Omega\times[0,T]$ we have

$$\begin{aligned}
& M_{\varepsilon,\delta}(\omega,t,y(\omega,t),\dot{y}(\omega,t)) \\
= \;& L^*_{\varepsilon,\delta}(\omega,t,\dot{y}(\omega,t),y(\omega,t)), \\
= \;& \xi^\delta(\dot{y}(\omega,t))+g_\varepsilon(f(\omega,t)-y(\omega,t)A,d(\omega,t))+\chi_{\mathbb{R}^K_-}(\dot{y}(\omega,t)),
\end{aligned}$$

Accordingly, the dynamics of the dual variable y for the problem $(\mathrm{D}^{\varepsilon,\delta})$ is given by

$$y(\omega,t)=y_0+\int_0^t \dot{y}(\omega,s)ds+M(\omega,t),$$

where M is a square integrable martingale, stopped at T, null at zero and $\dot{y}(\omega,t)\le 0$ for a.e. $(\omega,t)\in\Omega\times[0,T]$.

Finally, we derive the terminal conditions associated with the dual problem $(\mathrm{D}^{\varepsilon,\delta})$. To that end, define the functional $m_{\varepsilon,\delta}$ on $L_2^0\times L_2^T$ as follows:

$$m_{\varepsilon,\delta}(y_0,y_T)=(l_0^{\varepsilon,\delta})^*(y_0)+(l_T^{\varepsilon,\delta})^*(-y_T),$$

where $(l_0^{\varepsilon,\delta})^*$ and $(l_T^{\varepsilon,\delta})^*$ are the conjugates of $l_0^{\varepsilon,\delta}$ and $l_0^{\varepsilon,\delta}$. We have

$$(l_0^{\varepsilon,\delta})^*(y) = C \cdot y \quad \text{and} \quad (l_T^{\varepsilon,\delta})^*(y) = \chi_{\{0\}}(y).$$

The functional $m_{\varepsilon,\delta}$ for the dual problem is given by

$$\begin{aligned} m_{\varepsilon,\delta}(y_0, y_T) &= (l_0^{\varepsilon,\delta})^*(y_0) + (l_T^{\varepsilon,\delta})^*(-y_T), \\ &= C \cdot y_0 + \chi_{\{0\}}(y_T), \end{aligned}$$

where the expression $\chi_{\{0\}}(y_T)$ imposes that $y(\omega, T) = 0$ a.e. $\omega \in \Omega$.

The dual problem $(\mathrm{D}^{\varepsilon,\delta})$ associated with $(\mathrm{P}^{\varepsilon,\delta})$ is then to minimize

$$\mathbb{E} \int_0^T M_{\varepsilon,\delta}(\omega, t, y(\omega, t), \dot{y}(\omega, t))dt + m_{\varepsilon,\delta}(y_0, y_T),$$

which is equivalent to minimizing

$$\begin{aligned} &\mathbb{E}\left[\int_0^T [g_\varepsilon(f(\omega,t) - y(\omega,t)A, d(\omega,t)) + \xi^\delta(\dot{y}(\omega,t))]dt + C \cdot y_0(\omega)\right] \\ &\text{subject to} \\ &y(\omega,t) = y_0(\omega) + \int_0^t \dot{y}(\omega,s)ds + M(\omega,t), \quad (\omega,t) \in \Omega \times [0,T], \qquad (\mathrm{D}^{\varepsilon,\delta}) \\ &\dot{y}(\omega,t) \le 0, \quad (\omega,t) \in \Omega \times [0,T], \\ &y(\omega,T) = 0, \ \omega \in \Omega, \end{aligned}$$

where M is a square integrable martingale stopped at T, null at zero and adapted to the filtration $\{\mathcal{F}_t,\, t \in \mathbb{R}_+\}$. ■

Proof of Proposition 24. Let y be a feasible state trajectory for the problem ($\mathrm{D}^{\varepsilon,\delta}$). Suppose the statement is not true. Then, there exists a resource $k = 1, \ldots, K$ such that the set

$$B = \{(\omega, t) \in \Omega \times [0, T] : y_k(\omega, t) < 0\}$$

has strictly positive measure $d\mathbb{P} \otimes dt$. Now consider for $t \in [0, T]$, the sets

$$B_t = \{\omega \in \Omega : y_k(\omega, t) < 0\}.$$

B_t should have strictly positive measure for some t in a set that also has strictly positive Lebesgue measure. Fix such a time point t. We will next show that for $\omega \in B_t$, we have $y_j(\omega, T) < 0$ with strictly positive probability. First note that $B_t \in \mathcal{F}_t$ since $y_k(\omega, t) : \Omega \to \mathbb{R}$ is measurable with respect to $\mathcal{F}_t$. We have from the dual system dynamics in ($\mathrm{D}^{\varepsilon,\delta}$),

$$y_k(\omega, t) = (y_0)_k + \int_0^t \dot{y}_k(\omega, s)ds + M_k(\omega, t).$$

Then, $y_k(\omega, T)$ can be written as

$$y_k(\omega, T) = y_k(\omega, t) + \int_t^T \dot{y}_k(\omega, s)ds + M_k(\omega, T) - M_k(\omega, t).$$

Recall that M is a martingale adapted to $\{\mathcal{F}_t,\, t \in \mathbb{R}_+\}$ and is null at zero. Thus,

$$\mathbb{E}[(M_k(\omega, T) - M_k(\omega, t))\mathbf{1}_F] = 0$$

for all $F \in \mathcal{F}_t$ where $\mathbf{1}_F$ is the indicator function of the set F. Now consider

$$\begin{aligned}
\mathbb{P}\{\omega : \omega \in B_t,\ y_k(\omega, T) < 0\} &= \int_{B_t} \mathbf{1}_{\{\omega:\ y_k(\omega,T)<0\}} d\mathbb{P} \\
&\geq \int_{\Omega} \mathbf{1}_{B_t \cap \{\omega:\ M_k(\omega,T) - M_k(\omega,t) < -y_k(\omega,t)\}} d\mathbb{P}. \\
&> 0,
\end{aligned}$$

since the set

$$B_t \cap \{\omega :\ M_k(\omega, T) - M_k(\omega, t) < -y_k(\omega, t)\}$$

has strictly positive measure. To prove this, suppose not. Then, as M is a martingale adapted to $\{\mathcal{F}_t,\ t \in \mathbb{R}_+\}$ and $B_t \in \mathcal{F}_t$, we have

$$\begin{aligned}
0 &= \mathbb{E}\{(M_k(\omega, T) - M_k(\omega, t))\mathbf{1}_{B_t}\} \\
&= \int_{\Omega} \mathbf{1}_{B_t}(M_k(\omega, T) - M_k(\omega, t)) d\mathbb{P} \\
&= \int_{\Omega} \mathbf{1}_{B_t \cap \{\omega:\ M_k(\omega,T) - M_k(\omega,t) < -y_k(\omega,t)\}} (M_k(\omega, T) - M_k(\omega, t)) d\mathbb{P} \\
&\quad + \int_{\Omega} \mathbf{1}_{B_t \cap \{\omega:\ M_k(\omega,T) - M_k(\omega,t) \geq -y_k(\omega,t)\}} (M_k(\omega, T) - M_k(\omega, t)) d\mathbb{P} \\
&\geq (-y_k(\omega, t))\mathbb{P}(B_t) \\
&> 0,
\end{aligned}$$

which proves that every feasible dual state trajectory y is non-negative for a.e. $(\omega, t) \in \Omega \times [0, T]$. ■

Proof of Proposition 25. Theorem IV-1 of [**11**] implies that the primal problem ($\mathrm{P}^{\varepsilon,\delta}$) and its dual ($\mathrm{D}^{\varepsilon,\delta}$) have the same optimal objective value. Let $u^{\varepsilon,\delta}$ be a feasible control for ($\mathrm{P}^{\varepsilon,\delta}$) with the corresponding state trajectory $x^{\varepsilon,\delta}$, and $(y_0^{\varepsilon,\delta}, \dot{y}^{\varepsilon,\delta}, M^{\varepsilon,\delta})$ be a feasible control for ($\mathrm{D}^{\varepsilon,\delta}$) with the corresponding state trajectory $y^{\varepsilon,\delta}$. By Theorem IV-2 of [**11**], the controls $u^{\varepsilon,\delta}$ and $(y_0^{\varepsilon,\delta}, \dot{y}^{\varepsilon,\delta}, M^{\varepsilon,\delta})$ are optimal for ($\mathrm{P}^{\varepsilon,\delta}$) and ($\mathrm{D}^{\varepsilon,\delta}$), respectively, if and only if they satisfy the coextremality conditions stated in Definition IV-1 of [**11**]. We will next calculate the coextremality conditions stated in Definition IV-1 of [**11**] explicitly for the primal problem ($\mathrm{P}^{\varepsilon,\delta}$) and its dual ($\mathrm{D}^{\varepsilon,\delta}$). To that end, we calculate the subgradients of $L_{\varepsilon,\delta}$, $l_0^{\varepsilon,\delta}$ and $l_T^{\varepsilon,\delta}$, where $L_{\varepsilon,\delta}$ is a convex integrand and $l_0^{\varepsilon,\delta}$ and $l_T^{\varepsilon,\delta}$ are convex functionals as in the derivation of the dual ($\mathrm{D}^{\varepsilon,\delta}$).

The integrand $L_{\varepsilon,\delta}$ for ($\mathrm{P}^{\varepsilon,\delta}$) is given by (C.1) and for $(\omega, t) \in \Omega \times [0,T]$ and $x, \dot{x} \in \mathbb{R}^K$ can equivalently be expressed as

$$L_{\varepsilon,\delta}(\omega, t, x, \dot{x}) = L_\varepsilon(\omega, t, x, \dot{x}) + \sum_{k=1}^{K} \psi_\delta(x_k), \tag{C.4}$$

where L_ε is the normal convex integrand for the perturbed problem (P^ε) defined in (B.18). Note that by (C.4), the subgradient of $L_{\varepsilon,\delta}$ with respect to $\dot{x}$ is the same as the subgradient of L_ε with respect to $\dot{x}$ for $(\omega, t) \in \Omega \times [0,T]$. The coextremality condition in Definition IV-1 of [**11**] regarding a dual state trajectory $y^{\varepsilon,\delta}$ asserts that $d\mathbb{P} \otimes dt$ a.s. we have

$$y^{\varepsilon,\delta}(\omega,t) \in \partial_{\dot{x}} L_{\varepsilon,\delta}(\omega, t, x^{\varepsilon,\delta}(\omega,t), \dot{x}^{\varepsilon,\delta}(\omega,t)),$$

where $\partial_{\dot{x}} L_{\varepsilon,\delta}$ denotes the subgradient with respect to $\dot{x}$. The subgradient conditions in (2.18) together with the fact that the subgradient of $L_{\varepsilon,\delta}$ with respect to $\dot{x}$ is the same as the subgradient of L_ε with respect to $\dot{x}$ for $(\omega, t) \in \Omega \times [0, T]$, implies that the coextremality conditions in (3.4) should hold between $u^{\varepsilon,\delta}$ and $(y_0^{\varepsilon,\delta}, \dot{y}^{\varepsilon,\delta}, M^{\varepsilon,\delta})$.

Next we calculate the subgradients of $L_{\varepsilon,\delta}$ with respect to x. From (B.18) and the fact that $\sum_{k=1}^K \psi_\delta(x_k)$ is differentiable we have that

$$\partial_x L_{\varepsilon,\delta}(\omega, t, x, \dot{x}) = \left\{ \nabla \left(\sum_{k=1}^K \psi_\delta(x_k) \right) \right\}. \tag{C.5}$$

The coextremality condition in Definition IV-1 of [**11**] regarding $\dot{y}^{\varepsilon,\delta}$ states that $d\mathbb{P} \otimes dt$ a.s. we have

$$\dot{y}^{\varepsilon,\delta}(\omega, t) \in \partial_x L_{\varepsilon,\delta}(\omega, t, x^{\varepsilon,\delta}(\omega, t), \dot{x}^{\varepsilon,\delta}(\omega, t)). \tag{C.6}$$

Thus, the definition of ψ_δ, cf. (3.1) together with (C.5) and (C.6), imply that for a.e. $(\omega, t) \in \Omega \times [0, T]$,

$$\dot{y}_k^{\varepsilon,\delta}(\omega, t) = 0 \ \text{ if } x_k^{\varepsilon,\delta}(\omega, t) \geq \delta \ \text{ and } \dot{y}_k^{\varepsilon,\delta}(\omega, t) = R(\delta)(x_k^{\varepsilon,\delta}(\omega, t) - \delta) \ \text{ if } \ x_k^{\varepsilon,\delta}(\omega, t) < \delta,$$

which establishes the coextremality condition in (3.3). ■

Proof of Proposition 26. For $\varepsilon, \delta > 0$, the optimality of the control $\{u^{\varepsilon,\delta}(\omega, t) : (\omega, t) \in \Omega \times [0, T]\}$ given in (3.5) for $(\mathrm{P}^{\varepsilon,\delta})$ follows from Theorem IV-2 of [**11**] and the fact that for an optimal state trajectory $y^{\varepsilon,\delta}$ for $(\mathrm{D}^{\varepsilon,\delta})$, $u^{\varepsilon,\delta}$ and $y^{\varepsilon,\delta}$ satisfy the coextremality conditions stated in Proposition 25. We argue that for $\varepsilon, \delta > 0$, $(\mathrm{P}^{\varepsilon,\delta})$ has a unique solution. Suppose not. Then, there exists booking controls $\widetilde{u}$ and $\overline{u}$ that are optimal for $(\mathrm{P}^{\varepsilon,\delta})$ such that $\widetilde{u}$ and $\overline{u}$ are not equal on a set of strictly positive $d\mathbb{P} \otimes dt$ measure. From Theorem IV-2 of [**11**], both $\widetilde{u}$ and $\overline{u}$ should satisfy the coextremality conditions (3.3)-(3.4) with y^ε. This, in turn, implies that $\widetilde{u}(\omega,t)$ and $\overline{u}(\omega,t)$ are equal for a.e. $(\omega,t) \in \Omega \times [0,T]$. Contradiction. Thus, $(\mathrm{P}^{\varepsilon,\delta})$ has a unique solution and $\{u^{\varepsilon,\delta}(\omega,t) : (\omega,t) \in \Omega \times [0,T]\}$ is the unique optimal for $(\mathrm{P}^{\varepsilon,\delta})$.

We next prove that for $\varepsilon, \delta > 0$, $u^{\varepsilon,\delta}$ is feasible for the continuous-time network revenue management problem (P_{cont}). Clearly, $u^{\varepsilon,\delta}$ satisfies the demand restrictions of (P_{cont}). Hence, it suffices to check whether $u^{\varepsilon,\delta}$ satisfies the capacity restriction. That is, we prove that for a.e. $\omega \in \Omega$, $x^{\varepsilon,\delta}(\omega,T) \geq 0$, where $x^{\varepsilon,\delta}$ is the state trajectory corresponding to the booking control $u^{\varepsilon,\delta}$. Suppose not. Suppose that for some resource $k = 1,\ldots,K$, we have $x_k^{\varepsilon,\delta}(\omega,T) < 0$ on some set N of strictly positive measure. We will show that this contradicts the optimality of $u^{\varepsilon,\delta}$ for $(\mathrm{P}^{\varepsilon,\delta})$. First, note that

$$\mathbb{E}\int_0^T [f(\omega,t)\cdot u^{\varepsilon,\delta}(\omega,t) - \frac{1}{2}\sum_{j=1}^{J} \varepsilon_j(\omega,t)\,(u_j^{\varepsilon,\delta})^2(\omega,t)]\,dt \leq FJDT, \tag{C.7}$$

where F and D are the bounds on the fare and demand processes, respectively. We next provide a lower bound on the cost term

$$-\mathbb{E}\left[\int_0^T \sum_{k=1}^{K} \psi_\delta(x_k^{\varepsilon,\delta}(\omega,t))]\,dt\right].$$

For $k = 1,\ldots,K$, define the stopping time $\sigma_k^{\varepsilon,\delta}$ as the first time the capacity of resource k drops below zero under the control $u^{\varepsilon,\delta}$. Formally, for $k = 1,\ldots,K$ and $\omega \in \Omega$,

$$\sigma_k^{\varepsilon,\delta}(\omega) = \inf\{t \geq 0 : x^{\varepsilon,\delta}(\omega,t) \leq 0\},$$

where we set $\sigma_k^{\varepsilon,\delta}(\omega) = T$ if $x_k^{\varepsilon,\delta}(\omega, T) \geq 0$. For $\omega \in N$, we have $\sigma_k^{\varepsilon,\delta}(\omega) < T$ by the boundedness of the demand rate process. Then, the following holds.

$$\text{(C.8)} \qquad \mathbb{E}\left[\int_0^T \sum_{k=1}^K \psi_\delta(x_k^{\varepsilon,\delta}(\omega,t))\right] dt\Bigg] \geq \mathbb{E}\left[\int_{\sigma_k^{\varepsilon,\delta}(\omega)}^T \psi_\delta(x_k^{\varepsilon,\delta}(\omega,t))dt \mid N\right],$$

$$\text{(C.9)} \qquad = \mathbb{E}\left[\int_{\sigma_k^{\varepsilon,\delta}(\omega)}^T \frac{R(\delta)}{2}(x_k^{\varepsilon,\delta}(\omega,t) - \delta)^2 dt \mid N\right].$$

First, observe that since the demand rate is bounded, we have

$$T - \sigma_k^{\varepsilon,\delta}(\omega) \geq \frac{\delta}{DJ\overline{A}},$$

where D is the bound on the demand rate, and $\overline{A} = \sup_{k=1,\ldots,K}\{\sum_{j=1}^J A_{kj}\}$. Then, we can write (C.9) as

$$\mathbb{E}_N\left[\int_{\sigma_k^{\varepsilon,\delta}(\omega)}^T \frac{R(\delta)}{2}(x_k^{\varepsilon,\delta}(\omega,t) - \delta)^2 dt\right] \geq \frac{R(\delta)}{2}\mathbb{E}_N\left[\int_0^{\frac{\delta}{DJ\overline{A}}} z^2(\omega,t)dt\right],$$

where

$$z(\omega,0) = 0,\ z(\omega,t) = tDJ\overline{A} \text{ for } 0 \leq t \leq \frac{\delta}{DJ\overline{A}}.$$

Then, we have

$$\begin{aligned}
\frac{R(\delta)}{2}\mathbb{E}_N\left[\int_0^{\frac{\delta}{DJ\bar{A}}}(x_k^{\varepsilon,\delta}(\omega,t)-\delta)^2dt\right] &\geq \frac{R(\delta)}{2}\mathbb{E}_N\left[\int_0^{\frac{\delta}{DJ\bar{A}}} z^2(\omega,t)dt\right], \\
&\geq \delta^3\frac{R(\delta)}{6}\mathbb{P}(N), \\
&= TJFD\mathbb{P}(N),
\end{aligned}$$

since $R(\delta) = 6TJFD/\delta^3$. However, from (C.7), replacing $u^{\varepsilon,\delta}(\omega,t)$ with 0 for all $t \in [0,T]$ and $\omega \in N$ leads to a feasible control for ($\mathrm{P}^{\varepsilon,\delta}$) that has a higher objective function value than $u^{\varepsilon,\delta}$, contradicting the optimality of $u^{\varepsilon,\delta}$ for ($\mathrm{P}^{\varepsilon,\delta}$). Hence, for a.e. $\omega \in \Omega$, $x^{\varepsilon,\delta}(\omega,T) \geq 0$ and $u^{\varepsilon,\delta}$ is feasible for the continuous-time network revenue management problem (P_{cont}). ■

Proof of Theorem 27. The bid-price process $\pi^{\varepsilon,\delta}$ is defined as $\pi^{\varepsilon,\delta} = y^{\varepsilon,\delta}$, where $(y_0^{\varepsilon,\delta}, \dot{y}^{\varepsilon,\delta}, M^{\varepsilon,\delta})$ is an optimal control for the dual problem ($\mathrm{D}^{\varepsilon,\delta}$) and for $(\omega,t) \in \Omega\times[0,T]$, the state trajectory $y^{\varepsilon,\delta}$ is given by

$$y^{\varepsilon,\delta}(\omega,t) = y_0^{\varepsilon,\delta}(\omega) + \int_0^t \dot{y}^{\varepsilon,\delta}(\omega,s)ds + M^{\varepsilon,\delta}(\omega,t).$$

From Proposition 26, the booking controls $u^{\varepsilon,\delta}$ corresponding to the bid-price control $(\pi^{\varepsilon,\delta},\phi^{\varepsilon,\delta})$ is the unique optimal for the problem ($\mathrm{P}^{\varepsilon,\delta}$) for $\varepsilon,\delta > 0$. Letting $x^{\varepsilon,\delta}$ be the primal state trajectory associated with $u^{\varepsilon,\delta}$, the coextremality condition (3.3) requires that for $k = 1,\ldots,K$ and almost all $(\omega,t) \in \Omega\times[0,T]$,

(C.10)

$$\dot{y}_k^{\varepsilon,\delta}(\omega,t) = 0 \text{ if } x_k^{\varepsilon,\delta}(\omega,t) \geq \delta \text{ and } \dot{y}_k^{\varepsilon,\delta}(\omega,t) = R(\delta)(x_k^{\varepsilon,\delta}(\omega,t)-\delta) \text{ if } x_k^{\varepsilon,\delta}(\omega,t) < \delta.$$

Then, for each resource $k = 1, \ldots, K$ and $\varepsilon, \delta > 0$, the bid-price process $\pi_k^{\varepsilon,\delta}$ is a non-negative supermartingale adapted to the filtration $\{\mathcal{F}_t : t \in [0,T]\}$ with $\pi_k^{\varepsilon,\delta}(\omega, T) = 0$ for a.e. $\omega \in \Omega$. For $k = 1, \ldots, K$, define the stopping time $\sigma_k^{\varepsilon,\delta}$ as the first time the capacity of resource k drops below δ under the control $u^{\varepsilon,\delta}$. That is, for $k = 1, \ldots, K$ and $\omega \in \Omega$,

$$\sigma_k^{\varepsilon,\delta}(\omega) = \inf\{t \geq 0 : x_k^{\varepsilon,\delta}(\omega, t) \leq \delta\},$$

where $x^{\varepsilon,\delta}$ is the state trajectory associated with $u^{\varepsilon,\delta}$ and $\sigma_k^{\varepsilon,\delta}(\omega) = T$ if $x_k^{\varepsilon,\delta}(\omega, T) \geq \delta$. Then, $\{\pi_k^{\varepsilon,\delta}(\omega, t \wedge \sigma_k^{\varepsilon,\delta}(\omega)) : (\omega, t) \in \Omega \times [0,T]\}$ is a martingale adapted to the filtration $\{\mathcal{F}_t : t \in [0,T]\}$ since $\dot{y}_k^{\varepsilon,\delta}(\omega, t \wedge \sigma_k^{\varepsilon,\delta}(\omega)) = 0$ for a.e. $(\omega, t) \in \Omega \times [0,T]$ by (C.10). That is, $\pi_k^{\varepsilon,\delta}$ stopped at the first time the capacity of resource k drops below δ is a martingale and this completes the proof of the first part of Theorem 27.

Second, we prove that the bid-price policy $(\pi^{\varepsilon,\delta}, \phi^{\varepsilon,\delta})$ is $(\kappa\varepsilon + \rho\delta)$-optimal, where κ is given by (2.21) and ρ is defined in (3.6). Let u be an optimal booking control for the network revenue management problem (P_{cont}). For $k = 1, \ldots, K$, define the stopping time σ_k as the first time the capacity of resource k drops below δ under the control u. That is, for $k = 1, \ldots, K$ and $\omega \in \Omega$,

$$\sigma_k(\omega) = \inf\{t \geq 0 : x_k(\omega, t) \leq \delta\},$$

where x is the state trajectory associated with u and $\sigma_k(\omega) = T$ if $x(\omega, T) \geq \delta$. Define the stopping time σ as the first time the capacity of some resource drops below δ under the control u. Formally, for $\omega \in \Omega$,

$$\sigma(\omega) = \inf_{k=1,\ldots,K}\{\sigma_k(\omega)\}.$$

Then, consider the booking control $\widetilde{u}$ derived from u as follows: For $(\omega, t) \in \Omega \times [0, T]$, and $j = 1, \ldots, J$, define

$$\widetilde{u}_j(\omega, t) = \begin{cases} u(\omega, t) & \text{if } \ t \leq \min_{v \in \{k : A_{kj} > 0\}} \{\sigma_v(\omega)\}, \\ 0 & \text{otherwise.} \end{cases}$$

It is clear that $\widetilde{u}$ is a feasible control for the problem $(\mathrm{P}^{\varepsilon,\delta})$. From Proposition 26, $u^{\varepsilon,\delta}$ is the unique optimal control for the perturbed problem $(\mathrm{P}^{\varepsilon,\delta})$ for $\varepsilon, \delta > 0$. Hence, the objective value of $u^{\varepsilon,\delta}$ is greater than or equal to the objective value of $\widetilde{u}$ for the problem $(\mathrm{P}^{\varepsilon,\delta})$. Moreover, the expected revenue generated by $\widetilde{u}$ has the following lower bound.

$$\text{(C.11)} \quad \mathbb{E}\left[\int_0^T [f(\omega, t) \cdot u(\omega, t)]\, dt\right] - \frac{\delta KJF}{\min_{k,j}\{A_{kj} : A_{kj} > 0\}} \leq \mathbb{E}\left[\int_0^T [f(\omega, t) \cdot \widetilde{u}(\omega, t)]\, dt\right].$$

To see why (C.11) holds, observe that u and $\widetilde{u}$ generate the same revenue until time $\sigma(\omega)$ along each sample path ω. The differences may arise after time $\sigma(\omega)$, where for $j = 1, \ldots, J$ we have $\widetilde{u}_j(\omega, t) = 0$ by definition for $t > \min_{v \in \{k : A_{kj} > 0\}} \{\sigma_v(\omega)\}$. For each resource $k = 1, \ldots, K$, the expected revenue loss due to having $\widetilde{u}_j(\omega, t) = 0$ for all products j such that $A_{kj} > 0$ after time $\sigma_k(\omega)$ is less than or equal to ρ/K, and (C.11) holds. Then, letting $\widetilde{x}$ denote the state trajectory under $\widetilde{u}$, we have

$$\begin{aligned}
&\mathbb{E}\left[\int_0^T [f(\omega,t)\cdot u(\omega,t)]\,dt\right] - \varepsilon\kappa - \delta\rho \\
\leq\ & \mathbb{E}\left[\int_0^T [f(\omega,t)\cdot \widetilde{u}(\omega,t) - \frac{1}{2}\sum_{j=1}^{J}\varepsilon_j(\omega,t)\,\widetilde{u}_j^2(\omega,t) - \sum_{k=1}^{K}\psi_\delta(\widetilde{x}_k(\omega,t))]\,dt\right] \\
\leq\ & \mathbb{E}\left[\int_0^T [f(\omega,t)\cdot u^{\varepsilon,\delta}(\omega,t) - \frac{1}{2}\sum_{j=1}^{J}\varepsilon_j(\omega,t)\,(u_j^{\varepsilon,\delta})^2(\omega,t) - \sum_{k=1}^{K}\psi_\delta(x_k^{\varepsilon,\delta}(\omega,t))]\,dt\right], \\
\text{(C.12)}\quad \leq\ & \mathbb{E}\left[\int_0^T [f(\omega,t)\cdot u^{\varepsilon,\delta}(\omega,t)]\,dt\right].
\end{aligned}$$

The first inequality follows from (C.11), the definition of $\varepsilon_j(\omega,t)$, cf. (2.15) and the fact that $u(\omega,t) \leq d(\omega,t)$ for a.e. $(\omega,t) \in \Omega \times [0,T]$. The second inequality is given by feasibility of $\widetilde{u}$ for $(\mathrm{P}^{\varepsilon,\delta})$. The last inequality establishes the $(\kappa\varepsilon + \rho\delta)$-optimality of $u^{\varepsilon,\delta}$ and completes the proof of the second part of Theorem 27.

Finally, we show that every weak limit $\bar{u} \in \widetilde{\mathcal{U}}$ of the booking controls $\{u^{\varepsilon,\delta} : \varepsilon,\delta > 0\}$ is an optimal booking control for the continuous-time network revenue management problem (P_{cont}). Let $\{u^{\varepsilon_n,\delta_n} : n \geq 1\}$ be a sequence controls of feasible controls for the continuous-time network revenue management problem (P_{cont}) which converges to $\bar{u}$ in the weak* topology as $\varepsilon_n, \delta_n \searrow 0$. From Lemma 57, $\bar{u}$ is a feasible booking control for (P_{cont}) and, we have

$$\lim_{n\to\infty}\mathbb{E}\left[\int_0^T f(\omega,t)\cdot u^{\varepsilon_n,\delta_n}(\omega,t)dt\right] = \mathbb{E}\left[\int_0^T f(\omega,t)\cdot \bar{u}(\omega,t)dt\right].$$

The optimality of $\bar{u}$, then follows from (C.12). ■

Proof of Proposition 28. Combining the primal system dynamics equation $dx = -Audt$ with the coextremality condition (3.4) gives (3.7). Similarly, combining the dual system dynamics $dy = \dot{y}dt + dM$ with the coextremality condition (3.3) gives (3.8). Hence, $x^{\varepsilon,\delta}$ and $y^{\varepsilon,\delta}$ jointly solve (3.7)-(3.9).

Conversely, suppose $x^{\varepsilon,\delta}$ and $y^{\varepsilon,\delta}$ jointly give a solution to (3.7)-(3.9). Let $u^{\varepsilon,\delta}$ denote the booking process corresponding to the trajectory $x^{\varepsilon,\delta}$ such that for $(\omega, t) \in \Omega \times [0, T]$, and $j = 1, \ldots, J$

$$u_j^{\varepsilon,\delta}(\omega, t) = \phi^{\varepsilon,\delta}(y^{\varepsilon,\delta}(\omega, t)A^j, f_j(\omega, t), d_j(\omega, t)) \,.$$

Then, $u^{\varepsilon,\delta}$ is a feasible booking rate process for ($\mathrm{P}^{\varepsilon,\delta}$). Similarly, $(y_0^{\varepsilon,\delta}, \dot{y}^{\varepsilon,\delta}, M^{\varepsilon,\delta})$ is a feasible control for ($\mathrm{D}^{\varepsilon,\delta}$), where for $(\omega, t) \in \Omega \times [0, T]$, and $k = 1, \ldots, K$, $\dot{y_k}^{\varepsilon,\delta}$is given by

$$\dot{y_k}^{\varepsilon,\delta}(\omega, t) = \psi_\delta'(x_k^{\varepsilon,\delta}(\omega, t)),$$

and $M^{\varepsilon,\delta}$ is as in (3.8). Since, $u^{\varepsilon,\delta}$ and $(y_0^{\varepsilon,\delta}, \dot{y}^{\varepsilon,\delta}, M^{\varepsilon,\delta})$ satisfy the coextremality conditions in (3.3) and (3.4), from Proposition 25 they are optimal for ($\mathrm{P}^{\varepsilon,\delta}$) and ($\mathrm{D}^{\varepsilon,\delta}$), respectively. ■

Proof of Proposition 29. First, observe that the analysis in Section 3 can be extended for the case $\varepsilon = 0$ easily. One only needs to avoid division by ε. For $\varepsilon = 0$ and $\delta > 0$, the coextremality conditions associated with the perturbed problem ($\mathrm{P}^{\varepsilon,\delta}$) and its dual ($\mathrm{D}^{\varepsilon,\delta}$) can be stated as follows: Letting $u^{\varepsilon,\delta}$ be a feasible control for ($\mathrm{P}^{\varepsilon,\delta}$) with the corresponding state trajectory $x^{\varepsilon,\delta}$, and $(y_0^{\varepsilon,\delta}, \dot{y}^{\varepsilon,\delta}, M^{\varepsilon,\delta})$ be a feasible control for ($\mathrm{D}^{\varepsilon,\delta}$) with the corresponding state trajectory $y^{\varepsilon,\delta}$, the controls $u^{\varepsilon,\delta}$ and $(y_0^{\varepsilon,\delta}, \dot{y}^{\varepsilon,\delta}, M^{\varepsilon,\delta})$ are optimal for ($\mathrm{P}^{\varepsilon,\delta}$) and ($\mathrm{D}^{\varepsilon,\delta}$), respectively, if and only if they satisfy the following coextremality conditions (C.13) and (C.14): For $k = 1, \ldots, K$ and almost all $(\omega, t) \in \Omega \times [0, T]$,

(C.13)

$$\dot{y}_k^{\varepsilon,\delta}(\omega, t) = 0 \text{ if } x_k^{\varepsilon,\delta}(\omega, t) \geq \delta \text{ and } \dot{y}_k^{\varepsilon,\delta}(\omega, t) = R(\delta)(x_k^{\varepsilon,\delta}(\omega, t) - \delta) \text{ if } x_k^{\varepsilon,\delta}(\omega, t) < \delta,$$

and for $j = 1, \ldots, J$ and almost all $(\omega, t) \in \Omega \times [0, T]$ with $d_j(\omega, t) > 0$,

$$\text{(C.14)} \quad \begin{array}{lll} \text{if } u_j^{\varepsilon,\delta}(\omega, t) = 0, & \text{then } f_j(\omega, t) - y^{\varepsilon,\delta}(\omega, t)A^j \leq 0, \\ \text{if } u_j^{\varepsilon,\delta}(\omega, t) = d_j(\omega, t), & \text{then } f_j(\omega, t) - y^{\varepsilon,\delta}(\omega, t)A^j \geq 0, \\ \text{if } 0 < u_j^{\varepsilon,\delta}(\omega, t) < d_j(\omega, t), & \text{then } f_j(\omega, t) - y^{\varepsilon,\delta}(\omega, t)A^j = 0. \end{array}$$

For $\delta > 0$, suppose (x, y, M) jointly solve the FBSDE defined in (3.11)-(3.13). Let u denote the bookings resulting from the booking function ϕ in the forward equation (3.11). That is, for $j = 1, \ldots, J$ and $(\omega, t) \in \Omega \times [0, T]$,

$$u_j(\omega, t) = d_j(\omega, t)\mathbf{1}_{\{f_j(\omega,t) \geq y(\omega,t)A^j\}}.$$

Then, x, y and u satisfy the coextremality conditions (C.13) and (C.14) and, hence, u is an optimal control for the perturbed problem (P$^{\varepsilon,\delta}$) for $\varepsilon = 0$ and $\delta > 0$. From Theorem 27, the TvR bid-price control with bid-price process $\pi = y$ is $\rho\delta$-optimal. ■

APPENDIX D

Proofs of the Results in Chapter 4

D.1. Proofs of the Results in Section 4.1

Proof of Lemma 32. First of all, $\forall t \in [0, \bar{t}]$ and $t' > t$,

$$v = \arg\max_{v'} \{vy(v', t') - x(v', t')\}.$$

To see this, note that from the IC constraint at time t', we have

$$vy(v, t') - x(v, t') \geq vy(v', t) - x(v', t') \quad \text{for} \quad v, v' \in [\underline{v}, \overline{v}].$$

Second, we focus on

$$\max_{v'} \int_{\underline{v}}^{\overline{v}} [vy(v', t') - x(v', t')]\, g(v, t)\, dv, \ \forall t \in [0, \bar{t}],\ t' < t.$$

That is, we will consider what a consumer of type t would report as her valuation if she reports her type as $t' < t$. My conjecture is that she will report her best estimate of her valuation, i.e. $\mathbb{E}_t[v] = \int_{\underline{v}}^{\overline{v}} vg(v, t)\, dv$. Note that from the IC constraint at time t', we have

$$vy(v, t') - x(v, t') \geq vy(v', t) - x(v', t') \quad \text{for} \quad v, v' \in [\underline{v}, \overline{v}].$$

Moreover, we can write

$$
\begin{aligned}
& \max_{v'} \int_{\underline{v}}^{\overline{v}} \left[vy\left(v',t'\right) - x\left(v',t'\right)\right] g\left(v,t\right) dv \\
= \; & \max_{v'} \left\{ \int_{\underline{v}}^{\overline{v}} vy\left(v',t'\right) g\left(v,t\right) dv - \int_{\underline{v}}^{\overline{v}} x\left(v',t'\right) g\left(v,t\right) dv \right\} \\
= \; & \max_{v'} \left\{ y\left(v',t'\right) \int_{\underline{v}}^{\overline{v}} vg\left(v,t\right) dv - x\left(v',t'\right) \int_{\underline{v}}^{\overline{v}} g\left(v,t\right) dv \right\} \\
= \; & \max_{v'} \left\{ \mathbb{E}_t\left[v\right] y\left(v',t'\right) - x\left(v',t'\right) \right\},
\end{aligned}
$$

which is maximized at $v' = \mathbb{E}_t\left[v\right]$ due to the IC constraint at time t'. That is,

$$
\mathbb{E}_t\left[v\right] = \arg\max_{v'} \int_{\underline{v}}^{\overline{v}} \left[vy\left(v',t'\right) - x\left(v',t'\right)\right] g\left(v,t\right) dv, \; \forall t \in [0,\overline{t}], \; t' < t.
$$

This is as if the consumer knows that her valuation at time t' is $\mathbb{E}_t\left[v\right]$, in which case, she would have reported $\mathbb{E}_t\left[v\right]$ as her valuation. ■

Proof of Proposition 33. We first prove the "only if" part. Suppose a solution $\left\{x\left(v,t\right) : t \in [0,\overline{t}], \; v \in [\underline{v},\overline{v}]\right\}$ and $\left\{y\left(v,t\right) : t \in [0,\overline{t}], \; v \in [\underline{v},\overline{v}]\right\}$ to the mechanism design problem implements the first-best solution. Then, all the *expected* surplus is extracted from the consumers, i.e.

$$
\int_{\underline{v}}^{\overline{v}} \left[vy\left(v,t\right) - x\left(v,t\right)\right] g\left(v,t\right) dv \; = 0 \text{ for all } t.
$$

This implies that

$$
\int_{\underline{v}}^{\overline{v}} x\left(v,t\right) g\left(v,t\right) dv = \int_{c}^{\overline{v}} vg\left(v,t\right) dv = \mathbb{E}_t\left[v; v \geq c\right].
$$

For (IC_t) constraint at time t to be satisfied, we need the following conditions:

$$\begin{aligned}
\text{for } v &\geq c, \;\; v - x(v,t) \geq v - x(v',t) \;\text{ for } v' \geq c \text{ and } t \in [0,\bar{t}], \\
\text{for } v &\geq c, \;\; v - x(v,t) \geq -x(v',t) \;\text{ for } v' < c \text{ and } t \in [0,\bar{t}], \\
\text{for } v &< c, \;\; -x(v,t) \geq v - x(v',t) \;\text{ for } v' \geq c \text{ and } t \in [0,\bar{t}], \\
\text{for } v &< c, \;\; -x(v,t) \geq -x(v',t) \;\text{ for } v' < c \text{ and } t \in [0,\bar{t}].
\end{aligned}$$

First focus on the conditions under which the incentive compatibility constraints at time zero are satisfied.

Under the choice of y and x as above, the IC constraints at time 0 becomes

$$\begin{aligned}
0 &\geq \int_{\underline{v}}^{\overline{v}} \left[vy(v,t') - x(v,t')\right] g(v,t)\,dv, \; \forall t \in [0,\bar{t}],\; t' > t, \\
0 &\geq \int_{\underline{v}}^{\overline{v}} \left[vy(\mathbb{E}_t[v],t') - x(\mathbb{E}_t[v],t')\right] g(v,t)\,dv, \; \forall t \in [0,\bar{t}],\; t' < t,
\end{aligned}$$

which can be written as follows:

$$\begin{aligned}
0 &\geq \mathbb{E}_t[v; v \geq c] - \int_{\underline{v}}^{\overline{v}} x(v,t')\, g(v,t)\,dv, \; \forall t \in [0,\bar{t}],\; t' > t, \\
0 &\geq y(\mathbb{E}_t[v],t')\,\mathbb{E}_t[v] - x(\mathbb{E}_t[v],t'), \; \forall t \in [0,\bar{t}],\; t' < t.
\end{aligned}$$

Then, the types t such that $\mathbb{E}_t[v] < c$, would never want to pretend that they are some type $t' < t$, since $y(\mathbb{E}_t[v],t') = 0$. The types t such that $\mathbb{E}_t[v] < c$, would not want to pretend that they are some type $t' > t$ only if

$$\int_{\underline{v}}^{\overline{v}} x(v,t')\, g(v,t)\, dv \geq \mathbb{E}_t[v; v \geq c], \forall t' > t.$$

Now consider the types t such that $\mathbb{E}_t[v] \geq c$. The types t such that $\mathbb{E}_t[v] \geq c$, would not want to pretend that they are some type $t' < t$ only if

$$x(\mathbb{E}_t[v], t') \geq \mathbb{E}_t[v], \forall t' < t.$$

The types t such that $\mathbb{E}_t[v] > c$, would never want to pretend that they are some type $t' > t$ only if

$$\int_{\underline{v}}^{\overline{v}} x(v,t')\, g(v,t)\, dv \geq \mathbb{E}_t[v; v \geq c], \forall t' > t.$$

To summarize, the payments$\left\{x(v,t) : t \in [0,\overline{t}],\ v \in [\underline{v},\overline{v}]\right\}$ should satisfy the following conditions:

(1)

$$\int_{\underline{v}}^{\overline{v}} x(v,t)\, g(v,t)\, dv = \mathbb{E}_t[v; v \geq c],$$

(2) For IC constraint at time t to be satisfied, it must be that

$$\begin{aligned}
\text{for } v &\geq c, \quad -x(v,t) \geq -x(v',t) \text{ for } v' \geq c \text{ and } t \in [0,\overline{t}], \\
\text{for } v &\geq c, \quad v - x(v,t) \geq -x(v',t) \text{ for } v' < c \text{ and } t \in [0,\overline{t}], \\
\text{for } v &< c, \quad -x(v,t) \geq v - x(v',t) \text{ for } v' \geq c \text{ and } t \in [0,\overline{t}], \\
\text{for } v &< c, \quad -x(v,t) \geq -x(v',t) \text{ for } v' < c \text{ and } t \in [0,\overline{t}].
\end{aligned}$$

(3) For types t such that $\mathbb{E}_t[v] < c$,

$$\int_{\underline{v}}^{\overline{v}} x(v,t')\, g(v,t)\, dv \geq \mathbb{E}_t[v; v \geq c], \forall t' > t.$$

(4) For types t such that

$$\mathbb{E}_t[v] \geq c, \int_{\underline{v}}^{\overline{v}} x(v,t')\, g(v,t)\, dv \geq \mathbb{E}_t[v; v \geq c], \forall t' > t.$$

and

$$x(\mathbb{E}_t[v], t') \geq \mathbb{E}_t[v], \forall t' < t.$$

First observe that since for $v \geq c$, $x(v,t) \leq x(v',t)$ for $v' \geq c$ and $t \in [0,\overline{t}]$, we should have $x(v,t) = x(v',t)$ for $v', v \geq c$. Moreover, since for $v < c$, $-x(v,t) \geq -x(v',t)$ for $v' < c$ and $t \in [0,\overline{t}]$, we should have $x(v,t) = x(v',t)$ for $v', v < c$. That is, the payments can only depend on the type and whether the valuation v is greater than c or not. So define the payments $\underline{x}(t)$ and $\overline{x}(t)$ as the payment from type t if her valuation is below c and above c respectively. Then, we can write the conditions above as follows: Let $\pi(t)$ denote the probability that the valuation of type t is above c. Then,

(1)

$$\pi(t)\,\overline{x}(t) + (1-\pi(t))\,\underline{x}(t) = \mathbb{E}_t[v; v \geq c],$$

(2) For IC constraint at time t to be satisfied, we need the following conditions:

$$\text{for } v \geq c, \quad v - \overline{x}(t) \geq -\underline{x}(t) \text{ for } t \in [0,\overline{t}],$$

$$\text{for } v < c, \quad -\underline{x}(t) \geq v - \overline{x}(t) \text{ for } t \in [0,\overline{t}],$$

That is,

$$\text{for } v \geq c, \; v \geq \overline{x}(t) - \underline{x}(t) \; \text{ for } t \in [0, \overline{t}],$$

$$\text{for } v < c, \; \overline{x}(t) - \underline{x}(t) \geq v \text{ for } t \in [0, \overline{t}],$$

which implies that $\overline{x}(t) - \underline{x}(t) = c$.

(3) For types t such that $\mathbb{E}_t[v] < c$, we have

$$\pi(t)\,\overline{x}(t') + (1 - \pi(t))\,\underline{x}(t') \geq \mathbb{E}_t[v; v \geq c], \forall t' > t.$$

(4) For types t such that $\mathbb{E}_t[v] \geq c$, we have

$$\pi(t)\,\overline{x}(t') + (1 - \pi(t))\,\underline{x}(t') \geq \mathbb{E}_t[v; v \geq c], \forall t' > t.$$

$$\overline{x}(t') \geq \mathbb{E}_t[v], \forall t' < t.$$

Since $\overline{x}(t) - \underline{x}(t) = c$, the question is whether the payments $\{\underline{x}(t) : t \in [0, \overline{t}]\}$ satisfy the following conditions

(1)

$$\underline{x}(t) = \mathbb{E}_t[v; v \geq c] - c\pi(t),$$

(2) For types t such that $\mathbb{E}_t[v] < c$,

$$c\pi(t) + \mathbb{E}_{t'}[v; v \geq c] - c\pi(t') \geq \mathbb{E}_t[v; v \geq c], \forall t' > t.$$

For types t such that $\mathbb{E}_t[v] \geq c$,

$$c\pi(t) + \mathbb{E}_{t'}[v; v \geq c] - c\pi(t') \geq \mathbb{E}_t[v; v \geq c], \forall t' > t.$$

$$\mathbb{E}_{t'}[v; v \geq c] - c\pi(t') + c \geq \mathbb{E}_t[v], \forall t' < t,$$

which completes the "only if" part of the proof.

To prove the "if" part of the proof, consider the following solution to the mechanism design problem: For all t, let $y(v,t) = 1$ if $v \geq c$ and $y(v,t) = 0$ if $v < c$. Define the payments $x(v,t)$ as follows: $x(v,t) = \underline{x}(t)$ if $v < c$ and $x(v,t) = \underline{x}(t) + c$ for $v \geq c$ where $\underline{x}(t) = \mathbb{E}_t[v; v \geq c] - c\pi(t)$. Then, it is easy to check that this solution implements the first-best under Conditions 1 and 2 of Proposition 33. ■

D.2. Proofs of the Results in Section 4.2

Proof of Proposition 36. Since U is of bounded variation, we can write $U(t) = U_C(t) + \Delta U(t)$, where $\Delta U(t) = U(t) - U(t-)$. Then, using integration by parts and assuming without loss of generality that the cumulative type distribution $\{H(t) : 0 \leq t \leq \bar{t}\}$ is continuous, we get

$$\begin{aligned}
&\int_{v,t} f(v,t)[x(v,t) - cy(v,t)]\,dvdt \\
=\ &\int_{v,t} f(v,t)[(v-c)\,y(v,t)]\,dvdt - U(0) \\
&- \int_{[0,\bar{t}]} (1 - H(t))\,dU_C(t) - \sum_{0<s<\bar{t}} (1 - H(t))\,\Delta U(t),
\end{aligned}$$

where $\sum_{0<s<\bar{t}} (1 - H(t)) \Delta U(t)$ is at most a countable sum. To see this, note that

$$\int_0^{\bar{t}} h(t) U(t) dt = U(0) + \int_{[0,\bar{t}]} (1 - H(t)) dU_C(t) + \sum_{0<s<\bar{t}} (1 - H(t)) \Delta U(t).$$

We can rewrite the objective as follows: Since $u(v,t) = vy(v,t) - x(v,t)$, we have

$$x(v,t) - cy(v,t) = -u(v,t) + (v - c) y(v,t).$$

The objective is then to maximize

$$\begin{aligned}
& \int_{v,t} f(v,t) [x(v,t) - cy(v,t)] dvdt \\
= & \int_{v,t} f(v,t) [-u(v,t) + (v - c) y(v,t)] dvdt, \\
= & -\int_{v,t} f(v,t) u(v,t) dvdt + \int_{v,t} f(v,t) [(v - c) y(v,t)] dvdt, \\
= & -\int_0^{\bar{t}} h(t) U(t) dt + \int_{v,t} f(v,t) [(v - c) y(v,t)] dvdt, \\
= & \int_{v,t} f(v,t) [(v - c) y(v,t)] dvdt - U(0) \\
& - \int_{[0,\bar{t}]} (1 - H(t)) dU_C(t) - \sum_{0<s<\bar{t}} (1 - H(t)) \Delta U(t).
\end{aligned}$$

Then, we prove that

$$\limsup_{h \to 0} \frac{U(t+h) - U(t)}{h} \leq -\int_{\underline{v}}^{\overline{v}} y(v,t) \frac{\partial G(v,t)}{\partial t} dv.$$

By ($\overline{\text{IC}}_0$) constraint at time 0, we get for a type t and $h > 0$,

$$\begin{aligned} U(t) &\geq U(t+h) - \int_{\underline{v}}^{\overline{v}} [vy(v,t+h) - x(v,t+h)]\, g(v,t+h)\, dv \\ &\quad + \int_{\underline{v}}^{\overline{v}} [vy(v,t+h) - x(v,t+h)]\, g(v,t)\, dv, \end{aligned}$$

where the first line is just adding and subtracting $U(t+h)$, and the inequality follows because by the incentive constraints at time 0, type t would not want to pretend like type $t + h > t$. Then, we get

$$\begin{aligned} U(t+h) - U(t) &\leq \int_{\underline{v}}^{\overline{v}} [vy(v,t+h) - x(v,t+h)]\, g(v,t+h)\, dv \\ &\quad - \int_{\underline{v}}^{\overline{v}} [vy(v,t+h) - x(v,t+h)]\, g(v,t)\, dv, \end{aligned}$$

from which it follows that

$$\begin{aligned} &\limsup_{h \searrow 0} \frac{U(t+h) - U(t)}{h} \\ \leq\; &\lim_{h \searrow 0} \frac{1}{h} \int_{\underline{v}}^{\overline{v}} [vy(v,t+h) - x(v,t+h)]\, (g(v,t+h) - g(v,t))\, dv \\ =\; &\int_{\underline{v}}^{\overline{v}} u(v,t) \frac{\partial g(v,t)}{\partial t} dv, \\ =\; &-\int_{\underline{v}}^{\overline{v}} y(v,t) \frac{\partial G(v,t)}{\partial t} dv, \end{aligned}$$

where the last inequality follows from Lemma 34 and integration by parts. This provides an upper bound on the right-hand derivative of $U(t)$. Similarly using ($\underline{\text{IC}}_0$) constraint at time 0, we

$$\limsup_{h\searrow 0}\frac{U(t)-U(t-h)}{h}\leq -\int_{\underline{v}}^{\overline{v}} y(v,t)\frac{\partial G(v,t)}{\partial t}dv.$$

Then, since

$$\limsup_{h\to 0}\frac{U(t+h)-U(t)}{h}\leq -\int_{\underline{v}}^{\overline{v}} y(v,t)\frac{\partial G(v,t)}{\partial t}dv<\infty,$$

the expected utility function U cannot have upward jumps.

As our next step, we prove that any optimal U does not have downward jumps. Suppose not. Suppose that U has a downward jump at type τ. That is, $U(\tau-\varepsilon)>U(\tau)$ for $\varepsilon>0$, where ε is small. Then, as type τ does not find it profitable to pretend like some type $\tau'<\tau$, the types $t\in(\tau-\varepsilon,\tau)$ for $\varepsilon>0$ sufficiently small would not want it profitable to pretend like some type $\tau'<\tau-\varepsilon$. At the same time, the types $t\in(\tau-\varepsilon,\tau)$ for $\varepsilon>0$ sufficiently small would not find it profitable to pretend to be some type $t'\in[\tau,\bar{t}]$ as well since otherwise type τ would strictly prefer to deviate and pretend to be type t'. Let $\hat{U}$ denote a modified expected utility function such that $\hat{U}(t)=U(\tau-\varepsilon)$ for $t\in t\in(\tau-\varepsilon,\tau)$ and $\hat{U}(t)=U(t)$, otherwise. Then, $\hat{U}$ has a better objective function value than U and is still feasible. Contradiction.

Next, we show that U is nondecreasing. Suppose there exists some interval $[\tau_1,\tau_2]$ of types over which $U'(t)<0$. Since the right hand side of the global IC constraint is increasing in t, the global IC constraint is not binding for $t\in[\tau_1,\tau_2]$. Then, we can choose $\hat{U}$ such that $\hat{U}(t)=U(\tau_2)$ for $t\in[\tau_1,\tau_2]$ and this will strictly increase the objective. On the other hand, this modification creates a downward jump in the expected utilities at the point τ_1. In particular, $U(t)>U(\tau_1)$ for $t<\tau_1$ yet arbitrarily close to τ_1. However, since the global IC constraint is satisfied for τ_1 under $\hat{U}$, it must be the case that the global IC constraint is not binding for those types $t \leq \tau_1$ and $U(t) \geq \hat{U}(\tau_1)$. Thus, setting $\hat{U}(t) = U(\tau_2)$ for those types again strictly improves the objective. Carrying out this procedure repeatedly eliminates any point where $U' < 0$. Thus, $U'(t) \geq 0$ for all t.

An immediate implication is that

$$\sum_{0<s<\bar{t}} (1 - H(t)) \Delta U(t) = 0.$$

Moreover, U is everywhere differentiable and this concludes the proof of the fact that

$$0 \leq U'(t) \leq -\int_{\underline{v}}^{\overline{v}} y(v,t) \frac{\partial G(v,t)}{\partial t} dv.$$

Finally, we prove that in any optimal mechanism, the payments

$$\left\{x(v,t) : t \in \left[0, \bar{t}\right],\ v \in [\underline{v}, \overline{v}]\right\}$$

are nonnegative. We first prove that for a given t, $x(v,t)$ is increasing in v. We have

$$\partial u(v,t)/\partial v = \frac{\partial [vy(v,t) - x(v,t)]}{\partial v} = y(v,t) + v\frac{\partial y(v,t)}{\partial v} - x(v,t).$$

From Lemma 34, $\partial u(v,t)/\partial v = y(v,t)$ and $y(v,t)$ is non-decreasing in v. Thus, $x(v,t) = v\frac{\partial y(v,t)}{\partial v} \geq 0$.

Hence, it suffices to prove that $x(\underline{v},t) \geq 0$ for all $t \in \left[0, \bar{t}\right]$. We argue by contradiction. Suppose not. Suppose there exists a type $\hat{t}$ such that $x(\underline{v},t) < 0$. First consider the case that $\hat{t} > 0$. If $x\left(\underline{v},\hat{t}\right) < 0$, then $u\left(\underline{v},\hat{t}\right) > 0$ and since $\partial u\left(v,\hat{t}\right)/\partial v \geq 0$, we have

$u\left(v,\hat{t}\right)>0$ for all $v\in[\underline{v},\overline{v}]$. However, ($\overline{\text{IC}}_0$) implies that for all $t<\hat{t}$,

$$\begin{aligned} U(t) &= \int_{\underline{v}}^{\overline{v}} \left[vy(v,t)-x(v,t)\right] g(v,t)\,dv \\ &\geq \int_{\underline{v}}^{\overline{v}} \left[vy\left(v,\hat{t}\right)-x\left(v,\hat{t}\right)\right] g(v,t)\,dv, \\ &= \int_{\underline{v}}^{\overline{v}} u\left(v,\hat{t}\right) g(v,t)\,dv \\ &> 0. \end{aligned}$$

But then, we can decrease $U(t)$ for all $t\in\left[0,\hat{t}\right]$ by ε by increasing the payments and not changing the allocation y. This will strictly improve the objective while all the incentive compatibility constraints are still satisfied, contradicting the optimality of the solution with payments$\left\{x(v,t):t\in\left[0,\overline{t}\right],\ v\in[\underline{v},\overline{v}]\right\}$. Finally, suppose $x(\underline{v},0)<0$. Then, $U(0)>0$ and since the incentive compatibility constraints regarding downward deviations hold, there exists a type $\widetilde{t}>0$ such that $U(t)>0$ for all $t\in\left[0,\widetilde{t}\right]$.Then, we can decrease $U(t)$ for all $t\in\left[0,\widetilde{t}\right]$ by ε by increasing the payments and not changing the allocation y, which will strictly improve the objective while all the incentive compatibility constraints are still satisfied. This concludes the proof of the fact that in any optimal mechanism, the payments are nonnegative. ■

The following auxiliary optimal control problem (P_A) and its dual (D_A), and the coextremality conditions between the two formulations will facilitate the proof of Proposition

40. First, consider (P_A): Choose control $\{y(v) : v \in [\underline{v}, \bar{v}]\}$ so as to

$$\begin{aligned}
&\text{maximize } \int_{\underline{v}}^{\bar{v}} (v-c)\, g(v,t)\, y(v)\, dv \\
&\text{subject to} \\
&\dot{z}(v) = (1 - G(v,t))\, y(v) \;\text{ with } z(\underline{v}) = 0, \\
&\dot{s}(v) = -\frac{\partial G(v,t)}{\partial t} y(v) \;\text{ with } s(\underline{v}) = 0, \\
&\dot{u}(v) = y(v) \;\text{ with } u(\underline{v}) = u_0, \\
&z(\bar{v}) \geq U_*(t) - u_0, \\
&s(\bar{v}) \geq \dot{U}_*(t), \\
&u(\bar{v}) \leq u_*(\bar{v}), \\
&0 \leq y(v) \leq 1,
\end{aligned} \tag{P_A}$$

where z, s, u denote state trajectories and we treat $U_*(t)$, $\dot{U}_*(t)$ and $u_*(\bar{v})$ as given constants. The dual formulation (D_A) of (P_A) is derived in Appendix D.3 and is given as follows: Choose $\{\rho_z(v), \rho_s(v), \rho_u(v) : v \in [\underline{v}, \bar{v}]\}$ so as to

$$\begin{aligned}
&\min \int_{\underline{v}}^{\bar{v}} \max\left\{0, g(v,t)(v-c) - \frac{\partial G(v,t)}{\partial t}\rho_s(v) + (1 - G(v,t))\,\rho_z(v)\right\} dv + u_0 \rho_z(\underline{v}) \\
&\text{subject to} \\
&\dot{\rho}_z(v) = \dot{\rho}_s(v) = \dot{\rho}_u(v) = 0, \\
&\rho_u(\underline{v}) = 0, \\
&\rho_s(\bar{v}), \rho_z(\bar{v}) \geq 0.
\end{aligned} \tag{D_A}$$

The following lemma states the necessary and sufficient conditions for optimality and is proved in Appendix D.3.

Lemma 58. *Let* $\{y(v) : v \in [\underline{v}, \bar{v}]\}$ *be a feasible solution for* (P_A) *with the corresponding state trajectories* $\{z(v), s(v), u(v) : v \in [\underline{v}, \bar{v}]\}$ *and* $\{\rho_z(v), \rho_s(v), \rho_u(v) : v \in [\underline{v}, \bar{v}]\}$ *be a feasible control for* (D_A)*. The controls* y *and* ρ_z, ρ_s, ρ_u *are optimal for* (P_A) *and* (D_A)*, respectively, if and only if they satisfy the following coextremality conditions:*

(i) $\rho_s(v) = 0$ *for all* v*.*

(ii)

$$y(v) \in \arg\max_{0 \le z \le 1} \left\{ z \left(v - c + \frac{1 - G(v,t)}{g(v,t)} \rho_z(\bar{v}) \right) \right\} \text{ for all } v. \tag{D.1}$$

(iii) If $z(\bar{v}) > U(t) - u_0$*, then* $\rho_z(\bar{v}) = 0$*.*

Proof of Proposition 37. First, we show that there exists an optimal solution to $(\mathrm{P}^1_{relaxed})$ such that $y(v,t) \in \{0,1\}$ for all v, t. To this end, fix an optimal solution U_*, y_* (The corresponding ex-post utility function will be denoted by u_*) We will proceed by modifying this solution appropriately to reach at another solution U_*, y, which is of the desired form. In particular, we will keep the expected utility function U_* unchanged, while modifying the allocation probabilities. This will show that for each t, there exists a cut-off point $k(t)$ such that $y(v,t) = 1$ if $v \ge k(t)$ whereas $y(v,t) = 0$ if $v < k(t)$.

Fixing t, define the modified allocation $y(v,t)$ as the solution to the following problem. (For notational brevity, we suppress the dependence of y on t): Choose the control

$\{y(v) : v \in [\underline{v}, \bar{v}]\}$ so as to

$$\begin{aligned}
&\text{maximize } \int_{\underline{v}}^{\bar{v}} (v - c)\, g(v,t)\, y(v)\, dv \\
&\text{subject to} \\
&0 \leq y(v,t) \leq 1, \\
&\dot{u}(v) = y(v) \ \text{ with } u(\underline{v}) = u_0, \qquad\qquad (\widetilde{\mathrm{P}}_A) \\
&\int_{\underline{v}}^{\bar{v}} g(v,t)\, u(v)\, dv \geq U_*(t), \\
&\int_{\underline{v}}^{\bar{v}} \left[\frac{-\partial G(v,t)}{\partial t}\right] y(v)\, dv \geq \dot{U}_*(t), \\
&u(\bar{v}) \leq u_*(\bar{v}, t),
\end{aligned}$$

where $U_*(t)$, $\dot{U}_*(t)$ and $u_*(\bar{v}, t)$ are taken as constants. It follows from integration by parts that

$$\int_{\underline{v}}^{\bar{v}} g(v,t)\, u(v)\, dv = u_0 + \int_{\underline{v}}^{\bar{v}} (1 - G(v,t))\, y(v)\, dv,$$

which helps us rewrite the third constraint as

$$\text{(D.2)} \qquad \int_{\underline{v}}^{\bar{v}} (1 - G(v,t))\, y(v)\, dv \geq U_*(t) - u_0.$$

Then, $(\widetilde{\mathrm{P}}_A)$ reduces to the auxiliary problem (P_A) and it follows from Lemma D.23 that

$$y(v) \in \arg\max_{0 \leq z \leq 1} \left\{ z \left(v - c + \frac{1 - G(v,t)}{g(v,t)} \rho_z(\bar{v}) \right) \right\},$$

where $\rho_z(\bar{v}) \geq 0$. Under the standard monotone hazard rate condition that

$$(1 - G(v,t)) / g(v,t)$$

is nondecreasing in v, we have that

$$v - c + \frac{1 - G(v,t)}{g(v,t)} \rho_z(\bar{v})$$

is increasing in v, and that $y(v) \in \{0,1\}$ for all v. Indeed, for all t, there exists a cutoff $k(t)$ such that $y(v,t) = 1$ if $v \geq k(t)$ whereas $y(v,t) = 0$ if $v < k(t)$ so that $y(v)$ is nondecreasing in v.

As an aside, note that for each optimal solution to $(\widetilde{\mathrm{P}}_A)$, we can decrease u_0 and make the constraint (D.2) bind. Thus, without loss of generality, we focus only on solutions where (D.2) binds.

For type t, the modified solution will have $U_*(t)$ as the expected utility and $y(v,t)$ as the allocation. (The modified ex-post utility function $u(v,t)$ is also derived from the above optimal control problem.) This modified solution is clearly of the desired form and weakly improves the objective of $(\mathrm{P}^1_{relaxed})$. To establish that it is indeed an optimal solution to $(\mathrm{P}^1_{relaxed})$, we only need to check the global IC constraint. To check this, note that $u(\bar{v},t) \leq u_*(\bar{v},t)$ (for all t) by the last constraint of $(\widetilde{\mathrm{P}}_A)$ and that $\partial u(v,t)/\partial v = 1$ for v such that $u(v,t) \geq 0$, where the latter assertion follows since $u(\underline{v},t) \leq 0$ for all t. To see why $u(\underline{v},t) \leq 0$ for all t, notice that if $u(\underline{v}) > 0$ and the constraint that

$$\int_{\underline{v}}^{\bar{v}} g(v,t)\, u(v)\, dv \geq U_*(t)$$

does not bind in problem $(\widetilde{\mathrm{P}}_A)$, we can decrease $u(\underline{v})$ and increase the objective of the original mechanism design problem. If $u(\underline{v}) > 0$ and the constraint that

$$\int_{\underline{v}}^{\bar{v}} g(v,t)\, u(v)\, dv \geq U_*(t)$$

binds, then it should be that $U_*(t) > 0$. Then, we should have $U_*(t') > 0$ for all $t' < t$ as any type $t' < t$ could get a strictly positive surplus by pretending to be type t. As U_* is increasing, this would imply that $U_*(t) > 0$ for all t, in which case increasing $u(\underline{v}, t)$ uniformly for all types would increase the profits. Contradiction.

Since $u(\underline{v}, t) \leq 0$ for all t, it must be that $u(v,t) \leq u_*(v,t)$ for all v such that $u(v,t) \geq 0$. Thus, the global IC constraint holds since

$$\begin{aligned}
U_*(\bar{t}) &\geq \max_t \{u_*(\mathbb{E}_{\bar{t}}[v], t)\}, \\
&= \max_{t \in \{\tau : u_*(\mathbb{E}_{\bar{t}}[v], \tau) \geq 0\}} \{u_*(\mathbb{E}_{\bar{t}}[v], t)\}, \\
&\geq \max_{t \in \{\tau : u_*(\mathbb{E}_{\bar{t}}[v], \tau) \geq 0\}} \{u(\mathbb{E}_{\bar{t}}[v], t)\}, \\
&\geq \max_{t \in \{\tau : u(\mathbb{E}_{\bar{t}}[v], \tau) \geq 0\}} \{u(\mathbb{E}_{\bar{t}}[v], t)\}, \\
&= \max_t \{u(\mathbb{E}_{\bar{t}}[v], t)\},
\end{aligned}$$

which completes the proof. ■

Proof of Corollary 38. From Lemma 34, $\partial u(v,t)/\partial v = y(v,t) = 0$ for $v < k(t)$ and $\partial u(v,t)/\partial v = y(v,t) = 1$ for $v \geq k(t)$. This implies that for a given t, $x(v,t)$ is the same for all valuations $v < k(t)$. Similarly, since

$$u(v,t) = vy(v,t) - x(v,t) = v - x(v,t),$$

and $\partial u(v,t)/\partial v = y(v,t) = 1$ for $v \geq k(t)$, it also follows that given t, $x(v,t)$ is the same for all valuations $v \geq k(t)$. Any $x(v,t)$ that satisfies the (IC$_t$) constraint is of the form

$$x(v,t) = \begin{cases} \underline{x}(t) & \text{if} \quad v < k(t), \\ \overline{x}(t) & \text{if} \quad v \geq k(t). \end{cases}$$

Moreover, it follows that $\overline{x}(t) = \underline{x}(t) + k(t)$. To see this, note that for a consumer with valuation $k(t) - \varepsilon$ for $\varepsilon > 0$ small, to report her valuation truthfully, it must be that

$$-\underline{x}(t) \geq (k(t) - \varepsilon) - \overline{x}(t). \tag{D.3}$$

The left hand side is the utility that a type t consumer with valuation $k(t) - \varepsilon$ gets by reporting her type truthfully and obtaining the good with probability 0, in which case she makes a payment of $\underline{x}(t)$. The right hand side is the utility that the consumer with valuation $k(t) - \varepsilon$ gets by reporting her valuation as $k(t)$ and obtaining the good with probability 1 and paying $\overline{x}(t)$. Since (D.3) should hold for all $\varepsilon > 0$ small, we obtain $-\underline{x}(t) \geq k(t) - \overline{x}(t)$. Repeating the same argument for a consumer with valuation $k(t) + \varepsilon$, we also obtain $-\underline{x}(t) \leq k(t) - \overline{x}(t)$. Hence, $\overline{x}(t) = \underline{x}(t) + k(t)$.

Using Proposition 37 and $\overline{x}(t) = \underline{x}(t) + k(t)$,

$$\begin{aligned}
U(t') &= \int_{\underline{v}}^{\overline{v}} [vy(v,t') - x(v,t')]\, g(v,t')\, dv, \\
&= \int_{\underline{v}}^{k(t')} [vy(v,t') - x(v,t')]\, g(v,t')\, dv \\
&\quad + \int_{k(t')}^{\overline{v}} [vy(v,t') - x(v,t')]\, g(v,t')\, dv, \\
&= \int_{\underline{v}}^{k(t')} [-\underline{x}(t')]\, g(v,t')\, dv + \int_{k(t')}^{\overline{v}} [v - \overline{x}(t')]\, g(v,t')\, dv, \\
&= \int_{k(t')}^{\overline{v}} vg(v,t')\, dv - \int_{\underline{v}}^{k(t')} \underline{x}(t')\, g(v,t')\, dv \\
&\quad - \int_{k(t')}^{\overline{v}} (\underline{x}(t') + k(t'))\, g(v,t')\, dv, \\
&= \int_{k(t')}^{\overline{v}} vg(v,t')\, dv - \underline{x}(t') - (1 - G(k(t'),t))\, k(t').
\end{aligned}$$

This implies that

$$\underline{x}(t') = -U(t') + \int_{k(t')}^{\overline{v}} vg(v,t')\, dv - (1 - G(k(t'),t))\, k(t'),$$

and

$$k(t') + \underline{x}(t') = -U(t') + \int_{k(t')}^{\overline{v}} vg(v,t')\, dv + G(k(t'),t)\, k(t'). \tag{D.4}$$

Then, notice that a type t will not deviate to a type $t' < t$ for which $y(\mathbb{E}_t[v], t') = 0$. This is the case because $U(t) \geq 0$ and the payments are nonnegative for all t and v, cf.

Proposition 36. Hence, type t will get a payoff of

$$\mathbb{E}_t[v]\, y(\mathbb{E}_t[v], t') - x(\mathbb{E}_t[v], t') = -x(\mathbb{E}_t[v], t') \leq 0$$

if she pretends to be type $t' < t$. Thus, without loss of generality, consider only types $t' < t$ such that $y(\mathbb{E}_t[v], t') = 1$, in which case

$$x(\mathbb{E}_t[v], t') = k(t') + \underline{x}(t'). \tag{D.5}$$

Finally, using Proposition 37, (F), and the fact that type t will only consider deviating to types $t' < t$ such that $y(\mathbb{E}_t[v], t') = 1$, we can write

$$\begin{aligned}
& \max_{t'<t} \{\mathbb{E}_t[v]\, y(\mathbb{E}_t[v], t') - x(\mathbb{E}_t[v], t')\} \\
= & \max_{t' \in \{\tau: y(\mathbb{E}_t[v], \tau)=1,\ \tau<t\}} \{\mathbb{E}_t[v]\, y(\mathbb{E}_t[v], t') - x(\mathbb{E}_t[v], t')\} \\
= & \ \mathbb{E}_t[v] - \min_{t' \in \{\tau: y(\mathbb{E}_t[v], \tau)=1,\ \tau<t\}} \{x(\mathbb{E}_t[v], t')\},
\end{aligned}$$

Then, by (D.4) and (D.4),

$$\begin{aligned}
& \mathbb{E}_t[v] - \min_{t' \in \{\tau: y(\mathbb{E}_t[v], \tau)=1,\ \tau<t\}} \{x(\mathbb{E}_t[v], t')\} \\
= & \ \mathbb{E}_t[v] - \min_{t' \in \{\tau: y(\mathbb{E}_t[v], \tau)=1,\ \tau<t\}} \left\{-U(t') + \int_{k(t')}^{\overline{v}} v g(v, t')\, dv + G(k(t'), t)\, k(t')\right\}, \\
= & \ \mathbb{E}_t[v] - \min_{t' \in \{\tau: y(\mathbb{E}_t[v], \tau)=1,\ \tau<t\}} \left\{-U(t') + \int_{k(t')}^{\overline{v}} (v - k(t'))\, g(v, t')\, dv + k(t')\right\}, \\
= & \ \mathbb{E}_t[v] + \max_{t' \in \{\tau: y(\mathbb{E}_t[v], \tau)=1,\ \tau<t\}} \left\{U(t') - \int_{k(t')}^{\overline{v}} (v - k(t'))\, g(v, t')\, dv - k(t')\right\},
\end{aligned}$$

which concludes the proof. ■

Proof of Proposition 39. Proposition 37 and Corollary 38 immediately yields that there exists an optimal solution to the relaxed problem ($P^2_{relaxed}$) characterized by the cut-off points $\{k(t) : 0 \leq t \leq \bar{t}\}$ such that the allocation y satisfies $y(v,t) = 1$ for $v \geq k(t)$ and 0 otherwise. Hereafter, we will consider that optimal solution. The remainder of the proof consists of three major steps, each of which includes several steps.

The first step of the proof characterizes the expected utility function U. In particular, it shows that U is nondecreasing with $U(0) = 0$ and that there exists a threshold $t_2 \leq \bar{t}$ such that

$$U'(t) = \begin{cases} 0 & \text{if} \quad t < t_2, \\ -\int_{k(t)}^{\overline{v}} \frac{\partial G(v,t)}{\partial t} dv & \text{if} \quad t \geq t_2. \end{cases}$$

The second step characterizes the optimal cutoff points $\{k(t) : 0 \leq t \leq \bar{t}\}$, showing that for types $t \geq t_2$, $k(t)$ is the unique solution of $\phi(k(t), t) = 0$, where

$$\phi(v,t) = (v-c) + \frac{(1-H(t))}{h(t)} \frac{\partial G(v,t)/\partial t}{g(v,t)}.$$

It also shows that $k(t) \geq c$, and $k(t)$ is nonincreasing for $t \geq t_2$. Moreover, $k(t)$ nonincreasing on $[0, t_2)$ with $k(0) > c$. To be more specific, it shows that there exists another threshold $t_1 \leq t_2$ such that $k(t)$ is decreasing on $[0, t_1]$, while $k(t) = c$ on (t_1, t_2).

Finally, the third step shows that in the optimal solution we have $0 < t_2 < \bar{t}$, ruling out the possibilities $t_2 = \bar{t}$ and $t_2 = 0$.

Step 1. To establish the claims of Step 1 as stated above, we first prove two auxiliary results. The first auxiliary result states that if the constraint (IC_{global}) binds for some type t, then it should also bind for all types t' higher than t, i.e. $t' > t$. The second auxiliary claim proves that in any optimal solution, we have

$$U'(t) \in \left\{0, -\int_{k(t)}^{\overline{v}} \frac{\partial G(v,t)}{\partial t} dv\right\}.$$

In what follows, we establish these auxiliary result in Steps 1.A and 1.B. Finally, we show that if $U'(t') > 0$ for some t', then $U'(t) > 0$ for all $t \geq t'$, establishing the proof of Step 1.

Step 1.A. We start by proving that if the global constraint for downward deviations binds for some type t, then it should also bind for all $t' > t$. To see this, note that

(D.6) $$\frac{d\mathbb{E}_t[v]}{dt} = -\int_{\underline{v}}^{\overline{v}} \frac{\partial G(v,t)}{\partial t} dv \geq -\int_{k(t)}^{\overline{v}} \frac{\partial G(v,t)}{\partial t} dv \geq U'(t).$$

The derivative of the left hand side of (IC_{global}) is $U'(t)$. The derivative of the right hand side of (IC_{global}) is greater than or equal to $\partial\mathbb{E}_t[v]/\partial t$ since the derivative of the first term on the right hand side of (IC_{global}) is $\partial\mathbb{E}_t[v]/\partial t$ and the second term is also increasing in t. Hence, the derivative of the left hand side of (IC_{global}) is always less than the derivative of the right hand side. This implies that if (IC_{global}) binds for type t, it will bind for all types $t' > t$.

Step 1.B. Next, we prove that in any optimal solution to ($P^2_{relaxed}$), we should have either $U'(t) = 0$ or $U'(t) = -\int_{k(t)}^{\overline{v}} \frac{\partial G(v,t)}{\partial t} dv$. First recall that from Proposition 36, we have for all t that

$$0 \leq U'(t) \leq -\int_{k(t)}^{\overline{v}} \frac{\partial G(v,t)}{\partial t} dv.$$

We argue by contradiction. Suppose that there exists an interval $[\tau_1, \tau_2]$ such that

$$0 < U'(t) < -\int_{k(t)}^{\overline{v}} \frac{\partial G(v,t)}{\partial t} dv fort \in [\tau_1, \tau_2].$$

First notice that (IC_{global}) cannot not bind for any positive measure subset of $[\tau_1, \tau_2]$. Suppose not. Then, there exists a subinterval of $[\tau_1, \tau_2]$ over which (IC_{global}) binds. Without loss of generality, we suppose that (IC_{global}) binds for $t \in [\tau_1, \tau_2]$. Otherwise, we could have picked the subinterval to start with. Then,

$$\begin{aligned} U(\tau_1) &= \max_{t' \in \left\{\tau : y\left(\mathbb{E}_{\tau_1}[v], \tau\right) = 1,\ \tau < \tau_1\right\}} \left\{ U(t') - \int_{k(t')}^{\overline{v}} (v - k(t')) g(v, t') dv - k(t') \right\} \\ &\quad + \mathbb{E}_{\tau_1}[v]. \end{aligned}$$

Since $U'(\tau_1) < -\int_{k(\tau_1)}^{\overline{v}} \frac{\partial G(v,\tau_1)}{\partial t} dv$, and the derivative of the right hand side of (IC_{global}) at the point τ_1 is greater than or equal to $-\int_{\underline{v}}^{\overline{v}} \frac{\partial G(v,\tau)}{\partial t} dv$, we have

$$\begin{aligned} &\max_{t' \in \left\{\tau : y\left(\mathbb{E}_{\tau_1+\varepsilon}[v], \tau\right) = 1,\ \tau < \tau_1+\varepsilon\right\}} \left\{ U(t') - \int_{k(t')}^{\overline{v}} (v - k(t')) g(v, t') dv - k(t') \right\} \\ > &\ U(\tau_1 + \varepsilon) - \mathbb{E}_{\tau_1+\varepsilon}[v] \end{aligned}$$

for $\varepsilon > 0$ sufficiently small. That is, (IC_{global}) is violated for types $\tau_1 + \varepsilon$, which is a contradiction to the optimality of U.

Since (IC_{global}) does not bind for $[\tau_1, \tau_2]$, then the objective function can be improved by having $U'(t) = 0$ for $t \in [\tau_1, \tau_1 + \varepsilon]$ for $\varepsilon > 0$ sufficiently small and appropriately increasing $U'(t)$ (which can be done since $U'(t) < -\int_{k(t)}^{\overline{v}} \frac{\partial G(v,t)}{\partial t} dv$ for $t \in [\tau_1, \tau_2]$) such that $U(\tau_2)$ remains unchanged. This proves that in any optimal solution to ($P^2_{relaxed}$), we should have either $U'(t) = 0$ or $U'(t) = -\int_{k(t)}^{\overline{v}} \frac{\partial G(v,t)}{\partial t} dv$.

Having proved the auxiliary results for Step 1, we now conclude the proof of Step 1 by showing that if $U'(t') > 0$ for some t', then $U'(t) > 0$ for all $t \geq t'$. Suppose this statement is not true. Then, there exists an interval $[\tau_1, \tau_2]$ over which $U'(t) = 0$. For $t \in [\tau_1, \tau_2]$, the constraint (IC_{global}) cannot bind, cf. (D.6) of Step 1.A. Therefore, we can modify U such that $U'(t') = 0$ for $t \in [t', t' + \varepsilon]$ while keeping $U(\tau_2)$ the same as before, which improves the objective function. Hence, it must be that if $U'(t') > 0$ for some t', then $U'(t) > 0$ for all $t \geq t'$.

Finally, define $t_2 \in [0, \bar{t}]$ as the lowest type such that $U'(t) > 0$. Formally,

$$t_2 = \inf\left\{t \in [0, \bar{t}] : U'(t) > 0\right\},$$

where $t_2 = \bar{t}$ if $U(t) = 0$ for all t. Observe that since $U'(t)$ is either zero or $-\int_{k(t)}^{\bar{v}} \frac{\partial G(v,t)}{\partial t} dv$ by Step 1.B, we conclude that $U(0) = 0$ and $U'(t) = 0$ for $t \leq t_2$, while

$$U'(t) = -\int_{k(t)}^{\bar{v}} \frac{\partial G(v,t)}{\partial t} dv \quad \text{for} \quad t > t_2,$$

which completes the proof of Step 1.

Step 2. To establish Step 2 of the proof, we first prove two auxiliary results. The first one, proved in Step 2.A, states that the constraint (IC_{global}) binds for the highest type $\bar{t}$. The second auxiliary result, proved in Step 2.B, asserts that if $t_2 > 0$, then there exists a type $\widetilde{t} < t_2$ such that the constraint (IC_{global}) regarding type $\bar{t}$'s deviation to pretending to be type $\widetilde{t}$ binds. That is,

$$U(\bar{t}) = \mathbb{E}_{\bar{t}}[v] + U(\widetilde{t}) - \int_{k(\widetilde{t})}^{\bar{v}} \left(v - k(\widetilde{t})\right) g(v, \widetilde{t})\, dv - k(\widetilde{t}).$$

Then in Step 2.C, we characterize the optimal cutoff points $k(t)$ for $t < t_2$. Similarly, Step 2.D characterizes the optimal cutoff points $k(t)$ for $t \geq t_2$, ignoring the constraint (IC_{global}) for types $t \in [t_2, \bar{t})$ regarding their deviations to pretending to be types $t' \in [t_2, \bar{t}]$. Finally, in Step 2.E, we verify that the (IC_{global}) constraints ignored in Step 2.D indeed hold for the optimal cutoffs $k(t)$ for $t \geq t_2$ characterized in that step.

Step 2.A. We start the proof of Step 2 by showing that (IC_{global}) binds for $\bar{t}$. That is, we prove that

$$U(\bar{t}) = \mathbb{E}_{\bar{t}}[v] + \max_{t' \in \{\tau : y(\mathbb{E}_{\bar{t}}[v], \tau) = 1\}} \left\{ U(t') - \int_{k(t')}^{\overline{v}} (v - k(t'))\, g(v, t')\, dv - k(t') \right\}.$$

We argue by contradiction. Suppose not. Then, the global IC constraint does not bind for any type, cf. Step 1.A. There can be two cases. The first case is that $U(\bar{t}) = 0$. Then, $U(t) = 0$ for all t, by Step 1.B.

In this case, i.e. the case that (IC_{global}) does not bind for $\bar{t}$ and $U(\bar{t}) = 0$, we must have $k(t) > c$ for $t \in [0, \varepsilon]$ for $\varepsilon > 0$ sufficiently small. To prove this, first notice that under assumption that Condition 2 of Proposition 33 is violated, i.e. $\mathbb{E}_{\bar{t}}[v - c] > \mathbb{E}_0[v - c; v \geq c]$, there exists $\varepsilon > 0$ sufficiently small such that

$$\mathbb{E}_{\bar{t}}[v - c] > \mathbb{E}_t[v - c; v \geq c] \quad \text{for} \quad t \in [0, \varepsilon],$$

which follows by the continuity of the expectation $\mathbb{E}_t[v - c; v \geq c]$ is t. Moreover, for the global incentive compatibility constraint (IC_{global}) of $\bar{t}$ not to bind when type $\bar{t}$ pretends to be type $t \in [0, \varepsilon]$, it must be the case that

$$U(\bar{t}) > \mathbb{E}_{\bar{t}}[v] + \left\{ U(t) - \int_{k(0)}^{\overline{v}} (v - k(0))\, g(v, 0)\, dv - k(0) \right\}.$$

Then, since $U(t) = 0$ for all t,

$$\text{(D.7)} \qquad \mathbb{E}_{\bar{t}}[v - k(0)] < \int_{k(0)}^{\bar{v}} (v - k(0))\, g(v,0)\, dv.$$

Together with the fact that for all t,

$$k + \int_{k}^{\bar{v}} (v - k)\, g(v,t)\, dv$$

increases as k increases[1] and the assumption that Condition 2 of Proposition 33 is violated, i.e. $\mathbb{E}_{\bar{t}}[v - c] > \mathbb{E}_0[v - c; v \geq c]$, equation (D.7) implies that $k(0) > c$. Indeed, by continuity of $G(v,t)$ in t, we have

$$\mathbb{E}_{\bar{t}}[v - c] > \mathbb{E}_t[v - c; v \geq c]$$

for $t \in [0, \varepsilon]$ and $\varepsilon > 0$ sufficiently small. Thus, if the global incentive compatibility constraint (IC_{global}) of $\bar{t}$ does not bind when type $\bar{t}$ pretends to be type t, then it must be the case that $k(t) > c$ for all $t \in [0, \varepsilon]$. However, the objective function can be improved by lowering $k(t)$ slightly for types $[0, \varepsilon]$ so that the global IC constraints are not violated and we still set $U(t) = 0$ for all types. Hence, it cannot be the case that the global IC constraint does not bind for $\bar{t}$ and $U(\bar{t}) = 0$.

Next, we deal with the case that the global IC constraint does not bind for $\bar{t}$ and $U(\bar{t}) > 0$. In this case, the global IC constraint does not bind for any type, cf. Step 1.A, and we can uniformly decrease $U(t)$ slightly for those types such that $U(t) > 0$ so that the global

[1] To see this, observe that

$$\frac{\partial}{\partial k}\left[k + \int_{k}^{\bar{v}} (v - k)\, g(v,t)\, dv\right] = 1 - \int_{k}^{\bar{v}} g(v,t)\, dv = G(k,t) > 0.$$

IC constraints are not violated. Then, the objective function improves, contradicting the optimality of U. Hence, it cannot be the case that the global IC constraint does not bind for type $\bar{t}$.

Having proved that (IC_{global}) binds for $\bar{t}$, we proceed to characterizing the optimal cutoff points $\{k(t) : 0 \leq t \leq \bar{t}\}$ for a given $t_2 \in [0, \bar{t}]$.

Step 2.B. Recall that $t_2 \in [0, \bar{t}]$ is the lowest type such that $U'(t) > 0$, i.e.

$$t_2 = \inf \{t \in [0, \bar{t}] : U'(t) > 0\}.$$

We prove that if $t_2 > 0$, then there exists a type $\widetilde{t} < t_2$ such that the global IC constraint for $\bar{t}$ binds regarding the deviation of type $\bar{t}$ to pretend to be type $\widetilde{t}$. That is,

$$U(\bar{t}) = \mathbb{E}_{\bar{t}}[v] + U\left(\widetilde{t}\right) - \int_{k\left(\widetilde{t}\right)}^{\bar{v}} \left(v - k\left(\widetilde{t}\right)\right) g\left(v, \widetilde{t}\right) dv - k\left(\widetilde{t}\right).$$

We also prove that $k\left(\widetilde{t}\right) \geq c$. If $t_2 = \bar{t}$, then the result follows from Step 2.A. Suppose $t_2 < \bar{t}$. We argue by contradiction. Suppose that type $\bar{t}$ strictly prefers her contract offer to the contract offers of types $t \in [0, t_2]$. Then, there exists an $\varepsilon > 0$ sufficiently small such that (IC_{global}) does not bind for type $t \in [t_2, t_2 + \varepsilon)$. To prove this, suppose on the contrary that type $\bar{t}$ strictly prefers her contract offer to the contract offers of types $t \in [0, t_2]$ and that (IC_{global}) constraint of type $t = t_2$ binds, which in turn implies that (IC_{global}) constraints of all types $t \in [t_2, \bar{t}]$ binds. Since the (IC_{global}) constraint of type t_2 binds, it must be that type t_2 is indifferent between his contract offer and the contract offer of some type $\hat{t} < t_2$. That is,

$$U(t_2) = \mathbb{E}_{t_2}[v] + U(\hat{t}) - \int_{k(\hat{t})}^{\overline{v}} \left(v - k(\hat{t})\right) g(v, \hat{t})\, dv - k(\hat{t}) = 0. \tag{D.8}$$

Since the expected utility obtained by type $\bar{t}$ by pretending to be type $\hat{t}$ is strictly less than $U(\bar{t})$, we have

$$U(\bar{t}) > \mathbb{E}_{\bar{t}}[v] + U(\hat{t}) - \int_{k(\hat{t})}^{\overline{v}} \left(v - k(\hat{t})\right) g(v, \hat{t})\, dv - k(\hat{t}). \tag{D.9}$$

Since $U'(t) = -\int_{k(t)}^{\overline{v}} \frac{\partial G(v,t)}{\partial t} dv$ for $t \in [t_2, \bar{t}]$, by integrating and using the boundary condition $U(t_2) = 0$, we conclude

$$\begin{aligned} U(\bar{t}) &= \int_{t_2}^{\bar{t}} \int_{k(t)}^{\overline{v}} \left[-\frac{\partial G(v,t)}{\partial t}\right] dv dt \\ &\leq \int_{t_2}^{\bar{t}} \int_{\underline{v}}^{\overline{v}} \left[-\frac{\partial G(v,t)}{\partial t}\right] dv dt \\ &= \int_{t_2}^{\bar{t}} \frac{d\mathbb{E}_t[v]}{dt} dt \\ &= \mathbb{E}_{\bar{t}}[v] - \mathbb{E}_{t_2}[v]. \end{aligned} \tag{D.10}$$

However, combining (D.8) and (D.9), we get

$$\begin{aligned} & U(\bar{t}) - U(t_2) \\ = \; & U(\bar{t}), \\ > \; & \mathbb{E}_{\bar{t}}[v] + U(\hat{t}) - \int_{k(\hat{t})}^{\overline{v}} (v - k(\hat{t}))\, g(v, \hat{t})\, dv - k(\hat{t}) \\ & -\mathbb{E}_{t_2}[v] - U(\hat{t}) + \int_{k(\hat{t})}^{\overline{v}} (v - k(\hat{t}))\, g(v, \hat{t})\, dv + k(\hat{t}) \\ = \; & \mathbb{E}_{\bar{t}}[v] - \mathbb{E}_{t_2}[v], \end{aligned}$$

which is contradicts with (D.10), proving that there exists an $\varepsilon > 0$ sufficiently small, such that (IC_{global}) does not bind for type $t \in (t_2, t_2 + \varepsilon)$.

Then, we can set $U(t) = 0$ for $t \in [t_2, t_2 + \varepsilon)$ and also decrease $U(t)$ slightly for all types $t \in [t_2 + \varepsilon, \bar{t}]$, which will strictly improve the objective without violating the incentive compatibility constraints. This proves that there exists a type $\widetilde{t} < t_2$ such that

$$U(\bar{t}) = \mathbb{E}_{\bar{t}}[v] + U(\widetilde{t}) - \int_{k(\widetilde{t})}^{\overline{v}} (v - k(\widetilde{t}))\, g(v, \widetilde{t})\, dv - k(\widetilde{t}).$$

Finally, we prove that $k(\widetilde{t}) \geq c$. Again we argue by contradiction. Suppose $k(\widetilde{t}) < c$. The utility that type $\bar{t}$ gets by pretending to be type $\widetilde{t}$ is

$$\mathbb{E}_{\bar{t}}[v] + U(\widetilde{t}) - \int_{k(\widetilde{t})}^{\overline{v}} (v - k(\widetilde{t}))\, g(v, \widetilde{t})\, dv - k(\widetilde{t}),$$

where $U(\widetilde{t}) = 0$ since $\widetilde{t} \leq t_2$. However, the deviation utility is decreasing in $k(\widetilde{t})$. Moreover, the allocation is also getting more efficient as $k(\widetilde{t})$ increases since $k(\widetilde{t}) < c$.

Thus, increasing $k(\widetilde{t})$ slightly and keeping $U(\widetilde{t}) = 0$ improves the objective while (IC$_{global}$) constraint is still satisfied. Hence, we should have $k(\widetilde{t}) \geq c$.

Having established the auxiliary results for Step 2, we next characterize the optimal cutoffs $k(t)$ for $t < t_2$ in Step 2.C.

Step 2.C. Letting t_1 be the highest type such that the global IC constraint for $\bar{t}$ binds, i.e.

$$t_1 = \sup\left\{ t \in [0, \bar{t}] : U(\bar{t}) = \mathbb{E}_{\bar{t}}[v] - \int_{k(t)}^{\overline{v}} (v - k(t))\, g(v,t)\, dv - k(t) \right\},$$

we show that $t_1 < \bar{t}$ and type $\bar{t}$ is indifferent between his contract and the contract offer of any type $t \in [0, t_1]$. Moreover, we establish that $k(t)$ is (strictly) decreasing over the interval $[0, t_1]$. Finally, if $t_1 < t_2$, then $k(t) = c$ for $t \in (t_1, t_2)$.

We first show that $t_1 < \bar{t}$. If $t_2 < \bar{t}$, we have $t_1 \leq t_2 < \bar{t}$ by Step 2.B. If $t_2 = \bar{t}$, then optimality requires that $t_1 < \bar{t}$. To see this, note that if $t_2 = \bar{t}$ (i.e. $U(t) = 0$ for all t), then

$$U(\bar{t}) > \mathbb{E}_{\bar{t}}[v] - \mathbb{E}_t[\max\{v, c\}],$$

for t close enough to $\bar{t}$, and hence, setting $k(t) = c$ is optimal for t close enough to $\bar{t}$ since setting $k(t) = c$ yields an efficient allocation and the incentive constraints are still satisfied. This proves that $t_1 < \bar{t}$ when $t_2 = \bar{t}$ as well.

To prove that type $\bar{t}$ is indifferent between his contract and the contract offer of any type $t \in [0, t_1]$, we argue by contradiction. Suppose that type $\bar{t}$ strictly prefers his contract to the contract offer of some type $\hat{t} \in [0, t_1]$. Then, for types $t \in [\hat{t}, \hat{t} + \varepsilon]$ where $\varepsilon > 0$

sufficiently small, we must have

$$U(\bar{t}) > \mathbb{E}_{\bar{t}}[v] - \int_{k(t)}^{\overline{v}} (v - k(t))\, g(v, t)\, dv - k(t)\,.$$

First notice that since $k(t_1) \geq c$ by the previous step, and due to the fact that

$$U(\bar{t}) = \mathbb{E}_{\bar{t}}[v] - \int_{k(t_1)}^{\overline{v}} (v - k(t_1))\, g(v, t_1)\, dv - k(t_1)\,,$$

we should have $k(t) > c$ for $t \in \left[\hat{t}, \hat{t} + \varepsilon\right]$. Then, the objective function strictly improves by decreasing $k(t)$ slightly in a neighborhood of $\hat{t}$ and keeping U the same, which is a contradiction. Thus, the global IC constraint for type $\bar{t}$ binds regarding her deviation to any type $t \in \left[\hat{t}, \hat{t} + \varepsilon\right]$. Since $k(t_1) \geq c$ and the types are ordered by FSD, this argument also proves that $k(0) > c$. Together with Step 2.B, this also proves that the global incentive compatibility constraint (IC_{global}) binds for the highest type $\bar{t}$, who is indifferent between her contract and the contract choice of the lowest type, i.e.

$$U(\bar{t}) = \mathbb{E}_{\bar{t}}[v] - \mathbb{E}_0[\max\{v, k(0)\}]\,.$$

Moreover, since

$$U(\bar{t}) = \mathbb{E}_{\bar{t}}[v] - \int_{k(t)}^{\overline{v}} (v - k(t))\, g(v, t)\, dv - k(t)$$

for all $t \in [0, t_1]$ and

$$\int_{k}^{\overline{v}} (v - k)\, g(v, t)\, dv + k$$

is increasing in k for all t, $k(t)$ is strictly decreasing over the interval $[0, t_1]$ if $t_1 > 0$. If $t_1 < t_2$ we have $k(t) = c$ for $t \in [t_1, t_2]$ since the global incentive compatibility constraints are not binding for deviations to types in the interval $[t_1, t_2]$ and the efficiency in the allocation requires $k(t) = c$.

In summary, $k(0) > c$ with $k(t)$ decreasing[2] for $t \leq t_1$, while $k(t) = c$ for $t \in (t_1, t_2)$, characterizing the optimal cutoff $k(t)$ for $t < t_2$. Next, we characterize the optimal cutoffs $k(t)$ for $t \geq t_2$ in Step 2.D.

Step 2.D. In this step, we characterize the optimal cutoff $k(t)$ for $t \geq t_2$, ignoring the constraints (IC$_{global}$) for types $t \in [t_2, \bar{t})$ regarding their deviation to pretending to be types $t' \in [t_2, \bar{t}]$, which will be verified in Step 2.E. To be specific, we prove that if $t_2 < \bar{t}$, then for $t \geq t_2$, $k(t)$ satisfies $\phi(k(t), t) = 0$, where

$$\phi(v, t) = (v - c) + \frac{(1 - H(t))}{h(t)} \frac{\partial G(v, t) / \partial t}{g(v, t)}.$$

[2]Indeed, $k(t)$ is strictly decreasing for $t \leq t_1$ under strict FSD.

We proceed as follows: For a given $t_2 \in [0, \bar{t}]$, we solve for the optimal allocation solving the following problem, denoted (P_D): Given t_2, we choose $k(t)$ so as to maximize

$$\int_0^{t_2} \int_{k(t)}^{\bar{v}} f(v,t)(v-c)\, dvdt + \int_{t_2}^{\bar{t}} \int_{k(t)}^{\bar{v}} f(v,t)\, \phi(v,t)\, dvdt,$$

subject to

$$\begin{aligned}
&\mathbb{E}_{\bar{t}}[v] - S(t) \leq R(k(t), t) \quad \text{for all } t \leq t_2, \\
&U(t) - S(t) + \mathbb{E}_{\bar{t}}[v] \leq R(k(t), t) \quad \text{for all } t > t_2, \\
&\dot{S}(t) = 0 \text{ and } S(\bar{t}) = U(\bar{t}), \\
&\dot{U}(t) = 0 \text{ for } t \leq t_2, \\
&\dot{U}(t) = \int_{k(t)}^{\bar{v}} (-\partial G(v,t)/\partial t)\, dv \text{ for } t > t_2, \\
&k(t) \in [\underline{v}, \bar{v}],
\end{aligned} \qquad (\mathrm{P}_D)$$

where

$$R(k,t) = \int_k^{\bar{v}} (v-k)\, g(v,t)\, dv + k$$

and $S(t)$ is a state variable such that $S(t) = U(\bar{t})$ for all t. Moreover, $U(t)$ is viewed a state variable in the problem (P_D) and the objective function is rewritten using Proposition 36 and incorporating the fact that $\dot{U}(t) = 0$ for $t \leq t_2$ and

$$\dot{U}(t) = \int_{k(t)}^{\bar{v}} (-\partial G(v,t)/\partial t)\, dv \text{ for } t > t_2.$$

The constraint that

$$\mathbb{E}_{\bar{t}}[v] - S(t) \leq R(k(t), t) \quad \text{for all } \; t \leq t_2,$$

makes sure that type $\bar{t}$ does not find it profitable to pretend he is of type $t \in [0, t_2]$. Similarly, the constraint

$$U(t) - S(t) + \mathbb{E}_{\bar{t}}[v] \leq R(k(t), t) \quad \text{for all } t > t_2,$$

ensures that type $\bar{t}$ does not find it profitable to pretend he is of type $t \in (t_2, \bar{t})$.

Following [**53**], the dual problem of control associated with the above optimal control problem (D_D) is given as follows: (The derivation is along the same lines as the derivations of the dual problem (D_2) of optimal control problem (P_2) in the proof of Proposition 37 and hence, is skipped.) Choose $\{\rho_u(t) : t \in [0, \bar{t}]\}$ and $\{\rho_u(t) : t \in [0, \bar{t}]\}$ so as to

$$\begin{aligned}
&\int_{\underline{v}}^{\bar{v}} M(t, \rho_u(t), \rho_s(t), \dot{\rho}_u(t), \dot{\rho}_s(t))dt \\
&\text{subject to} \\
&\dot{\rho}_u(t) = 0, \\
&\rho_s(\bar{t}) = -\rho_u(\bar{t}), \\
&\rho_s(0) = 0.
\end{aligned} \tag{D_D}$$

where M is given as follows: For $t \leq t_2$,

$$\begin{aligned}
& M(t, \rho_u(t), \rho_s(t), \dot{\rho}_u(t), \dot{\rho}_s(t)) \\
= \; & L^*(t, \dot{\rho}_u(t), \dot{\rho}_s(t), \rho_u(t), \rho_s(t)), \\
= \; & \chi_{\{0\}}\{\dot{\rho}_u(t)\} \\
& + \sup_{s,k \in [\underline{v}, \bar{v}], \mathbb{E}_{\bar{t}}[v] - s \leq R(k,t)} \left\{ s\dot{\rho}_s(t) + \int_k^{\bar{v}} f(v,t)(v-c)\,dv \right\}
\end{aligned}$$

For $t > t_2$,

$$
\begin{aligned}
& M(t, \rho_u(t), \rho_s(t), \dot{\rho}_u(t), \dot{\rho}_s(t)) \\
= \;& L^*(t, \dot{\rho}_u(t), \dot{\rho}_s(t), \rho_u(t), \rho_s(t)), \\
= \;& \chi_{\{0\}} \{\dot{\rho}_u(t)\} \\
& + \sup_{s,k \in [\underline{v}, \bar{v}], \mathbb{E}_{\bar{t}}[v] - s \leq R(k,t)} \left\{ s\dot{\rho}_s(t) + \rho_u(t) \int_k^{\bar{v}} \frac{-\partial G(v,t)}{\partial t} dv + \int_k^{\bar{v}} f(v,t)\, \phi(v,t)\, dv \right\}.
\end{aligned}
$$

The dual problem of control (D_D) and the primal problem (P_D) are closely linked to each other. Above all, their objective function values are equal. Moreover, any optimal primal solution and any optimal dual solution satisfy a set of coextremality conditions, which are necessary and sufficient conditions for optimality. Using [**53**], the coextremality conditions between primal-dual solution pairs imply that for $t \geq t_2$, $k(t)$ should be an optimal solution to the following problem: Choose k so as to

$$
\text{maximize } U(\bar{t})\, \dot{\rho}_s(t) + \left(\int_k^{\bar{v}} (-\partial G(v,t) / \partial t)\, dv \right) \rho_u(t) + \int_k^{\bar{v}} f(v,t)\, \phi(v,t)\, dv.
$$

subject to

$$
U(t) - U(\bar{t}) + \mathbb{E}_{\bar{t}}[v] \leq R(k,t),
$$

$$
k \in [\underline{v}, \bar{v}].
$$

The coextremality conditions also require that if $U(t) > 0$, then $\rho_u(t) = 0$, which immediately implies that $\rho_u(t) = 0$ for $t > t_2$. Since, $\dot{\rho}_u(t) = 0$ for all t from the dual problem (D_D) stated above, we must have $\rho_u(t) = 0$ for all t. Since $\rho_s(0) = 0$ and

$\rho_s(\bar{t}) = -\rho_u(\bar{t})$ from (D_D), we have $\rho_s(0) = \rho_s(\bar{t}) = 0$. Moreover, $\dot{\rho}_s(t) = 0$ if

$$U(t) - U(\bar{t}) + \mathbb{E}_{\bar{t}}[v] < R(k(t), t),$$

which is the case when $k(t)$ satisfies $\phi(k(t), t) = 0$ for all $t \geq t_2$. Then, since we propose a solution with cutoff points $k(t)$ such that $\phi(k(t), t) = 0$ and the proposed cutoff $k(t)$ solves the relaxed problem of choosing k so as to

$$\begin{aligned}
&\text{maximize} \int_k^{\bar{v}} f(v,t)\,\phi(v,t)\,dv \\
&\text{subject to} \\
&U(t) - U(\bar{t}) + \mathbb{E}_{\bar{t}}[v] \leq R(k,t), \\
&k \in [\underline{v}, \bar{v}],
\end{aligned}$$

going back and checking the yields that the cutoff points $k(t)$ such that $\phi(k(t), t) = 0$ satisfy the coextremality conditions for all t. Since the coextremality conditions are necessary and sufficient for optimality, $k(t)$ is the optimal cutoff point.

Moreover, since $\phi(k(t), t) = 0$, we have

$$k(t) = c - \frac{(1 - H(t))}{h(t)} \frac{\partial G(v,t)/\partial t}{g(v,t)} \geq c.$$

As the virtual utility function $\phi(v,t)$ is increasing in t for a given v, we have $k'(t) \leq 0$ and hence, $y(v,t)$ is increasing in t for a given v.

Step 2.E. Finally, to conclude the proof of Step 2, we prove that for $t_2 < \bar{t}$, the constraint (IC_{global}) for types $t \in [t_2, \bar{t})$ regarding their deviations to pretending to be types $t' \in [t_2, \bar{t}]$ holds for the optimal cutoffs $k(t)$ for $t \geq t_2$ characterized in Step 2.D.

By Lemma 34, the transfer payments can be written as $x(v,t) = \underline{x}(t)$ if $v < k(t)$ and $x(v,t) = \overline{x}(t)$ if $v \geq k(t)$ with $k(t) = \overline{x}(t) - \underline{x}(t)$. Using integration by parts, the expected surplus of a type t consumer is given as follows

$$\begin{aligned} U(t) &= -\underline{x}(t) + \int_{k(t)}^{\overline{v}} vg(v,t)\,dv - k(t)(1 - G(k(t),t)), \\ &= -\underline{x}(t) + \int_{k(t)}^{\overline{v}} (1 - G(v,t))\,dv. \end{aligned} \tag{D.11}$$

Taking derivatives of both sides and using the fact that

$$U'(t) = -\int_{k(t)}^{\overline{v}} \frac{\partial G(v,t)}{\partial t} dv \quad \text{for} \quad t \geq t_2,$$

we get

$$-\underline{x}'(t) - k'(t)(1 - G(k(t),t)) = 0 \quad \text{for} \quad t \geq t_2.$$

For $t \in [t_2, \overline{t}]$, let $U(t,t')$ denote the expected utility of type t when he claims to be of type $t' \in (t, \overline{t}]$. Then $U(t,t')$ is given by

$$\begin{aligned} U(t,t') &= \int_{\underline{v}}^{\overline{v}} [vy(v,t') - x(v,t')]\, g(v,t)\,dv \\ &= \int_{k(t')}^{\overline{v}} [v - \overline{x}(t')]\, g(v,t)\,dv + \int_{\underline{v}}^{k(t')} [-\underline{x}(t')]\, g(v,t)\,dv \\ &= \int_{k(t')}^{\overline{v}} [v - (\underline{x}(t') + k(t'))]\, g(v,t)\,dv + \int_{\underline{v}}^{k(t')} [-\underline{x}(t')]\, g(v,t)\,dv \\ &= -\underline{x}(t') + \int_{k(t')}^{\overline{v}} [v - k(t')]\, g(v,t)\,dv. \end{aligned} \tag{D.12}$$

Since $k(t)$ satisfies $\phi(k(t),t) = 0$ and the virtual utility function $\phi(v,t)$ is increasing in t for a given v, we have $k'(t) \leq 0$. Then, taking the derivatives of both sides of (D.12), we obtain

$$\frac{\partial U\left(t,t'\right)}{\partial t'}=-\frac{d\underline{x}\left(t'\right)}{dt'}-\frac{dk\left(t'\right)}{dt'}\left(1-G\left(k\left(t'\right),t\right)\right). \tag{D.13}$$

Since $k'\left(t\right)\leq 0$, we get

$$\begin{aligned}\frac{\partial U\left(t,t'\right)}{\partial t'} &\leq -\frac{d\underline{x}\left(t'\right)}{dt'}-\frac{dk\left(t'\right)}{dt'}\left(1-G\left(k\left(t'\right),t'\right)\right),\\ &= U'\left(t'\right)+\int_{k(t')}^{\bar{v}}\frac{\partial G\left(v,t'\right)}{\partial t'}dv,\\ &\leq 0, \end{aligned} \tag{D.14}$$

where the first inequality is obtained using equation (D.13), $k'\left(t\right)\leq 0$ and

$$\left(1-G\left(k\left(t'\right),t'\right)\right)\geq\left(1-G\left(k\left(t'\right),t'\right)\right).$$

due to FSD. The second line follows from (D.11) and the third line is true since from Proposition 36, we have

$$U'\left(t'\right)\leq-\int_{k(t')}^{\bar{v}}\frac{\partial G\left(v,t'\right)}{\partial t'}dv.$$

Then it follows from the following identity

$$U\left(t,t'\right)=\int_{t}^{t'}\frac{\partial U\left(t,t'\right)}{\partial t'}+U\left(t\right)$$

and from (D.14) that $U\left(t,t'\right)\leq U\left(t\right)$. This proves that for $t_2<\bar{t}$, the constraint (IC_{global}) for types $t\in\left[t_2,\bar{t}\right)$ regarding their deviations to pretending to be types $t'\in\left[t_2,\bar{t}\right]$ is satisfied.

The constraint (IC_{global}) for $\bar{t}$ does not bind for deviations to types in the interval $[t_2, \bar{t}]$. Moreover, the global incentive compatibility constraint (IC_{global}) does not bind for type $t < \bar{t}$ since if (IC_{global}) had binded for some type $t' < \bar{t}$, it should have been that $k(t) = \underline{v}$ for $t \in [t', \bar{t}]$, which contradicts the fact that $k(t) \geq c$ for $t \leq t_2$ and $k(t)$ is the unique solution of $\phi(k(t), t) = 0$ for $t > t_2$, which implies $k(t) \geq c$ under FSD.

Since from Step 2.C the global incentive compatibility constraint (IC_{global}) binds for the highest type $\bar{t}$ and type $\bar{t}$ is indifferent between her contract and the contract choice of the lowest type, i.e.

$$U(\bar{t}) = \mathbb{E}_{\bar{t}}[v] - \mathbb{E}_0[\max\{v, k(0)\}],$$

it follows that the global incentive compatibility constraint (IC_{global}) binds for the highest type $\bar{t}$ only.

To summarize, in Steps 2.B through 2.E, we showed that for $t_2 \in [0, \bar{t}]$, the optimal cutoff points $\{k(t) : 0 \leq t \leq \bar{t}\}$ are given as follows. From Step 2.D, for $t \geq t_2$, $k(t)$ is nonincreasing and is the unique solution of $\phi(k(t), t) = 0$, where

$$\phi(v, t) = (v - c) + \frac{(1 - H(t))}{h(t)} \frac{\partial G(v, t) / \partial t}{g(v, t)}.$$

Also, $k(t)$ is nonincreasing with $k(t) \geq c$ for $t \leq t_2$. Similarly, $k(0) > c$ with $k(t)$ (strictly) decreasing for $t \leq t_1$, while $k(t) = c$ for $t \in (t_1, t_2)$. Moreover, the global incentive compatibility constraint (IC_{global}) binds for the highest type $\bar{t}$ only, who is indifferent between her contract and the contract choice of the lowest type, i.e.

$$U(\bar{t}) = \mathbb{E}_{\bar{t}}[v] - \mathbb{E}_0[\max\{v, k(0)\}].$$

Step 3. In this step, we prove that $0 < t_2 < \bar{t}$. We have already established the optimal allocation for a given t_2. We first prove that optimality requires $t_2 < \bar{t}$. To that end, consider the objective function value if we have $t_2 = \bar{t} - \varepsilon$. In that case, it follows from Step 2.D. that the optimal allocation dictates $\phi(k(t), t) = 0$ for $t > t_2$, and from Step 1.B we have that

$$\dot{U}(t) = -\int_{k(t)}^{\overline{v}} \frac{\partial G(v,t)}{\partial t} dv \quad \text{for} \quad t > t_2,$$

so that

$$U(\bar{t}) = -\int_{t_2}^{\bar{t}} \int_{k(t)}^{\overline{v}} \frac{\partial G(v,t)}{\partial t} dv.$$

Then, since the global IC constraint for $\bar{t}$ binds, cf. Step 2.A, $k(0)$ satisfies

$$U(\bar{t}) = \mathbb{E}_{\bar{t}}[v] - \int_{k(0)}^{\overline{v}} (v - k(0))\, g(v,0)\, dv - k(0).$$

Moreover, $k(t)$ is decreasing until the point $t_1 \le t_2$ where t_1 satisfies

$$U(\bar{t}) = \mathbb{E}_{\bar{t}}[v] - \int_{c}^{\overline{v}} (v - c)\, g(v,t_1)\, dv - c. \tag{D.15}$$

We also have for $t \in [t_1, t_2]$ that $k(t) = c$. Next, we rule out the possibility that $t_1 = \bar{t}$. To see this note that since

$$\mathbb{E}_{\bar{t}}[v] - c = \int_{\underline{v}}^{\overline{v}} (v - c)\, g(v,\bar{t})\, dv < \int_{\underline{v}}^{\overline{v}} (v - c)^+\, g(v,\bar{t})\, dv,$$

we have that

$$\int_{c}^{\overline{v}} (v - c)\, g(v,\bar{t})\, dv + c > \mathbb{E}_{\bar{t}}[v]. \tag{D.16}$$

Note also that if $t_2 = \bar{t}$, then $U(t) = 0$ for all t. Therefore, it follows from (D.15) and (D.16) that $t_1 < \bar{t}$. Clearly, we can think of t_1, t_2 and $k(t)$ for $t < t_2$ as a function of ε when $t_2 = \bar{t} - \varepsilon$. Moreover, just as argued for $\varepsilon = 0$, we can show that $t_1(\varepsilon) < t_2(\varepsilon) = \bar{t} - \varepsilon$ for $\varepsilon > 0$ sufficiently small.

The objective function of the monopolist as a function of ε is given by the following:

$$\text{(D.17)} \qquad \int_0^{\hat{t}(\varepsilon)} \int_{k(t,\varepsilon)}^{\bar{v}} f(v,t)(v-c)\,dvdt + \int_{\hat{t}(\varepsilon)}^{\bar{t}-\varepsilon} \int_c^{\bar{v}} f(v,t)(v-c)\,dvdt + \int_{\bar{t}-\varepsilon}^{\bar{t}} \int_{k(t)}^{\bar{v}} f(v,t)\,\phi(v,t)\,dvdt,$$

where $\phi(k(t), t) = 0$ for $t \geq t_2(\varepsilon) = \bar{t} - \varepsilon$, and $k(t,\varepsilon)$ for $t \leq t_1(\varepsilon)$ is characterized by the following

$$\text{(D.18)} \qquad \int_{\bar{t}-\varepsilon}^{\bar{t}} \int_{k(t)}^{\bar{v}} (-\partial G(v,t)/\partial t)\,dv = \mathbb{E}_{\bar{t}}[v] - \int_{k(t)}^{\bar{v}} (v - k(t))\,g(v,t)\,dv - k(t).$$

Similarly, $t_1(\varepsilon)$ is given by the following.

$$\int_{\bar{t}-\varepsilon}^{\bar{t}} \int_{k(t)}^{\bar{v}} (-\partial G(v,t)/\partial t)\,dv = \mathbb{E}_{\bar{t}}[v] - \int_c^{\bar{v}} (v-c)\,g(v, t_1(\varepsilon))\,dv - c.$$

Derivative of the objective given in (D.17) with respect to ε is as follows.

$$t_1'(\varepsilon) \int_c^{\bar{v}} f(v, t_1(\varepsilon))(v-c)\,dv - \int_0^{t_1(\varepsilon)} \frac{\partial k(t,\varepsilon)}{\partial \varepsilon} f(k(t,\varepsilon), t)(k(t,\varepsilon) - c)\,dt - \int_c^{\bar{v}} f(v, t_2(\varepsilon))(v-c)\,dv - t_1'(\varepsilon) \int_c^{\bar{v}} f(v, t_1(\varepsilon))(v-c)\,dv + \int_{k(t_2(\varepsilon))}^{\bar{v}} f(v, t_2(\varepsilon))\,\phi(v, t_2(\varepsilon))\,dv,$$

which is equal to

$$\int_0^{t_1(\varepsilon)} \left(-\frac{\partial k(t,\varepsilon)}{\partial \varepsilon}\right) f(k(t,\varepsilon),t)(k(t,\varepsilon)-c)\,dt - \int_c^{\bar{v}} f(v,t_2(\varepsilon))(v-c)\,dv$$
$$+\int_{k(t_2(\varepsilon))}^{\bar{v}} f(v,t_2(\varepsilon))\,\phi(v,t_2(\varepsilon))\,dv.$$

Calculating the derivative of the objective at $\varepsilon = 0$ gives the following

$$\int_0^{t_1(0)} \left(-\left.\frac{\partial k(t,\varepsilon)}{\partial \varepsilon}\right|_{\varepsilon=0}\right) f(k(t,0),t)(k(t,0)-c)\,dt - \int_c^{\bar{v}} f(v,\bar{t})(v-c)\,dv$$
$$+\int_{k(\bar{t})}^{\bar{v}} f(v,\bar{t})\,\phi(v,\bar{t})\,dv. \tag{D.19}$$

We want to show that this expression if positive. To this end, lets consider the last two terms. Since

$$\phi(v,t) = (v-c) + \frac{(1-H(t))}{h(t)}\frac{\partial G(v,t)/\partial t}{g(v,t)}$$

and $\phi(k(t),t) = 0$ for $t \geq t_2$, we have $k(\bar{t}) = c$. Thus,

$$\begin{aligned}
& \int_{k(\bar{t})}^{\bar{v}} f(v,\bar{t})\,\phi(v,\bar{t})\,dv - \int_c^{\bar{v}} f(v,\bar{t})(v-c)\,dv \\
= & \int_{k(\bar{t})}^{\bar{v}} f(v,\bar{t})\left[\phi(v,\bar{t}) - (v-c)\right]dv \\
= & \int_{k(\bar{t})}^{\bar{v}} f(v,\bar{t})\left[\frac{(1-H(\bar{t}))}{h(\bar{t})}\frac{\partial G(v,\bar{t})/\partial t}{g(v,\bar{t})}\right]dv \\
= & \; 0.
\end{aligned}$$

Thus, the expression in (D.19) reduces to the following

$$\int_0^{t_1(0)} \left(-\left.\frac{\partial k(t,\varepsilon)}{\partial \varepsilon}\right|_{\varepsilon=0}\right) f(k(t,0),t)(k(t,0)-c)\,dt.$$

To determine the sign of this expression, we next characterize $\partial k(t,\varepsilon)/\partial\varepsilon$ by differentiating both sides of the equation (D.18), which gives

$$\frac{dk(t,\varepsilon)}{d\varepsilon} = \frac{\int_{k(t_2(\varepsilon))}^{\overline{v}} \left(-\frac{\partial G(v,t_2(\varepsilon))}{\partial t}\right) dv}{\int_{k(t,\varepsilon)}^{\overline{v}} g(v,t)\,dv - 1} < 0.$$

In particular,

$$\left.\frac{dk(t,\varepsilon)}{d\varepsilon}\right|_{\varepsilon=0} = \frac{\int_{c}^{\overline{v}} \left(-\frac{\partial G(v,\bar{t})}{\partial t}\right) dv}{-G(k(t,0),t)} < 0.$$

Then, since $k(t,0) - c > 0$ on a set of positive measure and $\left.\frac{dk(t,\varepsilon)}{d\varepsilon}\right|_{\varepsilon=0} < 0$, we conclude that

$$\int_0^{t_1(0)} \left(-\left.\frac{\partial k(t,\varepsilon)}{\partial\varepsilon}\right|_{\varepsilon=0}\right) f(k(t,0),t)(k(t,0)-c)\,dt > 0.$$

Therefore, we must have $t_2 < \bar{t}$.

Similarly, we check whether it is optimal to have $t_2 = 0$. To that end, we write the expected profits of the firm as follows:

$$\text{(D.20)} \qquad \int_0^{\varepsilon}\int_{k(t,\varepsilon)}^{\bar{v}} f(v,t)(v-c)\,dvdt + \int_{\varepsilon}^{\bar{t}}\int_{k(t)}^{\bar{v}} f(v,t)\,\phi(v,t)\,dvdt,$$

Derivative of this objective with respect to ε is given by the following.

$$\int_{k(\varepsilon,\varepsilon)}^{\bar{v}} f(v,\varepsilon)(v-c)\,dv + \int_0^{\varepsilon}\left[-\frac{\partial k(t,\varepsilon)}{\partial\varepsilon} f(k(t,\varepsilon),t)(k(t,\varepsilon)-c)\right] dt$$
$$-\int_{k(\varepsilon)}^{\bar{v}} f(v,\varepsilon)\,\phi(v,\varepsilon)\,dv$$

Then evaluating this at $\varepsilon = 0$, it reduces to the following

$$\begin{aligned}
& \int_{k(0)}^{\bar{v}} f(v,0)(v-c)\,dv - \int_{k(0)}^{\bar{v}} f(v,0)\,\phi(v,0)\,dv \\
= & \int_{k(0)}^{\bar{v}} f(v,0)\left[(v-c) - \phi(v,0)\right] dv, \\
= & \int_{k(0)}^{\bar{v}} \frac{1}{h(0)} \frac{\partial G(v,0)/\partial t}{g(v,0)} dv, \\
> & \ 0.
\end{aligned}$$

Therefore, we must have $t_2 > 0$, which completes the proof of Step 3. In other words, it is not optimal to have $t_2 = \bar{t}$, i.e. $U(t) = 0$ for all t. Similarly, it is not optimal to have $t_2 = 0$, i.e. $U'(0) > 0$. This concludes the proof of Proposition 39. ■

Proof of Proposition 41. In order to analyze the behavior of the initial price path $\{\overline{x}(t) : 0 \leq t \leq \bar{t}\}$, we consider three regions. First, we have for $t \leq t_2$ that

$$\overline{x}(t) = \int_{k(t)}^{\bar{v}} (1 - G(v,t))\,dv + k(t).$$

Then,

$$\begin{aligned}
\overline{x}'(t) &= k'(t) - k'(t)(1 - G(k(t),t)) - \int_{k(t)}^{\bar{v}} \frac{\partial G(v,t)}{\partial t} dv, \\
&= k'(t)\,G(k(t),t) - \int_{k(t)}^{\bar{v}} \frac{\partial G(v,t)}{\partial t} dv, \\
&= k'(t)\,G(k(t),t) \\
&\quad - \left[\bar{v}\frac{\partial G(\bar{v},t)}{\partial t} - k(t)\frac{\partial G(k(t),t)}{\partial t} - \int_{k(t)}^{\bar{v}} v\frac{\partial g(v,t)}{\partial t} dv\right], \\
&= k'(t)\,G(k(t),t) + \left[k(t)\frac{\partial G(k(t),t)}{\partial t} + \int_{k(t)}^{\bar{v}} v\frac{\partial g(v,t)}{\partial t} dv\right],
\end{aligned}$$

since $\frac{\partial G(\bar{v},t)}{\partial t} = 0$.

First, focus on the types $t_1 < t \leq t_2$. On this interval, $k(t) = c$, and hence,

$$\begin{aligned} \overline{x}'(t) &= c\frac{\partial G(c,t)}{\partial t} + \int_c^{\bar{v}} v\frac{\partial g(v,t)}{\partial t}dv, \\ &= c\frac{\partial G(c,t)}{\partial t} + \left[-c\frac{\partial G(c,t)}{\partial t} - \int_c^{\bar{v}} v\frac{\partial G(v,t)}{\partial t}dv\right], \\ &= -\int_c^{\bar{v}} v\frac{\partial G(v,t)}{\partial t}dv, \end{aligned}$$

and hence, $\overline{x}'(t) > 0$.

For $t \in [0, t_1]$, we can find $k'(t)$ as follows: We know that

$$\mathbb{E}_t\left[\max\{v, k(t)\}\right]$$

is constant over the interval $[0, t_1]$. That is, derivative of

$$k(t)\int_{\underline{v}}^{k(t)} g(v,t)\,dv + \int_{k(t)}^{\bar{v}} vg(v,t)\,dv$$

is zero, i.e.

$$\begin{aligned}
0 &= k'(t)\int_{\underline{v}}^{k(t)} g(v,t)\,dv + k(t)\,k'(t)\,g(k(t),t) + k(t)\int_{\underline{v}}^{k(t)} \frac{\partial g(v,t)}{\partial t}dv \\
&\quad + \int_{k(t)}^{\bar{v}} v\frac{\partial g(v,t)}{\partial t}dv - k'(t)\,k(t)\,g(k(t),t) \\
&= k'(t)\int_{\underline{v}}^{k(t)} g(v,t)\,dv + k(t)\int_{\underline{v}}^{k(t)} \frac{\partial g(v,t)}{\partial t}dv + \int_{k(t)}^{\bar{v}} v\frac{\partial g(v,t)}{\partial t}dv \\
&= k'(t)\int_{\underline{v}}^{k(t)} g(v,t)\,dv + \int_{k(t)}^{\bar{v}} v\frac{\partial g(v,t)}{\partial t}dv \\
&\quad + k(t)\frac{\partial G(k(t),t)}{\partial t} - k(t)\frac{\partial G(\underline{v},t)}{\partial t} \\
&= k'(t)\int_{\underline{v}}^{k(t)} g(v,t)\,dv + \int_{k(t)}^{\bar{v}} v\frac{\partial g(v,t)}{\partial t}dv + k(t)\frac{\partial G(k(t),t)}{\partial t}.
\end{aligned}$$

That is,

$$\left[\int_{k(t)}^{\bar{v}} v\frac{\partial g(v,t)}{\partial t}dv + k(t)\frac{\partial G(k(t),t)}{\partial t}\right] = -k'(t)\int_{\underline{v}}^{k(t)} g(v,t)\,dv.$$

Then,

$$\begin{aligned}
\overline{x}'(t) &= k'(t)\,G(k(t),t) \\
&\quad - \left[\bar{v}\frac{\partial G(\bar{v},t)}{\partial t} - k(t)\,\frac{\partial G(k(t),t)}{\partial t} - \int_{k(t)}^{\bar{v}} v\frac{\partial g(v,t)}{\partial t}dv\right], \\
&= k'(t)\,G(k(t),t) - \bar{v}\frac{\partial G(\bar{v},t)}{\partial t} \\
&\quad + \left[k(t)\,\frac{\partial G(k(t),t)}{\partial t} + \int_{k(t)}^{\bar{v}} v\frac{\partial g(v,t)}{\partial t}dv\right], \\
&= k'(t)\,G(k(t),t) - \bar{v}\frac{\partial G(\bar{v},t)}{\partial t} - k'(t)\int_{\underline{v}}^{k(t)} g(v,t)\,dv, \\
&= k'(t)\,G(k(t),t) - \bar{v}\frac{\partial G(\bar{v},t)}{\partial t} - k'(t)\,G(k(t),t)\,dv, \\
&= -\bar{v}\frac{\partial G(\bar{v},t)}{\partial t} \\
&= 0.
\end{aligned}$$

Now consider the case that $t > t_2$. Then,

$$\underline{x}'(t) = -U'(t) - \frac{dk(t)}{dt}(1 - G(v,t)) - \int_{k(t)}^{\bar{v}} \frac{\partial G(v,t)}{\partial t}dv,$$

Therefore, we can calculate $\underline{x}(t)$ for $t \geq t_2$ from (4.3) and the boundary condition that

$$\underline{x}(t_2) = \int_{k(t_2)}^{\bar{v}} (1 - G(v,t_2))\,dv.$$

That is,

$$\begin{aligned}
\overline{x}'(t) &= \underline{x}'(t) + k'(t), \\
&= -U'(t) - \frac{dk(t)}{dt}(1 - G(k(t), t)) - \int_{k(t)}^{\bar{v}} \frac{\partial G(v,t)}{\partial t} dv + k'(t), \\
&= \int_{k(t)}^{\bar{v}} \frac{\partial G(v,t)}{\partial t} dv - \frac{dk(t)}{dt}(1 - G(k(t), t)) - \int_{k(t)}^{\bar{v}} \frac{\partial G(v,t)}{\partial t} dv + k'(t), \\
&= -\frac{dk(t)}{dt}(1 - G(k(t), t)) + k'(t), \\
&= k'(t)[-1 + G(k(t), t) + 1] \\
&= k'(t) G(k(t), t), \\
&< 0,
\end{aligned}$$

thus, the initial prices are decreasing over the interval $[t_2, \bar{t}]$.

The price $\overline{x}(\bar{t})$ charged to the highest type can be found as follows: We have

$$U(\bar{t}) = \mathbb{E}_{\bar{t}}[v] - \mathbb{E}_0[\max\{v, k(0)\}].$$

Moreover,

$$U(\bar{t}) = -\underline{x}(\bar{t}) + \int_c^{\bar{v}} (1 - G(v, \bar{t}))\, dv.$$

Then,

$$
\begin{aligned}
\overline{x}(\bar{t}) &= \underline{x}(\bar{t}) + c, \\
&= \int_c^{\bar{v}} (1 - G(v, \bar{t}))\, dv - U(\bar{t}) + c, \\
&= \int_c^{\bar{v}} (1 - G(v, \bar{t}))\, dv + c - \mathbb{E}_{\bar{t}}[v] + \mathbb{E}_0[\max\{v, k(0)\}], \\
&= \int_c^{\bar{v}} (1 - G(v, \bar{t}))\, dv + c - \mathbb{E}_{\bar{t}}[v] + \mathbb{E}_0[\max\{v, k(0)\}], \\
&= \underline{v} - \int_{\underline{v}}^{\bar{v}} (1 - G(v, \bar{t}))\, dv \\
&\quad + \int_c^{\bar{v}} (1 - G(v, \bar{t}))\, dv + c + \mathbb{E}_0[\max\{v, k(0)\}], \\
&= \underline{v} - \int_{\underline{v}}^{c} (1 - G(v, \bar{t}))\, dv + c + \mathbb{E}_0[\max\{v, k(0)\}].
\end{aligned}
$$

We can find $\overline{x}(t_2)$ as follows:

$$
\begin{aligned}
\overline{x}(t_2) &= \overline{x}(\bar{t}) - \int_{t_2}^{\bar{t}} k'(t)\, G(k(t), t)\, dt, \\
&= \underline{v} - \int_{\underline{v}}^{c} (1 - G(v, \bar{t}))\, dv + c + \mathbb{E}_0[\max\{v, k(0)\}] \\
&\quad - \int_{t_2}^{\bar{t}} k'(t)\, G(k(t), t)\, dt,
\end{aligned}
$$

where $\phi(k(t), t) = 0$ and hence,

$$
k(t) = c - \frac{(1 - H(t))}{h(t)} \frac{\partial G(k(t), t)/\partial t}{g(k(t), t)}.
$$

Next, we calculate $\overline{x}(t_2-)$. To that end, first consider the case that $t_1 = t_2$. Then, $\overline{x}(t_2) = \overline{x}(0)$. Moreover,

$$\begin{aligned}
\overline{x}(0) &= \underline{x}(0) + k(0), \\
&= \int_{k(0)}^{\bar{v}} (1 - G(v,0))\, dv + k(0), \\
&= -k(0)(1 - G(k(0),0)) + \int_{k(0)}^{\bar{v}} vg(v,0)\, dv + k(0), \\
&= \int_{k(0)}^{\bar{v}} vg(v,0)\, dv + k(0)\, G(k(0),0), \\
&= \mathbb{E}_0[\max\{v, k(0)\}].
\end{aligned}$$

This implies that

$$
\begin{aligned}
\overline{x}(t_2) - \overline{x}(0) &> \overline{x}(\bar{t}) - \overline{x}(0), \\
&= \int_c^{\bar{v}} (1 - G(v, \bar{t}))\, dv + c - \mathbb{E}_{\bar{t}}[v], \\
&= \int_c^{\bar{v}} (1 - G(v, \bar{t}))\, dv + c - \left[\int_{\underline{v}}^{\bar{v}} vg(v, \bar{t})\, dv\right] \\
&= \int_c^{\bar{v}} (1 - G(v, \bar{t}))\, dv + c + \left[\underline{v} - \int_{\underline{v}}^{\bar{v}} (1 - G(v, \bar{t}))\, dv\right], \\
&= \int_c^{\bar{v}} (1 - G(v, \bar{t}))\, dv + c + \underline{v} - \int_{\underline{v}}^{\bar{v}} (1 - G(v, \bar{t}))\, dv, \\
&= c + \underline{v} - \int_{\underline{v}}^{c} (1 - G(v, \bar{t}))\, dv, \\
&\geq c + \underline{v} - \left[\int_{\underline{v}}^{c} dv\right], \\
&= c + \underline{v} - [c - \underline{v}], \\
&= 2\underline{v} \\
&\geq 0.
\end{aligned}
$$

Thus, there is an upward jump in price at the point t_2 in the case that $t_1 = t_2$.

In the case that $t_1 < t_2$, we know that $k(t_2) = c$ and $U(t_2) = 0$. Thus, if $\overline{x}(t_2) > \overline{x}(\bar{t})$, type t_2 would pretend to be type $\bar{t}$. Thus, we have $\overline{x}(t_2) < \overline{x}(\bar{t}) < \overline{x}(t_2+)$ and there is an upward jump in the prices at the point t_2 as well.■

D.3. Technical Results

Derivation of the dual problem to ($\mathbf{P}_A$) and the coextremality conditions. We will follow the road map provided by [**53**] to derive the dual problem of control associated with (P_2). In particular, we first append the penalty expressions corresponding to the constraints in the objective function by defining the convex, extended real valued integrand L and the convex functional l. We also formulate the problem towards minimization. Next, we compute the conjugate convex functions associated with L and l so as to define the dual integrand M and the dual functional m. The dual problem of control is defined using M and m.

To facilitate the analysis to follow, define the indicator function $\chi_F(\cdot)$ for a given set F by

$$\chi_F(x) = \begin{cases} 0 & \text{if } x \in F, \\ \infty & \text{otherwise.} \end{cases}$$

We express (P_A) in terms of the convex integrand L and the convex lower semi-continuous functional l which are defined as follows. Define L on $[\underline{v}, \bar{v}] \times \mathbb{R}^6$ as follows:

$$L(v, s, z, u, \dot{s}, \dot{z}, \dot{u}) = -g\,(v,t)\,(v-c)\,y + \chi_{\mathbb{R}_-}(y-1) + \chi_{\mathbb{R}_+}(y) \tag{D.21}$$

if $\dot{s} = y\left[\frac{-\partial G(v,t)}{\partial t}\right], \dot{z} = y\,(1 - G\,(v,t))\,, \dot{u} = y$ and $L(v, s, z, u, \dot{s}, \dot{z}, \dot{u}) = \infty$ otherwise. The integrand L eliminates the hard constraints of (P_A) by appending them to the objective function as penalty expressions. In this sense, the penalty expression

$$\chi_{\mathbb{R}_-}(y-1)+\chi_{\mathbb{R}_+}(y)$$

makes sure that $0 \leq y(v) \leq 1$ for all $v \in [\underline{v}, \bar{v}]$. Notice also that we have reformulated the problem towards minimization. The system dynamics is incorporated in L by the fact that we require $\dot{s} = y\left[\frac{-\partial G(v,t)}{\partial t}\right]$, $\dot{z} = y(1-G(v,t))$ and $\dot{u} = y$, since otherwise the integrand L takes the value ∞.

Next step is to define the functional l on $\mathbb{R}^3 \times \mathbb{R}^3$ with values on $\mathbb{R} \cup \{\infty\}$ so as to initiate the problem with appropriate initial values of the state variables and impose the restrictions to be satisfied by the valuations $\bar{v}$. To be more specific, The functional l is defined as

$$l\left(s_{\underline{v}}, z_{\underline{v}}, u_{\underline{v}}, s_{\bar{v}}, z_{\bar{v}}, u_{\bar{v}}\right) = l_{\underline{v}}(s_{\underline{v}}, z_{\underline{v}}, u_{\underline{v}}) + l_{\bar{v}}(s_{\bar{v}}, z_{\bar{v}}, u_{\bar{v}}), \tag{D.22}$$

where the convex, lower semi-continuous functionals $l_{\underline{v}}$ and $l_{\bar{v}}$ are given by

$$l_{\underline{v}}(s_{\underline{v}}, z_{\underline{v}}, u_{\underline{v}}) = \chi_{\{0\}}(s_{\underline{v}}) + \chi_{\{u_0\}}(u_{\underline{v}}), \tag{D.23}$$

$$l_{\bar{v}}(s_{\bar{v}}, z_{\bar{v}}, u_{\bar{v}}) = \chi_{\mathbb{R}_+}(s_{\bar{v}} - \dot{U}_*(t)) + \chi_{\mathbb{R}_+}(z_{\bar{v}} + u_0 - U_*(t)) + \chi_{\mathbb{R}_-}(u(\bar{v}) - u_*(\bar{v})). \tag{D.24}$$

The functional l_0 dictates that $s(\underline{v}) = 0$ and $u(\underline{v}) = u_0$, which initiates the problem, whereas l_T imposes the constraints that

$$z(\bar{v}) \geq U_*(t) - u_0,\ \ s(\bar{v}) \geq \dot{U}_*(t) \text{ and } u(\bar{v}) \leq u_*(\bar{v}).$$

Then, the primal problem (P_A) can equivalently be stated as a problem of minimizing

$$\int_{\underline{v}}^{\bar{v}} L(v, s(v), z(v), u(v), \dot{s}(v), \dot{z}(v), \dot{u}(v))dv + l\left(s_{\underline{v}}, z_{\underline{v}}, u_{\underline{v}}, s_{\bar{v}}, z_{\bar{v}}, u_{\bar{v}}\right).$$

As our second step, we compute the conjugates to the functions L and l. Let L^* denote the conjugate to L. To be specific,

(D.25) $$L^*(v, p_s, p_z, p_u, q_s, q_z, q_u)$$

$$= \sup_{s,z,u,\dot{s},\dot{z},\dot{u}} \{sp_s + zp_z + up_u + \dot{s}q_s + \dot{z}q_z + \dot{u}q_u - L(v, s, z, u, \dot{s}, \dot{z}, \dot{u})\}.$$

We can express L^* more explicitly as follows. Note that $L(v, s, z, u, \dot{s}, \dot{z}, \dot{u}) < \infty$ only if there exists some $y \in [0, 1]$ such that $\dot{s} = y\left[\frac{-\partial G(v,t)}{\partial t}\right]$, $\dot{z} = yG(w, t)$ and $\dot{u} = y$. Then, we can write L^* as follows: for $p_s, p_z, p_u, q_s, q_z, q_u \in \mathbb{R}$,

$$\begin{aligned} & L^*(v, p_s, p_z, p_u, q_s, q_z, q_u) \\ = & \sup_{s,z,u,y\in[0,1]} \left\{ \begin{array}{c} sp_s + zp_z + up_u \\ +y\left[q_u + g(v,t)(v-c) + \left(-\partial G(v,t)/\partial t\right) q_s + \left(1 - G(w,t)\right) q_z\right] \end{array} \right\}, \\ = & \chi_{\{0\}}\{p_s\} + \chi_{\{0\}}\{p_z\} + \chi_{\{0\}}\{p_u\} \\ & + \max\left\{0, q_u + g(v,t)(v-c) + \left(-\partial G(v,t)/\partial t\right) q_s + G(w,t)\, q_z\right\}. \end{aligned}$$

The first equality is obtained by replacing $\dot{s} = y\left[\frac{-\partial G(v,t)}{\partial t}\right]$, $\dot{z} = yG(v, t)$ and $\dot{u} = y$ and noting that $L(v, s, z, u, \dot{s}, \dot{z}, \dot{u}) = -g(v, t)(v - c)\, y$. The second line follows from carrying out the maximization problem in the second line. To get the second equality, note that we have

$$\sup_{s,z,u}\{sp_s+zp_z+up_u\}=\chi_{\{0\}}\{p_s\}+\chi_{\{0\}}\{p_z\}+\chi_{\{0\}}\{p_u\},$$

since $\sup_{s,z,u}\{sp_s+zp_z+up_u\}$ takes the value ∞ unless $p_s=p_z=p_u=0$.

Using the conjugate L^* of the primal integrand L, we calculate the dual integrand M. For $v\in[\underline{v},\bar{v}]$ and $q_s,q_z,q_u,p_s,p_z,p_u\in\mathbb{R}$, the dual integrand M is given by

$$M(v,q_s,q_z,q_u,p_s,p_z,p_u)=L^*(v,p_s,p_z,p_u,q_s,q_z,q_u).$$

That is,

$$\begin{aligned}&M(v,\rho_s(v),\rho_z(v),\rho_u(v),\dot{\rho}_s(v),\dot{\rho}_z(v),\dot{\rho}_u(v))\\=\;&L^*(v,\dot{\rho}_s(v),\dot{\rho}_z(v),\dot{\rho}_u(v),\rho_s(v),\rho_z(v),\rho_u(v)),\\=\;&\chi_{\{0\}}\{\dot{\rho}_s(v)\}+\chi_{\{0\}}\{\dot{\rho}_z(v)\}+\chi_{\{0\}}\{\dot{\rho}_u(v)\}\\&+\max\{0,\rho_u(v)+g(v,t)(v-c)+(-\partial G(v,t)/\partial t)\rho_s(v)+(1-G(v,t))\rho_z(v)\}.\end{aligned}$$

This implies that $\dot{\rho}_s(v)=\dot{\rho}_z(v)=\dot{\rho}_u(v)=0$ for all v.

What remains is to derive the terminal conditions associated with the dual problem. To that end, define the functional m as follows:

$$\begin{aligned}&m(\rho_s(\underline{v}),\rho_z(\underline{v}),\rho_u(\underline{v}),\rho_s(\bar{v}),\rho_z(\bar{v}),\rho_u(\bar{v}))\\=\;&l^*_{\underline{v}}(\rho_s(\underline{v}),\rho_z(\underline{v}),\rho_u(\underline{v}))+l^*_{\bar{v}}(-\rho_s(\bar{v}),-\rho_z(\bar{v}),-\rho_u(\bar{v})),\end{aligned}$$

where $l_{\underline{v}}^*$ and $l_{\bar{v}}^*$ are the conjugates of $l_{\underline{v}}$ and $l_{\bar{v}}$. We calculate l_0^* as follows:

$$\begin{aligned} l_{\underline{v}}^*(\rho_s, \rho_z, \rho_u) &= \sup_{x_s, x_z, x_u} \{x_s\rho_s + x_z\rho_z + x_u\rho_u - l_{\underline{v}}(x_s, x_z, x_u)\}, \\ &= \sup_{x_u}\{x_u\rho_u\} + u_0\rho_z, \\ &= \chi_{\{0\}}(\rho_u) + u_0\rho_z. \end{aligned}$$

Since

$$\begin{aligned} & m(\rho_s(\underline{v}), \rho_z(\underline{v}), \rho_u(\underline{v}), \rho_s(\bar{v}), \rho_z(\bar{v}), \rho_u(\bar{v})) \\ = \quad & l_{\underline{v}}^*(\rho_s(\underline{v}), \rho_z(\underline{v}), \rho_u(\underline{v})) + l_{\bar{v}}^*(-\rho_s(\bar{v}), -\rho_z(\bar{v}), -\rho_u(\bar{v})), \end{aligned}$$

the coextremality condition impose the restriction that $\rho_u(\underline{v}) = 0$. Together with the fact that $\dot{\rho}_u(v) = 0$ for all v, this implies that $\rho_u(v) = 0$ for all v.

Similarly, since

$$l_{\bar{v}}(s_{\bar{v}}, z_{\bar{v}}, u_{\bar{v}}) = \chi_{\mathbb{R}_+}(s_{\bar{v}} - \dot{U}_*(t)) + \chi_{\mathbb{R}_+}(z_{\bar{v}} + u_0 - U_*(t)) + \chi_{\mathbb{R}_-}(u_{\bar{v}} - u_*(\bar{v})).$$

we get

$$\begin{aligned} l_{\bar{v}}^*(\rho_s, \rho_z, \rho_u) &= \sup_{x_s, x_z, x_u} \{x_s\rho_s + x_z\rho_z + x_u\rho_u - l_{\bar{v}}(x_s, x_z, x_u)\}, \\ &= \sup_{x_s \geq \dot{U}_*(t)} \{x_s\rho_s\} + \sup_{x_z \geq -u_0 + U_*(t)} \{x_z\rho_z\} + \sup_{x_u \leq u_*(\bar{v})} \{x_z\rho_z\}, \\ &= \chi_{\mathbb{R}_-}(\rho_s) + \chi_{\mathbb{R}_-}(\rho_z) + \chi_{\mathbb{R}+}(\rho_u). \end{aligned}$$

Then, the coextremality conditions impose the following restrictions:

$$\rho_s(\bar{v}), \rho_z(\bar{v}) \geq 0 \text{ and } \rho_u(\bar{v}) \leq 0.$$

The dual problem of control is then to minimize

$$\int_{\underline{v}}^{\bar{v}} M(v, \rho_s(v), \rho_z(v), \rho_u(v), \dot{\rho}_s(v), \dot{\rho}_z(v), \dot{\rho}_u(v))dv$$
$$+ m(\rho_s(\underline{v}), \rho_z(\underline{v}), \rho_u(\underline{v}), \rho_s(\bar{v}), \rho_z(\bar{v}), \rho_u(\bar{v})),$$

which is equivalent to minimizing

$$\int_{\underline{v}}^{\bar{v}} \max\left\{0, \rho_u(\underline{v}) + g\left(v,t\right)\left(v-c\right) - \frac{\partial G\left(v,t\right)}{\partial t}\rho_s\left(v\right) + \left(1 - G\left(v,t\right)\right)\rho_z\left(v\right)\right\} dv$$
$$+ u_0\rho_z\left(\underline{v}\right)$$

subject to $\qquad\qquad (\mathrm{D}_A)$

$$\dot{\rho}_s(v) = \dot{\rho}_z(v) = \dot{\rho}_u(v) = 0,$$

$$\dot{\rho}_u(\underline{v}) = 0,$$

$$\rho_s(\bar{v}), \rho_z(\bar{v}) \geq 0.$$

Since the primal problem (P_A) is trivially feasible (simply use the original allocation y_*), the objective function values of (P_A) and (D_A) are equal to each other, cf. Theorem 4 of [**53**]. This concludes the derivation of the dual problem to (P_A). ■

Proof of Lemma 58. We first establish part (i) of Lemma 58. To that end, notice that using the constraints that

$$\dot{\rho}_s(v) = \dot{\rho}_z(v) = \dot{\rho}_u(v) = 0,$$

$$\dot{\rho}_u(\underline{v}) = 0,$$

$$\rho_s(\bar{v}), \rho_z(\bar{v}) \geq 0,$$

we can rewrite the dual problem (D_A) as follows:

$$\begin{aligned} &\min_{\rho_z, \rho_s} \; u_0 \rho_z + \int_{\underline{v}}^{c} \max\left\{0, g\left(v,t\right)\left(v-c\right) - \frac{\partial G\left(v,t\right)}{\partial t}\rho_s + \left(1 - G\left(v,t\right)\right)\rho_z\right\} dv \\ &+ \int_c^{\bar{v}} g\left(v,t\right)\left(v-c\right) dv - \rho_s \int_c^{\bar{v}} \frac{\partial G\left(v,t\right)}{\partial t} dv + \rho_z \int_c^{\bar{v}} \left(1 - G\left(v,t\right)\right) dv \\ &\text{subject to} \qquad\qquad (\mathrm{D}_A) \\ &\rho_s, \rho_z \geq 0. \end{aligned}$$

Then, since $\partial G\left(v,t\right)/\partial t \leq 0$ for all v and $u_0 \leq 0$, it must be that $\rho_s = 0$. This completes the proof of part (i).

We next prove parts (ii) and (iii) of Lemma 58. The primal problem (P_A) and its dual (D_A) have the same optimal objective value by Theorem 4 of [**53**]. Moreover, by Theorem 5 of [**53**], letting$\{y\left(v\right) : v \in [\underline{v}, \bar{v}]\}$ be a feasible solution for (P_A) with the corresponding state trajectories $\{u\left(v\right), s\left(v\right), z\left(v\right) : v \in [\underline{v}, \bar{v}]\}$, and $\{\rho_s\left(v\right), \rho_z\left(v\right), \rho_u\left(v\right) : v \in [\underline{v}, \bar{v}]\}$ be a feasible control for (D_A), the controls y and ρ_s, ρ_z, ρ_u are optimal for (P_A) and (D_A), respectively, if and only if they satisfy the following coextremality conditions: For almost every $v \in [\underline{v}, \bar{v}]$,

$$\left(\rho_s\left(\underline{v}\right), \rho_z\left(\underline{v}\right), \rho_u\left(\underline{v}\right), -\rho_s\left(\bar{v}\right), -\rho_z\left(\bar{v}\right), -\rho_u\left(\bar{v}\right)\right)$$

is an element of

$$\partial l\left(s\left(\underline{v}\right), z\left(\underline{v}\right), u\left(\underline{v}\right), s\left(\bar{v}\right), z\left(\bar{v}\right), u\left(\bar{v}\right)\right) \tag{D.26}$$

and

$$\left(\dot{\rho}_s\left(v\right), \dot{\rho}_z\left(v\right), \dot{\rho}_u\left(v\right), \rho_s\left(v\right), \rho_z\left(v\right), \rho_u\left(v\right)\right)$$

is an element of

$$\partial L(v, s(v), z(v), u\left(v\right), \dot{s}(v), \dot{z}(v), \dot{u}(v)) \tag{D.27}$$

where ∂L and ∂l denote the subgradients of the convex integrand L and the the functional l, defined as in the derivation of the dual problem to (P_A) and the coextremality conditions.

To be more specific about the coextremality conditions, we derive the subgradients of L, $l_{\underline{v}}$ and $l_{\bar{v}}$, where L is a convex integrand and $l_{\underline{v}}$ and $l_{\bar{v}}$ are convex functionals as in the derivation of the dual problem (D_A). The theory of subgradients of convex functions on $\mathbb{R}^n$ is presented at length in Section 9 of [**55**]. This theory includes formulas to calculate subgradients in various situations.

First, we calculate the subgradient of L from its epigraphical normals. To that end, we use Theorem 8.9 of [**55**] which proves that for $h : \mathbb{R}^n \to [-\infty, +\infty]$ and any point $\overline{x}$

at which h is finite, one has

$$\partial h(\overline{x}) = \{v : (v, -1) \in N_{\text{epi } h}(\overline{x}, h(\overline{x}))\},$$

where, epi h denotes the epigraph of h defined as

$$\text{epi } h := \{(x, \alpha) \in \mathbb{R}^n \times \mathbb{R} : \alpha \geq h(x)\},$$

and $N_{\text{epi } h}(\overline{x}, h(\overline{x}))$ is the set of vectors normal to the set epi h at $(\overline{x}, h(\overline{x}))$ in the general sense as in Definition 6.3 of [**55**].

For $v \in [\underline{v}, \bar{v}]$, the epigraph of the integrand L is defined as follows: epi $L(v)$ consists of points $(s, z, u, \dot{s}, \dot{z}, \dot{u}, \alpha) \in \mathbb{R}^7$ such that

$$\dot{s} = y \left[\frac{-\partial G(v,t)}{\partial t} \right], \ \dot{z} = y\,(1 - G(v,t)), \dot{u} = y, \ \alpha \geq -g(v,t)\,(v-c)\,y \ \text{ and } 0 \leq y \leq 1,$$

since the points $(s, z, u, \dot{s}, \dot{z}, \dot{u}) \in \mathbb{R}^6$where $L(v, s, z, u, \dot{s}, \dot{z}, \dot{u}) = \infty$ are such that the vertical "line" $(s, z, u, \dot{s}, \dot{z}, \dot{u}) \in \mathbb{R}^6$ misses epi $L(v)$.

Then, we can write

$$\begin{aligned} & \partial L(v, \overline{s}, \overline{z}, \overline{u}, \overline{\dot{s}}, \overline{\dot{z}}, \overline{\dot{u}}) \\ = & \left\{ (v^1, v^2) \in \mathbb{R}^6 : (v^1, v^2, -1) \in N_{\text{epi } L(t)} \left(\overline{s}, \overline{z}, \overline{u}, \overline{\dot{s}}, \overline{\dot{z}}, \overline{\dot{u}}, L\left(v, \overline{s}, \overline{z}, \overline{u}, \overline{\dot{s}}, \overline{\dot{z}}, \overline{\dot{u}}\right)\right)\right\}. \end{aligned}$$

First, note that for $v \in [\underline{v}, \bar{v}]$, epi $L(v)$ is a convex set and the point

$$\left(\overline{s}, \overline{z}, \overline{u}, \overline{\dot{s}}, \overline{\dot{z}}, \overline{\dot{u}}, L\left(v, \overline{s}, \overline{z}, \overline{u}, \overline{\dot{s}}, \overline{\dot{z}}, \overline{\dot{u}}\right)\right)$$

is an element of epi $L(v)$ for $(\overline{s},\overline{z},\overline{u},\overline{\dot{s}},\overline{\dot{z}},\overline{\dot{u}})\in\mathbb{R}^6$. Let $\mathbf{v}$ denote an arbitrary element of $\mathbb{R}^7$, where the first 3 components of $\mathbf{v}$ is denoted as v^1, the subsequent 3 components by v^2 and the last component by v^α. That is, $\mathbf{v}=[v^1,v^2,v^\alpha]'$, where $v^1,v^2\in\mathbb{R}^3$ and $v^\alpha\in\mathbb{R}$. Then, Theorem 6.9 of [**55**], gives

$$N_{\text{epi }L(v)}\left(\overline{s},\overline{z},\overline{u},\overline{\dot{s}},\overline{\dot{z}},\overline{\dot{u}},L\left(v,\overline{s},\overline{z},\overline{u},\overline{\dot{s}},\overline{\dot{z}},\overline{\dot{u}}\right)\right) \tag{D.28}$$

$$= \left\{\begin{array}{c} \mathbf{v}\in\mathbb{R}^7:\forall(s,z,u,\dot{s},\dot{z},\dot{u},\alpha)\in\text{epi }L(v), \\ \left[(s,z,u,\dot{s},\dot{z},\dot{u},\alpha)-(\overline{s},\overline{z},\overline{u},\overline{\dot{s}},\overline{\dot{z}},\overline{\dot{u}},L\left(v,\overline{s},\overline{z},\overline{u},\overline{\dot{s}},\overline{\dot{z}},\overline{\dot{u}}\right))\right]\cdot\mathbf{v}\ \leq 0. \end{array}\right\}. \tag{D.29}$$

We next property will assist us in finding the subgradients of L and establishing part (ii) of Lemma 58.

Property A. *For $v\in[\underline{v},\bar{v}]$, if*

$$\mathbf{v}=(v^1,v^2,v_\alpha)'\in N_{\text{epi }L(v)}\left(\overline{s},\overline{z},\overline{u},\overline{\dot{s}},\overline{\dot{z}},\overline{\dot{u}},L\left(v,\overline{s},\overline{z},\overline{u},\overline{\dot{s}},\overline{\dot{z}},\overline{\dot{u}}\right)\right),$$

and if $\overline{\dot{s}}=\overline{y}\left[\frac{-\partial G(v,t)}{\partial t}\right]$, $\overline{\dot{z}}=\overline{y}\left(1-G\left(v,t\right)\right)$ and $\overline{\dot{u}}=\overline{y}$, for $\overline{y}$ such that $0\leq\overline{y}\leq 1$, then,

$$\bar{y}\in\arg\max_{0\leq z\leq 1}\left\{z\left(-v_\alpha\left(v-c\right)-\frac{\partial G\left(v,t\right)/\partial t}{g\left(v,t\right)}v_1^1+\frac{1-G\left(v,t\right)}{g\left(v,t\right)}v_2^1+\frac{v_3^1}{g\left(v,t\right)}\right)\right\}.$$

To establish Property A, first recall that for any $(s,z,u,\dot{s},\dot{z},\dot{u},\alpha)\in\text{epi }L(v)$, there exists some $y\in[0,1]$ such that

$$\dot{s}=y\left[\frac{-\partial G\left(v,t\right)}{\partial t}\right],\ \ \dot{z}=y\left(1-G\left(v,t\right)\right),\dot{u}=y,\ \alpha\geq -g\left(v,t\right)\left(v-c\right)y.$$

Consider now the following element of epi $L(v)$:

$$\left(\hat{s},\hat{z},\hat{u},\widehat{\dot{s}},\widehat{\dot{z}},\widehat{\dot{u}},\hat{\alpha}\right)=\left(\overline{s},\overline{z},\overline{u},\dot{s},\dot{z},\dot{u},L\left(v,\overline{s},\overline{z},\overline{u},\overline{\dot{s}},\overline{\dot{z}},\overline{\dot{u}}\right)-g\left(v,t\right)\left(v-c\right)\left(y-\bar{y}\right)\right).$$

where $y>\bar{y}$ and

$$\dot{s}=y\left[\frac{-\partial G\left(v,t\right)}{\partial t}\right],\ \dot{z}=y\left(1-G\left(v,t\right)\right)\ \text{and}\ \dot{u}=y.$$

Then, for

$$\mathbf{v}=(v^{1},v^{2},v_{\alpha})\in N_{\text{epi }L(v)}\left(\overline{s},\overline{z},\overline{u},\overline{\dot{s}},\overline{\dot{z}},\overline{\dot{u}},L\left(v,\overline{s},\overline{z},\overline{u},\overline{\dot{s}},\overline{\dot{z}},\overline{\dot{u}}\right)\right),$$

the following holds:

$$\begin{aligned}&\left[\left(\hat{s},\hat{z},\hat{u},\widehat{\dot{s}},\widehat{\dot{z}},\widehat{\dot{u}},\hat{\alpha}\right)-\left(\overline{s},\overline{z},\overline{u},\overline{\dot{s}},\overline{\dot{z}},\overline{\dot{u}},L\left(v,\overline{s},\overline{z},\overline{u},\overline{\dot{s}},\overline{\dot{z}},\overline{\dot{u}}\right)\right)\right]\cdot\mathbf{v}\\
=\ & v_{1}^{2}\left(\widehat{\dot{s}}-\overline{\dot{s}}\right)+v_{2}^{2}\left(\widehat{\dot{z}}-\overline{\dot{z}}\right)+v_{3}^{2}\left(\widehat{\dot{u}}-\overline{\dot{u}}\right)-v_{\alpha}g\left(v,t\right)\left(v-c\right)\left(y-\bar{y}\right),\\
=\ & \left(y-\bar{y}\right)\left\{v_{1}^{2}\left[\frac{-\partial G\left(v,t\right)}{\partial t}\right]+v_{2}^{2}\left(1-G\left(v,t\right)\right)+v_{3}^{2}-v_{\alpha}g\left(v,t\right)\left(v-c\right)\right\},\end{aligned}$$

Then, we have

$$\left[\left(\hat{s},\hat{z},\hat{u},\widehat{\dot{s}},\widehat{\dot{z}},\widehat{\dot{u}},\hat{\alpha}\right)-\left(\overline{s},\overline{z},\overline{u},\overline{\dot{s}},\overline{\dot{z}},\overline{\dot{u}},L\left(v,\overline{s},\overline{z},\overline{u},\overline{\dot{s}},\overline{\dot{z}},\overline{\dot{u}}\right)\right)\right]\cdot\mathbf{v}\ \leq 0,$$

only if

$$\bar{y}\in\arg\max_{0\leq z\leq 1-\bar{y}}\left\{z\left(v-c-\frac{\partial G\left(v,t\right)/\partial t}{g\left(v,t\right)}v_{1}^{1}+\frac{1-G\left(v,t\right)}{g\left(v,t\right)}v_{2}^{1}+\frac{v_{3}^{1}}{g\left(v,t\right)}\right)\right\}.$$

Similarly, considering now an element of epi $L(v)$:

$$\left(\hat{s},\hat{z},\hat{u},\widehat{\dot{s}},\widehat{\dot{z}},\widehat{\dot{u}},\hat{\alpha}\right)=\left(\overline{s},\overline{z},\overline{u},\dot{s},\dot{z},\dot{u},L\left(v,\overline{s},\overline{z},\overline{u},\overline{\dot{s}},\overline{\dot{z}},\overline{\dot{u}}\right)-g\left(v,t\right)\left(v-c\right)\left(y-\bar{y}\right)\right).$$

where $y<\bar{y}$ and

$$\dot{s}=y\left[\frac{-\partial G\left(v,t\right)}{\partial t}\right],\ \ \dot{z}=y\left(1-G\left(v,t\right)\right)\ \text{ and }\dot{u}=y,$$

we have

$$\left[\left(\hat{s},\hat{z},\hat{u},\widehat{\dot{s}},\widehat{\dot{z}},\widehat{\dot{u}},\hat{\alpha}\right)-\left(\overline{s},\overline{z},\overline{u},\overline{\dot{s}},\overline{\dot{z}},\overline{\dot{u}},L\left(v,\overline{s},\overline{z},\overline{u},\overline{\dot{s}},\overline{\dot{z}},\overline{\dot{u}}\right)\right)\right]\cdot\mathbf{v}\ \ \leq 0,$$

only if

$$\bar{y}\in\arg\max_{1-\bar{y}\leq z\leq 1}\left\{z\left(v-c-\frac{\partial G\left(v,t\right)/\partial t}{g\left(v,t\right)}v_1^1+\frac{1-G\left(v,t\right)}{g\left(v,t\right)}v_2^1+\frac{v_3^1}{g\left(v,t\right)}\right)\right\}.$$

Thus,

$$\bar{y}\in\arg\max_{0\leq z\leq 1}\left\{z\left(-v_{\alpha}\left(v-c\right)-\frac{\partial G\left(v,t\right)/\partial t}{g\left(v,t\right)}v_1^1+\frac{1-G\left(v,t\right)}{g\left(v,t\right)}v_2^1+\frac{v_3^1}{g\left(v,t\right)}\right)\right\},$$

and Property A is established.

To prove part (ii) of Lemma 58 recall that

$$\begin{aligned}&\partial L(v,\overline{s},\overline{z},\overline{u},\overline{\dot{s}},\overline{\dot{z}},\overline{\dot{u}})\\ =\ \ &\left\{(v^1,v^2)\in\mathbb{R}^6:(v^1,v^2,-1)\in N_{\text{epi }L(t)}\left(\overline{s},\overline{z},\overline{u},\overline{\dot{s}},\overline{\dot{z}},\overline{\dot{u}},L\left(v,\overline{s},\overline{z},\overline{u},\overline{\dot{s}},\overline{\dot{z}},\overline{\dot{u}}\right)\right)\right\}\end{aligned}$$

and

$$\left(\dot{\rho}_s(v), \dot{\rho}_z(v), \dot{\rho}_u(v), \rho_s(v), \rho_z(v), \rho_u(v)\right) \in \partial L(v, s(v), z(v), u(v), \dot{s}(v), \dot{z}(v), \dot{u}(v)).$$

Property A proves that if $\dot{s}(v) = y(v)\left[\frac{-\partial G(v,t)}{\partial t}\right]$, $\dot{z}(v) = y(v)\left(1 - G(v,t)\right)$ and $\dot{u}(v) = y(v)$, then

(D.30)
$$y(v) \in \arg\max_{0 \leq z \leq 1}\left\{z\left(v - c - \frac{\partial G(v,t)/\partial t}{g(v,t)}\rho_s(v) + \frac{1 - G(v,t)}{g(v,t)}\rho_z(v) + \frac{\rho_u(v)}{g(v,t)}\right)\right\}.$$

Since $\{\rho_s(v), \rho_z(v), \rho_u(v) : v \in [\underline{v}, \bar{v}]\}$ is a feasible control for (D_A), it must be that $\dot{\rho}_s(v) = \dot{\rho}_z(v) = 0$ for all v and $\rho_u(v) = 0$ for all v. Then, (D.30) and part (i) of Lemma 58 and the fact that $\rho_u(v) = 0$ proves that

$$y(v) \in \arg\max_{0 \leq z \leq 1}\left\{z\left(v - c + \frac{1 - G(v,t)}{g(v,t)}\rho_z(v)\right)\right\}$$

and establishes part (ii) of Lemma 58.

We next property will assist us in finding the subgradients of $l_{\bar{v}}$ and establishing part (iii) of Lemma 58.

Property B. *For* $(v^s, v^z, v^u, v_\alpha)' \in R^4$, *if*

$$\mathbf{v} = (v^s, v^z, v^u, v_\alpha) \in N_{\text{epi } l_{\bar{v}}}\left(\overline{s}, \overline{z}, \overline{u}, l_{\bar{v}}(\overline{s}, \overline{z}, \overline{u})\right),$$

and $\overline{z} > U_*(t) - u_0$, *then* $v^z = 0$.

To verify Property B, first note that , any $\mathbf{v} = (v^s, v^z, v^u, v_\alpha)$ such that $v^z < 0$ cannot be in $N_{\text{epi } l_{\bar{v}}}(\overline{s}, \overline{z}, \overline{u}, l_{\bar{v}}(\overline{s}, \overline{z}, \overline{u}))$. Suppose not. Then, we could find an element $(\hat{s}, \hat{z}, \hat{u}, \hat{\alpha})$

$$(\hat{s}, \hat{z}, \hat{u}, \hat{\alpha}) = (\overline{s}, \tilde{z}, \overline{u}, l_{\bar{v}}(\overline{s}, \tilde{z}, \overline{u}))$$

of epi $L(v)$ such that it is equal to $(\overline{s}, \overline{z}, \overline{u}, l_{\bar{v}}(\overline{s}, \overline{z}, \overline{u}))$ except that we have $\tilde{z} < \overline{z}$.and $\tilde{z} > U_*(t) - u_0$. However, we have

$$[(\hat{s}, \hat{z}, \hat{u}, \hat{\alpha}) - (\overline{s}, \overline{z}, \overline{u}, l_{\bar{v}}(\overline{s}, \overline{z}, \overline{u}))] \cdot \mathbf{v} \ > 0.$$

From Theorem 6.9 of [**55**], this contradicts the fact that

$$(v^s, v^z, v^u, v_\alpha) \in N_{\text{epi } l_{\bar{v}}}(\overline{s}, \overline{z}, \overline{u}, l_{\bar{v}}(\overline{s}, \overline{z}, \overline{u})).$$

Similarly, any $\mathbf{v} = (v^s, v^z, v^u, v_\alpha)$ such that $v^z > 0$ cannot be in

$$N_{\text{epi } l_{\bar{v}}}(\overline{s}, \overline{z}, \overline{u}, l_{\bar{v}}(\overline{s}, \overline{z}, \overline{u})).$$

This completes the proof of Property B.

Part (iii) of Lemma 58 then follows from Property B and the coextremality condition (D.27). This concludes the proof of Lemma 58. ■