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Topological Restriction Homology is Locally Even in Characteristic p

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Micah Darrell

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ABSTRACT

Topological Restriction Homology is Locally Even in Characteristic p

Micah Darrell

The goal of this thesis is to prove that topological restriction homology, TR, is locally even in the quasi-syntomic topology in characteristic p. This local evenness was already known for the other main trace theories, but is more subtle for TR.

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Dedication

To Janet

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CHAPTER 1

Introduction

In this dissertation we prove that the p-typical topological restriction homology, TR, is locally even in the quasisyntomic topology in characteristic p. This result is motivated by [2], where it is shown that the p-completions of HH, HC^-, THH, TC^-, TP, and TC all have this local evenness property, but it remained unknown for TR. For the sake of notational simplicity we suppress the p-completions going forward.

Topological restriction homology is an invariant of commutative rings with important connections to p-adic Hodge theory and algebraic K-theory. Given a commutative ring A, the topological restriction homology, $\mathrm{TR}(A) \in \mathrm{CycSp}$, is a cyclotomic spectrum, which we will define below. It has the amazing property that

$$\mathrm{TR}(A) \simeq \varprojlim_{e} \Omega \mathrm{K}\left(A[t]/t^e, (t)\right),$$

where the right hand side is known as the curves on K-theory of A [10]. The Dundas-Goodwillie-McCarthy theorem provides an identification

$$K_*(A[t]/t^e, (t)) \cong \mathrm{TC}_*(A[t]/t^e, (t)),$$

which allows us to compute $\mathrm{TR}(A)$ as

$$\mathrm{TR}(A) \simeq \varprojlim_{e} \Omega \mathrm{TC}\left(A[t]/t^e, (t)\right).$$

In [2] a filtration is constructed on TC, known as the motivic filtration, for $n \in \mathbb{N}$, the n'th graded piece of this filtration on $\mathrm{TC}(A)$ is denoted $\mathbb{Z}_p(n)(A)$ and

$$\mathbb{Z}_p(n)(A) \cong \mathrm{fib}\left(\phi - \mathrm{can} : \mathcal{N}^{\geq n}\hat{\Delta}_A\{n\} \to \hat{\Delta}_A\{n\}\right)[2n].$$

Here $\hat{\Delta}$ is the Nygaard-completed prismatic cohomology, $\mathcal{N}^{\geq n}\Delta$ is the n'th piece of the Nygaard filtration, and $\{n\}$ denotes the n'th Breuil-Kisin twist, which we will ignore moving forward by working over $\mathbb{F}_p$.

For an $\mathbb{F}_p$-algebra A, there is an identification $\hat{\Delta}_A \simeq \mathrm{LW}\Omega_A$, between the prismatic cohomology and the derived crystalline cohomology. If the ring A has a δ-lift to characteristic 0, it is explained in [18] how to compute the derived crystalline cohomology $\mathrm{LW}\Omega_A$ and its Nygaard filtration, using the divided power de Rham complex of the lift.

We write A_e for the $\mathbb{F}_p$-algebra

$$A_e \cong k[y^{1/p^\infty}_, x]/(y, x^e),$$

for a perfect $\mathbb{F}_p$-algebra k, this lets us identify

$$LW\Omega_{A_e} \simeq \left(W(k)\left[y^{1/p^\infty}, \frac{y^n}{n!}\right]\left[x, \frac{x^{em}}{m!}\right] \xrightarrow{d} W(k)\left[y^{1/p^\infty}, \frac{y^n}{n!}\right]\left[x, \frac{x^{em}}{m!}\right]\ dx\right)^\wedge_p$$

where m and n range across $\mathbb{N}$.

From here we can explicitly compute the cohomology of the complexes $\mathbb{Z}_p(i)(A_e)$ for all n, e. These complexes only have cohomology in degree 1, so the BMS spectral sequence, which is just the spectral sequence of the motivic filtration on TC, collapses and identifies

$$\mathrm{TC}_{2i-1}\left(A_e, (e)\right) \cong \mathrm{H}^1\left(\mathbb{Z}_p(i)(A_e)\right).$$

This identification allows us to explicitly compute the transition maps on the homotopy groups of the diagram

$$\mathrm{TR}(A_e) \simeq \varprojlim_e \Omega\mathrm{TC}\left(A[t]/t^e, (t)\right),$$

and from here we check by hand that these transition maps satisfy the Mittag-Leffler condition. This implies that $\mathrm{TR}(A_e)$ is concentrated in even degrees.

These calculations extend to the higher dimensional case

$$A_e^k = k[y_{1,}^{1/p^\infty} \ldots, y_{k,}^{1/p^\infty} x]/(y, \ldots, y_k, x^e),$$

where $k \in \mathbb{N} \cup \{\infty\}$, where we also verify that $\mathrm{TR}(A_e^k)$ is concentrated in even degrees.

Every characteristic p quasisyntomic ring admits a cover by a quasiregular semiperfect ring with a relatively perfect map to a ring of the form A_e^k, and using the filtration on TR from [21] this is enough to prove that every quasisyntomic ring in characteristic p has a cover by a ring where TR is even.

We attempt to make this dissertation as self contained as possible, below we summarize the contents of each chapter.

In chapter 2 we review the major trace theories, as well as the historical development of the field.

In chapter 3 we review the geometric approach to p-adic Hodge theory.

In chapter 4 we sketch the construction of prismatic cohomology in [2], and explain the quasisyntomic local evenness of all of the trace theories.

Chapter 5 is the paper *Local Evenness of Topological Restriction Homology in Characteristic p*, which is the heart of this dissertation.

CHAPTER 2

An Overview of Trace Theories

In recent years trace methods have been the centerpiece of many major results in p-adic Hodge theory and algebraic K-theory. In this section we attempt to give an introduction to the subject.

2.1. Hochschild Homology

In the 1940's Hochschild homology was introduced by Hochschild [15]. For a polynomial algebra A over a ring R, the Hochschild homology of A relative to R is defined as the homology of the Hochschild complex

$$\mathrm{HH}_n(A/R) = H_n(\ldots \xrightarrow{d_3} A \otimes_R A \otimes_R A \xrightarrow{d_2} A \otimes_R A \xrightarrow{d_1} A),$$

where the maps are

$$d_1(a \otimes b) = ab - ba,$$

$$d_2(a \otimes b \otimes c) = (ab \otimes c) - (a \otimes bc) + (ca \otimes b),$$

$$d_3(a \otimes b \otimes c \otimes d) = (ab \otimes c \otimes d) - (a \otimes bc \otimes d) + (a \otimes b \otimes cd) - (da \otimes b \otimes c),$$

etc.

When $R = \mathbb{Z}$ we denote $\mathrm{HH}(A/R) = \mathrm{HH}(A)$. If A is commutative, then $ab - ba = 0$ for all $a, b \in A$, and we see that $\mathrm{HH}_0(A/R) = A/(d_1(A \otimes_R A) = A$.
Still in the commutative case, $\mathrm{HH}_1(A/R) = \ker(d_1)/im(d_2)$, and since $d_1 = 0$ this is

$$\mathrm{HH}_1(A/R) = A \otimes A/(a \otimes bc = ab \otimes c + ca \otimes b).$$

The relation in the quotient is exactly the Leibniz rule, so we have

$$\mathrm{HH}_1(A/R) \cong \Omega^1_{A/R}.$$

To see this, define the map

$$\phi : A \otimes_R A \to \Omega^1_{A/R},$$

$$\phi(a \otimes b) = a\, db,$$

now $\phi(a \otimes bc) = \phi(ab \otimes c + ca \otimes b)$ takes the form $a\, d(bc) = ab\, dc + ac\, db$, which is precisely the Leibniz rule.
One can continue on and check by hand that

$$\mathrm{HH}_n(A/R) \cong \Omega^n_{A/R},$$

although it gets tedious in degrees higher than 4. By developing a little more theory we will obtain a less tedious proof of this result.

Recall the bar complex

$$B_\bullet(A/R) = (\dots A \otimes_R A \otimes_R A \otimes_R A \xrightarrow{d_2} A \otimes_R A \otimes_R A \xrightarrow{d_1} A \otimes_R A),$$

with

$$d_1(a \otimes b \otimes c) = ab \otimes c - a \otimes bc,$$

$$d_2(a \otimes b \otimes c \otimes d) = (ab \otimes c \otimes d) - (a \otimes bc \otimes d) + (a \otimes b \otimes cd),$$

etc.

This is almost the Hochschild complex, we just need to shift it up by a degree, adding A to degree 0, and add the wrapped around terms to the formula for d_n. These can both be done in one fell-swoop,

$$\mathrm{HH}_\bullet(A/R) \cong A \underset{A \underset{R}{\otimes} A}{\otimes} B_\bullet(A/R).$$

Since the bar complex is a resolution of A, we can write this is

$$\mathrm{HH}_\bullet(A/R) \cong A \underset{A \underset{R}{\otimes} A}{\otimes}{}^{\mathbb{L}} A \cong \mathrm{Tor}^{A \otimes_R A}(A, A).$$

Now since we already know that $\mathrm{HH}_1(A/R) \cong \Omega^1_{A/R}$, in order to prove that $\mathrm{HH}_n(A/R) \cong \Omega^n_{A/R}$ in general, we can show that $\wedge^n \mathrm{HH}_1(A/R) \cong \mathrm{HH}_n(A/R)$. This is equivalent to showing

$$\wedge^n \mathrm{Tor}_1^{A\otimes_R A}(A,A) \cong \mathrm{Tor}_n^{A\otimes_R A}(A,A),$$

which follows from computing the right hand side with the Koszul complex. In 1962, Hochschild, Kostant, and Rosenberg [16] used these ideas to prove:

Theorem (HKR). *For A a smooth R-algebra,*

$$\mathrm{HH}_n(A/R) \cong \Omega^n_{A/R}.$$

At this point we will abandon historical accuracy in exchange for the elegance that the modern theory provides. To this end, for any commutative ring R, and commutative R-algebra A, we define

$$\mathrm{HH}(A/R) := A \otimes^{\mathbb{L}}_{A\otimes^{\mathbb{L}}_R A} A,$$

where we take this tensor product in the ∞-category CAlg_R of $\mathbb{E}_\infty$-rings over R. In this setting $\mathrm{HH}(A/R)$ is now a spectrum as opposed to a chain complex, but in the case where A is a smooth algebra over R the homotopy groups of this spectrum agree with the homology of the chain complex defined above. In particular for A smooth over R

$$\pi_n \mathrm{HH}(A/R) \cong \Omega^n_{A/R}.$$

We will usually denote $\pi_n\mathrm{HH}(A/R)$ as $\mathrm{HH}_n(A/R)$.

In this derived setting there is now a version of the HKR for arbitrary commutative R-algebras A, where one replaces $\Omega^n_{A/R}$ with the cotangent complex $\wedge^n L_{A/R}$, which can no longer be $\mathrm{HH}_n(A/R)$ as it is an entire complex, but instead there is a filtration $\mathrm{Fil}^\bullet_{HKR}$ on $\mathrm{HH}(A/R)$ with

$$gr^i_{HKR}\mathrm{HH}(A/R) \simeq \wedge^i L_{A/R}.$$

This filtration is known as the HKR filtration, and it comes from left Kan extension of the Postnikov filtration.

There is an identification $\mathrm{HH}(A/R) \simeq A^{\otimes S^1}$, where this is the copower operation with the space S^1. To see this, write $S^1 \simeq * \sqcup_{*\sqcup *} *$ as the suspension of S^0, we can compute $A^{\otimes S^1}$ in CAlg_R as $A^{\otimes(*\sqcup_{*\sqcup *}*)}$ since the copower commutes with colimits, and this is precisely $\mathrm{HH}(A/R)$.

This perspective provides an S^1-action on $\mathrm{HH}(A/R)$, by acting on the S^1 factor in the tensor product. At the level of homotopy groups this S^1-action on $\mathrm{HH}(A/R)$ becomes an action of $\pi_*(S^1 \wedge R)$ on $\pi_*\mathrm{HH}(A/R)$, where $\pi_*(S^1 \wedge R) \cong \mathrm{H}_*(S^1, R) \cong R[\epsilon]/\epsilon^2$, $|\epsilon| = 1$. This gives us a differential on the homotopy groups of $\mathrm{HH}(A/R)$, given by multiplication by ϵ. This differential is referred to as the Connes' operator, and is usually denoted as $B : \mathrm{HH}_n(A/R) \to \mathrm{HH}_{n+1}(A/R)$.

When A is a smooth R-algebra the HKR-theorem identifies $\mathrm{HH}_n(A/R) \cong \Omega^n_{A/R}$, and under this identification the Connes' operator agrees with the de Rham differential d :

$\Omega^n_{A/R} \to \Omega^{n+1}_{A/R}$. Connes' constructed this operator as a generalization of the de Rham complex that makes sense for non-commutative rings, which has been very fruitful in the world of non-commutative geometry, but we will not explore these ideas here.

It is common to use the notation $\mathbb{T}$ instead of S^1 in this setting, and we will adopt this convention moving forward.

The $\mathbb{T}$-action on $\mathrm{HH}(A/R)$ lets us view $\mathrm{HH}(A/R)$ as an object of the ∞-category of $\mathbb{T}$-equivariant spectre $\mathrm{Sp}^{B\mathbb{T}} = \mathrm{Fun}(BS^1, \mathrm{Sp})$. There is a homotopy fixed point functor

$$(-)^{h\mathbb{T}} : \mathrm{Sp}^{B\mathbb{T}} \to \mathrm{Sp},$$

which is a higher analogue of group cohomology. We define the Negative Cyclic Homology

$$\mathrm{HC}^-(A/R) := \mathrm{HH}(A/R)^{h\mathbb{T}}.$$

There is also a functor known as the Tate construction $(-)^{t\mathbb{T}} : \mathrm{Sp}^{B\mathbb{T}} \to \mathrm{Sp}$, which is a higher analogue of Tate cohomology. We define the Periodic Cyclic Homology

$$\mathrm{HP}(A/R) := \mathrm{HH}(A/R)^{t\mathbb{T}}.$$

There is a natural transformation $(-)^{h\mathbb{T}} \to (-)^{t\mathbb{T}}$ which gives a map $\mathrm{HC}^-(A/R) \to \mathrm{HP}(A/R)$.

In the 1980's it was proven by many mathematicians that if R contains $\mathbb{Q}$ and A is a smooth R-algebra, then the Negative Cyclic Homology recovers the Hodge filtration on the de Rham complex

$$\mathrm{HC}^-(A/R) \simeq \prod_{i\in\mathbb{Z}} \Omega^{\geq i}_{A/R}[2i],$$

the Periodic Cyclic Homology recovers the de Rham complex,

$$\mathrm{HP}(A/R) \simeq \prod_{i\in\mathbb{Z}} \Omega^{\bullet}_{A/R}[2i],$$

and the map $(-)^{h\mathbb{T}} \to (-)^{t\mathbb{T}}$ which gives a map $\mathrm{HC}^-(A/R) \to \mathrm{HP}(A/R)$ is given by the inclusion of the Hodge filtration.

2.2. The Dennis Trace

The Hochschild homology receives a map from algebraic K-theory, known as the Dennis trace map

$$tr : \mathrm{K}(A) \to \mathrm{HH}(A).$$

Informally this map comes from the chain of maps

$$BGL_n(A) \to B^{cyc}GL_n(A) \to B^{cyc}M_n(A) \to B^{cyc}A \simeq \mathrm{HH}(A),$$

which are respectively the natural map from the bar construction to the cyclic bar construction, the map induced from the inclusion $GL_n \to M_n$, and then the map induced by the trace $M_n(A) \to A$. The cyclic bar construction is a modification of the bar construction that adds the wrap around maps mentioned above, and provides another construction of Hochschild homology.

This map is usually very far from being an isomorphism. For a trivial example, $\mathrm{HH}(\mathbb{Z}) \simeq \mathbb{Z}$, but $\mathrm{K}(\mathbb{Z})$ is extremely complicated. For a less trivial example, one can compute

$$\mathrm{HH}(\mathbb{F}_p) \simeq \mathbb{F}_p\langle x\rangle,$$

with $|x| = 2$. This is the divided power algebra on a class in degree 2. However $\mathrm{K}(\mathbb{F}_p)$ vanishes in positive even degrees, and $\pi_{2i-1}\mathrm{K}(\mathbb{F}_p) \cong \mathbb{Z}/(p^i - 1)$.

The Dennis trace map $\mathrm{K}(A) \to \mathrm{HH}(A)$ lifts through the canonical map $\mathrm{HC}^-(A) \to \mathrm{HH}(A)$ [9] to give a map called the Goodwillie-Jones trace

$$tr : \mathrm{K}(A) \to \mathrm{HC}^-(A).$$

This map is usually a more useful approximation than $\mathrm{K}(A) \to \mathrm{HH}(A)$.

Theorem (Goodwillie, 1986). *Let $I \subset A$ be a nilpotent ideal in a $\mathbb{Q}$-algebra A, then*

$$\begin{array}{ccc} \mathrm{K}(A) & \longrightarrow & \mathrm{K}(A/I) \\ \downarrow & & \downarrow \\ \mathrm{HC}^-(A) & \longrightarrow & \mathrm{HC}^-(A/I) \end{array}$$

is cartesian, where the vertical maps are the trace. Equivalently, the Goodwillie-Jones trace induces an equivalence on the fibers of the horizontal maps.

Around this time pioneered the idea that there should be a version of Hochschild homology relative to the sphere spectrum "$\mathrm{HH}(A/\mathbb{S})$", which would have a much closer relationship to algebraic K-theory.

2.3. Topological Hochschild Homology

In the 1980's Bökstedt constructed the above theory of Hochschild homology relative to the sphere spectrum and named it Topological Hochschild Homology, THH. This theory was studied extensively throughout the 90's and early 2000's by many mathematicians, but in 2017 Nikolaus and Scholze gave a new construction of THH using the formalism of higher algebra provided by the work of Lurie. Scholze's motivation for thinking about THH was inspired by a computation of Hesselholt [12], that

$$\pi_0 \mathrm{THH}(\mathcal{O}_C;\, \mathbb{Z}_p)^{h\mathbb{T}} \cong \mathrm{A}_{inf}.$$

This is the p-complete Topological Hochschild Homology, $\mathcal{O}_C$ is the ring of integers in the p-completion of the algebraic closure of $\mathbb{Q}_p$, and A_{inf} is Fontaine's infinitesimal period ring. At the time there were hopes that a cohomology theory could be constructed that would simultaneously be a generalization of étale, de Rham, and crystalline cohomology for p-adic varieties. The ring A_{inf} was a natural candidate as a base ring for such a cohomology theory due to its various relationships to the other period rings in p-adic Hodge theory, plus THH is closely related to HH, which is already closely related to the de Rham cohomology. All signs seemed to point at THH being related to this cohomology theory. This has all since been verified, THH was used to give the original construction of the prismatic cohomology!

For an $\mathbb{E}_\infty$-ring A, we define the Topological Hochschild Homology of A as

$$\mathrm{THH}(A) := A^{\otimes \mathbb{T}},$$

where the copower is now computed in the ∞-category $\mathrm{CAlg(Sp)}$ of $\mathbb{E}_\infty$-rings.

Nikolaus and Scholze also gave a new construction of the ∞-category CycSp, of cyclotomic spectra [20], where an object $X \in \mathrm{CycSp}$ consists of the data of a $\mathbb{T}$-equivariant spectrum X, and a map $\phi_p : X \to X^{tC_p}$ for every prime p. The Tate diagonal provides $\mathrm{THH}(A)$ with the structure of a cyclotomic spectrum, and the map $\phi_p : \mathrm{THH}(A) \to \mathrm{THH}(A)^{tC_p}$ is known as the cyclotomic Frobenius.

The unit object of CycSp is given by the sphere spectrum with the trivial $\mathbb{T}$-action, denoted $\mathbb{S}^{\mathrm{triv}}$. Then define the Topological Cyclic Homology of $X \in \mathrm{CycSp}$

$$\mathrm{TC}(X) := \mathrm{Map}_{\mathrm{CycSp}}(\mathbb{S}^{\mathrm{triv}}, X).$$

This is a drastic simplification of the classical definition of topological cyclic homology, which relied on methods from genuine equivariant homotopy theory.

Bökstedt was able to compute [6]

$$\mathrm{THH}(\mathbb{F}_p) \simeq \mathbb{F}_p[x],\ |x| = 2,$$

which is a much more palateable result than $\mathrm{HH}(\mathbb{F}_p) \simeq \mathbb{F}_p\langle x\rangle,\ |x| = 2$.

There are several other trace theories derived from THH, first by taking the homotopy fixed points we obtain the Topological Negative Cyclic Homology

$$\mathrm{TC}^-(A) := \mathrm{THH}(A)^{h\mathbb{T}},$$

and by taking the Tate construction we obtain the Topological Periodic Homology

$$\mathrm{TP}(A) := \mathrm{THH}(A)^{t\mathbb{T}}.$$

There is a canonical map

$$\mathrm{TC}^-(A) \to \mathrm{TP}(A),$$

coming from the canonical map $(-)^{h\mathbb{T}} \to (-)^{t\mathbb{T}}$.

From now on we will assume there is a fixed prime p, and we restrict attention to the p-completed Topological Hochschild Homology denoted $\mathrm{THH}(-\,;\,\mathbb{Z}_p)$.

The Tate orbit lemma gives an identification [20]

$$(\mathrm{THH}(A;\mathbb{Z}_p)^{tC_p})^{h\mathbb{T}} \simeq \mathrm{TP}(A;\mathbb{Z}_p).$$

So applying homotopy fixed points to the cyclotomic Frobenius map $\phi : \mathrm{THH}(A;\mathbb{Z}_p) \to \mathbb{THH}(A;\mathbb{Z}_p)^{tC_p}$, we get a map

$$\phi^{h\mathbb{T}} : \mathrm{TC}^{-}(A;\mathbb{Z}_p) \to \mathrm{TP}(A;\mathbb{Z}_p).$$

There is an equalizer diagram

$$\mathrm{TC}(A;\mathbb{Z}_p) \longrightarrow \mathrm{TC}^{-}(A;\mathbb{Z}_p) \underset{\phi}{\overset{can}{\rightrightarrows}} \mathrm{TP}(A;\mathbb{Z}_p).$$

This equalizer diagram is the standard way to approach calculations of $\mathrm{TC}(A;\mathbb{Z}_p)$, for example in [20, p. 4.4.10] it is shown that

$$\pi_i \mathrm{TC}^{-}(\mathbb{F}_p) \cong \mathbb{Z}_p[\tilde{u}, v]/(\tilde{u}v - p),$$

with $|\tilde{u}| = 2$, $|v| = -2$,

$$\pi_i \mathrm{TP}(\mathbb{F}_p) \cong \mathbb{Z}_p[v^{\pm 1}],$$

with $|v| = -2$, and

$$\pi_i \mathrm{TC}(\mathbb{F}_p) \cong \begin{cases} \mathbb{Z}_p & i = 0, -1 \\ 0 & \text{otherwise} \end{cases}$$

2.4. Topological Restriction Homology

For a cyclotomic spectrum X, and $n \in \mathbb{N}$, we inductively define the Truncated Topological Restriction Homology functors

$$\mathrm{TR}^1(X) := X,$$

$$\mathrm{TR}^2(X) := X^{hC_p} \times_{X^{tC_p}} X,$$

where the map $X^{hC_p} \to X^{tC_p}$ is the canonical map, and $X \to X^{tC_p}$ is the cycloctomic Frobenius, and the general formula

$$\mathrm{TR}^n(X) := (X^{hC_p})^{hC_{p^{n-2}}} \times_{(X^{tC_p})^{hC_{p^{n-2}}}} \mathrm{TR}^{n-1}(X).$$

It is enlightening to see one more example, so

$$\mathrm{TR}^3(X) \simeq (X^{hC_p})^{hC_p} \times_{(X^{tC_p})^{hC_p}} \left(X^{hC_p} \times_{X^{tC_p}} X\right),$$

the left hand map in the fiber is $(-)^{hC_p}$ applied to the canonical map $X^{hC_p} \to X^{tC_p}$, and the right hand map is $(-)^{hC_p}$ applied to the cyclotomic Frobenius $X \to X^{tC_p}$.

There are projection maps $\mathrm{TR}^n(X) \to \mathrm{TR}^{n-1}(X)$, and we define

$$\mathrm{TR}(X) := \varprojlim \mathrm{TR}^n(X).$$

For a ring A, it is typical to write $\mathrm{TR}(A) = \mathrm{TR}(\mathrm{THH}(A))$.

Restriction Homology and its truncations are analogous to a higher algebra version of the Witt-vector construction, in fact [13]:

Theorem (Hesselholt). *For a commutative ring A,*

$$\pi_0 \mathrm{TR}(A) \cong W(A),$$

and,

$$\pi_0 \mathrm{TR}^n(A) \cong W_n(A).$$

Where $W(A)$ and $W_n(A)$ are the p-typical Witt vectors of A, and the n-truncated p-typical Witt vectors of A, respectively.

Recall that for a ring A a Frobenius lift on A is a map $\psi : A \to A$ such that $\psi/p : A/p \to A/p$ is the Frobenius map $x \mapsto x^p$. There is a close connection between Frobenius lifts and delta structures on a ring A, indeed given any Frobenius lift ψ we obtain a delta structure by taking

$$\delta(x) = \frac{x^p - \psi(x)}{p},$$

and if A is p-torsion free this process is reversible so delta structres and Frobenius lifts are equivalent data. A delta structure on A is also equivalent to a section of the map $W_2(A) \to A$, i.e. given a map $\psi : A \to W_2(A)$ such that $A \to W_2(A) \to A$ is the identity, we can write $\psi(x) = (x, \delta(x))$ for some delta structure on A.

We say that a cyclotomic spectrum $X \in \mathrm{CycSp}$ has a Frobenius lift if the cyclotomic Frobenius map $X \to X^{tC_p}$ can be factored through the canonical map $X^{hC_p} \to X^{tC_p}$ to give a map $X \to X^{hC_p}$. This is tautologically equivalent to the projection map $\mathrm{TR}^2(X) \to X$ admitting a section $X \to \mathrm{TR}^2(X)$. Indeed, a section of $\mathrm{TR}^2(X) \to X$ is a map

$$X \to X^{hC_p} \times_{X^{tC_p}} X,$$

which is of the form (ψ, id), $\psi : X \to X^{hC_p}$, where $(\mathrm{can} \circ \psi) : X \to X^{tC_p} \simeq \phi : X \to X^{tC_p}$.

Interestingly, $\mathrm{TR}(A)$ itself has a Frobenius lift for all A, and in fact

Theorem (Krause-Nikolaus) [17]. *For a cyclotomic spectrum X, the map $\mathrm{TR}(X) \to X$ exhibits $\mathrm{TR}(X)$ as the universal co-free p-cyclotomic spectrum with Frobenius lifts over X.*

Topological restriction homology has deep connections to algebraic K-theory:

Theorem (Hesselholt, McCandless) [10][19]. *For a connective $\mathbb{E}_1$-ring R, there is a natural equivalence of spectra*

$$\mathrm{TR}(R) \simeq \varprojlim_{e} \Omega \mathrm{K}(R[t]/t^e, (t))$$

The spectrum $\mathrm{K}(R[t]/t^e, (t))$ is defined as the fiber of the map $\mathrm{K}(R[t]/t^e) \to \mathrm{K}(R)$, and the limit of these is referred to as the curves on K-theory.

Interestingly, in 1977 Bloch was studying the curves on K-theory as a potential approach to construct a cohomology theory that simultaneously generalizes the ℓ-adic étale cohomology and the crystalline cohomology [4]. In retrospect, it seems that Bloch had discovered an early predecessor of the prismatic cohomology.

2.5. Trace Maps

One of the original motivations for introducing topological hochschild homology was to refine the trace map $\mathrm{K}(A) \to \mathrm{HH}(A)$ to a map $\mathrm{K}(A) \to \mathrm{THH}(A)$ that was closer to an isomorphism. The trace theory functors can be extended to functors $\mathrm{Cat}_\infty^{stab} \to \mathrm{CycSp}$, where to recover for example $\mathrm{THH}(A)$ for a ring A, we take THH of the category Perf_A.

In [5] it is proven that algebraic K-theory is the universal additive invariant, i.e. if E is an additive invariant with values in spectra

$$E : \mathrm{Cat}_\infty^{ex} \to \mathrm{Sp},$$

then $\mathrm{Nat}_{add}(\mathrm{K}, E) \simeq E(\mathbb{S})$, where these are the natural transformations of additive invariants. So in particular,

$$\mathrm{Nat}_{add}(\mathrm{K}, \mathrm{THH}) \simeq \mathrm{THH}(\mathbb{S}),$$

and there is a canonical map $\mathrm{K} \to \mathrm{THH}$ corresponding to $1 \in \pi_0\mathrm{THH}(\mathbb{S}) \cong \mathbb{Z}$.

Since THH is an additive invariant, the functor

$$\mathrm{THH} : \mathrm{Cat}_\infty^{stab} \to \mathrm{CycSp},$$

has a factorization

$$\mathrm{Cat}_\infty^{stab} \xrightarrow{z} \mathrm{NMot} \xrightarrow{tr} \mathrm{CycSp},$$

where tr is an exact functor. Since algebraic K-theory is given by the mapping spectrum in the category of Non-commutative motives

$$\mathrm{K}(\mathcal{C}) \simeq \mathrm{map}_{\mathrm{NMot}}(z(\mathrm{Perf}_{\mathbb{S}}), z(C)),$$

the functor tr induces a map

$$\mathrm{map}_{\mathrm{NMot}}(z(\mathrm{Perf}_{\mathbb{S}}), z(C)) \to \mathrm{map}_{\mathrm{CycSp}}(tr(\mathbb{S}), tr(\mathcal{C})),$$

and the right hand side is $\mathrm{map}_{\mathrm{CycSp}}(\mathbb{S}^{triv}, \mathrm{THH}(\mathcal{C})) = \mathrm{TC}(\mathcal{C})$.
This is then a map known as the cyclotomic trace

$$tr : \mathrm{K}(\mathcal{C}) \to \mathrm{TC}(\mathcal{C}),$$

which is the most important trace map.

Let $\mathrm{K}^{inv}(A) := \mathrm{fib}(tr : \mathrm{K}(A) \to \mathrm{TC}(A))$, be the fiber of the cyclotomic trace map. The following theorem is one of the major results in the field.

Theorem [8]. *If $R \to R'$ is a surjective map of rings with nilpotent kernel, then $K^{inv}(R) \to K^{inv}(R')$ is an equivalence.*

The cyclotomic trace map is is étale locally an equivalence after p-completion, as made precise by the following theorem.

Theorem [7]. *Let R be a strictly Henselian local ring with residue characteristic p, then* $\mathrm{K}^{inv}(R)/p \simeq 0$*, so in particular*

$$\mathrm{K}(R;\mathbb{Z}_p) \simeq \mathrm{TC}(R;\mathbb{Z}_p).$$

There is also a trace map from the K-theory of endomorphisms to the topological restriction homology, although this map isn't currently as well understood. It is a result of Lindenstrauss-McCarthy that topological restriction homology and the reduced K-theory of endomorphisms have the same Goodwillie-Taylor tower. There also seems to be work in progress by Nikolaus where it is shown that a variant of the K-theory of endomorphisms known as the cyclic K-theory is equivalent to TR.

CHAPTER 3

A Brief Introduction to p-adic Hodge Theory

In this section we give a very brief introduction to p-adic Hodge theory, focusing on the geometric approach.

3.1. Motivation

In 1949 Weil put forth several conjectures concerning the number of $\mathbb{F}_q$-points of a projective variety X. The $\mathbb{F}_q$ points of X, denoted $X(\mathbb{F}_q)$ can be identified with the points of $X\left(\overline{\mathbb{F}_p}\right)$ that are fixed by Frob^r, where $q = p^r$.

In topology the Lefschetz fixed point theorem would let us count these fixed points in terms of the trace of the map induced on cohomology. This suggests that if there was a cohomology theory for varieties that behaved similarly to ordinary cohomology it would offer a solution to all of Weil's conjectures except for the Riemann hypothesis. Such a cohomology theory is known as a Weil cohomology theory.

The ℓ-adic étale cohomology is a Weil cohomology theory constructed in 1960 by Grothendieck, in 1974 Deligne proved Weil's Riemann Hypothesis using ℓ-adic cohomology, finally settling all of Weil's conjectures.

In topology, the ordinary cohomology is the uniquely classified by the Eilenberg-Steenrod axioms. In algebraic geometry however, several different Weil cohomology theories were discovered. Grothendieck proposed a theory of motives, intuitively motives should be some sort of linear type object, there should be a universal motive attached to any projective variety which will contain all cohomological information about that variety, and all of the known Weil cohomology theories should come from a specialization of the motive.

There are three primary Weil cohomology theories, these are

1. The ℓ-adic cohomology where ℓ is a prime number, written

$$\mathrm{H}^*_{et}(X, \mathbb{Q}_\ell),$$

which is only well behaved over a geometrically closed field of characteristic $p \neq \ell$,

2. the crystalline cohomology written

$$\mathrm{H}^*_{crys}(X/\mathbb{Z}_p) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p,$$

which is only well behaved in characteristic p, and

3. the de Rham cohomology written

$$H^*_{dR}(X),$$

which is only well behaved in characteristic 0.

The primary goal of the geometric approach to p-adic Hodge theory is to understand the relationships between these various cohomology theories. Intuitively, we expect that these cohomology theories contain roughly the same information about the variety they are applied to, but this information comes from different structure.

For example, if X/K is a smooth proper variety over a field K, one usually considers the étale cohomology of the geometric fiber $X_{\overline{K}} = X \times_{\mathrm{Spec}(K)} \mathrm{Spec}\left(\overline{K}\right)$,

$$\mathrm{H}^*_{et}(X_{\overline{K}}, \mathbb{Q}_\ell).$$

The étale cohomology has an action of the Galois group $\mathrm{Gal}\left(\overline{K}/K\right)$.

The de Rham cohomology $\mathrm{H}^*_{dR}(X)$ has no such canonical Galois action, but instead since the de Rham cohomology is the cohomology of a canonical chain complex, it has a filtration known as the Hodge filtration

$$\ldots \subseteq \mathrm{Fil}^2\mathrm{H}^*_{dR}(X) \subseteq \mathrm{Fil}^1\mathrm{H}^*_{dR}(X) \subseteq \mathrm{Fil}^0\mathrm{H}^*_{dR}(X) = \mathrm{H}^*_{dR}(X),$$

coming from the stupid filtration on the de Rham complex.

Since intuitively these cohomology theories contain the same information, one should expect a way to convert the étale cohomology into the de Rham cohomology, and vice versa, in a way such that the Galois action can be used to recover the Hodge filtration and vice versa.

3.2. Comparison Theorems

For simplicity, in this section we let $K = \mathbb{Q}_p$ be the field of p-adic numbers, so $\overline{K} = \overline{\mathbb{Q}_p}$, the ring of integers $\mathcal{O}_K = \mathbb{Z}_p$, and the residue field $k = \mathcal{O}_K/p = \mathbb{F}_p$.

Given a smooth proper variety $X/\mathcal{O}_K$, we expect the cohomology of X to be the same across all fibers, but since $X/\mathcal{O}_K$ is in mixed characteristic, each fiber is over a different field.

The special fiber of X,

$$X_k = X \times_{\mathcal{O}_K} (\mathcal{O}_K/p) \, ,$$

lives in characteristic p, so usually one is interested in the crystalline cohomology of the special fiber

$$\mathrm{H}^*_{crys} \left(X_k/\mathbb{Z}_p \right) .$$

We also consider the de Rham cohomology of X,

$$\mathrm{H}^*_{dR}(X) \in \mathrm{F} \left(K \right) ,$$

which takes values in the category of filtered $\mathcal{O}_K$-modules.

In this case, there is an equivalence

$$\mathrm{H}^*_{crys}\left(X_k/\mathbb{Z}_p\right) \cong \mathrm{H}^*_{dR}\left(X\right),$$

due to a result of Berthelot, which says that one can compute the crystalline cohomology of X/k by finding a lift of X to $\mathrm{W}(k)$ and computing the de Rham cohomology there.

The geometric generic fiber of X,

$$X_{\overline{K}} = X \times_{\mathcal{O}_K} \overline{K},$$

lives in characteristic 0, so we consider the p-adic étale cohomology (this is just the ℓ-adic cohomology where $\ell = p$),

$$\mathrm{H}^*_{et}\left(X_{\overline{K}}, \mathbb{Q}_p\right) \in \mathrm{Rep}_{G_K}\left(\mathbb{Q}_p\right),$$

which takes values in the category of $\mathbb{Q}_p$-vector spaces with an action of the Galois group $\mathrm{Gal}\left(\overline{K}/K\right) = G_K$, also known as the category of $\mathbb{Q}_p$-valued G_K-representations.

Fontaine constructed a ring called the de Rham period ring, B_{dR}, which is a field extension of K, equipped with a G_K-action such that the G_K-invariants of B_{dR} are K, i.e.

$$B_{dR}^{G_K} \cong K,$$

and a filtration.

Fontaine then conjectured

Conjecture (de Rham comparison conjecture)**.** *There is an equivalence*

$$\mathrm{H}^*_{dR}(X) \otimes_K B_{dR} \simeq \mathrm{H}^*_{et}\left(X_{\overline{K}}, \mathbb{Q}_p\right) \otimes_{\mathbb{Q}_p} B_{dR},$$

which is compatible with the Galois action (where we equip $\mathrm{H}^*_{dR}(X)$ *with the trivial Galois action), and the filtration (where we equip* $\mathrm{H}^*_{et}(X_{\overline{K}}, \mathbb{Q}_p)$ *with the trivial filtration).*

This would allow us to recover the de Rham cohomology from the p-adic étale cohomology, since

$$\begin{aligned}\mathrm{H}^*_{dR}(X) &\cong \mathrm{H}^*_{dR}(X) \otimes_K (B_{dR})^{G_K} \\ &\cong \left(\mathrm{H}^*_{dR}(X) \otimes_K B_{dR}^{G_K}\right)^{G_K} \\ &\cong \left(\mathrm{H}^*_{et}\left(X_{\overline{K}}, \mathrm{Q}_p\right) \otimes_{\mathbb{Q}_p} B_{dR}\right)^{G_K}_{.}\end{aligned}$$

It turns out that the Galois action on the étale cohomology cannot be recovered from the Hodge filtration alone, but the identification of crystalline and de Rham cohomology gives additional structure to the de Rham cohomology in the form of a Frobenius map.

In order to take advantage of the Frobenius map on the de Rham cohomology, we need to base change the étale cohomology up to a ring with a Frobenius map. There is no Frobenius map on B_{dR}, but Fontaine introduced a subring $B_{crys} \subseteq B_{dR}$ which does have a Frobenius morphism

$$\phi : B_{crys} \to B_{crys}.$$

This ring inherits the Galois action and filtration from B_{dR}, and has the property that

$$\mathrm{Fil}^0 \left(B_{crys}^{\phi=1} \right) = \mathbb{Q}_p,$$

where $B_{crys}^{\phi=1}$ is the subring of Frobenius fixed points.

Fontaine also conjectured

Conjecture (Crystalline comparison)**.** *There is an equivalence*

$$\mathrm{H}^*_{dR}(X) \otimes_K B_{crys} \simeq \mathrm{H}^*_{et} \left(X_{\overline{K}}, \mathbb{Q}_p \right) \otimes B_{crys},$$

which respects all structure, i.e. it is Galois equivariant, Frobenius equivariant, and preserves the filtration.

The name crystalline comparison comes from the previously mentioned equivalence between the de Rham and crystalline cohomology in this case.

Then to recover the étale cohomology from the de Rham cohomology, observe that

$$\begin{aligned}\mathrm{H}^*_{et}\left(X_{\overline{K}}, \mathbb{Q}_p\right) &\cong \mathrm{H}^*_{et}\left(X_{\overline{K}}, \mathbb{Q}_p\right) \otimes_{\mathbb{Q}_p} \mathrm{Fil}^0\left(B_{crys}^{\phi=1}\right)\\ &\cong \mathrm{Fil}^0\left(\mathrm{H}^*_{et}\left(X_{\overline{K}}, \mathbb{Q}_p\right) \otimes_{\mathbb{Q}_p} B_{crys}\right)^{\phi=1}\\ &\cong \mathrm{Fil}^0\left(\mathrm{H}^*_{dR}\left(X\right) \otimes_K B_{crys}\right)^{\phi=1}_{.}\end{aligned}$$

where we equip the étale cohomology with the trivial filtration and Frobenius action.

Both of these conjectures were proven by Faltings, but now have many different proofs using different approach.

3.3. Integral p-adic Hodge Theory

In the previous section we worked with the rationalized cohomology theories, but the integral theories contain more information about the variety they are applied to.

In [1], an integral version of the crystalline comparison theorem is proven

Theorem [BMS1]. *Let $\mathcal{X}$ be a proper smooth formal scheme over $\mathcal{O}_K$, where $\mathcal{O}_K$ is the ring of integers in a complete discretely valued nonarchimedean extension K of $\mathbb{Q}_p$ with perfect residue field k. Let C be a completed algebraic closure of K, and write $\mathcal{X}_C$ for the rigid-analytic generic fiber of $\mathcal{X}$. There are comparison isomorphisms*

$$\mathrm{H}^i_{et}\left(\mathcal{X}_C, \mathbb{Z}_p\right) \otimes_{\mathbb{Z}_p} B_{crys} \cong \mathrm{H}^i_{crys}\left(\mathcal{X}_k/\mathrm{W}(k)\right) \otimes_{\mathrm{W}(k)} B_{crys},$$

which is compatible with the Galois action, Frobenius, and filtration.

This implies that $\mathrm{H}^i_{et}(\mathcal{X}_C, \mathbb{Q}_p)$ is a crystalline-representation, which is just a fancy way of saying

$$\dim_{\mathbb{Q}_p}(\mathrm{H}^i_{et}(\mathcal{X}_C, \mathbb{Q}_p)) = \dim_{\mathbb{Q}_p}((\mathrm{H}^i_{et}(\mathcal{X}_C, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} B_{crys})^{G_K}).$$

There is a mixed characteristic version of the theory of Dieudonné modules, known as Breuil-Kisin-Fargues modules.

First let $\mathrm{A}_{inf} := \mathrm{W}(\mathcal{O}^\flat)$, where $\mathcal{O}$ is the ring of integers of C, and $\mathcal{O}^\flat$ is the tilt of $\mathcal{O}$, which is $\varprojlim_\phi \mathcal{O}/p$, the inverse limit perfection of $\mathcal{O}/p$. There is a canonical map $\theta : \mathrm{A}_{inf} \to \mathcal{O}$, which is surjective with kernel generated by an element ξ.

The period rings B_{dR} and B_{crys} can be derived from A_{inf} as follows.

To obtain B_{dR},

$$B^+_{dR} \cong \mathrm{A}_{inf}\left[\frac{1}{p}\right]^{\hat{}}_{\xi},\ B_{dR} \cong B^+_{dR}\left[\frac{1}{\xi}\right]^{\hat{}}_{\xi},$$

where this is the ξ-adic completion.

To obtain B_{crys}, let ϵ be a compatible system of p-power roots of unity in $\mathcal{O}$, so ϵ gives an element in $\mathcal{O}^\flat = \varprojlim_\phi \mathcal{O}/p$. Then let $\mu = [\epsilon]-1 \in \mathrm{A}_{inf}$. Then let A_{crys} be the PD-envelope of θ, i.e. the p-completion of the A_{inf}-algebra generated by $\frac{\xi}{n!}$, $n \geq 1$, inside of $\mathrm{A}_{inf}\left[\frac{1}{p}\right]$. Then

$$B_{crys} \cong A_{crys}\left[\frac{1}{\mu}\right].$$

Definition. *A Breuil-Kisin-Fargues Module is a finitely presented* A_{inf}*-module* M*, along with a a Frobenius semi-linear morphism*

$$\phi_M : M\left[\frac{1}{\xi}\right] \to M\left[\frac{1}{\phi(\xi)}\right],$$

such that $M\left[\frac{1}{p}\right]$ *is finite free over* $\mathrm{A}_{inf}\left[\frac{1}{p}\right]$.

Fargues proved that a Breuil-Kisin-Fargues module is equivalent to the data of a finite free $\mathbb{Z}_p$-module T, and a chosen B_{dR}^{+}-lattice inside of $T \otimes_{\mathbb{Z}_p} B_{dR}$.

The authors of [1] also prove an analogue of the de Rham comparison theorem, which gives an equivalence

$$\mathrm{H}^i_{crys}(X/B_{dR}^{+}) \otimes_{B_{dR}^{+}} B_{dR} \cong \mathrm{H}^i_{et}(X, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} B_{dR},$$

for proper smooth adic spaces X over C, and B_{dR}^{+} is another period ring, whose fraction field is B_{dR}.

This comparison tells us that the crystalline cohomology gives us a preferred B_{dR}^{+}-lattice inside the B_{dR}-module $\mathrm{H}_{et}^{i}(X,\mathbb{Z}_p)\otimes_{\mathbb{Z}_p} B_{dR}$, so combining these we get a Breuil-Kisin-Fargues module which knows the étale, crystalline, and de Rham cohomology of X.

These ideas suggest that we should look for a cohomology theory that takes values directly in the category of Breuil-Kisin-Fargues modules, which has all three of these cohomology theories as a specialization. In [1] such a cohomology theory was constructed, called the A_{inf}-cohomology, denoted

$$\mathrm{R}\Gamma_{\mathrm{A}_{inf}}(\mathcal{X}),$$

for a proper smooth formal scheme $\mathcal{X}$.

CHAPTER 4

Topological Hochschild Homology and p-adic Hodge Theory

In [2] topological Hochschild homology is used to give a construction of a new cohomology theory known as the prismatic cohomology. We outline this construction here.

4.1. Flat Descent for Trace Theories

Recall that the HKR-filtration on the Hochschild homology $\mathrm{Fil}^*\mathrm{HH}(A/R)$ has

$$\mathrm{gr}^i_{HKR} \simeq \wedge^i L_{A/R}[i].$$

In [2] it is shown that wedges of the cotangent complex have flat descent, i.e. for any base ring R the functors

$$A \mapsto \wedge^i L_{A/R},$$

are fpqc sheaves with values in the derived ∞-category of R-modules $\mathcal{D}(R)$.

The HKR-filtration on $\mathrm{HH}(A)$ then inductively implies that HH is an fpqc sheaf, and then one can prove that THH is an fpqc sheaf by observing that

$$\big(\mathrm{THH}(-/R) \otimes_{\mathrm{THH}(\mathbb{S})} \tau_{\leq n}\mathrm{THH}(\mathbb{Z})\big)$$

is a Postnikov tower with limit $\mathrm{THH}(-)$. The graded pieces of this tower are

$$\big(\mathrm{THH}(-) \otimes_{\mathrm{THH}(\mathbb{Z})} \mathbb{Z}\big) \otimes \pi_n \mathrm{THH}(\mathbb{Z}) \simeq \mathrm{HH}(-) \otimes \pi_n \mathrm{THH}(\mathbb{Z}),$$

and since $\pi_n \mathrm{THH}(\mathbb{Z})$ is a perfect complex this is an fpqc sheaf for each n, so THH is an fpqc sheaf as the limit of this tower.

This also implies fpqc descent for $\mathrm{HC}^-(-/R)$ and $\mathrm{TC}^-(-)$, since the limit of a sheaf is a sheaf. With a little more work it can also be shown that $\mathrm{HP}(-/R)$ and $\mathrm{TP}(-)$ are fpqc sheaves, and from here it follows that $\mathrm{TR}^n(-)$ and $\mathrm{TR}(-)$ are fpqc sheaves, as they can be written as a limit of fpqc sheaves.

4.2. Perfectoid rings and the Quasisyntomic site

In 2011 Scholze introduced the notion of a perfectoid space [22]. Since then these ideas have proven invaluable in p-adic Hodge theory. We use a slightly different notion of a perfectoid ring here, which are sometimes called integral perfectoid rings to distinguish them from the perfectoid fields of [22].

Given a ring R, Fontaine's infinitesimal period ring $A_{\mathrm{inf}}(R)$ is defined as

$$A_{\mathrm{inf}}(R) := W(R^\flat),$$

where $R^\flat$ is the inverse limit perfection $(R/p)^{perf}$ of R/p. If R is p-complete, the universal property of the Witt vectors tells us that to specify a map $A_{\mathrm{inf}}(R) \to R$ it is enough to specify a map $(R/p)_{perf} \to R/p$, and we let $\theta : A_{\text{inf}}(R) \to R$ be the map corresponding to the canonical projection map $(R/p)^{perf} \to R/p$.

A perfectoid ring is:

Definition. *A ring R is said to be a perfectoid ring if it is p-complete, there is an element $\pi \in R$ such that $\pi^p = pu$ for a unit u, the Frobenius map is surjective on R/p, and the kernel of the map $\theta : A_{\text{inf}}(R) \to R$ is generated by one element.*

We will see later that Bökstedt's calculation of $\text{THH}(\mathbb{F}_p)$ can be extended to give a calculation that $\text{THH}(R) \simeq R[x]$, $|x| = 2$, for any perfectoid ring R.

There is a close connection between the cotangent complex and topological Hochschild homology and the trace theories have flat descent; these two facts motivate one to work with a refinement of the flat site where the rings have particularly nice cotangent complexes, known as the quasisyntomic site.

Definition. *A ring A is quasisyntomic if A is p-adically complete, has bounded p^∞-torsion, and $L_{A/\mathbb{Z}_p}$ has p-complete Tor-amplitude in $[-1, 0]$.*

A ring has bounded p^∞-torsion if there exists $N \in \mathbb{N}$ such that all p^n-torsion elements of A, for $n \geq N$, are already p^N-torsion elements. The Tor-amplitude condition on $L_{A/\mathbb{Z}_p}$ just means that for any A/p-module N,

$$\left(L_{A/\mathbb{Z}_p} \otimes_A^L A/p\right) \otimes_{A/p}^L N$$

has homotopy groups concentrated in degrees $[-1, 0]$.

There is also a relative version of quasisyntomic, where a map $A \to B$ is a quasisyntomic map if B is p-completely flat over A, and $L_{B/A}$ has p-complete Tor-amplitude in $[-1, 0]$.

The category QSyn^{op} is a site, where the covers are quasisyntomic maps $A \to B$ which are p-completely faithfully flat.

The quasisyntomic site QSyn^{op} has a particularly nice basis, made up of the quasiregular semiperfectoid rings.

Definition. *A ring $S \in \mathrm{QSyn}$ is quasiregular semiperfectoid if there is a map $R \to S$ with R perfectoid, and the Frobenius is surjective on S/p.*

A quasiregular semiperfectoid ring of characteristic p is usually called a quasiregular semiperfect ring.

The subcategory of quasiregular semiperfectoid rings forms a site $\mathrm{QRSPerfd}^{op}$, and every element $A \in \mathrm{QSyn}$ admits a cover $A \to S$ with $S \in \mathrm{QRSPerfd}$, so QRSPerfd is a basis for QSyn.

Quasiregular semiperfectoid rings have particularly nice p-complete cotangent complexes. To see this, given $S \in$ QRSPerfd, pick a perfectoid ring R with a surjection $R \to S$. Then the transitivity triangle for $\mathbb{Z}_p \to R \to S$ takes the form

$$L_{R/\mathbb{Z}_p} \otimes_R S \to L_{S/R} \to L_{S/\mathbb{Z}_p}.$$

Since $R \to S$ is surjective $L_{S/R}$ is concentrated in degree -1, and we can use the transitivity triangle of $\mathbb{Z}_p \to \mathrm{A}_{inf}(R) \to R$, and since A_{inf} is relatively perfect over $\mathbb{Z}_p$, after p-completion $L_{\mathrm{A}_{inf}/\mathbb{Z}_p} \simeq 0$, so $L_{R/\mathrm{A}_{inf}} \simeq L_{R/\mathbb{Z}_p}$, but $\mathrm{A}_{inf}(R) \to R$ is surjective, so it is also concentrated in degree -1. Thus $L_{S/\mathbb{Z}_p}$ is p-completely concentrated in degree -1.

4.3. Recovering de Rham Cohomology From Negative Cyclic Homology

In the previous section we saw that the p-complete cotangent complex of a quasiregular semiperfectoid ring S is concentrated in degree -1. Recall that the HKR filtration on $\mathrm{HH}(S/\mathbb{Z}_p)$ has $gr^i_{HKR} \simeq \wedge^i L_{S/\mathbb{Z}_p}[i]$, if $S \in$ QRSPerfd this is concentrated in degree $2i$. So $\mathrm{HH}(S/\mathbb{Z}_p, \mathbb{Z}_p)$ is even.

Since QRSPerfd is a basis for QSyn, and HH is even on QRSPerfd, this proves that $\mathrm{HH}(-/\mathbb{Z}_p; \mathbb{Z}_p)$ is locally even in the quasisyntomic topology. This is an important result that is true for other trace theories as well, and will be a central focus of this dissertation.

The homotopy fixed point spectral sequence

$$E_2^{p,q} = \mathrm{H}^p(\mathbb{T}, \pi_{-q}\mathrm{HH}(S/\mathbb{Z}_p;\mathbb{Z}_p)) \Rightarrow \pi_{-p-q}\mathrm{HC}^-(S/\mathbb{Z}_p;\mathbb{Z}_p),$$

degenerates, since $B\mathbb{T} \cong \mathbb{CP}^\infty$, so the group cohomology $\mathrm{H}^p(\mathbb{T}, \pi_*\mathrm{HH}(S/\mathbb{Z}_p;\mathbb{Z}_p))$ can be computed as the singular cohomology $\mathrm{H}^p(\mathbb{CP}^\infty, \pi_*\mathrm{HH}(S/\mathbb{Z}_p;\mathbb{Z}_p))$, and the singular cohomology $\mathrm{H}^*(\mathbb{CP}^\infty, \mathbb{Z}) \cong \mathbb{Z}[x],\ |x| = 2$.

This shows that $\mathrm{HC}^-(-/\mathbb{Z}_p;\mathbb{Z}_p)$ is locally even in the quasisyntomic topology.

The degeneration of the homotopy fixed point spectral sequence yields a filtration on $\pi_0\mathrm{HC}^-(S/\mathbb{Z}_p;\mathbb{Z}_p)$ with the i'th graded piece being the p-completion of $\wedge^i L_{S/\mathbb{Z}_p}[-i]$. One can check that this is the p-completion of the Hodge-completed derived de Rham complex $L\Omega_{S/\mathbb{Z}_p}$.

Since QRSPerfd is a basis for QSyn, the sheaf $\pi_0\mathrm{HC}^-(-/\mathbb{Z}_p;\mathbb{Z}_p)$ is identified with the sheaf $\hat{L\Omega}_{-/\mathbb{Z}_p}$. This is a toy example of the construction of prismatic cohomology, which will be the sheaf $\pi_0\mathrm{TC}^-(-;\mathbb{Z}_p)$.

4.4. Local Evenness of THH and its variants

For a perfectoid ring R, in [2, p. 6] it is shown that

$$\pi_*\mathrm{THH}(R;\mathbb{Z}_p) \cong R[u],\ |u| = 2.$$

Using the homotopy fixed point spectral sequence, and the Tate spectral sequence, it is then derived that

$$\pi_*\mathrm{TC}^-(R;\mathbb{Z}_p) \cong \mathrm{A}_{inf}[u,v]/(uv-\xi),\ |u|=2,\ |v|=-2$$

$$\pi_*\mathrm{TP}(R;\mathbb{Z}_p) \cong \mathrm{A}_{inf}[\sigma^{\pm 1}],\ |\sigma|=2.$$

Given $A \in$ QRSPerfd, pick a perfectoid ring R with a surjection $R \to A$. Then $\mathrm{THH}(A;\mathbb{Z}_p)$ is a $\mathrm{THH}(R;\mathbb{Z}_p)$-algebra, so it is equipped with a degree 2 map $u : \mathrm{THH}(A;\mathbb{Z}_p)[2] \to \mathrm{THH}(A;\mathbb{Z}_p)$ given by multiplication by $u \in \pi_2\mathrm{THH}(R;\mathbb{Z}_p)$. This map fits into a cofiber sequence

$$\mathrm{THH}(A;\mathbb{Z}_p)[2] \to \mathrm{THH}(A;\mathbb{Z}_p) \to \mathrm{HH}(A/R;\mathbb{Z}_p).$$

If A is quasismooth then $\pi_i\mathrm{HH}(A/R;\mathbb{Z}_p) \cong \Omega^i_{A/R}$, and this cofiber sequence will let us inductively define a filtration on $\mathrm{THH}(A;\mathbb{Z}_p)$ with

$$gr^n = \bigoplus \wedge^i L_{A/R}[n],$$

where this sum ranges over all $i \leq n$ with $i - n$ even.

Since $A \in$ QRSPerfd, $\wedge^i L_{A/R}$ lives in degree i, so these graded pieces are concentrated in even degrees and the spectral sequence of this filtration tells us that $\mathrm{THH}(A;\mathbb{Z}_p)$ is concentrated in even degrees.

The homotopy fixed point spectral sequence and the Tate spectral sequence then imply that $\mathrm{TC}^{-}(A;\mathbb{Z}_p)$ and $\mathrm{TP}(A;\mathbb{Z}_p)$ are even, respectively. Evennes of TC then follows since TC is the fiber of a map $\mathrm{TC}^{-} \to \mathrm{TP}$.Since QRSPerfd is a basis for QSyn, evennes on QRSPerd implies local evennes on QSyn of these functors.

For $S \in$ QRSPerfd, the Nygaard-completed prismatic cohomology of S, written $\widehat{\mathbb{\Delta}}_S$ to be

$$\widehat{\mathbb{\Delta}}_S := \pi_0 \mathrm{TC}^{-}(S;\mathbb{Z}_p),$$

and the Nygaard filtration on $\widehat{\mathbb{\Delta}}_S$, written $\mathcal{N}^{\geq *}\widehat{\mathbb{\Delta}}_S$ is the filtration coming from the homotopy fixed point spectral sequence, so

$$gr^i \mathcal{N}^{\geq *}\widehat{\mathbb{\Delta}}_S \cong \pi_{2i}\mathrm{THH}(S;\mathbb{Z}_p).$$

For a general $A \in$ QSyn, we unfold the above constructions, i.e. since QRSPerfd is a basis for QSyn, we sheafify the above constructions to give constructions on all of QSyn.
Since $\mathrm{THH}(S;\mathbb{Z}_p)$ is even for $S \in$ QRSPerd, we have that $\mathrm{THH}(-;\mathbb{Z}_p)$ is locally even on QSyn. Consider the double speed Postnikov filtration on THH, written

$$\mathrm{Fil}^n \mathrm{THH}(-;\mathbb{Z}_p) = \tau_{\geq 2n}\mathrm{THH}(-;\mathbb{Z}_p),$$

this is a sheaf.

For a general $A \in$ QSyn, in [2] they define

$$\mathrm{Fil}^n\mathrm{THH}(A;\mathbb{Z}_p) := \mathrm{R}\Gamma_{syn}(A, \tau_{\geq 2n}\mathrm{THH}(-;\mathbb{Z}_p)),$$

$$\mathrm{Fil}^n\mathrm{TC}^-(A;\mathbb{Z}_p) := \mathrm{R}\Gamma_{syn}(A, \tau_{\geq 2n}\mathrm{TC}^-(-;\mathbb{Z}_p)),$$

$$\mathrm{Fil}^n\mathrm{TP}(A;\mathbb{Z}_p) := \mathrm{R}\Gamma_{syn}(A, \tau_{\geq 2n}\mathrm{TP}(-;\mathbb{Z}_p)).$$

Then for $A \in \mathrm{QSyn}$,

$$\widehat{\mathbb{\Delta}}_A := \mathrm{gr}^0\mathrm{TC}^-(A;\mathbb{Z}_p),$$

extends the definition of the Nygaard-completed prismatic cohomology, where

$$\mathcal{N}^{\geq n}\widehat{\mathbb{\Delta}}_A \simeq \mathrm{gr}^n\mathrm{TC}^-(A;\mathbb{Z}_p).$$

There is also a filtration on TC, since

$$\mathrm{TC}(A;\mathbb{Z}_p) \simeq \mathrm{fib}\left(\phi - \mathrm{can} : \mathrm{TC}^-(A;\mathbb{Z}_p) \to \mathrm{TP}(A;\mathbb{Z}_p)\right),$$

we have

$$\mathrm{Fil}^n\mathrm{TC}(A;\mathbb{Z}_p) \simeq \mathrm{fib}\left(\phi - \mathrm{can} : \mathrm{Fil}^n\mathrm{TC}^-(A;\mathbb{Z}_p) \to \mathrm{Fil}^n\mathrm{TP}(A;\mathbb{Z}_p)\right).$$

The graded pieces of this filtration are denoted $\mathrm{Z}_p(n)(A)$, and play a major role in this dissertation.

Conjecturely, this filtration is supposed to be compatible with the motivic filtration on K-theory under the trace map $\mathrm{K} \to \mathrm{TC}$.

There is now a construction of prismatic cohomology which avoids the use of topological Hochschild homology [3], but the connection between the two is still fruitful in p-adic Hodge theory and in homotopy theory.

CHAPTER 5

Local Evenness of Topological Restriction Homology in Characteristic p

5.1. Introduction

In [18] a method is given to compute the syntomic cohomology $\mathbb{Z}_p(i)(k[x]/x^2)$ for k a perfect $\mathbb{F}_p$-algebra. The idea is that $k[x]/x^2$ lifts to a quasisyntomic δ-ring of characteristic 0, $A = A_{crys}(k[x]/x^2)$, and then $\Delta_{k[x]/x^2} \simeq R\Gamma_{crys}(k[x]/x^2/\mathbb{Z}_p)$ can be computed using the derived de Rham cohomology $L\Omega_A$ – which can be computed explicitly using the divided power de Rham complex

$$W(k)\left[x, \frac{x^{2j}}{j!}\right]_{j\geq 0} \to W(k)\left[x, \frac{x^{2j}}{j!}\right]_{j\geq 0} dx.$$

The Nygaard filtration on this complex admits an explicit description, which allows one to compute $\mathbb{Z}_p(i)(k[x]/x^2) = \mathrm{fib}(\varphi/p^i - \mathrm{can}) : \mathcal{N}^{\geq i}L\Omega_A \to L\Omega_A$. In [23] it's explained how to use these methods to compute $\mathbb{Z}_p(i)(k[x]/x^e)$ and several clever notational conventions are introduced to simplify these computations.

In this article we will further use these methods to compute the syntomic cohomology of truncated polynomial algebras over some simple quasiregular semiperfect rings.

Theorem 2. *For* $R = k[y_1^{1/p^\infty}, ..., y_t^{1/p^\infty}, x]/(y_1, ..., y_t, x^e)$*,* k *a perfect* $\mathbb{F}_p$*-algebra,* $i \in \mathbb{N}$*,* $\mathbb{Z}_p(i)(R)$ *is concentrated in degree 1, and there is an isomorphism*

$$\mathrm{H}^1(\mathbb{Z}_p(i)(R)) \cong \bigoplus_{\substack{m \in I_p, \\ \alpha \in \mathbb{Z}[1/p]^t}} W(k)/\{p^{((s))}m, e\},$$

where I_p *is the set of integers coprime to* p*, with* $s = s(p, i, e, m, \alpha)$*. Here* $((s)) := \max(s, 0)$*, and* $\{a, e\} = a$ *if* $e \nmid a$*, and* $\{a, e\} = e$ *otherwise.*

We then explain how to identify

$$\mathrm{TR}\left(k[y_1^{1/p^\infty}, ..., y_t^{1/p^\infty}]/(y_1, ..., y_t)\right) \simeq \varprojlim_e \mathbb{Z}_p(i)\left(k[y_1^{1/p^\infty}, ..., y_t^{1/p^\infty}, x]/(y_1, ..., y_t, x^e)\right)$$

using [11] and [8].

Then by analyzing this limit we show that for quasiregular semiperfect rings of this form TR is concentrated in even degrees.

Theorem 4. *For* k *a perfect* $\mathbb{F}_p$*-algebra,* $\mathrm{TR}\left(k[y_1^{1/p^\infty}, ..., y_n^{1/p^\infty}]/y_1, ..., y_n\right)$ *is even.*

Finally using the filtration on TR coming from [21] we conclude that for quasisyntomic algebras in characteristic p, TR is locally even in the quasisyntomic topology.

Theorem 5. *Let* S *be a quasisyntomic algebra over a perfect* $\mathbb{F}_p$ *algebra* R*,* $\mathrm{TR}(S)$ *is locally even in the quasisyntomic topology.*

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5.2. $\mathbb{Z}_p(i)$ for some quasiregular semiperfect rings

The goal of this section is to compute the syntomic cohomology for rings of the form $R = k[y^{1/p^\infty}, x]/(y, x^e)$, where k is a perfect $\mathbb{F}_p$ algebra. This is the truncated polynomial algebra on x over the quasiregular semiperfect ring $S = k[y^{1/p^\infty}]/(y)$. The prismatic cohomology is symmetric monoidal, so we can compute

$$\Delta_R \simeq \Delta_S \otimes^L_{\mathbb{Z}_p} \Delta_{\mathbb{F}_p[x]/x^e}.$$

Since S is quasiregular semiperfect, there is an identification $\Delta_S \simeq \mathrm{A}_{crys}(S) \cong \mathrm{W}(k)[y^{1/p^\infty}, \frac{y^n}{n!}]$, which is a flat $\mathbb{Z}_p$-algebra. Then using the identification of $\Delta_{\mathbb{F}_p[x]/x^e}$ with the p-completed divided power de Rham complex as in [18] and [23] we get

$$\Delta_R \simeq \left(W(k) \left[y^{1/p^\infty}, \frac{y^n}{n!} \right] \left[x, \frac{x^{ej}}{j!} \right] \xrightarrow{d} W(k) \left[y^{1/p^\infty}, \frac{y^n}{n!} \right] \left[x, \frac{x^{ej}}{j!} \right] \ dx \right)^\wedge_p$$

From now on we take $n \in \mathbb{N}[1/p]$, so we can write $W(k)\left[y^{1/p^\infty}, \frac{y^n}{n!}\right]$ as $\mathrm{W}(k)\left[\frac{y^n}{\lfloor n \rfloor!}\right]$. We use the ideas and notation from [23] to write the above complex as

$$\bigoplus_{m,n} W(k) \left\langle \frac{y^n}{\lfloor n \rfloor!} \frac{x^m}{\lfloor m/e \rfloor!} \right\rangle \xrightarrow{d} \bigoplus_{m,n} W(k) \left\langle \frac{y^n}{\lfloor n \rfloor!} \frac{x^m}{\Gamma\lceil m/e \rceil} \mathrm{dlog} x \right\rangle,$$

where $\Gamma\lceil m/e \rceil = \lfloor \frac{m-1}{e} \rfloor!$, n ranges through $\mathbb{N}[1/p]$ and $m \in \mathbb{N}$.

Notice that this complex has a bi-grading by m, n with a generator in degree (m, n) given by $\frac{y^n}{\lfloor n \rfloor !}\frac{x^m}{\lfloor m/e \rfloor !}$, this is a $\mathbb{N} \times \mathbb{N}[1/p]$ grading. Also recall that d is the $W(k)\left[\frac{y^n}{\lfloor n \rfloor !}\right]$-linear differential, so $d\left(\frac{y^n}{\lfloor n \rfloor !}\frac{x^m}{\lfloor m/e \rfloor}\right) = \frac{y^n}{\lfloor n \rfloor !}d\left(\frac{x^m}{\lfloor m/e \rfloor}\right)$. The logarithmic differential $\mathrm{dlog}x$ satisfies $x * \mathrm{dlog}x = dx$, so $x^{m-1}\mathrm{d}x = x^m \mathrm{dlog}x$ which diagonalizes the differential. The point of these notations is to cleanly write the differential in terms of the generators

$$
\begin{aligned}
d\left(\frac{y^n}{\lfloor n \rfloor !}\frac{x^m}{\lfloor m/e \rfloor !}\right) &= m\frac{y^n}{\lfloor n \rfloor !}\frac{x^{m-1}}{\lfloor \frac{m}{e} \rfloor !}dx \\
&= m\frac{\lfloor \frac{m-1}{e} \rfloor !}{\lfloor m/e \rfloor !}\frac{y^n}{\lfloor n \rfloor !}\frac{x^{m-1}}{\lfloor \frac{m-1}{e} \rfloor !}dx \\
&= \{m, e\}\frac{y^n}{\lfloor n \rfloor !}\frac{x^m}{\Gamma\lceil m/e \rceil}\mathrm{dlog}x,
\end{aligned}
$$

where

$$
\{m, e\} := m\frac{\Gamma\lceil m/e \rceil}{\lfloor m/e \rfloor !} = \begin{cases} m & e \nmid m \\ e & e \mid m. \end{cases}
$$

To understand the Nygaard filtration we can identify $LW\Omega_R \simeq \mathrm{A}_{\mathrm{Crys}}(S) \otimes_{\mathbb{Z}_p} \mathbf{LW}\Omega_{\mathbb{F}_p[x]/x^e}$, and since the Nygaard filtration is symmetric monoidal $\mathcal{N}^* \simeq p^* \mathrm{A}_{\mathrm{Crys}_S} \otimes_{p^*\mathbb{Z}_p} \mathcal{N}^{\geq *}\mathbf{LW}\Omega_{\mathbb{F}_p[x]/x^e}$. Alternatively, $\mathcal{N}^{\geq i}$ is the subcomplex where the Frobenius map ϕ is divisible by p^i. The Frobenius sends the degree (m, n) generator $\frac{y^n}{\lfloor n \rfloor !}\frac{x^m}{\lfloor m/e \rfloor !}$ to $\frac{y^{pn}}{\lfloor n \rfloor !}\frac{x^{pm}}{\lfloor m/e \rfloor !}$ writing this in terms of the degree (pm, pn) generator we get

$$
\frac{\lfloor pn \rfloor !}{\lfloor n \rfloor !}\frac{\lfloor pm/e \rfloor !}{\lfloor m/e \rfloor !}\frac{y^{pn}}{\lfloor pn \rfloor !}\frac{x^{pm}}{\lfloor pm/e \rfloor !}.
$$

Then using the Legendre formula to compute the p-adic valuation of these coefficients we see that $\mathcal{N}^{\geq i}\mathbf{LW}\Omega_R$ is given by

$$\bigoplus_{m,n} p^{((i-\lfloor m/e\rfloor-\lfloor n\rfloor))}W(k)\left\langle \frac{y^n}{\lfloor n\rfloor!}\frac{x^m}{\lfloor m/e\rfloor!}\right\rangle \xrightarrow{d} \bigoplus_{m,n} p^{((i-\lceil m/e\rceil-\lfloor n\rfloor))}W(k)\left\langle \frac{y^n}{\lfloor n\rfloor!}\frac{x^m}{\Gamma\lceil m/e\rceil}\mathrm{dlog}x\right\rangle,$$

where $((a)) = \max(a,0)$.

We now need to compute the cohomology of $\mathbf{LW}\Omega_R$ and $\mathcal{N}^{\geq i}\mathbf{LW}\Omega_R$. In both cases $\mathrm{H}^0 = 0$, except in degrees $m = 0$ since the differential is injective. The bigrading also passes to cohomology, so we have

$$\mathrm{H}^1(\mathrm{LW}\Omega_R)_{(m,n)} = W(k)/\{m,e\}\left\langle \frac{y^n}{\lfloor n\rfloor!}\frac{x^m}{\Gamma\lceil m/e\rceil}\mathrm{dlog}x\right\rangle,$$

$$\mathrm{H}^1(\mathcal{N}^{\geq i}\mathbf{LW}\Omega_R)_{(m,n)} = W(k)/p^{\epsilon(i)}\{m,e\}\left\langle p^{((i-\lceil m/e\rceil-\lfloor n\rfloor))}\frac{y^n}{\lfloor n\rfloor!}\frac{x^m}{\Gamma\lceil m/e\rceil}\mathrm{dlog}x\right\rangle.$$

Where

$$\begin{aligned}\epsilon(i) &= ((i-\lfloor m/e\rfloor-\lfloor n\rfloor)) - ((i-\lceil m/e\rceil-\lfloor n\rfloor)) \\ &= \begin{cases} 1 & \lfloor m/e\rfloor < \lceil m/e\rceil \leq i-\lfloor n\rfloor \\ 0 & \text{otherwise.}\end{cases}\end{aligned}$$

Our goal is to compute $\mathbb{Z}_p(i)(R) = \mathrm{fib}(\phi/p^i - \mathrm{can}) : \mathcal{N}^{\geq i}\mathrm{LW}\Omega_R \to \mathrm{LW}\Omega_R$, so now we can use the long exact sequence in cohomology to determine $\mathrm{H}^k(\mathbb{Z}_p(i)(R))$ if we can determine the map $(\phi/p^i - \mathrm{can}) : \mathrm{H}^k(\mathcal{N}^{\geq i}\mathrm{LW}\Omega_R) \to \mathrm{H}^k(\mathrm{LW}\Omega_R)$.

The canonical map comes from the inclusion, so it sends a degree (m,n) generator

$$p^{((i-\lceil m/e\rceil-\lfloor n\rfloor))}\frac{y^n}{\lfloor n\rfloor!}\frac{x^m}{\Gamma\lceil m/e\rceil}\mathrm{dlog}x \in \mathrm{H}^1(\mathcal{N}^{\geq i}\mathbf{LW}\Omega_R)_{(m,n)}$$

to

$$p^{((i-\lceil m/e\rceil-\lfloor n\rfloor))}\frac{y^n}{\lfloor n\rfloor!}\frac{x^m}{\Gamma\lceil m/e\rceil}\mathrm{dlog}x \in \mathrm{H}^1(\mathrm{LW}\Omega_R)_{(m,n)}.$$

So

$$\mathrm{can}\left(\mathrm{H}^1(\mathcal{N}^{\geq i}\mathbf{LW}\Omega_R)_{(m,n)}\right) = p^{((i-\lceil m/e\rceil-\lfloor n\rfloor))}\mathrm{H}^1(\mathrm{LW}\Omega_R)_{(m,n)}.$$

Similarly, the divided Frobenius map sends a degree (m,n) generator

$$p^{((i-\lceil m/e\rceil-\lfloor n\rfloor))}\frac{y^n}{\lfloor n\rfloor!}\frac{x^m}{\Gamma\lceil m/e\rceil}\mathrm{dlog}x \in \mathrm{H}^1(\mathcal{N}^{\geq i}\mathbf{LW}\Omega_R)_{(m,n)}$$

to

$$p^{((i-\lceil m/e\rceil-\lfloor n\rfloor))-i+1}\ \frac{y^{pn}}{\lfloor n\rfloor!}\frac{x^{pm}}{\Gamma\lceil m/e\rceil}\mathrm{dlog}x \in \mathrm{H}^1(\mathbf{LW}\Omega_R)_{(pm,pn)}.$$

Where the $+1$ in the exponent of p comes from

$$\mathrm{dlog}(x^p) = \frac{px^{p-1}}{x^p}dx = p\frac{1}{x}dx = p\mathrm{dlog}x.$$

Now writing this element in terms of the degree (pm,pn) generator we get

$$p^{((i-\lceil m/e\rceil-\lfloor n\rfloor))-i+1}\frac{y^{pn}}{\lfloor n\rfloor!}\frac{x^{pm}}{\Gamma\lceil m/e\rceil}\mathrm{dlog}x = p^t\frac{y^{pn}}{\lfloor pn\rfloor!}\frac{x^{pm}}{\Gamma\lceil pm/e\rceil}\mathrm{dlog}x,$$

where t is still to be determined.

The Legendre formula and properties of a valuation tell us that

$$v_p\left(\frac{\lfloor pn\rfloor!}{\lfloor n\rfloor!}\right) = \lfloor n\rfloor + v_p\left(\lfloor n\rfloor!\right) - v_p\left(\lfloor n\rfloor!\right)$$
$$= \lfloor n\rfloor,$$

and

$$v_p\left(\frac{\Gamma\lceil pm/e\rceil}{\Gamma\lceil m/e\rceil}\right) = \lfloor m/e\rfloor + v_p\left(\frac{\{pm,e\}}{p\{m,e\}}\right)$$
$$= \lceil m/e\rceil - 1.$$

So $t = ((i - \lceil m/e\rceil - \lfloor n\rfloor)) - i + 1 + \lfloor n\rfloor + \lceil m/e\rceil - 1$. If $\lceil m/e\rceil + \lfloor n\rfloor > i$, then $((i - \lceil m/e\rceil - \lfloor n\rfloor)) = 0$ and $t = \lceil m/e\rceil + \lfloor n\rfloor - i$. Otherwise all of the terms cancel and $s = 0$. In other words $t = ((\lceil m/e\rceil + \lfloor n\rfloor - i))$, and

$$\phi/p^i\left(\mathrm{H}^1(\mathcal{N}^{\geq i}\mathbf{LW}\Omega_R)_{(m,n)}\right) = p^{((\lceil m/e\rceil+\lfloor n\rfloor-i))}\mathrm{H}^1(\mathbf{LW}\Omega_R)_{(pm,pn)}.$$

Since the canonical map is $((i - \lceil m/e\rceil - \lfloor n\rfloor))$ p-divisible, and the divided Frobenius is $((\lceil m/e\rceil + \lfloor n\rfloor - i))$ p-divisible, for any values of the variables at least one of these will be 0. So in any degree (m, n) that is mapped into by both the canonical map and the divided Frobenius, $(\phi/p^i - \mathrm{can})$ is surjective, since in this degree the map is a difference of a surjective map and a p-divisible map, which by p-completeness is still surjective. This lets us conclude that $(\phi/p^i - \mathrm{can})$ is surjective, since in any degree (m, n) where $H^1(\mathrm{LW}\Omega_R)_{(m,n)} \neq 0$ we must have $p|\{m, e\}$, but since $\{m, e\} = e$ only if $e|m$ we must have $p|m$ either way. Then since $n \in \mathbb{N}[1/p]$ it is also p-divisible, and degree (m,n) is in the image of the Frobenius. Therefore $\mathrm{H}^2(\mathbb{Z}_p(i)(R)) = 0$.

Now fix n, and an odd m such that $p \nmid m$. Using the idea from [14], we consider the following map

$$\bigoplus_{a\geq 0} \mathrm{H}^1\left(\mathcal{N}^{\geq i}\mathrm{LW}\Omega_R\right)_{(p^a m, p^a n)} \xrightarrow{\phi/p^i-\mathrm{can}} \bigoplus_{a\geq 0} \mathrm{H}^1\left(\mathrm{LW}\Omega_R\right)_{(p^a m, p^a n)}.$$

This is not a graded map, but is still a filtered map. For low values of a the divided Frobenius map is an isomorphism $(p^a m, p^a n) \mapsto (p^{a+1}m, p^{a+1}n)$, but for large enough values of a the canonical map will become an isomorphism $(p^a m, p^a n) \mapsto (p^a m, p^a n)$. The kernel is then $\mathrm{H}^1(\mathcal{N}^{\geq i}\mathrm{LW}\Omega_R)_{(p^s m, p^s n)}$ for s the smallest value of a such that the divided Frobenius map is no longer an isomorphism. Explicitly, s is the smallest positive integer such that

$$\left\lceil \frac{p^s m}{e} \right\rceil + \lfloor p^s n \rfloor > i,$$

which we denote $s(e,m,n)$, or sometimes for convenience just s with the dependence on e, m, n left implicit.

Theorem 1. *For $R = k[y^{1/p^\infty}, x]/(y, x^e)$, k a perfect $\mathbb{F}_p$-algebra, there is an isomorphism*

$$\mathrm{H}^1(\mathbb{Z}_p(i)(R)) \cong \bigoplus_{\substack{m\in I_p,\\ n\in\mathbb{Z}[1/p]}} W(k)/\{p^{((s))}m, e\},$$

where I_p is the set of integers coprime to p, and $s = s(p,i,e,m,n)$ is not a constant.

These same methods can be applied to the quasiregular semiperfect rings of the form $R = k[y_1^{1/p^\infty}, .., y_t^{1/p^\infty}, x]/(y_1, .., y_t, x)$ with $t \in \mathbb{N}$. To simplify things going forward we will use the multi-index notation $\alpha = (n_1, ..., n_t)$, which lets us write

$$\frac{y^\alpha}{\lfloor\alpha\rfloor!} = \frac{y_1}{\lfloor n_1\rfloor!}\frac{y_2}{\lfloor n_2\rfloor!}...\frac{y_t}{\lfloor n_t\rfloor!}.$$

With this new notation in hand we have

$$\mathrm{LW}\Omega_R \simeq \bigoplus_{m,\alpha} W(k)\left\langle \frac{y^\alpha}{\lfloor\alpha\rfloor!}\frac{x^m}{\lfloor m/e\rfloor!}\right\rangle \xrightarrow{d} \bigoplus_{m,\alpha} W(k)\left\langle \frac{y^\alpha}{\lfloor\alpha\rfloor!}\frac{x^m}{\Gamma\lceil m/e\rceil!}\right\rangle,$$

where α ranges over all length t multi-indexes of elements in $\mathbb{N}[1/p]$.

Just as before we get

$$\mathrm{H}^1(\mathrm{LW}\Omega_R)_{(m,\alpha)} = W(k)/\{m,e\}\left\langle \frac{y^\alpha}{\lfloor\alpha\rfloor!}\frac{x^m}{\Gamma\lceil m/e\rceil}\mathrm{dlog}x\right\rangle,$$

$$\mathrm{H}^1(\mathcal{N}^{\geq i}\mathbf{LW}\Omega_R)_{(m,\alpha)} = W(k)/p^{\epsilon(i)}\{m,e\}\left\langle p^{((i-\lceil m/e\rceil - \lfloor\alpha\rfloor_{\ell_1}))}\frac{y^\alpha}{\lfloor\alpha\rfloor!}\frac{x^m}{\Gamma\lceil m/e\rceil}\mathrm{dlog}x\right\rangle,$$

where $\lfloor\alpha\rfloor_{\ell_1} := \lfloor n_1\rfloor + \lfloor n_2\rfloor + ... + \lfloor n_t\rfloor$ is the ℓ_1 norm of $\lfloor\alpha\rfloor$.

Let $s(e, m, \alpha)$ be the minimal value of a such that the divided Frobenius map is no longer an isomorphism between degrees $(p^a m, p^a\alpha) \mapsto (p^{a+1}m, p^{a+1}\alpha)$, which is harder to give an explicit formula for in this case.

Theorem 2. *For $R = k[y_1^{1/p^\infty}, ..., y_t^{1/p^\infty}, x]/(y_1, ..., y_t, x^e)$, k a perfect $\mathbb{F}_p$-algebra, there is an isomorphism*

$$\mathrm{H}^1(\mathbb{Z}_p(i)(R)) \cong \bigoplus_{\substack{m \in I_p, \\ \alpha \in \mathbb{Z}[1/p]^t}} W(k)/\{p^{((s))}m, e\},$$

where I_p is the set of integers coprime to p, and $s = s(p, i, e, m, \alpha)$ is not a constant.

5.3. Computation of TR

In this section we compute TR for quasiregular semiperfect rings. The spectral sequence of the motivic filtration is given by

$$E_2^{i,j} = \mathrm{H}^{i-j}(\mathbb{Z}_p(-j)(R)) \Rightarrow \pi_{-i-j}\mathrm{TC}(R; \mathbb{Z}_p).$$

Since $\mathbb{Z}_p(i)(R)$ is concentrated in degree 1, this spectral sequence degenerates and we get

$$\widetilde{\mathrm{TC}}_{2i-1}(R) \cong \mathrm{H}^1(\mathbb{Z}_p(i)(R)).$$

Then from the Dundas-Goodwillie-McCarthy theorem [8] we have

$$K_{2i-1}(k[y^{1/p^\infty}, x]/(y, x^e), (x)) \cong \widetilde{\mathrm{TC}}_{2i-1}(k[y^{1/p^\infty}, x]/(y, x^e)),$$

and we would like to use the description of TR due to Hesselholt [11] and McCandless [19]

$$\mathrm{TR}(k[y^{1/p^{\infty}}]/y) \simeq \varprojlim_{e} \Omega K(k[y^{1/p^{\infty}}, x]/(y, x^e), (x))$$

to understand $\mathrm{TR}(k[y^{1/p^{\infty}}]/y)$.

First we understand this limit diagram for the toy example $\mathrm{TR}(\mathbb{F}_p)$, where we have

$$\mathrm{TR}(\mathbb{F}_p) \simeq \varprojlim_{e} \Omega K(\mathbb{F}_p[x]/x^e, (x), \mathbb{Z}_p) \simeq \mathbb{Z}_p$$

For a fixed p, i, e, with $(e, p) = 1$, and any $m \in I_p$ recall that

$$\mathrm{H}^1\left(\mathbb{Z}_p(i)(\mathbb{F}_p[x]/x^e)\right)_m \cong \mathbb{Z}/p^{((s+1))}$$

In order to describe a generator of $\mathrm{H}^1\mathbb{Z}_p(i)(R)$ consider the diagram from [23] illustrating the map

$$\bigoplus_{a\geq 0} \mathrm{H}^1\left(\mathcal{N}^{\geq i}\mathrm{LW}\Omega_R\right)_{(p^a m)} \xrightarrow{\phi/p^i - \mathrm{can}} \bigoplus_{a\geq 0} \mathrm{H}^1\left(\mathrm{LW}\Omega_R\right)_{(p^a m)}.$$

$$
\begin{array}{ccc}
\mathcal{N}^{\geq i}\mathrm{H}^1(\mathrm{LW}\Omega_{R_e})_m & & \mathrm{H}^1(\mathrm{LW}\Omega_{R_e})_m \\
\mathbb{Z}_p/p\left\langle p^{((i-\lceil m/e\rceil))}\frac{x^m}{\Gamma\lceil m/e\rceil}\right\rangle & \dashrightarrow & 0 \\
 & \searrow & \\
\mathbb{Z}_p/p^2\left\langle p^{((i-\lceil pm/e\rceil))}\frac{x^{pm}}{\Gamma\lceil pm/e\rceil}\right\rangle & \dashrightarrow & \mathbb{Z}_p/p\left\langle \frac{x^{pm}}{\Gamma\lceil pm/e\rceil}\right\rangle \\
 & \searrow & \\
\vdots & \dashrightarrow & \vdots \\
 & \searrow & \\
\mathbb{Z}_p/p^s\left\langle p^{((i-\lceil p^{s-1}m/e\rceil))}\frac{x^{p^{s-1}m}}{\Gamma\lceil p^{s-1}m/e\rceil}\right\rangle & \dashrightarrow & \mathbb{Z}_p/p^{s-1}\left\langle \frac{x^{p^{s-1}m}}{\Gamma\lceil p^{s-1}m/e\rceil}\right\rangle \\
 & \searrow & \\
\mathbb{Z}_p/p^{s+1}\left\langle p^{((i-\lceil p^{s}m/e\rceil))}\frac{x^{p^{s}m}}{\Gamma\lceil p^{s}m/e\rceil}\right\rangle & \dashrightarrow & \mathbb{Z}_p/p^{s}\left\langle \frac{x^{p^{s}m}}{\Gamma\lceil p^{s}m/e\rceil}\right\rangle \\
 & \searrow & \\
\mathbb{Z}_p/p^{s+1}\left\langle p^{((i-\lceil p^{s+1}m/e\rceil))}\frac{x^{p^{s+1}m}}{\Gamma\lceil p^{s+1}m/e\rceil}\right\rangle & \longrightarrow & \mathbb{Z}_p/p^{s+1}\left\langle \frac{x^{p^{s+1}m}}{\Gamma\lceil p^{s+1}m/e\rceil}\right\rangle
\end{array}
$$

Where we use the same notation as [23] and have taken the quotient by summands with $a \geq s+1$.

To build a generator of the kernel, start by taking a generator in degree $s+1$, in this degree $can - \phi/p^i = can$ is an isomorphism, so this element hits the generator of $\mathrm{H}^1(\mathrm{LW}\Omega)_{p^{s+1}m}$, we must subtract off the preimage of this generator under ϕ/p^i coming from the degree s term, which is the generator in degree s as ϕ/p^i is an isomorphism in degree s. In degree s the canonical map is no longer an isomorphism, so $can(gen_s)$ lifts to $p^{((i-\lceil p^s m/e\rceil))}gen_{s-1}$. Repeating this process we find that a generator of the kernel is given by the element

$$(gen_{s+1}, -gen_s,\ p^{((i-\lceil p^s m/e\rceil))}gen_{s-1}, -p^{((i-\lceil p^s m/e\rceil))+((i-\lceil p^{s-1}m/e\rceil))}gen_{s-2}, ...)$$

In particular, these elements get highly p-divisible in low degrees, so for degrees much lower than $s(e)$ they are 0. This will let us show that the transition map $tr_{fe} = 0$ for large values of f, since $s(f) \to \infty$ as $f \to \infty$.

The transition map $tr_{fe} : \mathrm{H}^1\mathbb{Z}_p(i)(\mathbb{F}_p[x]/x^f)_m \to \mathrm{H}^1\mathbb{Z}_p(i)(\mathbb{F}_p[x]/x^e)_m$ is a graded map, and in degree k is given by multiplication by

$$\frac{\Gamma\lceil p^k m/e\rceil}{\Gamma\lceil p^k m/f\rceil}p^{((i-\lceil p^k m/f\rceil))-((i-\lceil p^k m/e\rceil))} \tag{5.1}$$

Proposition 1. *For any e, for all f large enough $tr_{fe} = 0$. So in particular $\mathrm{TR}(\mathbb{F}_p) \simeq \mathbb{Z}_p$.*

Proof. Since $\lim_{f\to\infty} s(f) = \lim_f \lfloor \log_p \frac{fi}{m} \rfloor = \infty$, choose f large enough such that $s(f,m) > 2s(e,m)$ for all m – this will make the terms in degree $\leq s(e,m)$ of $\mathrm{H}^1\mathbb{Z}_p(\mathbb{F}_p[x]/x^f)$ at least p^{s+1}-divisible, and so the map is 0. □

5.3.1. Evenness of $\mathrm{TR}(k[y_1^{1/p^\infty}, \ldots, y_t^{1/p^\infty}]/(y_1, \ldots, y_t)$

Now we would like to compute $\mathrm{TR}\left(k[y_1^{1/p^\infty}, \ldots, y_t^{1/p^\infty}]/(y_1, \ldots, y_t\right)$ by using the identifications

$$\mathrm{TR}\left(k[y_1^{1/p^\infty}, \ldots, y_t^{1/p^\infty}]/(y_1, \ldots, y_t)\right) \simeq \varprojlim_e \Omega K\left(k[y_1^{1/p^\infty}, \ldots, y_t^{1/p^\infty}, x]/(y_1, \ldots, y_t, x^e), (x), \mathbb{Z}_p\right),$$

and

$$\pi_{2i-1}K\left(k[y_1^{1/p^\infty},\dots,y_t^{1/p^\infty},x]/(y_1,\dots,y_t,x^e),(x),\mathbb{Z}_p\right)\cong \mathrm{H}^1\mathbb{Z}_p(i)\left(k[y_1^{1/p^\infty},\dots,y_t^{1/p^\infty},x]/(y_1,\dots,y_t,x^e\right).$$

This case is slightly more difficult than that of $\mathrm{TR}(\mathbb{F}_p)$, because it is no longer the case that $s(e)\to\infty$ as $e\to\infty$, meaning the transition maps are non-trivial in some degrees.

Lemma 3. *The limit diagram*

$$\varprojlim_e \mathrm{H}^1\mathbb{Z}_p(i)(k[y_1^{1/p^\infty},\dots,y_t^{1/p^\infty},x]/(y_1,\dots,y_t,x^e)),$$

for $t\in\mathbb{N}\cup\{\infty\}$ *is Mittag-Leffler.*

Proof. Fix an $e\in\mathbb{N}$, we need to show that for some large enough value of f, we have $\mathrm{im}(tr_{fe})=\mathrm{im}(tr_{f'e})$ for all $f'\geq f$, i.e. the image of the transition maps stabilizes. For fixed e, there are only finitely many values of m such that

$$\mathrm{H}^1\mathbb{Z}_p(i)(k[y_1^{1/p^\infty},\dots,y_t^{1/p^\infty},x]/(y_1,\dots,y_t,x^e))_{(m,n)}\neq 0.$$

This lets us reduce to the case where there is only a single value of m.

As before there are two things contributing to the map tr_{fe}, these are the coefficients which in degree (p^km,p^kn) are given by

$$\frac{\Gamma\lceil p^km/e\rceil}{\Gamma\lceil p^km/f\rceil}p^{((i-\lceil p^km/f\rceil-\lfloor p^kn\rfloor))-((i-\lceil p^km/e\rceil-\lfloor p^kn\rfloor))},$$

since there are only finitely many values of m, each with only finitely many values of $k\leq s(e,m)$, these coefficients eventually stabilize in all degrees for large values of f.

Choose an f'' large enough for these coefficients to be stable for all $f \geq f''$.

The other way for the image to change as f grows is for $s(f,m,n)$ to change, since if $s(f,m,n) \neq s(f-1,m,n)$ then in degree (m,n) the transition map tr_{fe} is more p-divisible than $tr_{(f-1)e}$. The transition map becoming more p-divisible is of course not an issue if $tr_{(f-1)e} = 0$, so we must only show that f can be chosen large enough to stabilize the transition maps in degrees (m,n) that do not eventually become 0, and simultaneously in degrees (m,n) where the maps do eventually become 0, f is large enough for $tr_{fe} = 0$.

Recall that $s(f,m,n)$ is the minimal value of a such that

$$\left\lceil \frac{p^a m}{f} \right\rceil + \lfloor p^a n_1 \rfloor + \ldots + \lfloor p^a n_t \rfloor \geq i.$$

So s is a step function that is decreasing in m and each n, and increasing in f.

Let $s' = s(e,m,0)$, i.e. s' is the minimal value of a such that $\lceil \frac{p^a m}{e} \rceil \geq i$. This is the maximal possible value of s at stage e since for any n, $s(e,m,n) \leq s(e,m,0)$. Now choose f' large enough such that

$$\left\lceil \frac{p^{2s'} m}{f'} \right\rceil = 1,$$

this ensures that for values of $f \geq f'$,

$$\left\lceil \frac{p^k m}{f} \right\rceil + \lfloor p^k n_1 \rfloor + \ldots + \lfloor p^k n_t \rfloor = 1 + \lfloor p^k n_1 \rfloor + \ldots + \lfloor p^k n_t \rfloor$$

no longer depends on f for values of $k \leq 2s'$, and if $k > 2s'$ then the transition map is 0 anyways.

Now for all $f \geq \max(f', f'')$ the transition maps are stable, so the limit diagram is Mittag-Leffler

$\square$

So in particular

Theorem 4. *For k a perfect $\mathbb{F}_p$-algebra,* $\mathrm{TR}\left(k[y_1^{1/p^\infty}, \ldots, y_t^{1/p^\infty}]/(y_1, \ldots, y_t\right)$ *is even.*

Proof. Using the above identifications we find that

$$\pi_{2(i-1)}\mathrm{TR}\left(k[y_1^{1/p^\infty}, \ldots, y_t^{1/p^\infty}]/(y_1, \ldots, y_t)\right) \simeq \varprojlim_e \mathrm{H}^1\mathbb{Z}_p(i)\left(k[y_1^{1/p^\infty}, \ldots, y_t^{1/p^\infty}, x]/(y_1, \ldots, y_t, x^e\right).$$

Only $\varprojlim^1$ terms can contribute to odd degrees in TR, but this diagram is Mittag-Leffler so there can be no $\lim^1$ terms. $\square$

5.3.2. Local Evenness of TR

Theorem 5. *For S a quasi-syntomic algebra over a perfect $\mathbb{F}_p$ algebra k,* $\mathrm{TR}(S)$ *is locally even in the quasisyntomic topology.*

Proof. Given a presentation of S,

$$k[x_1, ..., x_n] \to S,$$

with n possibly ∞, we obtain a quasisyntomic cover $S \to S'$ from the following pushout square

$$\begin{array}{ccc} k[x_1, ..., x_n] & \longrightarrow & S \\ \downarrow & & \downarrow \\ k[x_1^{1/p^\infty}, ..., x_n^{1/p^\infty}] & \longrightarrow & S', \end{array}$$

where S' is of the form $k[x_1^{1/p^\infty}, ..., x_n^{1/p^\infty}]/(f_1, ..., f_m)$, and $(f_1, ..., f_m)$ is a regular sequence. Now letting $R = k[x_1^{1/p^\infty}, ..., x_n^{1/p^\infty}]$ and defining $y_i \mapsto f_i$ gives a relatively perfect map

$$S'' := R[y_1^{1/p^\infty}, ..., y_m^{1/p^\infty}]/(y_1, ..., y_m) \to S'.$$

Since $S'' \to S'$ is relatively perfect, $L_{S'/S''}$ vanishes after p-completion, and $L_{S''/\mathbb{Z}_p} \otimes S' \simeq L_{S'/\mathbb{Z}_p}$. Using the Nygaard filtration further gives that $\Delta_{S''} \to \Delta_{S'}$ is surjective (since S' and S'' are quasiregular semiperfect, their prismatic cohomology is concentrated in degree 0).

Now using the filtration on TR from [21, p. 6.2], evenness for S' will follow from showing that the equalizer diagram,

$$\text{Eq}\left(\prod (\mathcal{N}^{\geq i}\Delta_{S'}/\mathcal{N}^{\geq i+1}\Delta_{S'})/p^n \rightrightarrows \prod \Delta_{S'}/p^n\right),$$

is surjective which follows from evenness of S'' as

$$\left(\mathcal{N}^{\geq i}\Delta_{S''}/\mathcal{N}^{\geq i+1}\Delta_{S''}\right) \otimes_{S''} S' \simeq \mathcal{N}^{\geq i}\Delta_{S'}/\mathcal{N}^{\geq i+1}\Delta_{S'},$$

and surjectivity of $\Delta_{S''} \to \Delta_{S'}$, so surjectivity in the equalizer diagram for S'' implies surjectivity of the equalizer for S'.

□

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