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Scaling Asymptotics of Szego Kernels and Concentration of Husimi Distributions on Grauert Tubes

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ABSTRACT

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We study the complexification of Laplace Eigenfunctions on the Grauert tube of a compact real analytic manifold. Our main results concern scaling asymptotics of Fourier coefficients of the Szego kernel on the Grauert tube boundary in a Heisenberg frequency scaled neighborhood of the geodesic flow. We show that in the high frequency limit the Fourier Coefficients are to first order the kernel associated to the quantization of the derivative of the geodesic flow. As an application we provide sharp $L^p \to L^q$ mapping estimates for partial Szego kernels on the Grauert Tube boundary and describe their relationship to Husimi distributions of Laplace eigenfunctions.

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CHAPTER 1

Introduction

Husimi Distributions are quasiprobability distributions in complex phase space associated to the wavefunction of a particle. They play the role of quantum states in the phase space formulation of quantum mechanics. In such a framework one works on phase space directly rather than using the Fourier Transform to go between position and momentum space. This picture allows one to make more direct use of intuition from classical mechanics in order to study the concentration and evolution of Husimi Distributions. This thesis is concerned with the phase space formulation on curved space and collects several results related to the high frequency asymptotics of Husimi Distributions associated to Laplace Eigenfunctions on a compact Riemannian Manifold. We use this chapter to briefly describe the phase space formulation of quantum mechanics on $T^*\mathbb{R}^n$ and to motivate and introduce the main results of this thesis

1.1. Quantum Mechanics

In the Hamiltonian formulation of mechanics the motion of a particle in the phase space $\mathbb{R}^{2n} \simeq T^*\mathbb{R}^n$ is governed by the classical Hamiltonian $H(q,p) \in C^\infty(\mathbb{R}^{2n})$ or the total energy. For simplicity we assume that H takes the form as the sum of a kinetic term and a smooth potential term

$$H(q,p) = |p|^2 + V(q)$$

The trajectories of a particle with starting position and momentum (q_0, p_0) is determined as a solution to the system of differential equations

$$\begin{cases} q'(t) &= \frac{\partial H}{\partial p}\left(q(t), p(t)\right) \\ p'(t) &= -\frac{\partial H}{\partial q}\left(q(t), p(t)\right) \end{cases} \qquad q(0) = q_0, p(0) = p_0$$

In this framework the time evolution of a state can be determined precisely from the Once a starting position and momenta is specified, the location of the object is known for any time in the future and may be measured precisely. Such theory plays well to human intuition and is successful in describing the motion of macroscopic objects which do not travel at speeds approaching that of light. However, this theory makes predictions at odds with experimental evidence for the behavior of subatomic particles. The most successful mathematical framework for studying the motion of subatomic particles is *quantum mechanics*. Instead of a state being represented by a point $(q, p) \in \mathbb{R}^{2n}$ it is represented by a square integrable function $\varphi(x) \in L^2(\mathbb{R}^n)$ of unit norm called a *wavefunction*. To measure the energy of a state one replaces the function H with it's quantization $\widehat{H}_h$ [1]

$$\widehat{H}_h = -h^2\Delta + V(q)$$

where $V(q)$ denotes multiplication by $V(q)$. We focus on the wavefunctions which solve the eigenequation of $\widehat{H}_h$

[1] Here h is Planck's constant. It relates the frequency and momentum by $f = \frac{p}{h}$

$$\widehat{H}_h\varphi(x) = E(h)\varphi(x), \quad \|\varphi\|_{L^2(\mathbb{R}^n)} = 1$$

$\varphi(x)$ is the position representation of the wavefunction associated to a particle of energy level E for the quantum system determined by $\widehat{H}_h$. The semiclassical Fourier transform is used to go between the position and the momentum representation of φ

$$\widehat{\varphi}(p) = \frac{1}{(2\pi h)^n}\int_{\mathbb{R}^n} e^{\frac{i}{h}\langle q,p\rangle}dq$$

Since $\|\varphi\|_{L^2(\mathbb{R}^n)} = 1 = \|\widehat{\varphi}\|_{L^2(\mathbb{R}^n)}$, the squared moduli $|\varphi(q)|^2, |\widehat{\varphi}(p)|^2$ determine probability densities on position and momentum space. Given measurable sets $A, B \subseteq \mathbb{R}^n$ the probabilities of the particle having position in A (resp. having momentum in B) are given by

$$\int_A |\varphi(q)|^2 dq, \quad \int_B |\widehat{\varphi}(p)|^2 dp$$

This prompts the question of whether there exists a phase space formulation of quantum mechanics in which we can work with one wavefunction defined on phase space. One would like to assign to φ a corresponding function $f_\varphi \in L^2(\mathbb{R}^{2n})$ to serve as the joint probability density of $|\varphi(q)|^2, |\widehat{\varphi}(p)|^2$. However, the existence of such a distribution is precluded by the uncertainty principle. Typically[2], such a distribution cannot simultaneously be non-negative and have $|\varphi|^2, |\widehat{\varphi}|^2$ as its marginal densities. Nevertheless, there are suitable candidates for *quasiprobability* distributions on phase space for which one of the

[2]Specifically, a joint probability distribution whose marginals are not dependent. The only time such a distribution exists is in the case of minimal uncertainty states such as the Gaussian.

two conditions hold. We focus on the *Husimi* quasiprobability distributions. They are non-negative probability distributions on the complex phase space $\mathbb{C}^n$ but do not have position and momentum as its marginal densities . For a detailed discussion we point the reader to [**37**].

1.2. Coherent States and the Bargmann Fock space

Given a point $z = (q, p) \in \mathbb{R}^{2n}$ we define the coherent space centered at z to be

$$\phi_z(x) = \hbar^{-\frac{n}{4}} e^{\frac{i}{\hbar} p\cdot(x-q)} e^{-\frac{|x-q|^2}{2\hbar}}$$

The ϕ_z are translations in phase space of the minimal uncertainty state which altogether form an overcomplete system on $L^2(\mathbb{R}^n)$. More precisely a function $f \in L^2(\mathbb{R}^n)$ may be expanded in coherent states by

$$f(x) = \int_{\mathbb{R}^{2n}} \langle f, \phi_z \rangle \phi_z(x) dz$$

In the following we identify $\mathbb{R}^{2n}$ with $\mathbb{C}^n$ by the map

$$(q, p) \to \zeta = \frac{q - ip}{\sqrt{2}}$$

We define the Bargmann-Fock space by

$$\mathcal{H}(\mathbb{C}^n) = \{ f(\zeta) e^{-\frac{|\zeta|^2}{2\hbar}} \in L^2(\mathbb{C}^m) \,\text{s.t.}\; f \text{ is holomorphic} \}$$

This is a Hilbert space with inner product given by

$$\langle f_1, f_2\rangle_{\mathcal{H}} = \int_{\mathbb{C}^n} f_1(\zeta)\overline{f_2\left(\zeta\right)}e^{-\frac{|\zeta|^2}{\hbar}}d\zeta$$

For $f \in L^2\left(\mathbb{R}^n\right)$ we define the *Bargmann-Fock Transform*

$$BF(f)(\zeta) = (2\pi\hbar)^{-\frac{n}{2}}\, e^{\frac{|\zeta|^2}{2\hbar}} \int_{\mathbb{R}^n} f(x)\phi_\zeta(x)dx$$

The Bargmann-Fock transform is an isometry from $L^2\left(\mathbb{R}^n\right)$ onto $\mathcal{H}\left(\mathbb{C}^n\right)$. If f is a wave-function, then $|BF(f)|^2e^{-\frac{|\zeta|^2}{2\hbar}}$ determines a probability distribution on $\mathbb{C}^n$ which we call the *Husimi Distirbution* associated to f. The orthogonal projection $\Pi : L^2\left(\mathbb{C}^n\right) \to \mathcal{H}\left(\mathbb{C}^n\right)$ is known as the *Bergman Projection*. It's integral kernel is the *Bergman Kernel* which can be written as

$$\Pi_h(\zeta, \eta) = (2\pi\hbar)^{-n}\, e^{\frac{-|\zeta|^2-|\eta|^2+2\zeta\cdot\overline{\eta}}{2\hbar}}$$

The Bergman Projection is used to quantize functions and symplectic maps in a natural way. Given a function $a(z) \in L^\infty(\mathbb{C}^n)$ its quantization is given by the operator

$$T_a : \mathcal{H}\left(\mathbb{C}^n\right) \to \mathcal{H}\left(\mathbb{C}^n\right)$$

$$T_a(f) = \Pi M_a \Pi f$$

where M_a is the operator given by multiplication by a. We say that T_a is a *Toeplitz Operator* with symbol a. Finally quantization of symplectic mappings have a nice form in the phase space picture. Let F^t be a time-dependent family of symplectomorphisms given by a Hamiltonian flow and $(F^t)^*$ denote their pullbacks. We define the operators

$$V_{F^t} : \mathcal{H}(\mathbb{C}^n) \to \mathcal{H}(\mathbb{C}^n)$$

$$V_{F^t}(f) = \sigma_t \Pi (F^t)^* \Pi f$$

where σ_t is a family of Toeplitz operators chosen to make the family V_{F^t} unitary. We note that in the phase space framework the time evolution is constructed directly from the classical dynamics F^t making clear the relationship between classical and quantum dynamics.

1.3. Outline of this thesis

In chapter 2 we introduce and give context for the main theorems Theorem 2.2.1, Theorem 2.2.9, Theorem 2.2.3 that make up the papers [**9**], [**10**] which were written in collaboration with Robert Chang. These results form a study of the asymptotic behavior of the quantum evolution operator in a neighborhood of the geodesic flow and their applications to studying the concentration of Husimi Distributions. In chapter 3 we give a crash course on the theory of Grauert Tubes. Roughly speaking, the Grauert Tube of a Riemannian manifold is a complex manifold isomorphic to an open ball of the zero section in the Cotangent bundle. It is a way of giving T^*M a complex structure and plays the role of $\mathbb{C}^n$. Similar to the Bargmann-Fock transform one uses the Poisson transform to analytically continue functions to the Grauert Tube. The analytic continuations of Laplace eigenfunctions are the analogue of the Husimi Distributions. The image of the Poisson Transform restricted to a Grauert Tube boundary plays the role of Bargmann Fock space. This space comes equipped with a projection kernel onto this image. This allows us to define Toeplitz operators as well as the time evolution operator arising from the geodesic flow. The proofs of the main results are given in Chapters 4 and 5. We note that due to lack of orthogonality we often don't work with the Husimi Distributions directly. Instead we work with an orthonormal basis of eigenfunctions associated to a closely related operator which has the same principal symbol as $\sqrt{\Delta}$ under the Grauert Tube identification. Finally, in chapter 6 we show how Husimi Distributions are approximate eigenfunctions and show that the two families coincide in the model cases of the flat torus and the round sphere.

CHAPTER 2

Discussion of the main results

This chapter gives a more detailed description of the results of this thesis. We encourage the reader who is unfamiliar with analysis on Grauert Tubes and Heisenberg normal coordinates to consult Section 3.1.1 and the references therein in tandem with this section to clarify any unfamiliar concepts and terminology.

2.1. Results on Near Diagonal Scaling Asymptotics of Szego Kernels

Throughout this chapter we work on (M, g) a closed, real analytic manifold of dimension $m \geq 2$. Its Grauert tube M_τ is a Kähler manifold with boundary that is diffeomorphic to the co-ball bundle consisting of co-vectors of length at most τ. The *Szegő projector* Π_τ associated to the boundary of a Grauert tube is the orthogonal projection

$$\Pi_\tau \colon L^2(\partial M_\tau) \to H^2(\partial M_\tau) \tag{2.1}$$

onto the Hardy space of boundary values of holomorphic functions in the tube.

We introduce the Toeplitz operator

$$\Pi_\tau D_{\sqrt{\rho}} \Pi_\tau \colon H^2(\partial M_\tau) \to H^2(\partial M_\tau), \tag{2.2}$$

where $D_{\sqrt{\rho}} = \frac{1}{i} \Xi_{\sqrt{\rho}}$ is a constant multiple of the Hamilton vector field of the Grauert tube function $\sqrt{\rho}$ acting as a differential operator.(2.2) is a positive, self-adjoint, elliptic operator on $H^2(\partial M_\tau)$ in the sense of [**5**], so has a discrete spectrum

$$0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \cdots \to \infty$$

with associated L^2-normalized eigenfunctions

$$\Pi_\tau D_{\sqrt{\rho}} \Pi_\tau e_{\lambda_j} = \lambda_j e_{\lambda_j}, \qquad \|e_{\lambda_j}\|_{L^2(\partial M_\tau)} = 1. \tag{2.3}$$

Fix a positive, even Schwartz function χ whose Fourier transform is compactly supported with $\widehat{\chi}(0) = 1$. We study the smoothed out spectral localizations

$$\Pi_{\chi,\lambda} := \Pi_\tau \chi(\Pi_\tau D_{\sqrt{\rho}} \Pi_\tau - \lambda) = \int_{\mathbb{R}} \widehat{\chi}(t) e^{-it\lambda} \Pi_\tau e^{it\Pi_\tau D_{\sqrt{\rho}} \Pi_\tau} \, dt \tag{2.4}$$

of the Toeplitz operator .

As explained in Section 2.1.1, the operator (2.4) is an analogue of the Fourier components (2.13) of the Szegő kernel in the line bundle setting. We state our scaling asymptotics in Heisenberg coordinates on ∂M_τ in the sense of [**16**, Section 14, 18]; see Definition 3.2.2.

Theorem 2.1.1 (Scaling asymptotics for $\Pi_{\chi,\lambda}$, Chang-Rabinowitz [**9**])**.** *Let* $\Pi_{\chi,\lambda}$ *be the spectral projection* (2.4) *associated to the Grauert tube boundary* ∂M_τ *of a closed, real analytic manifold* M *of dimension* m*. Fix* $p \in \partial M_\tau$*. Then, in Heisenberg coordinates* *centered at $p = 0$, the distribution kernel has the following scaling asymptotics:*

$$\begin{aligned}
&\Pi_{\chi,\lambda}\left(\frac{\theta}{\lambda}, \frac{u}{\sqrt{\lambda}}; \frac{\phi}{\lambda}, \frac{v}{\sqrt{\lambda}}\right) \\
&\quad = \frac{C_{m,M}}{\tau}\left(\frac{\lambda}{\tau}\right)^{m-1} e^{\frac{1}{\tau}\left(\frac{i}{2}(\theta-\phi)-\frac{|u|^2}{2}-\frac{|v|^2}{2}+v\cdot\overline{u}\right)} \\
&\qquad \times \left(1+\sum_{j=1}^{N}\lambda^{-\frac{j}{2}}P_j(p,u,v,\theta,\phi)+\lambda^{-\frac{N+1}{2}}R_N\left(p,\theta,u,\phi,v,\lambda\right)\right),
\end{aligned}$$

where $\|R_N(p,\theta,u,\phi,v,\lambda)\|_{C^j(\{|\theta|+|\phi|+|u|+|v|\leq\rho\})} \leq C_{K,j,\rho}$ *for* $\rho > 0,\ j = 1,2,3,\ldots$ *and* P_j *is a polynomial in* u, v, θ, ϕ.

To give some context to our scaling asymptotics, recall the reduced Heisenberg group $\mathbf{H}^{m-1}_{\text{red}} = \mathbf{H}^{m-1}/\{(0, 2\pi k) : k \in \mathbb{Z}\} = S^1 \times \mathbb{C}^{m-1}$ of degree $m-1$, which is obtained as a discrete quotient of the Heisenberg group $\mathbf{H}^{m-1} = \mathbb{R} \times \mathbb{C}^{m-1}$. The group law on $\mathbf{H}^{m-1}_{\text{red}}$ is given by

$$(e^{it}, \zeta) \cdot (e^{is}, \eta) = \left(e^{i(t+s+\text{Im}(\zeta\cdot\overline{\eta}))}, \zeta + \eta\right).$$

The level one Szegő projector $\Pi^{\mathbf{H}}_1 \colon L^2(\mathbf{H}^{m-1}_{\text{red}}) \to H^2_1$ is the orthogonal projection onto the space H^2_1 of CR holomorphic functions square integrable with respect to $e^{-|z|^2}$. Its Schwartz kernel has an exact formula:

$$\Pi^{\mathbf{H}}_1(\theta, z, \varphi, w) = \frac{1}{\pi^{m-1}} e^{i(\theta-\varphi)} e^{z\cdot\overline{w}-\frac{1}{2}|z|^2-\frac{1}{2}|w|^2}. \tag{2.5}$$

We refer to [**4**] for more details. It is well-known (see (2.17)) that, in correctly chosen coordinates, the Szegő kernel in the line bundle case is to leading order some multiple of

(2.5). Theorem 2.1.1 shows that the same phenomenon holds in the Grauert tube setting:

$$\Pi_{\chi,\lambda}\left(\frac{\theta}{\lambda},\frac{u}{\sqrt{\lambda}};\frac{\phi}{\lambda},\frac{v}{\sqrt{\lambda}}\right)=\frac{C_{m,M}}{\tau}\left(\frac{\lambda}{\tau}\right)^{m-1}\Pi_1^{\mathbf{H}}\left(\frac{\theta}{2\tau},\frac{u}{\sqrt{\tau}},\frac{\phi}{2\tau},\frac{v}{\sqrt{\tau}}\right)$$
$$\times\left(1+\sum_{j=1}^{N}\lambda^{-\frac{j}{2}}P_j(p,u,v,\theta,\phi)+\lambda^{-\frac{N+1}{2}}R_N\left(p,\theta,u,\phi,v,\lambda\right)\right)$$

Another natural operator to study is the spectral projection kernel obtained by analytically continuing Laplace eigenfunctions to the Grauert tube M_τ. Let

$$\Delta=-\frac{1}{\sqrt{|g|}}\frac{\partial}{\partial x_j}\left(\sqrt{|g|}g^{ij}\frac{\partial}{\partial x_k}\right)$$

be the (positive) metric Laplacian. It is well-known that Δ has a discrete spectrum:

$$0=\lambda_0^2<\lambda_1^2\leq\ldots\lambda_j^2\leq\cdots\to\infty$$

with associated L^2-normalized eigenfunctions

$$\Delta\varphi_{\lambda_j}=\lambda_j^2\varphi_{\lambda_j},\quad\|\varphi_{\lambda_j}\|_{L^2(M)}=1. \tag{2.6}$$

Our next result concerns the complex analogue of the partial spectral projection kernel

$$E_\lambda(x,y)=\sum_{\lambda_j\leq\lambda}\varphi_{\lambda_j}(x)\overline{\varphi_{\lambda_j}(y)}. \tag{2.7}$$

The Poisson wave operator is used to analytically continue eigenfunctions. One starts with the half-wave operator $U(t)=e^{it\sqrt{\Delta}}\colon L^2(M)\to L^2(M)$ whose Schwartz kernel can be expressed as a sum of eigenfunctions as well as an oscillatory integral (for instance the

Lax–Hörmander parametrix):

$$U(t,x,y)=\sum_{j=0}^{\infty}e^{it\lambda_j}\varphi_{\lambda_j}(x)\overline{\varphi_{\lambda_j}(y)}=\int_{T_y^*M}A(t,x,y,\xi)e^{it|\xi|_y}e^{i\langle\xi,\exp_y^{-1}(x)\rangle}\,d\xi. \tag{2.8}$$

Analytically continuing the oscillatory integral representation in the time variable $t\mapsto t+i\tau\in\mathbb{C}$ and the spatial variable $x\to z\in\partial M_\tau$ yields the Poisson wave kernel (see Theorem 3.1.4):

$$U(i\tau)\colon L^2(M)\to\mathcal{O}(\partial M_\tau). \tag{2.9}$$

Here, $\mathcal{O}(\partial M_\tau)=\ker(\bar\partial_b)$ denotes the space of CR holomorphic functions on the Grauert tube boundary. Comparing (2.8) with (2.9), we see

$$U(i\tau,z,y)=\sum_{j=0}^{\infty}e^{-\tau\lambda_j}\varphi_{\lambda_j}^{\mathbb{C}}(z)\overline{\varphi_{\lambda_j}(y)},$$

so that the analytic continuation $\varphi_{\lambda_j}^{\mathbb{C}}(z)$ to the Grauert tube boundary of an eigenfunction $\varphi_{\lambda_j}(x)$ is given by

$$\varphi_{\lambda_j}^{\mathbb{C}}=e^{\tau\lambda_j}U(i\tau)\varphi_{\lambda_j}.$$

We define tempered spectral projections P_λ by the eigenfunction partial sums

$$P_\lambda\colon\mathcal{O}(\partial M_\tau)\to\mathcal{O}(\partial M_\tau),\quad P_\lambda(z,w)=\sum_{j:\lambda_j\le\lambda}e^{-2\tau\lambda_j}\varphi_{\lambda_j}^{\mathbb{C}}(z)\overline{\varphi_{\lambda_j}^{\mathbb{C}}(w)}.$$

The prefactor $e^{-2\tau\lambda_j}$ is introduced because of the exponential growth estimate ([**51**, Corollary 3]) on complexified eigenfunctions

$$\lambda_j^{-\frac{m-1}{2}} e^{\tau\lambda_j} \lesssim \sup_{\zeta \in M_\tau} |\varphi_{\lambda_j}^{\mathbb{C}}(z)| \lesssim \lambda_j^{\frac{m-1}{2}} e^{\tau\lambda_j}.$$

As before, fix a positive, even Schwartz function χ whose Fourier transform is compactly supported with $\widehat{\chi}(0) = 1$. The techniques for proving Theorem 2.1.1 are easily adapted to prove scaling asymptotics for the smoothed sum

$$P_{\chi,\lambda}(z,w) = \chi * d_\lambda P_\lambda(z,w) = \sum_{j:\lambda_j \leq \lambda} \chi(\lambda - \lambda_j) e^{-2\tau\lambda_j} \varphi_{\lambda_j}^{\mathbb{C}}(z) \overline{\varphi_{\lambda_j}^{\mathbb{C}}(w)}. \tag{2.10}$$

Theorem 2.1.2 (Scaling asymptotics for P_λ^τ, Chang-Rabinowitz [**9**])**.** *Let $P_{\chi,\lambda}$ be the spectral projection* (2.25) *associated to the Grauert tube boundary ∂M_τ of a closed, real analytic manifold M of dimension m. Fix $p \in \partial M_\tau$. Then, in Heisenberg coordinates centered at $p = 0$, the distribution kernel has the following scaling asymptotics:*

$$\begin{aligned} &P_{\chi,\lambda}\left(\frac{\theta}{\lambda}, \frac{u}{\sqrt{\lambda}}; \frac{\phi}{\lambda}, \frac{v}{\sqrt{\lambda}}\right) \\ &\quad = \frac{C_{m,M}}{\tau^m} \lambda^{\frac{m-1}{2}} \Pi_1^{\mathbf{H}} \left(\frac{\theta}{2\tau}, \frac{u}{\sqrt{\tau}}, \frac{\phi}{2\tau}, \frac{v}{\sqrt{\tau}}\right) \\ &\quad\quad \times \left(1 + \sum_{j=1}^{N} \lambda^{-\frac{j}{2}} P_j(p,u,v,\theta,\phi) + \lambda^{-\frac{N+1}{2}} R_N(p,\theta,u,\phi,v,\lambda)\right) \end{aligned}$$

where $\|R_N(p,\theta,u,\phi,v,\lambda)\|_{C^j(\{|\theta|+|\phi|+|u|+|v| \leq \rho\})} \leq C_{K,j,\rho}$ for $\rho > 0$, $j = 1,2,3,\ldots$ and P_j is a polynomial in u, v, θ, ϕ.

Remark 2.1.3. The leading order asymptotics of Theorem 2.1.1 and Theorem 2.1.2 differ by a factor of $\lambda^{-\frac{m-1}{2}}$ because the analytic extensions $\varphi_{\lambda_j}^{\mathbb{C}}$ are not L^2-normalized on ∂M_τ.

Theorem 2.1.2 is the complex analogue of [**8**, Proposition 10], where it was shown that the rescaled real spectral projection kernel $E_\lambda(x+\frac{u}{\lambda}, y+\frac{v}{\lambda})$ from (2.7) exhibits Bessel-type scaling asymptotics as $\lambda \to \infty$. The scaling we use is the Heisenberg-type scaling with $\sqrt{\lambda}$ rather than λ. The complex geometry of the Grauert tube win over the real Riemannian geometry since as $\tau \to 0$ our scaling asymptotics do not resemble Bessel asymptotics.

2.1.1. Comparison to the line bundle setting

Let $(L, h) \to (M, \omega)$ be a positive Hermitian line bundle over a closed Kähler manifold. The *Bergman projections* are orthogonal projections

$$\Pi_{h^k} \colon L^2(M, L^k) \to H^0(M, L^k) \tag{2.11}$$

from the space of L^2 sections of the kth tensor power of the line bundle onto the space of square integrable holomorphic sections.

The Bergman projections are related to the Szegő projection

$$\Pi_h \colon L^2(\partial D) \to H^2(\partial D), \tag{2.12}$$

where $\partial D = \{\ell \in L^* : \|\ell\|_{h^*} = 1\}$ is the unit co-circle bundle. Indeed, let r_θ denote the circle action on ∂D, then the map

$$H^0(M, L^k) \to \{f \in H^2(\partial D) : f(r_\theta x) = e^{ik\theta} f(x)\}$$

$$s_k(z) \mapsto f(x) = (\ell^{\otimes k}, s_k(z)) \quad \text{where } x = (\ell, z) \in \pi^{-1}(z) \times M$$

defines a unitary equivalence between the space of holomorphic sections of L^k and the subspace of equivariant functions in $H^2(\partial D)$. Since the circle action on ∂D commutes with $\overline{\partial}_b$, we conclude that the kernel of (2.11) are Fourier coefficients of the kernel of (2.12), that is,

$$\Pi_{h^k}(x, y) = \frac{1}{2\pi} \int_0^{2\pi} e^{-ik\theta} \Pi_h(r_\theta x, y)\, d\theta. \tag{2.13}$$

Note that the Fourier decomposition of Π_h coincides with the spectral decomposition of $D_\theta = \frac{1}{i}\frac{\partial}{\partial\theta}$ on ∂D.

In the Grauert tube setting, ∂M_τ plays the role of the strongly pseudoconvex CR hypersurface $\partial D \subseteq L^*$ and the Hamilton vector field $\Xi_{\sqrt{\rho}}$ of the Grauert tube function on ∂M_τ plays the role of the Reeb vector field $\frac{\partial}{\partial\theta}$ on ∂D. With this analogy in mind, the direct analogue of (2.13) in the Grauert tube setting appears to be

$$\int_{\mathbb{R}} \widehat{\chi}(t) e^{-i\lambda t} \Pi_\tau(G_\tau^t(z), w)\, dt = \left(\int_{\mathbb{R}} \widehat{\chi}(t) e^{-i\lambda t} (G_\tau^t)^* \Pi_\tau\, dt \right)(z, w), \tag{2.14}$$

where χ is a suitably chosen smooth function and G_τ^t is the Hamilton flow (3.4) of $\Xi_{\sqrt{\rho}}$ on the Grauert tube.

However, this is not quite correct. Unlike the circle action on ∂D generated by $\frac{\partial}{\partial\theta}$, the geodesic flow on a Grauert tube is never holomorphic, nor does its orbits form a fiber bundle over a quotient space. In particular, G_τ^t need not commute with Π_τ, so (2.14) fails to be CR holomorphic in the z variable. This issue of holomorphy can be fixed: It is shown in [**48**, Proposition 5.3] (see Proposition 3.1.7 for the statement) that there exists a polyhomogeneous pseudodifferential operator $\widehat{\sigma}$ on ∂M_τ so that

$$\Pi_\tau e^{it\Pi_\tau D_{\sqrt{\rho}}\Pi_\tau} \cong \Pi_\tau \widehat{\sigma}(G_\tau^t)^*\Pi_\tau \quad \text{modulo smoothing Toeplitz operators.} \tag{2.15}$$

Evidently, the kernel of (2.15) is CR holomorphic in both variables. Thus, we are led to the CR holomorphic analogue of (2.13) that is

$$\int_{\mathbb{R}} \widehat{\chi}(t)e^{-i\lambda t}\Pi_\tau\widehat{\sigma}(G_\tau^t)^*\Pi_\tau\, dt \cong \int_{\mathbb{R}} \widehat{\chi}(t)e^{-i\lambda t}\Pi_\tau e^{it\Pi_\tau D_{\sqrt{\rho}}\Pi_\tau}\, dt. \tag{2.16}$$

We note that (2.16) is essentially the spectral decomposition of the elliptic Toeplitz operator $\Pi_\tau D_{\sqrt{\rho}}\Pi_\tau$ introduced in (2.2). An interesting problem is to compare the spectrum of the Toeplitz operator with that of $\sqrt{\Delta}$ (note that both operators have the same principal symbol under the identification discussed in Section 3.1.1). We include some observations in Section 6.2 for certain model spaces.

2.1.2. Related results in the line bundle setting

Scaling asymptotics in the line bundle setting have been proved in varying degrees of generality; see [**4, 39, 31, 32, 30, 22**]. The simplest setup is a closed Kähler manifold M polarized by an ample line bundle L. Endow L with a Hermitian metric h so that the curvature two-form Ω is positive, and let $\omega = \frac{1}{2}\Omega$ be the Kähler form on M.

Choose coordinates $z = (z_1, \ldots, z_m)$ in a neighborhood U centered at $p \in M$ so that the Kähler potential is locally of the form $\varphi(z) = |z|^2 + O(|z|^3)$. We identify the Bergman kernel with the Fourier components of the Szegő kernel using (2.12) and write

$$\Pi_{h^k}(x, y) = \Pi_{h^k}(z, \theta, w, \varphi),$$

where $x = (z, \theta)$ are linear coordinates on $T_z M \times S^1$. Then the Bergman kernel scaling asymptotics is of the form

$$\begin{aligned} k^{-m}\Pi_{h^k}\left(\frac{z}{\sqrt{k}}, \frac{\theta}{k}, \frac{w}{\sqrt{k}}, \frac{\varphi}{k}\right) &= \frac{1}{\pi^m} e^{i(\theta-\varphi)+i\Im(z\cdot\bar{w})-\frac{1}{2}|z-w|^2} \\ &\quad \times \left[1 + \sum_{n=1}^{N} k^{-\frac{n}{2}} b_k(z, w) + k^{-\frac{N+1}{2}} R_N(z, w)\right] \end{aligned} \tag{2.17}$$

with

$$\|R_N\|_{C^j(\{|z|+|w|\leq\delta\})} \leq C(N, j, \delta).$$

The remainder estimate implies Gaussian decay of the Bergman kernel in a neighborhood of the diagonal:

$$\left|\Pi_{h^k}(z, \theta, w, \varphi)\right| \leq \frac{k^m}{\pi^m} e^{\frac{-kD(z,w)}{2}} \left(1 + O\left(\frac{1}{k}\right)\right) \quad \text{whenever } d(z, w) \leq \sqrt{\frac{\log k}{k}}.$$

Here, $D(z, w)$ is the Calabi diastasis (3.13), which is bounded above and below by the square $d^2(z, w)$ of the Riemannian distance function.

We take this opportunity to remark that even though useful asymptotics are unknown when $d(z,w) \gg \sqrt{(\log k)/k}$, global upper bounds of the form

$$\left|\Pi_{h^k}(z,\theta,w,\varphi)\right| \leq Ce^{-c\sqrt{k}d(z,w)} \quad \text{for all } z,w \in M$$

have been established in [**11, 15, 29, 1, 33, 53, 12**]; under suitable hypotheses, this Agmon-type estimate can be strengthened considerably. There has also been a lot of recent work [**38, 13, 23, 14**] showing that the Bergman kernel is an *analytic kernel* when the Kähler potential is real analytic. Because Grauert Tubes are analytic, we expect the Szego Kernel to be an analytic kernel and therefore have improved exponentially decaying remainder estimates. However, we only make use of rapid decay afforded by working in the smooth category and leave the study of analyticity to future investigation.

2.2. Further results on scaling asymptotics and applications to mapping norms of spectral projections

In this section we outline our results pertaining to the study of $L^p \to L^q$ mapping norms of the spectral projections $\Pi_{\chi,\lambda}$ (2.4) and $\Pi_{[\lambda,\lambda+1]}$ (2.20) associated to the Szegő projector Π_τ (2.1) on the boundary ∂M_τ a of Grauert tube M_τ. These norm estimates, which are sharp, are stated in Theorem 2.2.3 and Theorem 2.2.5. A key ingredient of the proof is the on-shell off-diagonal scaling asymptotics of $\Pi_{\chi,\lambda}$ on ∂M_τ (Theorem 2.2.1). These scaling asymptotics are an extension of those described in the previous section. The ideas used in their proof and formulation are nearly identical to the near diagonal case. As applications, we deduce sharp L^p estimates for analytic continuations (2.23) of Laplace eigenfunctions (Theorem 2.2.3), as well as for eigenfunctions of the Toeplitz operator

(2.3), whose principal symbol coincides with that of $\sqrt{-\Delta}$ transported to the Grauert tube boundary (Proposition 2.2.6). Unlike the Sogge estimates in the real domain, there is no 'critical exponent' p separating low and high L^p norms.

The classical dynamics associated to $e^{it\Pi_\tau D_{\sqrt{\rho}}\Pi_\tau}$ on ∂M_τ is the Hamilton flow

$$G^t_\tau \colon \partial M_\tau \to \partial M_\tau, \quad G^t_\tau = \exp t\Xi_{\sqrt{\rho}}. \tag{2.18}$$

This flow coincides with the pullback of the Riemannian geodesic flow on $S^*_\tau M$ under the diffeomorphism (3.3). Our scaling asymptotics for (2.4) is stated in Heisenberg coordinates in the sense of [**16**] centered at $p \in \partial M_\tau$ and $G^s_\tau(p) \in \partial M_\tau$. In these coordinates, the derivative of the flow (2.18) takes the form

$$DG^s_\tau : T_p\partial M_\tau \to T_{G^s_\tau(p)}\partial M_\tau, \qquad DG^s_\tau = \begin{pmatrix} 1 & 0 \\ 0 & M_s \end{pmatrix},$$

where M_s is a symplectic matrix on $\mathbb{R}^{2(m-1)}$. Let $\widehat{\Pi}_{\mathcal{H},M_s}$ denote the lift to the reduced Heisenberg group $\mathbf{H}^{m-1}_{\text{red}} = S^1 \times \mathbb{C}^{m-1}$ of the metaplectic representation of M_s acting on the model Bargmann–Fock space $\mathcal{H}(\mathbb{C}^{m-1})$. See Section 3.1.6 for details. The following theorem states that under a parabolic λ-rescaling near p and $G^s_\tau(p)$, the kernel of (2.4) behaves like $\lambda^{m-1}\widehat{\Pi}_{\mathcal{H},M_s}$ to leading order as $\lambda \to \infty$.

Theorem 2.2.1 (On-shell scaling asymptotics for $\Pi_{\chi,\lambda}$, Chang-Rabinowitz [**10**])**.** *Let M be a closed, real analytic Riemannian manifold of dimension $m \geq 2$. Let $\Pi_{\chi,\lambda}$ be as in (2.4) and G^t_τ be as in (2.18). Fix $p \in \partial M_\tau$ and $s \in \operatorname{supp}\widehat{\chi}$. Let (θ, u) and (ϕ, v) be*

Heisenberg coordinates centered at p *and* $G_\tau^s(p)$*, respectively. Then, we have*

$$\begin{aligned}&\Pi_{\chi,\lambda}\left(p+\left(\frac{\theta}{\lambda},\frac{u}{\sqrt{\lambda}}\right);G_\tau^s(p)+\left(\frac{\phi}{\lambda},\frac{v}{\sqrt{\lambda}}\right)\right)\\&\quad=\frac{C_{m,M}}{\tau^m}e^{is\lambda}\lambda^{m-1}\widehat{\Pi}_{\mathcal{H},M_s}\left(\frac{\theta}{2\tau},\frac{u}{\sqrt{\tau}},\frac{\phi}{2\tau},\frac{v}{\sqrt{\tau}}\right)\left[1+\sum_{j=1}^{N}\lambda^{-\frac{j}{2}}P_j(p,s,u,v,\theta,\phi)\right]\\&\quad\quad+\lambda^{-\frac{N+1}{2}}R_N\left(p,s,\theta,u,\phi,v,\lambda\right),\end{aligned}$$

where P_j *is a polynomial in* θ, u, ϕ, v*, the remainder* R_N *satisfies*

$$\|R_N(p,s,\theta,u,\phi,v,\lambda)\|_{C^j(\{|(\theta,u)|+|(\phi,v)|\leq\rho\})}\leq C_{N,j,\rho}\quad \textit{for } \rho>0,\ j=1,2,3,\ldots,$$

and all quantities vary smoothly with p *and* s*.*

Remark 2.2.2. When $s=0$ so that $M_s=I$ is the identity matrix,

$$\widehat{\Pi}_{\mathcal{H},I}(\theta,u;\phi,v)=\frac{1}{\pi^{m-1}}e^{i(\theta-\phi)+u\cdot\bar{v}-\frac{1}{2}|u|^2-\frac{1}{2}|v|^2}$$

coincides with the Szegő kernel of level one on $\mathbf{H}^{m-1}_{\text{red}}$, and we recover near diagonal asymptotics computed in the previous section.

An argument similar to that found in [**40**] allows us to deduce the following sharp $L^p\to L^q$ mapping norm estimate for (2.4).

Theorem 2.2.3 ($L^p\to L^q$ mapping estimate for $\Pi_{\chi,\lambda}$, Chang-Rabinowitz [**10**])**.** *Let* M *be a closed, real analytic Riemannian manifold of dimension* $m\geq 2$*. Let* $\Pi_{\chi,\lambda}$ *be as in*

(2.4). *Then we have the sharp estimate*

$$\|\Pi_{\chi,\lambda} f\|_{L^q(\partial M_\tau)} \leq C_{\partial M_\tau} \lambda^{(m-1)(\frac{1}{p}-\frac{1}{q})} \|f\|_{L^p(\partial M_\tau)} \qquad (2 \leq p, q \leq \infty)\,.$$

An immediate consequence of the eigenfunction expansion

$$\Pi_{\chi,\lambda} = \sum_{j:\lambda_j \leq \lambda} \chi(\lambda - \lambda_j) e_{\lambda_j} \otimes \overline{e_{\lambda_j}}, \tag{2.19}$$

of the spectral projection (2.4) together with Theorem 2.2.3 in the case $p = 2$ is the following.

Corollary 2.2.4 (L^p estimates for eigenfunctions of $\Pi_\tau D_{\sqrt{\rho}} \Pi_\tau$,Chang-Rabinowitz [**10**])**.** *Let M be a closed, real analytic Riemannian manifold of dimension $m \geq 2$. Let e_{λ_j} be L^2-normalized eigenfunctions of $\Pi_\tau D_{\sqrt{\rho}} \Pi_\tau$ as in* (2.3)*. Then we have*

$$\|e_{\lambda_j}\|_{L^q(\partial M_\tau)} \leq C_{\partial M_\tau} \lambda_j^{(m-1)(\frac{1}{2}-\frac{1}{q})} \qquad (2 \leq q \leq \infty)$$

The conclusion of Theorem 2.2.3 also holds for spectral projections onto short spectral intervals

$$\Pi_{[\lambda,\lambda+1]} = \sum_{j:\lambda \leq \lambda_j \leq \lambda+1} e_{\lambda_j} \otimes \overline{e_{\lambda_j}}, \tag{2.20}$$

which we state below in Theorem 2.2.5. This result may be viewed as a Grauert tube analogue, with $\Pi_\tau D_{\sqrt{\rho}} \Pi_\tau$ replacing $\sqrt{-\Delta}$, of Sogge's L^p estimate [**41**] for spectral projections of the Lapalcian.

Theorem 2.2.5 (Short Window Estimates, Chang-Rabinowitz [**10**])**.** *Let M be a closed, real analytic Riemannian manifold of dimension $m \geq 2$. Then we have the sharp estimate*

$$\|\Pi_{[\lambda,\lambda+1]} f\|_{L^q(\partial M_\tau)} \leq C_{\partial M_\tau} \lambda^{(m-1)(\frac{1}{2}-\frac{1}{q})} \|f\|_{L^2(\partial M_\tau)} \qquad (2 \leq q \leq \infty)$$

Our next set of results concern analytic continuations to the Grauert tube boundary of Laplace eigenfunctions on M. To distinguish from the spectra λ_j of $\Pi_\tau D_{\sqrt{\rho}} \Pi_\tau$ let $0 = \mu_0 < \mu_1 \leq \mu_2 \leq \cdots$ be eigenvalues of $\sqrt{-\Delta}$ with associated L^2-normalized eigenfunctions

$$-\Delta \varphi_{\mu_j} = \mu_j^2 \varphi_{\mu_j}, \qquad \|\varphi_{\mu_j}\|_{L^2(M)} = 1. \tag{2.21}$$

The analytic extensions $\varphi_{\mu_j}^{\mathbb{C}}$, which are CR holomorphic functions on ∂M_τ, are defined by

$$\varphi_{\mu_j}^{\mathbb{C}} = e^{\tau \mu_j} U(i\tau) \varphi_{\mu_j}, \tag{2.22}$$

where $U(i\tau) = \exp(-\tau\sqrt{-\Delta})$ is the Poisson operator. The probability amplitudes

$$\tilde{\varphi}_{\mu_j}^{\mathbb{C}} = \frac{\varphi_{\mu_j}^{\mathbb{C}}}{\|\varphi_{\mu_j}^{\mathbb{C}}\|_{L^2(\partial M_\tau)}} \tag{2.23}$$

are *Husimi distributions*, that is, *microlocal lifts* of φ_{μ_j} to phase space $\partial M_\tau \cong S_\tau^* M$. They are "approximate eigenfunctions" of $\Pi_\tau D_{\sqrt{\rho}} \Pi_\tau$ in the following sense.

Proposition 2.2.6. *Let $\{\lambda_j\}$ and $\{\mu_j\}$ be the eigenvalues of $\Pi_\tau D_{\sqrt{\rho}} \Pi_\tau$ and $\sqrt{-\Delta}$, respectively. Let $\Pi_{\chi,\lambda}$ be as in* (2.4) *and $\tilde{\varphi}_{\mu_j}^{\mathbb{C}}$ be as in* (2.23)*. Then, $\mu_j = \lambda_j + O(1)$ as*

$j \to \infty$ and

$$\|\Pi_{\chi,\lambda}\tilde{\varphi}^{\mathbb{C}}_{\mu_j} - \tilde{\varphi}^{\mathbb{C}}_{\mu_j}\|_{L^2(\partial M_\tau)} = O(1),$$

$$\|\Pi_\tau D_{\sqrt{\rho}}\Pi_\tau\tilde{\varphi}^{\mathbb{C}}_{\mu_j} - \lambda_j\tilde{\varphi}^{\mathbb{C}}_{\mu_j}\|_{L^2(\partial M_\tau)} = O(1).$$

Furthermore, we have the sharp estimate

$$\|\tilde{\varphi}^{\mathbb{C}}_{\mu_j}\|_{L^p(\partial M_\tau)} \leq C_{\partial M_\tau}\lambda_j^{(m-1)(\frac{1}{2}-\frac{1}{p})} \qquad (2 \leq p \leq \infty). \tag{2.24}$$

The L^p bound (2.24) may also be deduced from the sup norm bound of Zelditch [**48**] and log-convexity of L^p norms. In Section 5.3, we show that the bound is saturated by complexified Gaussian beams. This result is yet another Grauert tube analogue, with the analytically continued $\tilde{\varphi}^{\mathbb{C}}_{\lambda_j}$ replacing φ_{λ_j}, of Sogge's L^p estimate for eigenfunctions. Note that, unlike in the real domain, there are no separate estimates for "high" versus "low" L^p; see Section 2.2.1.2 for further discussion.

Remark 2.2.7. We note that it is shown in [**26, 51**] that $e^{-\tau\mu_j}\varphi^{\mathbb{C}}_{\mu_j}$ is a Riesz basis (but not an orthonormal basis) in general, so they fail to be reproduced by $P_{\chi,\mu}$. This is why it is preferable to work with e_λ which give access to orthogonality arguments.

Remark 2.2.8. In Section 6.2 we demonstrate that analytically continued eigenfunctions of $\sqrt{\Delta}$ and eigenfunctions of $\Pi_\tau D_{\sqrt{\rho}}\Pi_\tau$ coincide in the model cases ot a flat torus and the round sphere. Furthermore their spectra are asymptotic to one another.

For completeness we also state the more general scaling asymptotics for tempered sums of complexified eigenfunctions.

$$P_{\chi,\mu} = \sum_{j:\mu_j \leq \mu} \chi(\mu - \mu_j) e^{-2\tau\mu_j} \varphi^{\mathbb{C}}_{\mu_j} \otimes \overline{\varphi^{\mathbb{C}}_{\mu_j}}. \tag{2.25}$$

using (2.22).

The proof of Theorem 2.2.1 is easily adapted to prove scaling asymptotics for (2.25). Comparing the statements of Theorem 2.2.1 and Theorem 2.2.9 below, we see the leading order asymptotics of $\Pi_{\chi,\lambda}$ and $P_{\chi,\mu}$ differ only in the powers of the frequency parameters λ or μ.

Theorem 2.2.9 (On-shell asymptotics for $P_{\chi,\mu}$, Chang-Rabinowitz [**10**]). *Let $P_{\chi,\mu}$ be as in* (2.4). *Under the same hypotheses as Theorem 2.2.1, we have*

$$\begin{aligned} &P_{\chi,\mu}\left(p + \left(\frac{\theta}{\mu}, \frac{u}{\sqrt{\mu}}\right); G^s_\tau(p) + \left(\frac{\phi}{\mu}, \frac{v}{\sqrt{\mu}}\right)\right) \\ &\quad = \frac{C_{m,M}}{\tau^m} e^{is\mu} \mu^{\frac{m-1}{2}} \widehat{\Pi}_{\mathcal{H},M_s}\left(\frac{\theta}{2\tau}, \frac{u}{\sqrt{\tau}}, \frac{\phi}{2\tau}, \frac{v}{\sqrt{\tau}}\right)\left[1 + \sum_{j=1}^{N} \mu^{-\frac{j}{2}} P_j(p,s,u,v,\theta,\phi)\right] \\ &\qquad + \mu^{-\frac{N+1}{2}} R_N\left(p,s,\theta,u,\phi,v,\mu\right), \end{aligned}$$

where P_j is a polynomial in θ, u, ϕ, v, the remainder R_N satisfies

$$\|R_N(p,s,\theta,u,\phi,v,\mu)\|_{C^j(\{|(\theta,u)|+|(\phi,v)|\leq\rho\})} \leq C_{N,j,\rho} \quad \text{for } \rho > 0,\ j = 1,2,3,\ldots,$$

and all quantities vary smoothly with p and s.

Remark 2.2.10. Our techniques for proving Theorem 2.2.1 and Theorem 2.2.3 hold in the more general setting of a compact, strictly pseudoconvex CR manifold X for which $\Box_b$ has closed range. In particular, the Boutet de Monvel–Sjöstrand description of the Szegő projector remains valid and quantization of the geodesic flow (2.18) can be replaced by that of the Reeb flow. The main interest in the Grauert tube setting is the manifestation of the underlying Riemannian geometry as well as the connection between analytic extensions and microlocal lifts of eigenfunctions.

2.2.1. Comparison to prior results

Section 2.2.1.1 recalls some results on L^p estimates on Bergman kernels associated to line bundles, as well as asymptotic expansions of quantized Hamiltonian symplectomorphisms. We return to the real domain in Section 2.2.1.2 with a comparison of L^p norms of eigenfunctions on the manifold M versus those of analytically continued eigenfunctions on the tube ∂M_τ.

2.2.1.1. L^p estimates and quantized Hamiltonians on line bundles. In the line bundle setting $(L,h) \to (X,\omega)$, Shiffman–Zelditch [**40**, Lemma 4.1] proved $\|\Pi_{h^k}\|_{L^p \to L^q} \leq Ck^{m(1/p-1/q)}$ by combining the Shur–Young inequality with a near-diagonal Gaussian estimate [**39**, Lemma 5.2]. Consequently, $\|s\|_{L^p} = O(k^{m(1/2-1/p)})$ for all L^2-normalized holomorphic sections $s \in H^0(X, L^k)$.

The proof techniques of our Grauert tube analogue, Theorem 2.2.3, are similar. But, in place of a near-diagonal scaling asymptotics, we need the full strength of Theorem 2.2.1, which gives an asymptotic expansion in a $\lambda^{-\frac{1}{3}}$-neighborhood of the orbit $p \mapsto G_\tau^s(p)$. This finer control of the time evolution under the Reeb flow (i.e., G_τ^t on ∂M_τ or r_θ on the circle bundle) is unnecessary in the line bundle setting because rotations of the fiber introduce only an overall phase factor to the near-diagonal scaling asymptotics.

There has also been prior work on quantized Hamiltonian flows on line bundles. More precisely, let $f \in C^\infty(X)$ be a Hamiltonian on the classical phase space (Kähler manifold) X that induces a 1-parameter group of symplectomorphisms $\varphi_t \colon X \to X$ which lifts to a family of contactomorphisms $\tilde{\varphi}_t \colon \partial D \to \partial D$. As shown by Zelditch [**49**], these contactomorphisms may be quantized as unitary maps

$$\Phi_t \colon L^2(\partial D) \to L^2(\partial D), \quad \Phi_t = R_t \Pi_h (\tilde{\varphi}_{-t})^* \Pi_h,$$

in which R_{-t} is a zeroth order Toeplitz operator chosen to ensure the unitarity of Φ_t. In a series of papers, Paoletti [**34, 35, 36**] computed scaling asymptotics for the Fourier coefficients (with respect to the S^1 action) of Φ_t near points on the graph of $\tilde{\varphi}_t$ for fixed t. When $t = 0$, this specializes to the scaling asymptotics of [**4, 39**]. We emphasize that in contrast, G_τ^t is simultaneously playing the role of the S^1 action and the Hamiltonian flow in our set up. Nevertheless, our main argument borrows heavily from that of [**35**]. We also mention the works of Zelditch–Zhou [**54, 55, 56**], which treat other types of asymptotics for partial Bergman kernels of quantized Hamiltonian flows on line bundles.

2.2.1.2. L^p norms of eigenfunctions in the real domain. In this section we discuss how the L^p estimates of Theorem 2.2.3 and Corollary 2.2.4 compare with those of Sogge eigenfunctions in the real domain. Let $P_{[\lambda,\lambda+1]}$ denote the orthogonal projection onto the span of Laplace eigenfunctions φ_{λ_j} with frequencies $\lambda \leq \lambda_j < \lambda + 1$.

Theorem 2.2.11 (Sogge [**41**], see also [**42, 25**]). *Let (M,g) be a closed Riemannian manifold of dimension n, then the following estimates are sharp.*

$$\|P_{[\lambda,\lambda+1]}f\|_{L^q(M)} \leq \begin{cases} C\lambda^{(\frac{n-1}{2})(\frac{1}{2}-\frac{1}{q})}\|f\|_{L^2(M)} & \text{for } 2 \leq q \leq \frac{2(n+1)}{n-1}, \\ C\lambda^{n(\frac{1}{2}-\frac{1}{q})-\frac{1}{2}}\|f\|_{L^2(M)} & \text{for } \frac{2(n+1)}{n-1} \leq q < \infty. \end{cases}$$

Consequently, for L^2-normalized Laplace eigenfunctions φ_{λ_j} with frequencies λ_j, we have

$$\|\varphi_{\lambda_j}\|_{L^q(M)} \leq \begin{cases} C\lambda_j^{(\frac{n-1}{2})(\frac{1}{2}-\frac{1}{q})} & \text{for } 2 \leq q \leq \frac{2(n+1)}{n-1}, \\ C\lambda_j^{n(\frac{1}{2}-\frac{1}{q})-\frac{1}{2}} & \text{for } \frac{2(n+1)}{n-1} \leq q < \infty. \end{cases}$$

Note the presence of a critical exponent $q_n = \frac{2(n+1)}{n-1}$ at which the sharp estimates change. Roughly speaking, high L^q norms measure concentration around single points, whereas low L^q norms measure concentration around larger sets such as geodesics and hypersurfaces. It is well known on the round sphere S^n that the sequence of zonal spherical harmonics at a pole saturate the estimate for $q > q_n$. On the other hand, the sequence of highest weight spherical harmonics, that is Gaussian beams along a stable elliptic geodesic, saturate the estimate for $q < q_n$. However, these bounds are rarely sharp on other manifolds. For example, on the flat torus all eigenfunctions have L^q norms bounded by $O(1)$. An interesting question in this direction is which manifolds admit sequences of eigenfunctions that saturate these bounds. We point the readers to [**43, 45, 46, 44**] for research in this topic.

Although we project onto the orthonormal basis consisting of eigenfunctions of $\Pi_\tau D_{\sqrt{\rho}}\Pi_\tau$ rather than onto the span $\tilde{\varphi}^{\mathbb{C}}_{\lambda_j}$, thanks to Proposition 2.2.6 we can interpret Theorem 2.2.5

as a complexified version of the theorem above. In the complex setting there is no critical exponent q_n differentiating the behavior between the low and high L^q norms. Indeed, the exponent in our sharp estimate is analogous to that of Sogge's in the low L^q regime, and we show in Section 5.3 that complexified Gaussian beams are extremals for all p.

Our main result has an interpretation as measuring concentration in phase space. As discussed in [**48**], the squares of $\tilde{\varphi}^{\mathbb{C}}_{\lambda_j}$ are microlocal lifts of φ_{λ_j} to $\partial M_\tau \cong S^*_\tau M$, so they may be viewed as probability densities of finding a quantum particle at a phase space point in ∂M_τ. Their marginals are given by the pushforward $\pi_*(\tilde{\varphi}^{\mathbb{C}}_{\lambda_j})^2$ under the natural projection $\pi\colon S^*_\tau M \to M$. It is natural to ask how the marginal densities of these Husimi distributions relate to eigenfunction concentration on M. Other types of phase space norms of eigenfunctions have been studied by Blair–Sogge [**2, 3**]. We also mention the work of Galkowski [**18**], which uses defect measures to study eigenfunction concentration. It would be interesting to compare the results and techniques with those of complexification.

CHAPTER 3

Grauert Tube Preliminaries

3.1. Geometry of and analysis on Grauert tubes

We assume throughout that (M, g) is a closed, real analytic Riemannian manifold of dimension m. Grauert tubes were first studied by [**20, 21, 27, 28**] in relation to the complex Monge–Ampère equation and complexified geodesics. The Poisson wave operator and the tempered spectral projection (3.17), as well as other related operators, have been studied in [**50, 51, 48**].

3.1.1. Grauert tube and the cotangent bundle

Bruhat–Whitney proved that a real analytic manifold M admits a complexification $M_{\mathbb{C}}$ such that $M \subseteq M_{\mathbb{C}}$ is a totally real submanifold, that is, $T_pM \cap JT_pM = \{0\}$ for all $p \in M$.

In a neighborhood of M in $M_{\mathbb{C}}$, there exists a unique strictly plurisubharmonic function $\rho \colon U \subseteq M_{\mathbb{C}} \to [0, \infty)$ whose square root is the solution of the Monge–Ampère equation

$$\det\left(\frac{\partial^2 \sqrt{\rho}}{\partial z_j \partial \overline{z}_k}\right) = 0 \quad \text{on } U \setminus M,$$

with the initial condition that the metric induced by the Kähler form $i\partial\overline{\partial}\rho$ restricts to the Riemannian metric g on M. In fact, $\sqrt{\rho}$ is given by

$$\sqrt{\rho}\colon U \subseteq M_{\mathbb{C}} \to \mathbb{R}, \quad \sqrt{\rho}(z) = \frac{1}{2i}\sqrt{r_{\mathbb{C}}^2(z,\overline{z})}, \tag{3.1}$$

where $r_{\mathbb{C}}^2(z,\overline{w})$ is the analytic extension of the square of the Riemannian distance function $r\colon M \times M \to \mathbb{R}$ to a neighborhood of the diagonal in $M_{\mathbb{C}} \times \overline{M}_{\mathbb{C}}$. We call (3.1) the *Grauert tube function.* For each $0 < \tau < \tau_{\max}$, the sublevel set

$$M_\tau = \{z \in M_{\mathbb{C}} : \sqrt{\rho}(z) < \tau\} \tag{3.2}$$

is called the *Grauert tube of radius* τ.

The complexified exponential map can be used to identify (3.2) with the co-ball bundle of radius τ:

$$B_\tau^* M = \{(x,\xi) : |\xi| < \tau\}.$$

To explain this, we introduce notation for geodesic flows.

- The (geometer's) geodesic flow $g^t\colon T^*M \to T^*M$ is the Hamilton flow of the metric norm squared function $H(x,\xi) = |\xi|_x^2$.
- The homogeneous geodesic flow $G^t\colon T^*M - 0 \to T^*M - 0$ is the Hamilton flow of the metric norm function $\sqrt{H}(x,\xi) = |\xi|_x$. Note that $G^t(x,\varepsilon\xi) = \varepsilon G^t(x,\xi)$.

Let $\pi\colon T^*M \to M$ be the natural projection. The exponential map $\exp\colon T^*M \to M$ is defined by $\exp_x \xi = \pi g^1(x,\xi)$. Analyticity of the metric allows us to complexify $t \mapsto t + i\tau$ in the time variable. For $0 < \tau < \tau_{\max}$, the imaginary time geodesic flow E is a diffeomorphism ([**52**, Lemma 1.1])

$$E\colon B^*_\tau M \to M_\tau, \quad E(x,\xi) = \exp^{\mathbb{C}}_x i\xi. \tag{3.3}$$

With $\sqrt{H}(x,\xi) = |\xi|$ and $\sqrt{\rho}$ the Grauert tube function, we have

$$H = \rho \circ E \quad \text{and} \quad \sqrt{H} = \sqrt{\rho} \circ E.$$

It follows that E^{-1} conjugates the geodesic flow G^t (resp. g^t) on the cotangent bundle to the Hamilton flow of Hamiltonian vector field $\Xi_{\sqrt{\rho}}$ (resp. Ξ_ρ) on the Grauert tube. In particular, the transfer

$$G^t_\tau\colon \partial M_\tau \to \partial M_\tau, \quad G^t_\tau = E \circ G^t \circ E^{-1}|_{\partial M_\tau} \tag{3.4}$$

of the geodesic flow from the energy surface $\partial B^*_\tau M$ to the Grauert tube boundary ∂M_τ coincides with the restriction of the Hamilton flow of $\Xi_{\sqrt{\rho}}$ to ∂M_τ.

Finally, if we write $\alpha_{T^*M} = \xi\, dx$ and $\omega_{T^*M} = d\xi \wedge dx$ for the canonical 1-form and the symplectic form on the cotangent bundle, then we also have

$$(E^{-1})^* \alpha_{T^*M} = \operatorname{Im} \partial\rho = d^c\sqrt{\rho} \quad \text{and} \quad (E^{-1})^*\omega_{T^*M} = -i\partial\bar{\partial}\rho.$$

Note that the right-hand sides of the two equalities are the canonical 1-form and the Kähler form on the Grauert tube M_τ.

Remark 3.1.1. It is useful to think of the Grauert tube as the co-ball bundle endowed with an *adapted complex structure* $J = J_g$ induced from the Riemannian metric on M.

This complex structure is characterized as follows. Let $\gamma\colon \mathbb{R} \to M$ be a geodesic (i.e., a g^t orbit) and let $\gamma_{\mathbb{C}}\colon \{t + i\tau \in \mathbb{C} : \tau < \tau_{\max}\} \to M_{\mathbb{C}}$ be its analytic continuation to a strip. Then J_g is the complex structure on $M_\tau \cong B^*_\tau M$ so that $\gamma_{\mathbb{C}}$ is a holomorphic map; see [**27**].

3.1.2. Contact and CR structure on the boundary of a Grauert tube

The boundary of a Grauert tube, being a level set of the strictly plurisubharmonic function ρ, is strongly pseudoconvex and real analytic. We endow ∂M_τ with the volume form $d\mu_\tau$ obtained by pulling back the standard Liouville form $\alpha \wedge \omega_{T^*M}^{m-1}$ on $\partial B^*_\tau M_\tau$ under the symplectic diffeomorphism (3.3):

$$d\mu_\tau = (E^{-1})^*(\alpha \wedge \omega_{T^*M}^{m-1})\Big|_{\partial M_\tau}. \tag{3.5}$$

Let $J = J_g$ denote the adapted complex structure on a Grauert tube (see Remark 3.1.1). The boundary ∂M_τ, being a real hypersurface of the complex manifold M_τ, carries a CR structure. The tangent space admits the decomposition

$$T\partial M_\tau = H \oplus \mathbb{R}T,$$

where $H = \ker \alpha = JT\partial M_\tau \cap T\partial M_\tau$ is a real, J-invariant hyperplane bundle and $T = \Xi_{\sqrt{\rho}}$, called the *Reeb vector field*, is the Hamilton vector field of the Grauert tube function. Equivalently, $JT = \nabla\sqrt{\rho}$.

Complexifying the tangent space yields

$$T^{\mathbb{C}}\partial M_\tau = H^{(1,0)}\partial M_\tau \oplus H^{(0,1)}\partial M_\tau \oplus \mathbb{C}T$$

where $H^{(1,0)}$ and $H^{(0,1)}$ are the J-holomorphic and J-antiholomorphic subspaces. The boundary Cauchy–Riemann operators are defined by

$$\partial_b f = df|_{H^{(1,0)}} \quad \text{and} \quad \overline{\partial}_b f = df|_{H^{(0,1)}}.$$

3.1.3. The Szegő projector and the Boutet de Movel–Sjöstrand parametrix

The Szegő projector is a complex Fourier integral operator (FIO) with a positive complex canonical relation. We recall the Boutet de Monvel–Sjöstrand parametrix construction for the Szegő kernel in the context of Grauert tubes, but the same construction holds with ∂M_τ replaced by any bounded, strongly pseudoconvex domain.

Definition 3.1.2. The Szegő projector Π_τ is the orthogonal projection

$$\Pi_\tau \colon L^2(\partial M_\tau, d\mu_\tau) \to H^2(\partial M_\tau, d\mu_\tau)$$

onto the Hardy space consisting of boundary values of holomorphic functions in M_τ that are square integrable with respect to the volume form (3.5). Its distributional kernel $\Pi_\tau(z,w)$ is defined by the relation

$$\Pi_\tau f(z) = \int_{\partial M_\tau} \Pi_\tau(z,w) f(w)\, d\mu_\tau(w) \quad \text{for all } f \in L^2(\partial M_\tau). \tag{3.6}$$

The Szegő projector is a Fourier integral operator with a positive complex canonical relation. The real points of this canonical relation is the graph of the identity map on the symplectic cone Σ_τ spanned by the contact form $\alpha := d^c\sqrt{\rho}|_{T*\partial M_\tau}$, i.e.,

$$\Sigma_\tau = \{(\zeta, r\alpha_\zeta) : r \in \mathbb{R}_+\} \subseteq T^*(\partial M_\tau). \tag{3.7}$$

Using the imaginary time geodesic flow E in (3.3), we can construct the symplectic equivalence ι_τ between the cotangent bundle and the symplectic cone:

$$\iota_\tau : T^*M - 0 \to \Sigma_\tau, \quad \iota_\tau(x,\xi) = \left(E\Big(x, \tau\frac{\xi}{|\xi|}\Big), |\xi|\alpha_{E(x,\tau\frac{\xi}{|\xi|})}\right). \tag{3.8}$$

We now briefly describe the symbol of the Szegő projector. Details can be found in [**5**, Theorem 11.2] or [**49**, Section 3]. Let $\Sigma_\tau^\perp \otimes \mathbb{C}$ be the complexified normal bundle of (3.7). The symbol $\sigma(\Pi_\tau)$ of Π_τ is a rank one projection onto a ground state e_{Λ_τ}, which is annihilated by a Lagrangian system of Cauchy–Riemann equations corresponding to a Lagrangian subspace $\Lambda_\tau \subseteq \Sigma_\tau^\perp \otimes \mathbb{C}$. The time evolution $\Pi_\tau \mapsto G_\tau^{-t}\Pi_\tau G_\tau^t$ under the Hamilton flow (3.4) yields another a rank one projection onto some time-dependent ground state $e_{\Lambda_\tau^t}$, where Λ_τ^t is the pushforward of Λ_τ under the flow. The quantity

$$\sigma_{t,\tau,0} = \langle e_{\Lambda_\tau^t}, e_{\Lambda_\tau}\rangle^{-1} \tag{3.9}$$

appears in Proposition 3.1.5 and Proposition 3.1.7. It was show in [**49**] that $\langle e_{\Lambda_\tau^t}, e_{\Lambda_\tau}\rangle$ is the L^2 inner product of two Gaussians, hence nowhere vanishing.

Finally, we discuss an oscillatory integral representation of the Szegő kernel. Recall from Section 3.1.1 that ρ is the real analytic Kähler potential on the Grauert tube. We introduce the defining function for the boundary ∂M_τ of a Grauert tube:

$$\varphi_\tau\colon M_{\tau_{\max}} \to [0,\infty), \quad \varphi_\tau(z) := \rho(z) - \tau^2 \tag{3.10}$$

Let $\varphi_\tau(z,\overline{w})$ be the analytic extension of $\varphi_\tau(z) = \varphi_\tau(z,\overline{z})$ to $M_\tau \times \overline{M}_\tau$ obtained by polarization. Put (c.f. the formula (3.1) for the Grauert tube function)

$$\psi_\tau(z,w) = \frac{1}{i}\varphi_\tau(z,\overline{w}) = \frac{1}{i}\left(-\frac{1}{4}r_{\mathbb{C}}^2(z,\overline{w}) - \tau^2\right). \tag{3.11}$$

By construction, ψ is homolomorphic in z, antiholomorphic in w, and satisfies $\psi(z,w) = -\overline{\psi(z,w)}$.

Theorem 3.1.3 (The Boutet de Monvel–Sjöstrand parametrix, [**7**, Theorem 1.5])**.** *With ψ as in* (3.11)*, there exists a classical symbol*

$$s \in S^{m-1}(\partial M_\tau \times \partial M_\tau \times \mathbb{R}^+) \quad \text{with} \quad s(z,w,\sigma) \sim \sum_{k=0}^{\infty} \sigma^{m-1-k} s_k(z,w)$$

so that the Szegő kernel (3.6) *has the oscillatory integral representation*

$$\Pi_\tau(z,w) = \int_0^\infty e^{i\sigma\psi_\tau(z,w)} s(z,w,\sigma)\, d\sigma \quad \text{modulo a smoothing kernel.} \tag{3.12}$$

We record a key estimate for φ_τ (or equivalently for the phase function ψ). Define the *Calabi diastatis function* by

$$D(z,w) = \varphi_\tau(z,\overline{z}) + \varphi_\tau(w,\overline{w}) - \varphi_\tau(z,\overline{w}) - \varphi_\tau(w,\overline{z}). \tag{3.13}$$

On ∂M_τ, the diastatis simplifies to

$$D(z,w) = -\varphi_\tau(z,\overline{w}) - \varphi_\tau(w,\overline{z}) = -2\,\mathrm{Re}\,\varphi_\tau(z,\overline{w}) \quad \text{for } z,w \in \partial M_\tau.$$

In the closure of the Grauert tube, [**7**, Corollary 1.3] gives the lower bound

$$D(z,w) \geq C\big(d(z,\partial M_\tau) + d(w,\partial M_\tau) + d(z,w)^2\big) \quad \text{for } z,w \in \overline{M_\tau}. \tag{3.14}$$

3.1.4. Poisson wave operator and tempered spectral projection

Recall the eigenequation (2.21) for the Laplacian on M. The eigenfunction expansion of the Schwartz kernel of the half-wave operator $U(t) := e^{it\sqrt{\Delta}}$ is given by

$$U(t,x,y) = \sum_j e^{it\lambda_j}\varphi_{\lambda_j}(x)\overline{\varphi_{\lambda_j}(y)}. \tag{3.15}$$

It is well-known (see for instance [**21**, Theorem]) that for $0 \leq \tau \leq \tau_{\max}$, the Schwartz kernel $U(t,x,y)$ admits an analytic extension $U(t+i\tau,x,y)$ in the time variable $t \mapsto t+i\tau \in \mathbb{C}$. Note that the corresponding operator $U(i\tau) = e^{-\tau\sqrt{\Delta}}$ is the Poisson operator. Moreover, for y and τ fixed, the Poisson kernel can be further analytically extended in the spacial variable $x \mapsto z \in M_\tau$. The resulting operator is a Fourier integral operator of complex type. More precisely, denote by $\mathcal{O}(\partial M_\tau)$ the space of CR holomorphic functions on the Grauert tube boundary.

Theorem 3.1.4 ([**6, 21, 19, 51**]). *For $0 < \tau < \tau_{\max}$, the Poisson operator*

$$U(i\tau) = e^{-\tau\sqrt{\Delta}} \colon L^2(M) \to \mathcal{O}(\partial M_\tau)$$

whose Schwartz kernel $U(i\tau, z, y)$ is obtained from analytically continuing the half-wave kernel $U(t, x, y)$ of (3.15) *is a Fourier integral operator of order $-(m-1)/4$ with complex phase associated to the canonical relation*

$$\{(y, \eta, \iota_\tau(y, \eta)\} \subseteq T^*M \times \Sigma_\tau,$$

where ι_τ and Σ_τ are defined in (3.8) *and* (3.7)*.*

To introduce the complexified spectral projection kernels, we need to further continue $U(i\tau, z, y)$ anti-holomorphically in the y variable. Consider the operator

$$U^{\mathbb{C}}(t + 2i\tau) = U(i\tau)U(t)U(i\tau)^*: \mathcal{O}(\partial M_\tau) \to \mathcal{O}(\partial M_\tau)$$

with Schwartz kernel

$$U^{\mathbb{C}}(t + 2i\tau, z, w) = \sum_{\lambda_j} e^{i(t+2i\tau)j} \varphi^{\mathbb{C}}_{\lambda_j}(z) \overline{\varphi^{\mathbb{C}}_{\lambda_j}(w)}.$$

Proposition 3.1.5 ([**48**, Proposition 7.1])**.** *Let $(G^t_\tau)^*$ denote the pullback by the Hamilton flow* (3.4) *of $\Xi_{\sqrt{\rho}}$ on ∂M_τ. There exists a classical polyhomogeneous pseudodifferential operator $\widehat{\sigma}_{t,\tau}(w, D_{\sqrt{\rho}})$ on ∂M_τ so that*

$$U^{\mathbb{C}}(t + 2i\tau) \cong \Pi_\tau \widehat{\sigma}_{t,\tau} \left(G^t_\tau\right)^* \Pi_\tau \quad \textit{modulo a smoothing Toeplitz operator.}$$

The symbol $\sigma_{t,\tau}$ of $\widehat{\sigma}_{t,\tau}$ admits a complete asymptotic expansion

$$\sigma_{t,\tau}(w, r) \sim \sum_{j=0}^{\infty} \sigma_{t,\tau,j}(w)\, r^{-\frac{m-1}{2} - j}, \tag{3.16}$$

in which $\sigma_{t,\tau,0} = \langle e_{\Lambda_\tau^t}, e_{\Lambda_\tau}\rangle^{-1}$ *is the reciprocal of the overlap of two Gaussians, as in* (3.9).

Finally, we define the tempered spectral projection kernel by

$$P_\lambda(z,w) = \sum_{j:\lambda_j\leq\lambda} e^{-2\tau\lambda_j}\varphi_{\lambda_j}^{\mathbb{C}}(z)\overline{\varphi_{\lambda_j}^{\mathbb{C}}(w)}. \tag{3.17}$$

Fix $\varepsilon > 0$ and let χ be a positive even Schwartz function such that $\widehat{\chi}(0) = 1$ and $\operatorname{supp}\widehat{\chi} \subseteq [-\varepsilon, \varepsilon]$. Then

$$P_{\chi,\lambda}(z,w) = \chi * d_\lambda P_\lambda(z,w) = \int_{\mathbb{R}} \widehat{\chi}(t) e^{-it\lambda} U^{\mathbb{C}}(t+2i\tau, z, w)\, dt \tag{3.18}$$

Remark 3.1.6. It was shown in [**51**] that the complexified spectral function

$$\sum_{\lambda_j\leq\lambda} |\varphi_{\lambda_j}^{\mathbb{C}}(\zeta)|^2 \qquad (\zeta \in M_\tau)$$

grows exponentially at a rate of $e^{2\lambda\sqrt{\rho}(\zeta)}$. We introduce the exponentially damping prefactor to obtain polynomial growth in the tempered version (3.17).

3.1.5. The Toeplitz operator $\Pi_\tau D_{\sqrt{\rho}}\Pi_\tau$

Let $D_{\sqrt{\rho}} = \frac{1}{i}\Xi_{\sqrt{\rho}}$ denote the Hamilton vector field of the Grauert tube function acting as a differential operator. The symbol of $D_{\sqrt{\rho}}$ is nowhere vanishing on $\Sigma_\tau - 0$, where Σ_τ is the symplectic cone (3.7). Thus, $\Pi_\tau D_{\sqrt{\rho}}\Pi_\tau$ is an elliptic Toeplitz operator with discrete spectrum.

As discussed in Section 2.1.1, the Grauert tube analogue of Fourier coefficients of the Szegő kernel are given by the spectral localization operator

$$\Pi_{\chi,\lambda} = \Pi_\tau \chi(\Pi_\tau D_{\sqrt{\rho}} \Pi_\tau - \lambda) = \int_{\mathbb{R}} \widehat{\chi}(t) e^{-it\lambda} \Pi_\tau e^{it\Pi_\tau D_{\sqrt{\rho}} \Pi_\tau} \, dt$$

Proposition 3.1.7 ([**48**, Proposition 5.3])**.** *Let $(G_\tau^t)^*$ denote the pullback by the Hamilton flow* (3.4) *of $\Xi_{\sqrt{\rho}}$ on ∂M_τ. There exists a classical polyhomogeneous pseudodifferential operator $\widehat{\sigma}_{t,\tau}(w, D_{\sqrt{\rho}})$ on ∂M_τ so that*

$$\Pi_\tau e^{it\Pi_\tau D_{\sqrt{\rho}} \Pi_\tau} \cong \Pi_\tau \widehat{\sigma}_{t,\tau} \left(G_\tau^t\right)^* \Pi_\tau \quad \textit{modulo a smoothing Toeplitz operator.}$$

The symbol $\sigma_{t,\tau}$ of $\widehat{\sigma}_{t,\tau}$ admits a complete asymptotic expansion

$$\sigma_{t,\tau}(w, r) \sim \sum_{j=0}^{\infty} \sigma_{t,\tau,j}(w) r^{-j}, \tag{3.19}$$

in which $\sigma_{t,\tau,0} = \langle e_{\Lambda_\tau^t}, e_{\Lambda_\tau} \rangle^{-1}$ is the reciprocal of the overlap of two Gaussians, as in (3.9)*.*

Remark 3.1.8. The difference between Proposition 3.1.7 and Proposition 3.1.5 is that the symbol (3.16) is of order zero and (3.19) is of order $-(m-1)/2$. This is because the partial sums (3.17) are not divided through by $\|e^{-\tau\lambda}\varphi_\lambda\|^2_{L^2(\partial M_\tau)} \sim \lambda^{-\frac{m-1}{2}}$. For an in-depth discussion of complexified eigenfunctions and dynamical Toeplitz operators see [**26, 51, 48**].

We recall the following lemma which we will use to relate the complexified eigenfunctions to the eigenfunctions of $\Pi_\tau D_{\sqrt{\rho}} \Pi_\tau$

Lemma 3.1.9 ([**51**, Lemma 8.2]). *Let Ψ^s denote the class of psuedodifferential operators of order s. Then,*

(1) $U(i\tau)^* U(i\tau) \in \Psi^{-\frac{m-1}{2}}(M)$ *with principal symbol* $|\xi|_g^{-\frac{m-1}{2}}$.

(2) $U(i\tau) U(i\tau)^* = \Pi_\tau A_\tau \Pi_\tau$ *where* $A_\tau \in \Psi^{\frac{m-1}{2}}(\partial M_\tau)$ *has principal symbol* $|\sigma|_g^{\frac{m-1}{2}}$ *as a function on* Σ_τ.

3.1.6. Quantization of linear symplectic maps on Bargmann–Fock space

The proofs of our theorems involve Taylor expansions in appropriate coordinates to reduce the geometry to the model linear space, so we briefly review the metaplectic representation on Bargmann–Fock space used to quantize symplectic linear mappings. Details can be found [**47, 17**]

The Bargmann–Fock space on $\mathbb{C}^m$ is

$$\mathcal{H}(\mathbb{C}^m) = \left\{ f(z) e^{-\frac{|z|^2}{2}} \in L^2(\mathbb{C}^m, dz) \;\middle|\; f \in \mathcal{O}(\mathbb{C}^m) \right\}.$$

The reproducing Bergman kernel has the exact formula

$$\Pi_{\mathcal{H}}(z, w) = (2\pi)^{-m} \, e^{-\frac{|z|^2}{2} - \frac{|w|^2}{2} + z\bar{w}}.$$

Let $Sp(m, \mathbb{R})$ denote the space of real symplectic matrices on $\mathbb{R}^{2m} = \mathbb{R}^m_x \times \mathbb{R}^m_y$ with respect to the standard symplectic form. Then matrix multiplication $M \in Sp(m, \mathbb{R})$ in real coordinates takes the form

$$M \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x' \\ y' \end{pmatrix}. \tag{3.20}$$

We map $\mathbb{R}^{2m}$ into $\mathbb{C}^{2m}$ via $(x,y) \mapsto (x+iy, x-iy) =: (z,\bar{z})$. Under this mapping, (3.20) becomes

$$\mathcal{M}\begin{pmatrix} z \\ \bar{z} \end{pmatrix} = \begin{pmatrix} P & Q \\ \bar{Q} & \bar{P} \end{pmatrix}\begin{pmatrix} z \\ \bar{z} \end{pmatrix} = \begin{pmatrix} z' \\ \bar{z}' \end{pmatrix}, \tag{3.21}$$

where the holomorphic component P and antiholomorphic component Q of the symplectic mapping are given by

$$\begin{pmatrix} P & Q \\ \bar{Q} & \bar{P} \end{pmatrix} = \mathcal{W}^{-1}\begin{pmatrix} A & B \\ C & D \end{pmatrix}\mathcal{W}, \quad \mathcal{W} = \frac{1}{\sqrt{2}}\begin{pmatrix} I & I \\ -iI & iI \end{pmatrix}.$$

(The choice of normalization is taken so that $\mathcal{W}$ is unitary.) The explicit formula for the holomorphic component is

$$P = \frac{1}{2}(A + D + i(C - B)).$$

The metaplectic representation on $\mathcal{H}(\mathbb{C}^m)$ is defined by $M \mapsto \Pi_{\mathcal{H},M}$, the latter being a unitary operator with kernel

$$\Pi_{\mathcal{H},M}(z,w) = (\det P)^{-\frac{1}{2}} \int_{\mathbb{C}^m} \Pi_{\mathcal{H}}(z, \mathcal{M}v)\Pi_{\mathcal{H}}(v,w)\, dv, \tag{3.22}$$

in which we set $\mathcal{M}v := Pv + Q\bar{v}$. (The ambiguity of the sign of $(\det P)^{-\frac{1}{2}}$ is determined by the lift to the double cover.) Explicit computations involving standard Gaussian integrals show

$$\Pi_{\mathcal{H},M}(z,w) := \mathcal{K}_{\mathcal{M}}(z,w)e^{-\frac{|z|^2}{2}-\frac{|w|^2}{2}},$$
$$\mathcal{K}_{\mathcal{M}}(z,w) := (2\pi)^{-m}(\det P)^{-\frac{1}{2}}\exp\left\{\frac{1}{2}\left(z\bar{Q}P^{-1}z + 2\bar{w}P^{-1}z - \bar{w}P^{-1}Q\bar{w}\right)\right\}.$$

The principal term of Theorem 2.2.1 contains the lift of $\Pi_{\mathcal{H},M}$ to the reduced Heisenberg group $\mathbb{H}^m_{\mathrm{red}} \cong S^1 \times \mathbb{C}^m$, which is given by

$$\widehat{\Pi}_{\mathcal{H},M}(\theta,z,\phi,w) = e^{i(\theta-\phi)}\Pi_{\mathcal{H},M}(z,w).$$

3.2. Heisenberg coordinates on the boundary of a Grauert tube

This section reviews the construction and properties of Heisenberg coordinates following [**16**, Section 18]. Roughly speaking, in these coordinates, the strongly pseudoconvex boundary $\partial M_\tau \subseteq M_{\mathbb{C}}$ is, to a first approximation, the Heisenberg group viewed as a hypersurface in complex Euclidean space. We briefly recall the CR structure on the Heisenberg group.

The Heisenberg group $\mathbf{H}^{m-1}$ of degree $m-1$ is the Lie group $\mathbb{R} \times \mathbb{C}^{m-1}$ with group law

$$(\theta, z) \cdot (\phi, w) = (\theta + \phi + 2\,\mathrm{Im}(z \cdot w), z + w). \tag{3.23}$$

Remark 3.2.1. It is more common to use t for the $\mathbb{R}$-coordinate but, since t is already reserved for the time parameter of the geodesic or Hamiltonian flow, we use θ instead.

For $1 \leq j \leq m-1$, the vector fields

$$Z_j = \frac{\partial}{\partial z_j} + i\overline{z}_j \frac{\partial}{\partial \theta}, \qquad \overline{Z}_j = \frac{\partial}{\partial \overline{z}_j} + iz_j \frac{\partial}{\partial \theta}, \qquad T = \frac{\partial}{\partial \theta}$$

satisfy the commutation relations

$$[Z_j, \overline{Z}_k] = -2i\delta_{jk}T \quad \text{and} \quad [Z_j, Z_k] = [\overline{Z}_j, \overline{Z}_k] = [Z_j, T] = [\overline{Z}_j, T] = 0.$$

Furthermore, the 1-form

$$\alpha = d\theta - i \sum_{j=1}^{m-1} (\overline{z}_j \, dz_j - z_j \, d\overline{z}_j)$$

annihilates Z_j and $\overline{Z}_j$ for all $1 \leq j \leq m-1$. Hence, the complexified tangent space of the Heisenberg group admits the decomposition

$$T^{\mathbb{C}}\mathbf{H}^{m-1} = H^{(1,0)} \oplus H^{(0,1)} \oplus \mathbb{C}T,$$

in which $H^{(1,0)}$ and $H^{(0,1)}$ are spanned by the holomorphic vector fields Z_j and the anti-holomorphic vector fields $\overline{Z}_j$, respectively. Direct verification shows that the subspace $H^{(1,0)}$ defines a CR structure on $\mathbf{H}^{m-1}$.

The Levi form $\langle \, , \, \rangle_L$ is the Hermitian form on $H^{(1.0)}$ defined by

$$\langle Z, W \rangle_L = -i \langle d\alpha, Z \wedge \overline{W} \rangle \quad \text{for all } Z, W \in H^{(1,0)}.$$

Direct computation shows that the Levi form with respect to the basis Z_j is the identity. Thus, $\mathbf{H}^{m-1}$ is strongly pseudoconvex.

Set

$$D_m = \left\{ \zeta = (\zeta_0, \ldots, \zeta_{m-1}) \in \mathbb{C}^m : \sum_{j=1}^{m-1} |\zeta_j|^2 < \operatorname{Im} \zeta_0 \right\}.$$

$$\partial D_m = \left\{ \zeta = (\zeta_0, \ldots, \zeta_{m-1}) \in \mathbb{C}^m : \sum_{j=1}^{m-1} |\zeta_j|^2 = \operatorname{Im} \zeta_0 \right\}. \tag{3.24}$$

The Heisenberg group $\mathbf{H}^{m-1}$ acts on $\mathbb{C}^m$ by holomorphic affine transformations; given $(\theta, z) = (\theta, z_1, \ldots, z_{m-1}) \in \mathbf{H}^{m-1}$ and $\zeta = (\zeta_0, \ldots, \zeta_{m-1}) \in \mathbb{C}^m$, we have $(\theta, z) \cdot \zeta = (\zeta_0', \ldots, \zeta_{m-1}')$, where

$$\begin{aligned} \zeta_0' &= \zeta_0 + \theta + i|z|^2 + 2i \sum_{j=1}^{m-1} \zeta_j \overline{z}_j, \\ \zeta_j' &= \zeta_j + z_j \quad \text{for } 1 \leq j \leq m-1. \end{aligned} \tag{3.25}$$

Note that the group action preserves D_m (which is biholomorphic to the unit ball in $\mathbb{C}^m$) and ∂D_m. Indeed, $\mathbf{H}^{m-1}$ acts simply and transitively on ∂D_m. Thus, comparing (3.23) and (3.25), we find that the Heisenberg group may be identified to ∂D_m via the correspondence

$$H_{m-1} \ni (\theta, z) \leftrightarrow (\theta, z) \cdot 0 = (\theta + i|z|^2, z_1, \ldots, z_{m-1}) \in \partial D_m.$$

On any nondegenerate CR manifold (in particular the boundary of a Grauert tube), there exists a Heisenberg-like coordinate system $(\theta, z_1, \ldots, z_{m-1})$ so that $Z_j = (\partial/\partial z_j) + i\overline{z}_j(\partial/\partial\theta)$ and $T = (\partial/\partial\theta)$ up to lower order terms. An explicit construction of such "normal coordinates" was presented in [**16**]. We call the same coordinates "Heisenberg coordinates" in this paper to emphasize the osculating Heisenberg structure (see Remark 3.2.5).

Definition 3.2.2. By *Heisenberg coordinates* (relative to the orthonormal frame $Z_1, \dots, Z_{m-1}$) at $T_p^{1,0}\partial M_\tau$, we mean holomorphic coordinates $z_0, \dots, z_{m-1}$ in a neighborhood of $p \in M_\tau$ with the following properties.

(i) If we set $\theta = \operatorname{Re} z_0$ and $z_j = x_j + iy_j$, then $x_1, \dots x_{m-1}, y_1, \dots, y_{m-1}, \theta$ form a coordinate system in a neighborhood of p in ∂M_τ.

(ii) Moreover, we have

$$Z_j = \frac{\partial}{\partial z_j} + i\overline{z}_j \frac{\partial}{\partial t} + \sum_{k=1}^{m-1} O^1 \frac{\partial}{\partial z_k} + O^2 \frac{\partial}{\partial \theta},$$
$$T = \frac{\partial}{\partial \theta} + \sum_{k=1}^{m-1} O^1 \frac{\partial}{\partial z_k} + \sum_{k=1}^{m-1} O^1 \frac{\partial}{\partial \overline{z}_k} + O^1 \frac{\partial}{\partial \theta}.$$

Here, the Heisenberg-type order O^k is defined inductively by

$$f = O^1 \text{ if } f(\eta) = O\left(\sum_{k=1}^{m-1} (|x_k(\eta)| + |y_k(\eta)|) + |\theta(\eta)|^{\frac{1}{2}} \right) \text{ as } \eta \to p, \tag{3.26}$$
$$f = O^k \text{ if } f = O(O^1 \cdot O^{k-1}).$$

Remark 3.2.3. If f is smooth, then

$$f = O^1 \text{ if and only if } f(\eta) = O\left(\sum_{k=1}^{m-1} (|x_k(\eta)| + |y_k(\eta)|) + |\theta(\eta)| \right),$$
$$f = O^2 \text{ if and only if } f(\eta) = O\left(\sum_{k=1}^{m-1} (|x_k(\eta)|^2 + |y_k(\eta)|^2) + |\theta(\eta)| \right).$$

3.2.1. Taylor expansions in Heisenberg coordinates

We fix notation. Let $p \in \partial M_\tau$ and let z, w be two points in $M_{\tau_{\max}}$ that lie in a Heisenberg coordinate patch containing p. We write

$$\begin{aligned} z &= (z_0, \ldots, z_{m-1}), \quad \theta = \operatorname{Re} z_0, \quad u = (z_1, \ldots, z_{m-1}), \\ w &= (w_0, \ldots, w_{m-1}), \quad \phi = \operatorname{Re} w_0, \quad v = (w_1, \ldots, w_{m-1}). \end{aligned} \tag{3.27}$$

Note that if $z \in \partial M_\tau$, then $z = (\theta, u)$.

We record the Taylor expansions of the boundary defining function (3.10) and the Hamilton flow (3.4) on the boundary in these coordinates. Detailed computations in the setting of a real hypersurface in a complex manifold whose induced CR structure is strongly pseudoconvex (as is the case of $\partial M_\tau \subseteq M_{\mathbb{C}}$) are found in [**16**, Section 18].

Lemma 3.2.4. *Let φ_τ be the defining function* (3.10) *of the Grauert tube boundary. Denote by $\varphi_\tau(z, \overline{w})$ be its analytic extension obtained by polarization. Then, in Heisenberg coordinates near p in M_τ, we have*

$$\varphi_\tau(z, \overline{w}) = \frac{i}{2}(z_0 - \overline{w}_0) + \sum_{j=1}^{m-1} z_j \overline{w}_j + R(z, \overline{w}).$$

The remainder R takes the form

$$R(z, \overline{w}) = R^{\varphi_\tau}(z_0, u, \overline{w}_0, \overline{v}) = R_2(z_0, \overline{w}_0) + R_2(z_0, u, \overline{w}_0, \overline{v}) + R_3(u, \overline{v}),$$

where

- *$R_2(z_0, \overline{w}_0)$ contains only of mixed terms of the form $z_0^\alpha \overline{w}_0^\beta$ with $|\alpha| + |\beta| \geq 2$.*

- *$R_2(z_0, u, \overline{w}_0, \overline{v})$ contains only of mixed terms of the form $z_0^\alpha \overline{v}^\beta$ or $\overline{w}_0^\alpha u^\beta$ with $|\alpha| + |\beta| \geq 2$.*
- *$R_3(u, \overline{v})$ contains only of mixed terms of the form $u^\alpha \overline{v}^\beta$ with $|\alpha| + |\beta| \geq 3$.*

Proof. Direct computation (see [**16**, (18.4)]) yields

$$\varphi_\tau(z) = \varphi_\tau(z_0, u) = -\operatorname{Im} z_0 + |u|^2 + O(|z_0||u| + |z_0|^2 + |u|^3), \tag{3.28}$$

so

$$d\varphi_\tau\Big|_p = -\operatorname{Im} dz_0\Big|_p \quad \text{and} \quad \frac{\partial^2 \varphi_\tau}{\partial z_j \partial \overline{z}_k}\Bigg|_p = \delta_{jk} \quad (1 \leq j, k \leq m-1).$$

Combined with the near-diagonal Taylor expansion

$$\varphi_\tau(p + h, p + k) = \sum_{\alpha,\beta} \frac{\partial^{\alpha+\beta} \varphi_\tau}{\partial z^\alpha \partial \overline{z}^\beta}(p) \frac{h^\alpha}{\alpha!} \frac{\overline{k}^\beta}{\beta!}$$

of φ_τ, we obtain the desired result. □

Remark 3.2.5. Note that the formula (3.28) for the defining function implies ∂M_τ is highly tangent at p to the hypersurface $\operatorname{Im} z_0 = \sum_{k=1}^{m-1} |z_k|^2$, which is the geometric model (3.24) for the Heisenberg group.

Lemma 3.2.6. *Let G_τ^t be the Hamilton flow* (3.4) *of the Grauert tube function on ∂M_τ. Then, in Heisenberg coordinates near p in ∂M_τ, we have*

$$G_\tau^t(z) = G_\tau^t(\theta, u) = \Big(\theta + 2\tau t + t \cdot O^1 + O(t^2), u + t \cdot O^1 + O(t^2)\Big),$$

where O^1 denotes the Heisenberg-type order (3.26).

Proof. By [**16**, Theorem 18.5], the Reeb vector field is of the form

$$\frac{1}{2\tau}H_{\sqrt{\rho}} = \frac{\partial}{\partial\theta} + \sum_{j=1}^{m-1} O^1 \frac{\partial}{\partial z_j} + \sum_{j=1}^{m-1} O^1 \frac{\partial}{\partial \overline{z}_j} + O^1 \frac{\partial}{\partial\theta}.$$

The Lemma then follows by Taylor expanding G_τ^t near $t = 0$. □

We now record several Taylor expansions in Heisenberg coordinates pertaining to the evaluation of ψ_τ in a Heisenberg scaled neighborhood of the graph of G_τ^t. In the following $\lambda \in \mathbb{R}^+$ is a parameter tending to ∞. We state all of the identities in rescaled form as they appear in the main argument.

Lemma 3.2.7 (Expansion of the rescaled phase function)**.** *Let* ψ_τ *be as in* (3.11)*. In Heisenberg coordinates centered at* $p \in \partial M_\tau$ *we have*

$$i\lambda\psi_\tau\left(\left(\frac{\theta}{\lambda}, \frac{u}{\sqrt{\lambda}}\right), \frac{w}{\sqrt{\lambda}}\right) = -\sqrt{\lambda}\frac{i}{2}\operatorname{Re} w_0 + \tilde{R}\left(\frac{\theta}{\lambda}, \frac{\operatorname{Re} w_0}{\sqrt{\lambda}}, \frac{u}{\sqrt{\lambda}}, \frac{w'}{\sqrt{\lambda}}\right),$$

where

$$\tilde{R} = \frac{i}{2}\theta - \frac{|u|^2}{2} - \frac{|w|^2}{2} + u \cdot \overline{w} + \lambda Q\left(\frac{\theta}{\lambda}, \frac{u}{\sqrt{\lambda}}, \frac{\operatorname{Re} w_0}{\sqrt{\lambda}}, \frac{w'}{\sqrt{\lambda}}\right) \tag{3.29}$$

and Q *takes the form*

$$Q\left(\frac{\theta}{\lambda}, \frac{u}{\sqrt{\lambda}}, \frac{\operatorname{Re} w_0}{\sqrt{\lambda}}, \frac{w'}{\sqrt{\lambda}}\right) = O\left(\frac{|\operatorname{Re} w_0||u|}{\lambda} + \frac{|\operatorname{Re} w_0||w'|}{\lambda}\right) + O\left(\lambda^{-\frac{3}{2}}\right)$$

Proof. This is an immediate consequence of Lemma 3.2.4 combined with the fact that $\varphi_\tau = 0$ on ∂M_τ and the expansion Equation 3.28. Keeping track of the powers of $\sqrt{\lambda}$ under parabolic rescaling results in the statement of the lemma. □

To prove our scaling asymptotics we will simultaneously be working with two sets of Heisenberg coordinate systems, one centered at p and another centered at $G^s_\tau(p)$. We recall that G^s_τ is the Hamiltonian flow of the characteristic vector field which also preserves $T^{1,0}\partial M_\tau \oplus T^{0,1}\partial M_\tau$. Therefore, its derivative DG^s_τ is a linear map $T_p\partial M_\tau \to T_{G^s_\tau(p)}\partial M_\tau$ such that in Heisenberg coordinates at p and $G^s_\tau(p)$, we have

$$DG^s_\tau = \begin{pmatrix} 1 & 0 \\ 0 & M_s \end{pmatrix} \quad \text{with} \quad M_s \in Sp(m-1,\mathbb{R}). \tag{3.30}$$

We denote its complexification by $\mathcal{M}_s$ as in (3.21) and use the same notation P, Q for its holomorphic and anti-holomorphic components. We have the following identities for Heisenberg coordinates centered at $G^s_\tau(p)$.

Lemma 3.2.8 (Expansion of the rescaled geodesic flow)**.** *Let $w = (w_0, w')$ be a point in a Heisenberg coordinate chart centered at $p \in \partial M_\tau$. Then, in Heisenberg coordinates centered at $G^s_\tau(p) \in \partial M_\tau$, we have*

$$G_\tau^{s+\frac{r}{\sqrt{\lambda}}}\left(\frac{w}{\sqrt{\lambda}}\right) = \left(\frac{\operatorname{Re} w_0}{\sqrt{\lambda}} + \frac{2\tau r}{\sqrt{\lambda}} + O\left(\frac{|r|}{\lambda}\right) + O\left(\lambda^{-3/2}\right),\right.$$
$$\left.\frac{\mathcal{M}_s w'}{\sqrt{\lambda}} + O\left(\frac{|r|}{\lambda}\right) + O\left(\lambda^{-3/2}\right)\right).$$

Proof. This follows from the Taylor expansion $G^t_\tau(z_0, z') = (z_0 + 2\tau t + t\cdot O^1 + O(t^2), z' + t\cdot O^1 + O(t^2))$ combined with (3.30) ◻

Lemma 3.2.9 (Combined expansion of the rescaled phase and flow)**.** *In Heisenberg coordinates centered at $G^s_\tau(p) \in \partial M_\tau$ we have*

$$i\lambda\psi_\tau\left(G_\tau^{s+\frac{r}{\sqrt{\lambda}}}\left(\frac{w}{\sqrt{\lambda}}\right), \left(\frac{\varphi}{\lambda}, \frac{v}{\sqrt{\lambda}}\right)\right) = \sqrt{\lambda}\frac{i}{2}\left(\operatorname{Re} w_0 + 2\tau r\right) + \tilde{S}\left(\frac{\phi}{\lambda}, \frac{\operatorname{Re} w_0}{\sqrt{\lambda}}, \frac{r}{\sqrt{\lambda}}, \frac{v}{\sqrt{\lambda}}, \frac{w'}{\sqrt{\lambda}}\right),$$

where

$$\tilde{S} = -\frac{i}{2}\phi - \frac{|v|^2}{2} - \frac{|\mathcal{M}_s w|^2}{2} + \overline{v}\cdot(\mathcal{M}_s w) + \lambda T\left(\frac{\phi}{\lambda}, \frac{v}{\sqrt{\lambda}}, \frac{r}{\sqrt{\lambda}}, \frac{\operatorname{Re} w_0}{\sqrt{\lambda}}, \frac{w'}{\sqrt{\lambda}}\right) \tag{3.31}$$

and T takes the form

$$T\left(\frac{\phi}{\lambda}, \frac{v}{\sqrt{\lambda}}, \frac{r}{\sqrt{\lambda}}, \frac{\operatorname{Re} w_0}{\sqrt{\lambda}}, \frac{w'}{\sqrt{\lambda}}\right) = O\left(\frac{|r||v|}{\lambda} + \frac{|r||w'|}{\lambda} + \frac{|\operatorname{Re} w_0||v|}{\lambda} + \frac{|\operatorname{Re} w_0||w'|}{\lambda}\right) + O\left(\lambda^{-\frac{3}{2}}\right)$$

Proof. This follows from theLemma 3.2.7 and Lemma 3.2.8. □

CHAPTER 4

Proof of Scaling Asymptotics

In this chapter we give a proof of the scaling asymptotics Theorem 2.2.1 and Theorem 2.2.9. Theorem 2.1.1 and Theorem 2.1.2 follow as immediate corollaries. We briefly outline the proof of Theorem 2.1.1 before delving into the computations. As indicated in a subsequent Section, the same techniques work almost verbatim for proving Theorem 2.1.2.Since we are interested in scaling asymptotics, variables are rescaled according to their Heisenberg-type order (cf (3.26)). Namely, given $(\theta, u) \in \partial M_\tau$ in Heisenberg coordinates near $p \in \partial M_\tau$ (recall Definition 3.2.2 and (3.27)), we consider

$$(\theta, u) \mapsto \left(\frac{\theta}{\lambda}, \frac{u}{\sqrt{\lambda}}\right) \in \partial M_\tau.$$

We rewrite the (rescaled) spectral localization kernel of (2.4) in terms of the kernel of a "dynamical Toeplitz operator" using Proposition 3.1.7. Then, substituting the Boutet de Monvel–Sjöstrand parametrix (3.12) for the Szegő projectors, we are left with an oscillatory integral. Taylor expanding the phase function using Lemma 3.2.4, Lemma 3.2.6, and then applying the method of stationary phase complete the proof.

4.1. Proof of Theorem 2.2.1: asymptotic expansion for $\Pi_{\chi,\lambda}$

From (2.4) and Proposition 3.1.7, we have

$$\Pi_{\chi,\lambda} \cong \int_{\mathbb{R}} \widehat{\chi}(t)\Pi_\tau \widehat{\sigma}_{t,\tau}(G_\tau^t)^*\Pi_\tau \, dt$$

Here $\cong$ denotes equality modulo a smoothing Toeplitz operator. Fix $p \in \partial M_\tau$ and let $(\theta, u), (\phi, v) \in \partial M_\tau$ be two points in Heisenberg coordinates centered at p and $G_\tau^s(p)$ respectively. Then, after substituting the parametrix (3.12) for each instance of Π_τ above and composing the resulting kernels, we arrive at the oscillatory integral representation

$$\text{(4.1)} \quad \Pi_{\chi,\lambda}\left(p + \left(\frac{\theta}{\lambda}, \frac{u}{\sqrt{\lambda}}\right), G_\tau^s(p) + \left(\frac{\phi}{\lambda}, \frac{v}{\sqrt{\lambda}}\right)\right) \sim \int_{\mathbb{R}\times\partial M_\tau\times\mathbb{R}^+\times\mathbb{R}^+} e^{i\lambda\Psi} A \, d\sigma_1 d\sigma_2 d\mu_\tau(w) dt,$$

in which the phase Ψ and the amplitude A are given by

$$\text{(4.2)} \quad \begin{aligned} \Psi &= -t + \frac{1}{\lambda}\sigma_2\psi_\tau\left(p + \left(\frac{\theta}{\lambda}, \frac{u}{\sqrt{\lambda}}\right), w\right) + \frac{1}{\lambda}\sigma_1\psi_\tau\left(G_\tau^t(w), G_\tau^s(p) + \left(\frac{\phi}{\lambda}, \frac{v}{\sqrt{\lambda}}\right)\right), \\ A &= \widehat{\chi}(t) s\left(p + \left(\frac{\theta}{\lambda}, \frac{u}{\sqrt{\lambda}}\right), w, \sigma_2\right) s\left(G_\tau^t(w), G_\tau^t(p) + \left(\frac{\phi}{\lambda}, \frac{v}{\sqrt{\lambda}}\right), \sigma_1\right)\sigma_{t,\tau}(w, \sigma_1). \end{aligned}$$

From now on, we suppress p and $G_\tau^s(p)$ from the notation, keeping in mind that they are the origin in each of their respective coordinates. Make the change-of-variables $\sigma_j \mapsto \lambda\sigma_j$. Homogeneity of the symbols implies

$$\text{(4.3)} \quad \Pi_{\chi,\lambda}\left(\frac{\theta}{\lambda}, \frac{u}{\sqrt{\lambda}}, \frac{\phi}{\lambda}, \frac{v}{\sqrt{\lambda}}\right) \sim \lambda^{2m} \int_{\mathbb{R}\times\partial M_\tau\times\mathbb{R}^+\times\mathbb{R}^+} e^{i\lambda\tilde{\Psi}} \tilde{A} \, d\sigma_1 d\sigma_2 d\mu_\tau(w) dt,$$

in which the phase $\tilde{\Psi}$ and the amplitude $\tilde{A}$ are given by

$$\text{(4.4)} \quad \begin{aligned} \tilde{\Psi} &= -t + \sigma_2\psi_\tau\left(\left(\frac{\theta}{\lambda}, \frac{u}{\sqrt{\lambda}}\right), w\right) + \sigma_1\psi_\tau\left(G_\tau^t(w), \left(\frac{\phi}{\lambda}, \frac{v}{\sqrt{\lambda}}\right)\right), \\ \tilde{A} &= \lambda^{-2m} A. \end{aligned}$$

We begin by localizing in $(w,t) \in \partial M_\tau \times \mathbb{R}$. Fix $C > 0$ and $0 < \delta < \frac{1}{2}$. Set

$$V_\lambda = \Big\{(w,t) : \max \Big\{ d\big(w, (\tfrac{\theta}{\lambda}, \tfrac{u}{\sqrt{\lambda}})\big), d\big(G_\tau^t(w), (\tfrac{\phi}{\lambda}, \tfrac{v}{\sqrt{\lambda}})\big)\Big\} < \tfrac{2C}{3}\lambda^{\delta - \frac{1}{2}}\Big\},$$

$$W_\lambda = \Big\{(w,t) : \max \Big\{ d\big(w, (\tfrac{\theta}{\lambda}, \tfrac{u}{\sqrt{\lambda}})\big), d\big(G_\tau^t(w), (\tfrac{\phi}{\lambda}, \tfrac{v}{\sqrt{\lambda}})\big)\Big\} > \tfrac{C}{2}\lambda^{\delta - \frac{1}{2}}\Big\}.$$

Let $\{\varrho_\lambda, 1 - \varrho_\lambda\}$ be a partition of unity subordinate to the cover $\{V_\lambda, W_\lambda\}$ and decompose the integral (4.3) into

$$\Pi_{\chi,\lambda}\left(\frac{\theta}{\lambda}, \frac{u}{\sqrt{\lambda}}, \frac{\phi}{\lambda}, \frac{v}{\sqrt{\lambda}}\right) \sim I_1 + I_2,$$

$$I_1 = \lambda^{2m} \int e^{i\lambda\tilde{\Psi}} \varrho_\lambda(t,w) \tilde{A}\, d\sigma_1 d\sigma_2 d\mu_\tau(w) dt,$$

$$I_2 = \lambda^{2m} \int e^{i\lambda\tilde{\Psi}} (1 - \varrho_\lambda(t,w)) \tilde{A}\, d\sigma_1 d\sigma_2 d\mu_\tau(w) dt.$$

Lemma 4.1.1. *We have $I_2 = O(\lambda^{-\infty})$.*

Proof. By definition of W_λ, on the support of $1 - \varrho_\lambda$ either

$$|d_{\sigma_2}\tilde{\Psi}| = \left|\psi_\tau\left(\left(\frac{\theta}{\lambda}, \frac{u}{\sqrt{\lambda}}\right), w\right)\right| \geq 2D\left(\left(\frac{\theta}{\lambda}, \frac{u}{\sqrt{\lambda}}\right), w\right) \geq C'\lambda^{2\delta-1}$$

or

$$|d_{\sigma_1}\tilde{\Psi}| = \left|\psi_\tau\left(\left(G_\tau^t(w), \frac{\varphi}{\lambda}, \frac{v}{\sqrt{\lambda}}\right)\right)\right| \geq 2D\left(\left(G_\tau^t(w), \frac{\varphi}{\lambda}, \frac{v}{\sqrt{\lambda}}\right)\right) \geq C'\lambda^{2\delta-1}$$

where D is the Calabi diastasis (3.13) and the inequalities follow from (3.14). Repeated integration by parts in σ_1 or σ_2 as appropriate completes the proof. $\square$

In preparation for stationary phase we make the following change of variables

$$t \mapsto s + \frac{r}{\sqrt{\lambda}} \quad \text{and} \quad (\operatorname{Re} w_0, w') \mapsto \left(\frac{\operatorname{Re} w_0}{\sqrt{\lambda}}, \frac{w'}{\sqrt{\lambda}}\right).$$

Substituting in our formulas from lemmas 3.2 and 3.4 we obtain the following oscillatory integral with parameter $\sqrt{\lambda}$.

$$\Pi_{\chi,\lambda}\left(\frac{\theta}{\lambda}, \frac{u}{\sqrt{\lambda}}, \frac{\varphi}{\lambda}, \frac{v}{\sqrt{\lambda}}\right) \sim e^{-is\lambda}\lambda^m \int e^{i\lambda\tilde{\Psi}}\tilde{A}d\sigma_1 d\sigma_2 dw dr \tag{4.5}$$

where

$$\begin{aligned} \tilde{\Psi} &= -r - \frac{\sigma_2}{2}\operatorname{Re} w_0 + \frac{\sigma_1}{2}(\operatorname{Re} w_0 + 2\tau r), \\ \tilde{A} &= e^{\sigma_2 \tilde{R} + \sigma_1 \tilde{S}} \varrho_\lambda \tilde{A}\left(\frac{\theta}{\lambda}, \frac{u}{\sqrt{\lambda}}, \frac{\phi}{\lambda}, \frac{v}{\sqrt{\lambda}}, \frac{\operatorname{Re} w_0}{\sqrt{\lambda}}, \frac{w'}{\sqrt{\lambda}}, \frac{r}{\sqrt{\lambda}}, \sigma_1, \sigma_2\right) J\left(\frac{w'}{\sqrt{\lambda}}\right), \end{aligned} \tag{4.6}$$

with $J(\,\cdot\,)$ the volume density in Heisenberg coordinates.

We may further localize this integral in the σ_1, σ_2 variables. Let $\{\eta, 1-\eta\}$ be a partition of unity subordinate to the cover

$$\left\{(\sigma_1, \sigma_2) : 0 < \sigma_1, \sigma_2 < \frac{2}{\tau}\right\} \quad \text{and} \quad \left\{(\sigma_1, \sigma_2) : \sigma_1, \sigma_2 > \frac{3}{2\tau}\right\}.$$

Decompose (4.5) into two integrals:

$$\Pi_{\chi,\lambda}\left(\frac{\theta}{\lambda}, \frac{u}{\sqrt{\lambda}}, \frac{\phi}{\lambda}, \frac{v}{\sqrt{\lambda}}\right) \sim I_1' + I_2',$$

$$I_1' = e^{-is\lambda}\lambda^m \int e^{i\lambda\tilde{\Psi}}\eta(\sigma_1,\sigma_2)\tilde{A}\, d\sigma_1 d\sigma_2 d\mu_\tau(w) dr$$

$$I_2' = e^{-is\lambda}\lambda^m \int e^{i\lambda\tilde{\Psi}}(1-\eta(\sigma_1,\sigma_2))\tilde{A}\, d\sigma_1 d\sigma_2 d\mu_\tau(w) dr$$

with $\tilde{A}$ and $\tilde{\Psi}$ as in (4.6).

Lemma 4.1.2. *We have* $I_2' = O(\lambda^{-\infty})$.

Proof. Notice that

$$\left|\nabla_{\operatorname{Re} w_0, r}\tilde{\Psi}\right|^2 \geq \left(\frac{\sigma_1}{2} - \frac{\sigma_2}{2}\right)^2 + (\tau\sigma_1 - 1)^2 \geq \frac{1}{4}$$

on the support of $1-\eta$. Thus, the lemma follows from repeated integration by parts in $(\operatorname{Re} w_0, r)$. □

We have reduced the spectral localization kernel to the oscillatory integral

$$\Pi_{\chi,\lambda}\left(\frac{\theta}{\lambda}, \frac{u}{\sqrt{\lambda}}, \frac{\phi}{\lambda}, \frac{v}{\sqrt{\lambda}}\right) \sim e^{-is\lambda}\lambda^m \int e^{i\sqrt{\lambda}\tilde{\tilde{\Psi}}}\tilde{\tilde{A}}\, dw' d(\operatorname{Re} w_0) d\sigma_1 d\sigma_2 dr, \tag{4.7}$$

with phase and amplitude

$$\tilde{\tilde{\Psi}} = -r - \frac{\sigma_2}{2}\operatorname{Re} w_0 + \frac{\sigma_1}{2}(\operatorname{Re} w_0 + 2\tau r),$$

$$\tilde{\tilde{A}} = e^{\sigma_2\tilde{R} + \sigma_1\tilde{S}}\eta\varrho_\lambda\widehat{\chi}\tilde{A}J.$$

Since the exponential of the terms of order $\lambda^{-\frac{1}{2}}$ appearing in $\tilde{R}, \tilde{S}$ is bounded it may be absorbed into the main amplitude. We will now reduce (4.7) to a Gaussian integral over $\mathbb{C}^{m-1}$ by integrating out the variables $\operatorname{Re} w_0, \sigma_1, \sigma_2, r$ using the method of stationary phase. We note the following derivatives:

$$\partial_{\sigma_2}\tilde{\tilde{\Psi}} = -\frac{1}{2}\operatorname{Re} w_0, \qquad \partial_{\sigma_1}\tilde{\tilde{\Psi}} = \frac{1}{2}(\operatorname{Re} w_0 + 2\tau r),$$
$$\partial_t\tilde{\tilde{\Psi}} = -1 + \tau\sigma_1, \qquad \partial_{\operatorname{Re} w_0}\tilde{\tilde{\Psi}} = -\frac{\sigma_2}{2} + \frac{\sigma_1}{2}.$$

The critical set of the phase is the point $C = \{\operatorname{Re} w_0 = 0, r = 0, \sigma_1 = \sigma_2 = \frac{1}{\tau}\}$. The Hessian matrix and its inverse at the critical point are

$$\tilde{\tilde{\Psi}}''_C = \left(\begin{array}{c|cccc} & r & \sigma_1 & \sigma_2 & \operatorname{Re} w_0 \\ \hline r & 0 & \tau & 0 & 0 \\ \sigma_1 & \tau & 0 & 0 & \frac{1}{2} \\ \sigma_2 & 0 & 0 & 0 & -\frac{1}{2} \\ \operatorname{Re} w_0 & 0 & \frac{1}{2} & -\frac{1}{2} & 0 \end{array}\right), \quad (\tilde{\tilde{\Psi}}''_C)^{-1} = \begin{pmatrix} 0 & \frac{1}{\tau} & \frac{1}{\tau} & 0 \\ \frac{1}{\tau} & 0 & 0 & 0 \\ \frac{1}{\tau} & 0 & 0 & -2 \\ 0 & 0 & -2 & 0 \end{pmatrix}.$$

Set

$$L_{\tilde{\tilde{\Psi}}} = \left\langle (\tilde{\tilde{\Psi}}''_C)^{-1} D, D \right\rangle = \frac{2}{\tau}\partial_{\sigma_1}\partial_r + \frac{2}{\tau}\partial_{\sigma_2}\partial_r - 4\partial_{\sigma_2}\partial_{\operatorname{Re} w_0}$$

By the method of stationary phase ([**24**, Theorem 7.75]), we have

$$\begin{aligned} (4.8) \quad & e^{-is\lambda}\lambda^m \int_{\mathbb{R}\times\mathbb{R}^+\times\mathbb{R}^+\times\mathbb{R}} e^{i\sqrt{\lambda}\tilde{\tilde{\Psi}}}\tilde{\tilde{A}}\, dw' d\sigma_1 d\sigma_2 dr \\ & = \frac{8\pi^2}{\lambda\tau} e^{-is\lambda}\lambda^m \sum_{j=0}^{N-1} \lambda^{-\frac{j}{2}} \sum_{\nu-\mu=j} \sum_{2\nu\geq 3\mu} \frac{1}{i^j 2^\nu} L^\nu_{\tilde{\tilde{\Psi}}} \left[e^{\mu(\sigma_2\tilde{R}+\sigma_1\tilde{S})} \eta\varrho_\lambda\widehat{\chi}\tilde{A}J \right]_C + \widehat{R}_N, \end{aligned}$$

with the remainder term satisfying

$$\int_{\mathbb{C}^{m-1}} |\widehat{R}_N|\, dw' \leq \lambda^{-\frac{N}{2}} \int_{\mathbb{C}^{m-1}} \sum_{|\alpha|\leq 2N} \sup |D^\alpha(\eta\rho_\lambda\widehat{\chi}\tilde{A}J)|\, dw' \leq C_N \lambda^{-\frac{N}{2}}.$$

(Here, the supremum and the derivative D^α are taken over $t, \sigma_1, \sigma_2, \operatorname{Re} w_0$ and the integral is with respect to the remaining variable w'. Note that $\tilde{A}$, defined in (4.4), is a symbol of order zero.)

Thanks to the remainder estimate, we may integrate the asymptotic expansion (4.8) term-by-term in w' to obtain (4.7). Upon substituting expressions (3.29) and (3.31) the leading term is given by the following Gaussian integral

$$\begin{aligned}(4.9)\quad & C_m \frac{\lambda^{m-1}}{\tau} e^{-is\lambda} \sigma_{s,\tau,0}(p) e^{\frac{i}{2\tau}(\theta-\phi)} \\ & \times \int_{\mathbb{C}^{m-1}} \exp\left\{ \frac{1}{\tau}\left(-\frac{|u|^2}{2} - \frac{|w'|^2}{2} + u\cdot\bar{w}' - \frac{|v|^2}{2} - \frac{|\mathcal{M}_s w'|^2}{2} + \bar{v}\cdot\mathcal{M}_s w' \right) \right\} dw'\end{aligned}$$

The symbol $\sigma_{s,\tau,0}(p)$ can be computed as in [**48**] to be $(\det P_s)^{-\frac{1}{2}}$. This is precisely the same integral as (3.22) and so we obtain the leading term

$$\frac{C_m}{\tau}\left(\frac{\lambda}{\tau}\right)^{m-1} e^{-is\lambda} \widehat{\Pi}_{\mathcal{H},M_s}\left(\frac{\theta}{2\tau}, \frac{u}{\sqrt{\tau}}, \frac{\phi}{2\tau}, \frac{v}{\sqrt{\tau}}\right).$$

The lower order terms have the form

$$\frac{C_m}{\tau^m}\lambda^{m-1-\frac{j}{2}}e^{-is\lambda}e^{\frac{i}{2\tau}(\theta-\phi)} \times \int_{\mathbb{C}^{m-1}} P_j(u,v,w,s,\theta,\phi)e^{\frac{1}{\tau}\left(-\frac{|u|^2}{2}-\frac{|w'|^2}{2}+u\cdot\bar{w}'-\frac{|v|^2}{2}-\frac{|\mathcal{M}_s w'|^2}{2}+\bar{v}\cdot\mathcal{M}_s w'\right)}\,dw',$$

with j a positive integer and $P_j(u,v,w,s,\theta,\phi)$ a polynomial. This can be rewritten as

$$\frac{C_m}{\tau^m}\lambda^{m-1-\frac{j}{2}}e^{-is\lambda}e^{\frac{i}{2\tau}(\theta-\phi)} \times \int_{\mathbb{C}^{m-1}} e^{\frac{1}{\tau}\left(-\frac{|u|^2}{2}-\frac{|v|^2}{2}+u\cdot\bar{w}'+\bar{v}\cdot\mathcal{M}_s w'\right)}\widehat{P}_j(u,v,s,\theta,\phi,D)e^{\frac{1}{\tau}\left(-\frac{|w'|^2}{2}-\frac{|\mathcal{M}_s w'|^2}{2}\right)}\,dw',$$

where $\widehat{P}_j$ is a differential operator with polynomial coefficients. We integrate by parts with the $\widehat{P}_j$ operator from which we obtain the same Gaussian integral as (4.9) against a polynomial independent of w'. As a result, the lower order terms in the asymptotic expansion take the form

$$\frac{C_m}{\tau}\left(\frac{\lambda}{\tau}\right)^{m-1-\frac{j}{2}} e^{-is\lambda}P_j(p,s,\tau,u,v,\theta,\phi)\widehat{\Pi}_{\mathcal{H},M_s}\left(\frac{\theta}{2\tau},\frac{u}{\sqrt{\tau}},\frac{\phi}{2\tau},\frac{v}{\sqrt{\tau}}\right).$$

4.2. Proof of Theorem 2.2.9: asymptotic expansion for $P_{\chi,\lambda}$

We can also study the on-shell scaling asymptotics for the tempered spectral projection kernel (3.18) under Heisenberg-type rescaling. The proof is nearly identitical to that of Theorem 2.2.1. We first write out the kernel using Proposition 3.1.5 and (3.12):

$$P_{\chi,\lambda}\left(\frac{\theta}{\lambda},\frac{u}{\sqrt{\lambda}},\frac{\phi}{\lambda},\frac{v}{\sqrt{\lambda}}\right) \sim \int_{\mathbb{R}\times\partial M_\tau\times\mathbb{R}^+\times\mathbb{R}^+} e^{i\lambda\Psi}B\,d\sigma_1 d\sigma_2 d\mu_\tau(w)dt, \tag{4.10}$$

in which the phase Ψ and the amplitude B are given by

$$\Psi = -t + \frac{1}{\lambda}\sigma_2\psi_\tau\left(\left(\frac{\theta}{\lambda}, \frac{u}{\sqrt{\lambda}}\right), w\right) + \frac{1}{\lambda}\sigma_1\psi_\tau\left(G_\tau^t(w), \left(\frac{\phi}{\lambda}, \frac{v}{\sqrt{\lambda}}\right)\right),$$
$$B = \widehat{\chi}(t) s\left(\frac{\theta}{\lambda}, \frac{u}{\sqrt{\lambda}}, w, \sigma_2\right) s\left(G_\tau^t(w), \frac{\phi}{\lambda}, \frac{v}{\sqrt{\lambda}}, \sigma_1\right)\sigma_{t,\tau}(w, \sigma_1).$$

The only modification is that despite identical notation, the unitarization symbol $\sigma_{t,\tau}$ for B is now of order $-(m-1)/2$, whereas $\sigma_{t,\tau}$ in the expression for A in (4.2) is of order zero. Hence, the oscillatory integral expression (4.10) is exactly $\lambda^{-(m-1)/2}$ times the expression (4.1). The rest of the computations proceed in the same manner.

CHAPTER 5

Proofs of L^p mapping norms

In this chapter we prove L^p estimates of the Szegő kernel, namely Theorem 2.2.3 and Theorem 2.2.5. Corollary 2.2.4, an L^p upper bound for normalized eigenfunctions of $\Pi_\tau D_{\sqrt{\rho}}\Pi_\tau$, is then deduced. The idea is to use Theorem 2.2.1 to give Gaussian estimates for $\Pi_{\chi,\lambda}(z,w)$ which we use with the Shur-Young ineqlaity to give sharp estimates. We show that sharpness of the mapping norms is verified by computing the L^p norm of $\Pi_{\chi,\lambda}$ applied to a "coherent state". We then show that L^p norms of Husimi distributions are sharp in the case of complexified Gaussian on a round sphere. We finish by providing a heuristic explanation for why complexified Gaussian beams extremize L^p norms for all p rather than just the low L^p norms their real counterparts extremize.

5.1. Proof of Theorem 2.2.3: sharp norm estimates for $\Pi_{\chi,\lambda}$

We begin by establishing the following Gaussian decay estimate for $\Pi_{\chi,\lambda}(z,w)$ away from a small neighborhood of the graph $(z,w) = (p, G^s_\tau(p))$.

Lemma 5.1.1 (Gaussian decay estimate)**.** *Fix $z \in \partial M_\tau$. Set $\delta = |\operatorname{supp}\widehat{\chi}| = 2\varepsilon$ and $T_\delta(z) = \{G^t_\tau(z) : |t| < \delta\}$. Then, after possibly shrinking* $\operatorname{supp}\widehat{\chi}$*, there exists $C > 0$ such that whenever $d(T_\delta(z), w) < C\lambda^{-\frac{1}{3}}$ we have*

$$|\Pi_{\chi,\lambda}(z,w)| \leq C(1+o(1))\lambda^{m-1}e^{-\frac{1-\varepsilon}{2\tau}\lambda d(T_\delta(z),w)^2} + O(\lambda^{-\infty}).$$

Proof. Let $|s| < \varepsilon = \delta/2$. In Heisenberg coordinates centered at $G_\tau^s(z)$, consider points of the form $w = G_\tau^s(z) + (\frac{\phi}{\lambda}, \frac{u}{\sqrt{\lambda}})$ with $|(\phi, u)| < \lambda^{\frac{1}{6}}$. We repeat the stationary phase computation in the proof of Theorem 2.2.1:

$$\Pi_{\chi,\lambda}\left(z, G_\tau^s(z) + \left(\frac{\phi}{\lambda}, \frac{v}{\sqrt{\lambda}}\right)\right) \sim e^{-is\lambda}\lambda^m \int e^{i\sqrt{\lambda}\tilde{\tilde{\Psi}}}\tilde{\tilde{A}}\, dw' d(\operatorname{Re} w_0) d\sigma_1 d\sigma_2 dt,$$

with phase and amplitude defined in the same way as (4.7). Keeping track of the first order remainder term, we find

$$\left|\Pi_{\chi,\lambda}\left(z, G_\tau^s(z) + \left(\frac{\phi}{\lambda}, \frac{v}{\sqrt{\lambda}}\right)\right)\right| = \lambda^{m-1} e^{-\frac{|u|^2}{2\tau} - \frac{\bar{u} P_s^{-1} Q_s \bar{u}}{2\tau}}$$
$$+ \lambda^{m-1-\frac{1}{2}} e^{-\frac{1-\varepsilon}{2\tau}\left(|u|^2 + \bar{u} P_s^{-1} Q_s \bar{u}\right)} R(p, s, u, \lambda),$$

where P_s, Q_s are matrices defined in (3.21) and $R(p, s, u, N) \leq C(s)|u|$ for $|u| < \lambda^{\frac{1}{6}}$.

Since $P_s^{-1} Q_s = o(s)$ and $|u| \approx \sqrt{\lambda} d(G_\tau^s(z), G_\tau^s(z) + \frac{u}{\sqrt{\lambda}})$, we get uniformly for $|s| < \varepsilon$ and $|u| < \lambda^{\frac{1}{6}}$ that

$$\left|\Pi_{\chi,\lambda}\left(z, G_\tau^s(z) + \left(\frac{\phi}{\lambda}, \frac{v}{\sqrt{\lambda}}\right)\right)\right| \leq C\lambda^{m-1}\left(1 + \frac{|u|}{\sqrt{\lambda}}\right) e^{-\frac{1-\varepsilon}{2} d\left(G_\tau^s(z), G_\tau^s(z) + \frac{u}{\sqrt{\lambda}}\right)},$$

as desired. □

To establish sharpness we will also need the following lower bound on $\Pi_{\chi,\lambda}$ in a $\lambda^{-\frac{1}{2}}$ of the graph.

Lemma 5.1.2. *Fix $z \in \partial M_\tau$ and $D > 0$. Set $\delta = |\operatorname{supp} \widehat{\chi}| = 2\varepsilon$ and $T_\delta(z)$ as in Lemma 5.1.1. Then, after possibly shrinking* $\operatorname{supp} \widehat{\chi}$*, there exists $C > 0$ such that* *whenever* $d(T_\delta(z), w) < D\lambda^{-\frac{1}{2}}$ *we have*

$$|\Pi_{\chi,\lambda}(z,w)| \geq C(1-o(1))\lambda^{m-1} e^{-\frac{1+\varepsilon}{2\tau}\lambda d(T_\delta(z),w)^2}.$$

Proof. This is an immediate corollary of Theorem 2.2.1 when we take $w = G_\tau^s(z) + (\frac{\phi}{\lambda}, \frac{u}{\sqrt{\lambda}})$ with $|(\phi, u)| < D$. □

We are now in a position to prove the main result. We invoke the Shur-Young inequality

$$\|\Pi_{\chi,\lambda}\|_{L^p \to L^q} \leq C_p \left[\sup_z \int_{\partial M_\tau} |\Pi_{\chi,\lambda}(z,w)|^r \, dw\right]^{\frac{1}{r}}, \quad \frac{1}{r} = 1 - \frac{1}{p} + \frac{1}{q}.$$

With T_δ as in Lemma 5.1.1, we break up the integral

$$\int_{\partial M_\tau} |\Pi_{\chi,\lambda}(z,w)|^r \, dw = \int_{d(T_\delta(z),w) \leq \lambda^{-\frac{1}{3}}} |\Pi_{\chi,\lambda}(z,w)|^r \, dw \tag{5.1}$$

$$+ \int_{d(T_\delta(z),w) \geq \lambda^{-\frac{1}{3}}} |\Pi_{\chi,\lambda}(z,w)|^r \, dw. \tag{5.2}$$

The integration by parts argument for Lemma 4.1.1 can be adapted to show that (5.2) is $O(\lambda^{-\infty})$. We use Lemma 5.1.1 to see that (5.1) is to leading order

$$C\lambda^{r(m-1)} \int_{\mathbb{R}^{2(m-1)}} e^{-r\lambda \frac{|u|^2}{4\tau}} \, du \leq C\lambda^{(r-1)(m-1)}. \tag{5.3}$$

Combining these estimates establishes the desired upper bound. To see that this upper bound is sharp define the "coherent state"

$$\Phi_{\chi,\lambda}^w(z) := \frac{\Pi_{\chi,\lambda}(z,w)}{\|\Pi_{\chi,\lambda}(\,\cdot\,,w)\|_{L^p(\partial M_\tau)}}.$$

Note that $\|\Phi^w_{\chi,\lambda}\|_{L^p(\partial M_\tau)} = 1$ and by (2.19),

$$\Pi_{\chi,\lambda}(\Phi^w_{\chi,\lambda})(z) = \frac{\Pi_{\chi^2,\lambda}(z,w)}{\|\Pi_{\chi,\lambda}(\,\cdot\,,w)\|_{L^p(\partial M_\tau)}}. \tag{5.4}$$

To estimate the numerator of (5.4), we observe

$$\int_{\partial M_\tau} |\Pi_{\chi,\lambda}(z,w)|^q\,dw \geq \int_{d(T_\delta(z),w)\leq D\lambda^{-\frac{1}{2}}} |\Pi_{\chi,\lambda}(z,w)|^q\,dw,$$

so by applying Lemma 5.1.2 to the integrand and a using similar argument used to show (5.3), we may conclude $\|\Pi_{\chi^2,\lambda}(\cdot,w)\|_{L^q(\partial M_\tau)} \geq C\lambda^{(m-1)(1-\frac{1}{q})}$. Therefore, $\|\Pi_{\chi^2,\lambda}\|_{L^p(\partial M_\tau)} \sim \lambda^{(m-1)(1-\frac{1}{q})}$.

Similarly, the denominator of (5.4) is asymptotically $\|\Pi_{\chi,\lambda}(\,\cdot\,,w)\|_{L^p(\partial M_\tau)} \sim \lambda^{(m-1)(1-\frac{1}{p})}$. Together we have

$$\left\|\Pi_{\chi,\lambda}(\Phi^w_{\chi,\lambda})(z)\right\|_{L^q(\partial M_\tau)} \sim \lambda^{(m-1)(1-\frac{1}{q})-(m-1)(1-\frac{1}{p})} = \lambda^{(m-1)(\frac{1}{p}-\frac{1}{q})},$$

which shows $\Phi^w_{\chi,\lambda}$ saturates the upper bound.

5.2. Proof of Theorem 2.2.5: norm estimates for $\Pi_{[\lambda,\lambda+1]}$

A standard argument [**42**, Chapter 5] converts the $L^p \to L^q$ estimate for $\Pi_{\chi,\lambda}$ to that for the projection $\Pi_{[\lambda,\lambda+1]}$ onto a short spectral interval $[\lambda,\lambda+1]$ as defined in (2.20). We include a proof here for the readers' convenience.

Theorem 2.2.5 is equivalent to sharpness of the dual inequality

$$\|\Pi_{[\lambda,\lambda+1]}f\|_{L^2(\partial M_\tau)} \leq C\lambda^{(m-1)(\frac{1}{p}-\frac{1}{2})}\|f\|_{L^p(\partial M_\tau)},$$

which we now establish. For the upper bound, we compute

$$\begin{aligned}
\|\Pi_{[\lambda,\lambda+1]}f\|_{L^2}^2 &= \sum_{\lambda\le\lambda_j<\lambda+1} |\langle f, e_j\rangle|^2 \\
&\le C\sum_{j=1}^{\infty} \chi(\lambda-\lambda_j)^2|\langle f, e_j\rangle|^2 \\
&= C\|\Pi_{\chi,\lambda}f\|_{L^2}^2 \\
&\le \lambda^{2(m-1)(\frac{1}{p}-\frac{1}{2})}\|f\|_{L^p}^2
\end{aligned}$$

To show this upper bound is saturated, fix $w \in \partial M_\tau$ and set $f_{\chi,\lambda}(z) = \Pi_{\chi,\lambda}(z,w)$. We compute

$$\begin{aligned}
\|\Pi_{[\lambda,\lambda+1]}f_{\chi,\lambda}\|_{L^2}^2 &= \sum_{\lambda\le\lambda_j<\lambda+1} |\chi(\lambda-\lambda_j)|^2|e_{\lambda_j}(w)|^2\|e_{\lambda_j}\|_{L^2}^2 \\
&\ge C\sum_{\lambda\le\lambda_j<\lambda+1} |e_j(w)|^2 \\
&\ge \frac{1}{\mathrm{vol}(\partial M_\tau)}\int_{\partial M_\tau}\sum_{\lambda\le\lambda_j<\lambda+1} |e_j(z)|^2\,dz \\
&\ge C(N(\lambda+1)-N(\lambda)) \\
&\sim C\lambda^{m-1},
\end{aligned}$$

where $N(\lambda) = \#\{j : \lambda_j < \lambda\}$ is the eigenvalue counting function.

It follows from the proof of the sharpness of Theorem 2.2.3 that $\|f_{\chi,\lambda}\|_{L^p} \sim \lambda^{(m-1)(1-\frac{1}{p})}$, so

$$\sup_{f\in L^p(\partial M_\tau)} \lambda^{-(m-1)(\frac{1}{p}-\frac{1}{2})} \frac{\|\Pi_{[\lambda,\lambda+1]}f\|_{L^2}}{\|f\|_{L^p}} \geq C\lambda^{-\frac{(m-1)}{2}}(N(\lambda+1)-N(\lambda))^{\frac{1}{2}}$$
$$\geq C + o(1).$$

Taking the lim sup of both sides we get

$$\limsup_{\lambda\to\infty} \sup_{f\in L^p(\partial M_\tau)} \lambda^{-(m-1)(\frac{1}{p}-\frac{1}{2})} \frac{\|\Pi_{[\lambda,\lambda+1]}f\|_{L^2}}{\|f\|_{L^p}} > 0$$

which shows that upper bound is sharp.

5.3. Complexified Gaussian beams as extremals: direct computation

In this section, we show that the L^p estimate of Proposition 2.2.6 on complexified Laplace eigenfunctions is saturated by analytic continuations of Gaussian beams on the round S^2. We use spherical coordinates

$$x = \sin\varphi\cos\theta, \quad y = \sin\varphi\sin\theta, \quad z = \cos\varphi,$$

where $0 \leq \varphi \leq \pi$ and $0 \leq \theta < 2\pi$. The standard spherical harmonics are the joint eigenfunctions

$$-\Delta_{S^2} Y_N^m = N(N+1)Y_N^m, \quad \frac{1}{i}\frac{\partial}{\partial\theta}Y_N^m = mY_N^m, \quad -N \leq m \leq N.$$

of the spherical Laplacian and the angular momentum operator. The highest weight spherical harmonic (Gaussian beam) is of the form

$$Y_N^N(\theta,\varphi) = c_N \sin^N(\varphi) e^{iN\theta}, \quad c_N = \frac{(-1)^N}{2^N N!}\sqrt{\frac{(2N+1)!}{4\pi}} \sim N^{\frac{1}{4}}.$$

It is convenient to transfer the computations of Guillemin–Stenzel [**20**] from Cartesian coordinates to spherical coordinates. In terms of complexified Cartesian coordinates, the Grauert tube $S^2_{\mathbb{C}}$ of the sphere is the set

$$\Big\{(x+i\xi_x, y+i\xi_y, z+i\xi_z) : (x+i\xi_x)^2 + (y+i\xi_y)^2 + (z+i\xi_z)^2 = 1\Big\},$$

and the Grauert tube function is

$$\sqrt{\rho}(x+i\xi_x, y+i\xi_y, z+i\xi_z) = \sinh^{-1}\Big[(\xi_x^2+\xi_y^2+\xi_z^2)^{\frac{1}{2}}\Big]. \tag{5.5}$$

In terms of complexified spherical coordinates, we have

$$\begin{aligned}
x+i\xi_x &= \sin(\varphi+i\xi_\varphi)\cos(\theta+i\xi_\theta)\\
&= \cos(\varphi)\sinh(\xi_\varphi)\sin(\theta)\sinh(\xi_\theta) + \sin(\varphi)\cosh(\xi_\varphi)\cos(\theta)\cosh(\xi_\theta)\\
&\quad + i\Big[\cos(\varphi)\sinh(\xi_\varphi)\cos(\theta)\cosh(\xi_\theta) - \sin(\varphi)\cosh(\xi_\varphi)\sin(\theta)\sinh(\xi_\theta)\Big]\\
y+i\xi_y &= \sin(\varphi+i\xi_\varphi)\sin(\theta+i\xi_\theta)\\
&= -\cos(\varphi)\sinh(\xi_\varphi)\cos(\theta)\sinh(\xi_\theta) + \sin(\varphi)\cosh(\xi_\varphi)\sin(\theta)\cosh(\xi_\theta)\\
&\quad + i\Big[\cos(\varphi)\sinh(\xi_\varphi)\sin(\theta)\cosh(\xi_\theta) + \sin(\varphi)\cosh(\xi_\varphi)\cos(\theta)\sinh(\xi_\theta)\Big]\\
z+i\xi_z &= \cos(\varphi)\cosh(\xi_\varphi) - i\sin(\varphi)\sinh(\xi_\varphi).
\end{aligned}$$

The formula for $\sqrt{\rho}$ in spherical coordinates is complicated. Since we will be simplifying our expressions by picking special values, it suffices to note

$$\xi_x^2+\xi_y^2+\xi_z^2=\cos^2(\varphi)\sinh^2(\xi_\varphi)\cosh^2(\xi_\theta) + \sin^2(\varphi)(\cosh^2(\xi_\varphi)\sinh^2(\xi_\theta)+\sinh^2(\xi_\varphi)). \tag{5.6}$$

We also note that the analytically continued highest weight spherical harmonic is of the form

$$(Y_N^N)^{\mathbb{C}}(\theta+i\xi_\theta,\varphi+i\xi_\varphi)=c_N\Big[\sin(\varphi)\cosh(\xi_\varphi)+i\cos(\varphi)\sinh(\xi_\varphi)\Big]^N e^{iN\theta}e^{-N\xi_\theta}. \tag{5.7}$$

To simplify (5.6) and (5.7), we fix $\varphi=\pi/2$ so that

$$\begin{aligned}\xi_x^2+\xi_y^2+\xi_z^2&=\cosh^2(\xi_\varphi)\sinh^2(\xi_\theta)+\sinh^2(\xi_\varphi),\\ (Y_N^N)^{\mathbb{C}}(\theta+i\xi_\theta,i\xi_\varphi)&=c_N\cosh^N(\xi_\varphi)e^{iN\theta}e^{-N\xi_\theta}\quad (c_N\sim N^{\frac{1}{4}}).\end{aligned} \tag{5.8}$$

We additionally set $\tau=1$, so that (5.5) and (5.8) imply the Grauert tube boundary is given by $\sinh^2(1)=\cosh^2(\xi_\varphi)\sinh^2(\xi_\theta)+\sinh^2(\xi_\varphi)$. Direct computation shows the equality is satisfied whenever

$$-1<\xi_\varphi<1 \quad\text{and}\quad \xi_\theta=-\sinh^{-1}\Big[\operatorname{sech}(\xi_\varphi)\big(\sinh^2(1)-\sinh^2(\xi_\varphi)\big)^{\frac{1}{2}}\Big].$$

Note that $\xi_\theta < 0$, so

$$\begin{aligned}
\|(Y_N^N)^{\mathbb{C}}\|_{L^p}^p &= c_N^p \int_{\partial S_1^2} |\cosh^N(\xi_\varphi)|^p e^{-Np\xi_\theta} d\xi_\theta d\xi_\varphi \\
&\geq c_N^p \int_{-1}^{1} |\cosh^N(\xi_\varphi)|^p e^{Np\sinh^{-1}\left[\operatorname{sech}(\xi_\varphi)\left(\sinh^2(1)-\sinh^2(\xi_\varphi)\right)^{\frac{1}{2}}\right]}\, d\xi_\varphi \\
&\geq c_N^p \int_{-1}^{1} |\cosh^N(\xi_\varphi)|^p\, d\xi_\varphi \\
&\geq c_N^p \int_{0}^{1} e^{Np\xi_\varphi}\, d\xi_\varphi \\
&\sim N^{\frac{p}{4}} \frac{e^{Np}}{Np}.
\end{aligned}$$

In the last line we used $c_N \sim N^{\frac{1}{4}}$. Combined with the universal asymptotics $\|(Y_N^N)^{\mathbb{C}}\|_{L^2(\partial M_\tau)} = N^{-\frac{1}{4}} e^N (1 + O(N^{-\frac{1}{2}}))$ proved in [**48**, Lemma 0.2], we conclude

$$\frac{\|(Y_N^N)^{\mathbb{C}}\|_{L^p(\partial S_1^2)}}{\|(Y_N^N)^{\mathbb{C}}\|_{L^2(\partial S_1^2)}} \sim \frac{N^{\frac{1}{4}} N^{-\frac{1}{p}} e^N}{N^{-\frac{1}{4}} e^N} = N^{\frac{1}{2}-\frac{1}{p}},$$

showing that Proposition 2.2.6 is sharp.

5.4. Complexified Gaussian beams as extremals: geometric explanation

As mentioned earlier, in the real domain Gaussian beams only saturate the low L^p norms whereas zonal spherical harmonics saturate the high L^p norms. As in [**48**], we give a heuristic symplectic geometry explanation for why complexifications of Gaussian beams are also extremals for high L^p norms on ∂S_τ^2.

Let P be the north pole and let $\frac{\partial}{\partial \theta}$ denote the generator of rotation about the z-axis. The zonal spherical harmonics denoted Y_N^0 are semiclassical Lagrangian distributions associated to

$$\Lambda_P = \{g^t(S^*_P S^2) : t \in \mathbb{R}\}$$

Under the natural projection $S^*S^2 \to S^2$ there is a blowdown singularity at P, which leads to peaking of sup norms. This is a heuristic explanation for the zonal harmonics saturating high L^p norms in the real domain.

Now let E denote the equator. The Gaussian beams Y_N^N associated to E have wave-front set

$$\Gamma = \{g^t(p, d\theta) : p \in E,\ t \in \mathbb{R}\}$$

The symplectic cone $\Sigma_\tau = (\partial M_\tau, \mathbb{R}^+ d\alpha) \cong (S^*_\tau M, \mathbb{R}^+ d\xi)$ from (3.7) is the phase space of the Grauert tube boundary. Under the identification (3.8), Λ_P is a Lagrangian sub-manifold embedded in ∂M_τ and no blowdown singularities occur, suggesting that zonal harmonics are longer extremals in the complex domain. Instead, the geodesic flow and the lift of rotations to S^*S^2 coincide on Γ, and Γ is a singular leaf of the foliations of ∂M_τ generated by the geodesic flow together with rotations. This singularity suggests that Gaussian extremizes L^p norms in the complex domain.

CHAPTER 6

Comparison of complexified Laplace eigenfunctions and eigenfunctions of $\Pi_\tau D_{\sqrt{\rho}} \Pi_\tau$

In this chapter we study the relationship between complexified Laplace eigenfunction and eigenfunctions of $\Pi_\tau D_{\sqrt{\rho}} \Pi_\tau$. We first show that Husimi distributions are approximate eigenfunctions for the $\Pi_\tau D_{\sqrt{\rho}} \Pi_\tau$ operator on a general Riemannian manifold. We then provide examples when the two coincide in the case of the flat torus and the round sphere. These results appear in the appendix and final chapters of [**9, 10**]

6.1. Proof of Proposition 2.2.6: complexified Laplace eigenfunctions and eigenfunctions of $\Pi_\tau D_{\sqrt{\rho}} \Pi_\tau$

Here we give a proof of Proposition 2.2.6. In the following we use the parameter μ for the frequencies of $-\Delta$ to distinguish it from the spectral parameter λ used for $\Pi_\tau D_{\sqrt{\rho}} \Pi_\tau$. Set

$$\tilde{\varphi}_\mu^{\mathbb{C}}(z) = \frac{\varphi_\mu^{\mathbb{C}}(z)}{\|\varphi_\mu^{\mathbb{C}}\|_{L^2(\partial M_\tau)}}$$

to be the $L^2(\partial M_\tau)$ normalized complexified Laplace eigenfunction.

On one hand, by the first part of Lemma 3.1.9, we may write $U(i\tau)^*U(i\tau) = (-\Delta)^{-\frac{m-1}{4}} + R$ for some $R \in \Psi^{-\frac{m+1}{2}}(M)$. It follows that

$$\begin{aligned} U(i\tau)\sqrt{-\Delta}^{\frac{m+1}{2}}U(i\tau)^*\tilde{\varphi}_\mu^{\mathbb{C}} &= \frac{U(i\tau)\sqrt{-\Delta}^{\frac{m+1}{2}}U(i\tau)^*U(i\tau)\varphi_\mu}{\sqrt{\langle U(i\tau)^*U(i\tau)\varphi_\mu, \varphi_\mu\rangle}} \\ &= \frac{U(i\tau)\sqrt{-\Delta}\varphi_\mu + U(i\tau)\sqrt{-\Delta}^{\frac{m+1}{2}}R\varphi_\mu}{\sqrt{\langle U(i\tau)^*U(i\tau)\varphi_\mu, \varphi_\mu\rangle}} \\ &= \mu\tilde{\varphi}_\mu^{\mathbb{C}} + O_{L^2(\partial M_\tau)}(1). \end{aligned}$$

In the last equality, the first term follows from the definition of complexification and the eigenvalue equation; the second term follows from L^2 boundedness of $(-\Delta)^{\frac{m+1}{4}}R$ as a zeroth order ΨDO and $U(i\tau)$ being a continuous isomorphism $L^2(M) \to \mathcal{O}^{-\frac{m-1}{4}}(\partial M_\tau)$

On the other hand, by the second part of Lemma 3.1.9, there exists $A \in \Psi^1(\partial M_\tau)$ such that that

$$U(i\tau)\sqrt{-\Delta}^{\frac{m+1}{2}}U(i\tau)^* = \Pi_\tau A \Pi_\tau \quad \text{and} \quad \sigma(A)\big|_{\Sigma_\tau} = \sigma(D_{\sqrt{\rho}}).$$

Therefore, we may write $\Pi_\tau A\Pi_\tau = \Pi_\tau D_{\sqrt{\rho}}\Pi_\tau + \Pi_\tau B \Pi_\tau$ for some $B \in \Psi^0(\partial M_\tau)$. It follows from L^2 boundedness of B that

$$\Pi_\tau D_{\sqrt{\rho}}\Pi_\tau\tilde{\varphi}_\mu^{\mathbb{C}} = \Pi_\tau A\Pi_\tau\tilde{\varphi}_\mu^{\mathbb{C}} - \Pi_\tau B\Pi_\tau\tilde{\varphi}_\mu^{\mathbb{C}} = \mu\tilde{\varphi}_\mu^{\mathbb{C}} + O_{L^2(\partial M_\tau)}(1).$$

By a standard theorem giving the distance to the spectrum (see for example [**57**, Theorem C.11]), if $\lambda_j \in \mathrm{spec}(\Pi_\tau D_{\sqrt{\rho}}\Pi_\tau)$ and $\mu_j \in \mathrm{spec}(\Pi_\tau A \Pi_\tau)$ then there exists $M > 0$ such that for $|\mu_j - \lambda_j| < M$ for all j sufficiently large. Therefore, we can view the $\tilde{\varphi}_\mu^{\mathbb{C}}$ as approximate eigenfunctions for $\Pi_\tau D_{\sqrt{\rho}}\Pi_\tau$.

Additionally, we know $\|\tilde{\varphi}^{\mathbb{C}}_{\lambda_j} - e_{\lambda_j}\|_{L^\infty(\partial M_\tau)} = O(\lambda_j^{\frac{m-1}{2}})$ thanks to [**48**, Theorem 0.1]. Since $\|\tilde{\varphi}^{\mathbb{C}}_{\lambda_j} - e_{\lambda_j}\|_{L^2(\partial M_\tau)} = O(1)$, by the log convexity of L^p norms we get $\|\tilde{\varphi}^{\mathbb{C}}_{\lambda_j} - e_{\lambda_j}\|_{L^p(\partial M_\tau)} = O(\lambda^{(m-1)(\frac{1}{2}-\frac{1}{p})})$.

6.2. Eigenfunctions on Flat Torii

Eigenfunctions of the Laplacian on the flat torus $\mathbb{T}^m = \mathbb{R}^m/\mathbb{Z}^m$ are exponentials $e^{i\langle k,x\rangle}$, where $k \in \mathbb{Z}^m$. The complexification of the torus is $\mathbb{T}^m_{\mathbb{C}} = \mathbb{C}^m/\mathbb{Z}^m$. The square of the Riemannian distance may be analytically continued to $r^2_{\mathbb{C}}(z,\overline{w}) = (z-\overline{w})^2$ on $\mathbb{T}^m_{\mathbb{C}}$, so the Grauert tube function is

$$\sqrt{\rho}(z) = |\mathrm{Im}\, z| = |y| \qquad (z = x+iy).$$

The boundary of the Grauert tube of radius τ is therefore given by

$$\partial\mathbb{T}^m_\tau = \{z = x+iy \in \mathbb{C}^m/\mathbb{Z}^m : |y| = \tau\} \cong \mathbb{T}^m \times S^{m-1}(\tau).$$

Directly computing the Hamilton vector field of $\sqrt{\rho}$ yields

$$D_{\sqrt{\rho}} = \frac{1}{i}\sum_{j=1}^{m} \frac{y_j}{\tau}\frac{\partial}{\partial x_j}. \tag{6.1}$$

Proposition 6.2.1. *For $k \in \mathbb{Z}^m$, complexified exponentials $e^{i\langle k,z\rangle}$ are eigenfunctions of the Toeplitz operator $\Pi_\tau D_{\sqrt{\rho}}\Pi_\tau \colon H^2(\partial\mathbb{T}^m_\tau) \to H^2(\partial\mathbb{T}^m_\tau)$.*

Proof. It was shown in [**26**] that an orthonormal basis for the Hardy space of the Grauert tube boundary $\partial\mathbb{T}^m_\tau$ is given by the set of $L^2(\partial\mathbb{T}^m_\tau)$-normalized complexified eigenfunctions

$$\varphi_k(z) = e^{-\tau|k|}\Gamma_m^{-\frac{1}{2}}(\tau k)(2\pi)^{-\frac{m}{2}} e^{i\langle k,z\rangle} \qquad (k \in \mathbb{Z}^m), \tag{6.2}$$

where

$$\Gamma_m(\eta) = \frac{1}{\mathrm{vol}(S^{m-1})}\int_{S^{m-1}} e^{-2(|\eta|+x\cdot\eta)}\, d\sigma(x).$$

It follows that

$$\Pi_\tau(z,w) = \sum_{k\in\mathbb{Z}^m} \varphi_k(z)\overline{\varphi_k(w)}. \tag{6.3}$$

Combining (6.1), (6.2), and (6.3), we find

$$\begin{aligned}
\Pi_\tau D_{\sqrt{\rho}}\Pi_\tau\varphi_k(w) &= \Pi_\tau\left(\frac{\langle k,y\rangle}{\tau}\varphi_k(z)\right)\\
&= \int_{\partial\mathbb{T}^m_\tau} \varphi_k(w)\overline{\varphi_k(z)}\frac{\langle k,y\rangle}{\tau}\varphi_k(z)\,dz\\
&= \left[\int_{S^{m-1}(\tau)}\int_{\mathbb{T}^m}\frac{\langle k,y\rangle}{\tau}\varphi_k(x+iy)\overline{\varphi_k(x+iy)}\,dxd\sigma(y)\right]\varphi_k(w)\\
&= \left[\frac{1}{(2\pi)^m\Gamma_m(\tau k)}\int_{S^{m-1}(\tau)}\frac{\langle k,y\rangle}{\tau}e^{-2(\tau|k|+\langle k,y\rangle)}\,d\sigma(y)\right]\varphi_k(w),
\end{aligned}$$

which shows that $\varphi_k(w)$ is an eigenfunction of $\Pi_\tau D_{\sqrt{\rho}}\Pi_\tau$, as desired.

The integral representing the eigenvalue of φ_k may be computed asymptotically by Laplace's method to be $\sim |k|$. Hence, the eigenvalues for $\Pi_\tau D_{\sqrt{\rho}}\Pi_\tau$ are asymptotic to those of $\sqrt{\Delta}$. □

6.3. Eigenfunctions on Round Spheres

The unit sphere $S^m = \{x = (x_1, \dots, x_{m+1}) \in \mathbb{R}^{m+1} : x_1^2 + \cdots + x_{m+1}^2 = 1\}$ is complexified as the complex quadric

$$S^m_{\mathbb{C}} = \{z = (z_1, \dots, z_{m+1}) \in \mathbb{C}^{m+1} : z_1^2 + \cdots + z_{m+1}^2 = 1\}. \tag{6.4}$$

The Grauert tube function is computed in [**20**, Section 4] to be

$$\sqrt{\rho}(z) = 2\sin^{-1}(i|\Im z|) = 2i\sinh^{-1}|\Im z| = i\cosh^{-1}|z|^2. \tag{6.5}$$

This is globally well-defined on $S^m_{\mathbb{C}}$, i.e., the maximal Grauert tube of the sphere has infinite radius. Henceforth, we fix some τ and consider the Grauert tube boundary

$$\partial S^m_\tau = S^m_{\mathbb{C}} \cap \{\sqrt{\rho} = \tau\}.$$

We now recall spherical harmonics on S^m. Let

$$\mathcal{P}_\ell = \left\{ \begin{matrix} \text{complex-valued harmonic homogeneous poly-} \\ \text{nomials } p(x) \text{ of degree } \ell \text{ in } n+1 \text{ real variables} \end{matrix} \right\}.$$

A spherical harmonic of degree ℓ is then an element of

$$\mathcal{H}_\ell = \{p|_{S^m} : S^m \to \mathbb{C} : p \in \mathcal{P}_\ell\}.$$

For all $Y_\ell \in \mathcal{H}_\ell$, we have $\Delta_{S^m} Y_\ell = \ell(\ell + n - 1)Y_\ell$.

The analytic extension of a spherical harmonic $Y_\ell = p|_{S^m}$ to the Grauert tube ∂S^m_τ of some fixed radius τ is given by

$$Y_\ell^{\mathbb{C}} = p^{\mathbb{C}}|_{\partial S^m_\tau}(z_1, \ldots, z_{m+1}), \tag{6.6}$$

that is, one analytically continues the polynomial $p\colon \mathbb{R}^{m+1} \to \mathbb{C}$ to the polynomial $p^{\mathbb{C}}\colon \mathbb{C}^{m+1} \to \mathbb{C}$, and then restricts $p^{\mathbb{C}}$ to the intersection of the complex quadric (6.4) and the τ level set of (6.5).

Proposition 6.3.1. *Complexified spherical harmonics* (6.6) *are eigenfunctions of* $\Pi_\tau D_{\sqrt{\rho}} \Pi_\tau \colon H^2(\partial S^m_\tau) \to H^2(\partial S^m_\tau)$.

Proof. We present a representation theoretic argument. The Possion wave operator (2.9) conjugates the $\mathrm{SO}(m+1)$ action on S^m to a transitive and CR holomorphic action on $\partial S^{\mathbb{C}}_\tau$, and hence determines a unitary representation of $\mathrm{SO}(m+1)$ on $H^2(\partial S^{\mathbb{C}}_\tau)$. In particular, the Hardy space decomposes into a direct sum of irreducible finite-dimensional unitary representations of $\mathrm{SO}(m+1)$:

$$H^2(\partial S^{\mathbb{C}}_\tau) = \bigoplus_\ell \mathcal{H}^{\mathbb{C}}_\ell = \bigoplus_\ell \mathrm{span}\{Y^{\mathbb{C}}_\ell\}.$$

Because the $\mathrm{SO}(m+1)$ action is CR holomorphic, it commutes with Π_τ. The action also commutes with $D_{\sqrt{\rho}}$ because the $\mathrm{SO}(m+1)$ orbits preserve level sets of the Grauert tube function. Hence, $\Pi_\tau D_{\sqrt{\rho}} \Pi_\tau$ commutes with the $\mathrm{SO}(m+1)$ action on $\partial S^{\mathbb{C}}_\tau$. Schur's lemma then implies $\Pi_\tau D_{\sqrt{\rho}} \Pi_\tau$ acts by scalars on $\mathcal{H}^{\mathbb{C}}_\ell$ for each ℓ, i.e., $Y^{\mathbb{C}}_\ell$ are eigenfunctions of $\Pi_\tau D_{\sqrt{\rho}} \Pi_\tau$, as desired. $\square$

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