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ABSTRACT

Fiberwise Poincaré–Hopf Theory and Exotic Smooth Structures on Manifold Bundles

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We prove the Rigidity Conjecture of Goette and Igusa, which states that, after rationalizing, there are no stable exotic smoothings of manifold bundles with closed even dimensional fibers. The key ingredients of the proof are fiberwise Poincaré–Hopf theorems generalizing earlier such results about the Becker–Gottlieb transfer. These theorems show how to compute the smooth structure class, an invariant of smooth structures on fiber bundles, using the data of a fiberwise generalized Morse function. We use these results to prove a duality theorem for the smooth structure class, from which the conjecture directly follows. This duality theorem generalizes Milnor's duality theorems for Reidemeister and Whitehead torsion, as well as similar results for higher Franz–Reidemeister torsion due to Igusa.

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CHAPTER 1

Introduction

A *topologically trivial family of smooth h-cobordisms* is a bundle of h-cobordisms that is fiberwise homeomorphic but not fiberwise diffeomorphic to a bundle of product h-cobordisms. In [**GIW14**], Goette, Igusa, and Williams study an equivalent notion: *exotic smooth structures on smooth manifold bundles.* Briefly, a bundle $M' \to B$ is an exotic smoothing of a bundle $M \to B$ if these bundles form the boundary of a topologically trivial family of smooth h-cobordisms.

In [**GIW14**], the authors define the *smooth structure class* $\Theta(M, M')$, an element of the real homology of the total space that distinguishes exotic smooth structures on manifold bundles and is closely related to the higher Franz–Reidemeister torsion. In a subsequent paper, [**GI14**], Goette and Igusa give a procedure to construct exotic smooth structures on a bundle with closed odd dimensional fibers. In other words, they construct topologically trivial families of smooth h-cobordisms with odd dimensional boundaries for which the smooth structure class $\Theta(M, M')$ is nonzero. Furthermore, for fixed M, they show that their procedure for constructing M' generates all possible values of $\Theta(M, M')$, up to linear combinations of rational multiples. They go on to conjecture that when the fibers of M are closed even dimensional manifolds, there are no rationally nontrivial exotic smooth structures on M. I.e., the smooth structure class always vanishes in this case.

In this paper, we prove the Rigidity Conjecture of Goette and Igusa [**GI14**] (See Appendix A for an explanation of how the main theorem below compares to the statement of the original conjecture in [**GI14**]).

Main Theorem (Theorem H)**.** *If the fibers of $p_0 : M \to B$ are even dimensional and closed, then for any topologically trivial family of smooth h-cobordisms $p : W \to B$ from M to M', $\Theta(M \times I, M' \times I)$ is trivial.*

The main theorem stated above has several antecedents for related invariants. These include vanishing theorems for the Euler characteristic, Reidemeister torsion, the Becker–Gottlieb transfer, and the higher Franz–Reidemeister torsion. Vanishing theorems for each of these invariants are proven in stylistically equivalent ways, though the technical ingredients vary widely. They generally follow by applying a Poincaré–Hopf type theorem in combination with Poincaré duality. In order to motivate the contents of this paper, we will briefly describe these theorems and their proofs. This discussion is summarized in the table below. At the end of this section we summarize the proof of our main theorem and give an outline of the contents of this paper.

Recall the classical proof that the Euler characteristic of a closed odd dimensional manifold is zero: The Poincare–Hopf theorem states that the Euler characteristic of a closed manifold M is equal to the sum of the indices of isolated critical points of a Morse function f on M:

$$\chi(M) = \sum_{z \in Z} (-1)^{\mathsf{Ind}_{\nabla f}(z)}$$

On an odd dimensional manifold, $\mathsf{Ind}_{\nabla(-f)}(z)$ and $\mathsf{Ind}_{\nabla f}(z)$ have opposite parity. It follows that $\chi(M) = 0$.

	Euler characteristic	Reidemeister torsion	Becker–Gottlieb transfer	higher torsion τ^{IK}	smooth structure class Θ
Poincaré–Hopf theorem	$\chi(M) = \sum(-1)^{\mathsf{Ind}_X(z)}$	Compute torsion from explicit choice of triangulation	Fiberwise Poincaré–Hopf theorem	Framing Principle	Theorem F
Poincaré duality	$\mathsf{Ind}_X(z) = -\mathsf{Ind}_{-X}(z)$ for odd dimensional manifolds	Recompute for dual cell complex	Recompute fiberwise index map after negating Morse function and compare	Apply the framing principle for a fiberwise GMF f and compare with results for $-f$	Theorem G
⇒Vanishing result	$\chi(M) = 0$ when $\dim M$ is odd	Torsion is trivial on even dimensional manifolds	transfer map on real cohomology vanishes for odd dimensional fibers	higher torsion of even dimensional manifold bundles is MMM class⇒relative torsion of fiberwise tangentially homeomorphic manifold bundles vanishes	Rigidity Conjecture, Theorem H

Table 1.1. Proofs of vanishing theorems by analogy.

We adopt a stylized view of this proof: the vanishing result for the Euler characteristic is proven by applying the Poincaré–Hopf theorem in combination with Poincaré duality.

The Reidemeister torsion is a K-theoretic generalization of the Euler characteristic that admits an analogous vanishing theorem: the Reidemeister torsion of a closed even dimensional manifold is zero. To prove this, recall that the Reidemeister torsion of a manifold is computed in terms of a triangulation of the manifold that can be obtained from a Morse function. We can compare the torsion of one cell decomposition to the torsion of the dual cell decomposition obtained by inverting the Morse function. The specific computation is due to Milnor [**Mil62**], and when the dimension is even it follows that the torsion must be zero. Stylistically this proof is the same as the proof of the vanishing result for the Euler characteristic: the formula for torsion in terms of the data of a triangulation is an instance of a Poincaré–Hopf theorem. The comparison to the dual cell complex is an instance of Poincaré duality. A nearly identical argument is also used to prove a duality theorem for Whitehead torsion [**Mil66**].

The Becker–Gottlieb transfer is a generalization of the Euler characteristic to families. For a smooth bundle of smooth manifolds $p : E \to B$, one can associate a wrong way map of spectra $\Sigma^\infty B_+ \to \Sigma^\infty E_+$. If the base is a point, this is equivalently a map of infinite loop spaces from S^0 to $\Omega^\infty \Sigma^\infty E_+$. On components, if E is connected, we have a map $S^0 \to \mathbb{Z}$, and the non-basepoint element maps to $\chi(E) \in \mathbb{Z}$.

One can easily prove a vanishing result for the Becker–Gottlieb transfer using a small amount of technology. Fiberwise Poincaré–Hopf theorems for the Becker–Gottlieb transfer have been proven by [**BM76**,**Dou06**]. Briefly, assume that X is a smooth nondegenerate vertical vector field on the total space of a smooth bundle $p : E \to B$. This vector field might be obtained by computing the gradient of a fiberwise Morse function, so long as such a function exists. Let Z be the vanishing locus of the vector field, which forms a covering space π over B. Then the fiberwise Poincaré–Hopf theorem is expressed in terms of the following homotopy commutative diagram of spectra:

$$\begin{tikzcd}
\Sigma^\infty B_+ \arrow[r, "\mathsf{tr}_p"] \arrow[d, "\mathsf{tr}_\pi"'] & \Sigma^\infty M_+ \\
\Sigma^\infty Z_+ \arrow[r, "\mathsf{Ind}_X"] & \Sigma^\infty Z_+ \arrow[u, "+"']
\end{tikzcd}$$

In the diagram above, tr_π and tr_p denote the transfers associated to π and p. The map Ind_X denotes a fiberwise index map associated to the vertical vector field X. On real cohomology one can easily prove from the definitions that $(\mathsf{Ind}_X)^* = (-1)^d(\mathsf{Ind}_{-X})^*$, where d denotes the fiber dimension. It follows that $(\mathsf{tr}_p)^* = (-1)^d(\mathsf{tr}_p)^*$. Thus the transfer map on cohomology vanishes when the fiber dimension is odd, if p admits a fiberwise Morse function. In this case the classical Poincaré–Hopf theorem was replaced by a parametrized version, and Poincaré duality arose in the comparison of the vector field and its negative.

A common generalization of the Euler characteristic to both the K-theoretic and parametrized settings is the higher Franz–Reidemeister torsion. This invariant is a characteristic class in the cohomology of the base of a smooth fiber bundle. The primary tool that enables computations of this invariant is Igusa's framing principle [**Igu05**]. The framing principle describes the higher Franz–Reidemeister torsion as the sum of an 'exotic' class and a 'tangential' term

A consequence of the framing principle is that for smooth manifold bundles with closed even dimensional fibers, the torsion class is congruent to a Miller–Morita–Mumford class. To prove this, Igusa compares the framing principle for a fiberwise generalized Morse function f to the analogous formula for $-f$. By studying the canonical involution on the Whitehead space, one can prove that the exotic term is two-torsion. Thus we are left only with the tangential term which agrees with a Miller–Morita–Mumford class.

Once again, this proof is analogous to those from above: the framing principle resembles a Poincaré–Hopf theorem, and the comparison of the formulas for f and $-f$ resembles an application of Poincaré duality. However, the proof of this vanishing theorem requires significantly more technology than those which came previously. In particular, to define the exotic term in the framing principle, Igusa uses a Waldhausen category model for the Whitehead space which encodes the combinatorics of colliding critical points of fiberwise generalized Morse functions. The proof of the framing principle requires an understanding of the deformation properties of the critical loci of fiberwise generalized Morse functions.

We postpone giving a precise definition of the smooth structure class until Section 3.4, however we point out that the smooth structure class is closely related to the invariants discussed above. One explanation for this is that the higher Franz–Reidemeister torsion and the smooth structure class can both be defined in terms of nullhomotopies of maps that factor through the Becker–Gottlieb transfer. An explicit relationship between these invariants at the level of homology groups of the base is proven in [**GI14**]: the pushdown class $p_*\Theta(M, M')$ is congruent to $D\tau^{IK}(M, M')$, the Poincaré dual of the relative higher torsion. Thus one should expect that a proof of the main theorem above should follow from a sufficiently general version of a Poincaré–Hopf theorem along with an application of Poincaré duality. This paper provides such a proof, which is summarized in the next section.

1.1. Proof Summary

In this paper we prove a vanishing result for the smooth structure class, an invariant of smooth structures on fiber bundles introduced by Goette, Igusa, and Williams in [**GIW14**], after work of Dwyer, Weiss, and Williams [**DWW03**]. In analogy with the examples above, the proof is an application of a Poincaré–Hopf type theorem in combination with Poincaré duality. In this section we give a precise outline of the proof.

By a fiberwise Poincaré–Hopf theorem we broadly mean a computation of a fiberwise characteristic, e.g. the Becker–Gottlieb transfer, the excisive A-theory Euler characteristics, etc., in terms of the critical locus of a fiberwise generalized Morse function. Examples of such theorems can be found in [**BM76**, **CJ98**, **Dou06**]. These theorems generalize the classical Poincaré–Hopf theorem, which computes the Euler characteristic in terms of local data at the isolated critical points of a Morse function.

The following is a bullet-pointed outline of the proof of the main theorem.

(0) These background items are necessary for this outline:

- The smooth structure class $\Theta(M, M')$ is an element of $\pi_0\Gamma_B\mathcal{H}^{\%}_B(M)\otimes\mathbb{Q}$. The space $\Gamma_B\mathcal{H}^{\%}_B(M)$ is the space of sections of the fiberwise homology bundle obtained by taking fiberwise smash products with the stable h-cobordism space of a point. See Section 3.4 for a precise definition.

- By the stable parametrized h-cobordism theorem, $\Gamma_B\mathcal{H}^{\%}_B(M)$ is the homotopy fiber of the map $\Gamma_B Q_B(M) \to \Gamma_B A^{\%}_B(M)$, which is induced by the unit map from the sphere spectrum to $A(*)$. The spectrum $A(*)$ is the Waldhausen K-theory of spaces functor, otherwise known as A-theory, evaluated at a point. The functor $A^{\%}$ is the excisive approximation to A-theory.

- All smooth bundles admit fiberwise generalized Morse functions by [**Igu84**, **Lur09**, **EM12**]. In stark contrast, smooth bundles rarely admit fiberwise Morse functions.

- If a family of h-cobordisms $p : W \to B$ with boundaries $p_0 : M_0 \to B$ and $p_1 : M_1 \to B$ is *topologically trivial*, then W is fiberwise tangentially homeomorphic to $M_0 \times I$. This data produces a nullhomotopy of the excisive A-theory Euler characteristic of $\chi^{\%}(W, \partial_0 W)$, which are used to define the smooth structure characteristics $\theta(W, \partial_0 W)$ in Definition 4.1. Likewise, we define the smooth structure characteristic $\theta(W, \partial_1 W)$.

(1) We prove a fiberwise Poincaré–Hopf theorem for the Becker–Gottlieb transfer, an element of $\Gamma_B Q_B(W)$. This result, Theorem A, expresses the transfer in terms of the critical locus of a fiberwise generalized Morse function on W.

(2) In Theorem B, we further generalize the fiberwise Poincaré–Hopf theorem for the Becker–Gottlieb transfer to the excisive A-theory characteristic, an element of $\Gamma_B A_B^{\%}(W)$. Furthermore, this factorization is compatible with the factorization of the Becker–Gottlieb transfer in the previous theorem. This is Theorem C.

(3) We generalize Theorems A, B, and C to be invariant under any *stratified deformation* of the critical locus of a fiberwise generalized Morse function. This yields Theorems D and E.

(4) Theorem F gives a fiberwise Poincaré–Hopf theorem for the smooth structure characteristic. This theorem is different from those preceding it because it is a rational statement. This theorem relies on Theorem D, in that the smooth structure class is expressed in terms of a particular stratified deformation of the critical locus of a fiberwise generalized Morse function. The stratified deformation that we use encodes a parametrized handle cancellation argument used by Hatcher in [**Hat75**] and Igusa in [**Igu84**, **Igu88**, **Igu02**, **Igu05**]. We summarize this construction in Subsection 4.3.

(5) Theorem F is used to prove a duality theorem for the smooth structure class, Theorem G, by inverting the Morse function. Theorem H, equivalent to the Rigidity Conjecture, follows from this duality theorem.

To complete the analogy in the exposition from the previous section, we will indicate how the proof of the main theorem, Theorem H, can be thought of as an application of a Poincaré–Hopf type theorem and Poincaré duality. The Poincaré–Hopf type theorem that we ultimately apply is Theorem F, and as the outline indicates, this is a generalization of other Poincaré–Hopf theorems that we prove along the way. The duality theorem for the smooth structure class, Theorem G, is our instance of Poincaré duality.

CHAPTER 2

Related Work

Two main ideas arising in this paper, fiberwise Poincaré–Hopf theorems and the Hatcher construction, are of independent interest. In this section we give a brief survey of the literature and recent progress pertaining to both keywords. We omit a discussion of the most immediate literature pertaining to the Rigidity Conjecture, including [**DWW03**, **GIW14**, **GI14**], as detailed descriptions of these works appear elsewhere in this paper.

Hatcher's construction associates to an element of the kernel of the J-homomorphism a disk bundle which is fiber homotopy trivial but not smoothly trivial. The construction should be interpreted as a stable map from G/O to $\Omega \mathrm{Wh}^{\mathrm{Diff}}(*)$. Waldhausen gave a different formulation of the same map in [**Wal82**], and in [**Bök84**] Bokstedt proved this map to be a rational homotopy equivalence. Later Igusa gave another proof using parametrized Morse theory. Exciting new developments by Kragh [**Kra18**] have identified the homotopy fiber of the Hatcher–Waldhausen map as a certain functional space $\mathcal{M}_\infty$ considered by Eliashberg and Gromov in [**EG98**], establishing a connection between the study of Lagrangians to algebraic K-theory of spaces. Kragh associates to every exact Lagrangian an element of $\pi_*(\mathcal{M}_\infty)$. If any of these examples were proven to be nontrivial, they would be counterexamples to the nearby Lagrangian conjecture in symplectic topology. In essence, counter examples to the nearby Lagrangian conjecture might be found in the kernel of Hatcher's construction. Recent work by Igusa and Alvarez-Gavela elaborates further on these developments [**ÁGI21**].

Goodwillie, Igusa, and Ohrt have developed an equivariant version of Hatcher's construction [**GIO15**]. Ordinarily, the space G/O classifies vector bundles whose spherical fibrations are fiber homotopy trivial. In the equivariant version, G/O is replaced by the space G_n/U, which is the colimit of spaces $G_n(N)/U(N)$, classifying rank N complex vector bundles together with a C_n-equivariant fiber homotopy trivialization of the associated sphere bundle. The equivariant Hatcher construction is then a map $G_n/U \to \mathcal{H}^s(BC_n)$, where the target is the stable h-cobordism space of the classifying space of C_n. The geometric outcome of the construction is no longer a disk bundle, but instead a bundle of h-cobordisms of the product of a disk with lens spaces.

Bunke and Gepner have reformulated the Becker–Gottlieb transfer in the context of derived algebraic K-theory [**BG13**]. Their work leads to the Transfer Index Conjecture, essentially a derived version of the parametrized index theorem of Dwyer, Weiss, and Williams. This conjecture suggests as a corollary the existence of certain classes in algebraic K-theory of a ring of integers in a number field. The authors prove that the Hatcher construction produces nontrivial representatives for these classes in special cases.

The Farell–Hsiang [**FH78**] results on diffeomorphism groups of disks relative to their boundary prove that in the pseudoisotopy stable range,

$$\pi_i \mathrm{BDiff}_\partial(D^n) \otimes \mathbb{Q} = \begin{cases} \mathbb{Q} & \text{if } i = 0 \mod 4 \text{ and } n \text{ odd} \\ 0 & \text{otherwise} \end{cases} \tag{2.1}$$

According to [**Igu02**], the nontrivial generator for odd dimensional disks can be obtained from Hatcher's construction. There has been much recent progress outside of the stable range by a number of authors, including distinct but related work by Kupers, Randal-Williams, Watanabe, and Weiss. In [**Wei15**] Weiss identifies nontrivial Pontryagin classes $p_{n+k} \in H^{4n+4k}(\mathrm{BTop}(2n);\mathbb{Q})$. These are shown to evaluate nontrivially on $\pi_{4n+4k}(\mathrm{BTop}(2n);\mathbb{Q})$. It then follows by the Morlet equivalence that there must be nonzero rational homotopy in $\pi_*(\mathrm{BDiff}_\partial(D^{2n}))$ outside of the stable range. One might reasonably suspect that these examples could produce unstable exotic smoothings of manifold bundles. However, stably these classes are trivial, so they do not provide counterexamples to the Rigidity Conjecture.

Fiberwise Poincaré–Hopf theorems first appeared in work by Brumfiel and Madsen [**BM76**], and have proven to be a useful computational tool with many applications. For instance, in [**MT01**] the authors then apply the theorem towards early progress on the Mumford conjecture. Similar theorems are used in [**RW08**] to compute the $\mod 2$ homology of the stable nonorientable mapping class group, as well as in [**Rei19**] to establish the existence of 'non-kinetic' smooth bundles over the classifying space $BSU(2)$. Douglas [**Dou06**] gave alternative proofs to [**BM76**] using Dold's Euclidean neighborhood rectracts. This paper establishes fiberwise Poincaré–Hopf theorems for smooth manifold bundles admitting fiberwise Morse functions, and provided motivation for the present work. Our results provide fiberwise Poincaré–Hopf theorems for bundles that admit fiberwise generalized Morse functions. This is a significant strengthening, as all bundles admit fiberwise generalized Morse functions, but bundles rarely admit fiberwise Morse functions.

This paper is concerned with a characteristic in the homotopy fiber of the fibration

$$\Gamma_B \mathcal{H}^{\%}_B(M) \to \Gamma_B Q_B(M) \to \Gamma_B A^{\%}_B(M)$$

that arises from a nullhomotopy of the excisive A-theory characteristic. One could naturally ask about characteristics in the homotopy fiber of the fibration

$$\Gamma_B \Omega \mathrm{Wh}^{\mathrm{PL}}_B(M) \to \Gamma_B A^{\%}_B(M) \to \Gamma_B A_B(M)$$

given by a nullhomotopy of the ordinary A-theory characteristic. This is the premise of Steimle's PhD thesis [**Ste10**], in which the author studies the 'parametrized excisive characteristic'. One of the main technical results of their work is an additivity theorem for the parametrized excisive characteristic, which parallels the Poincaré–Hopf theorems in the present work, if they were restricted to bundles admitting fiberwise Morse decompositions. For comparison, the smooth structure class appearing in $\Gamma_B \mathcal{H}^{\%}_B(M)$ concerns the existence of stable exotic smoothings of fiber bundles, whereas the parametrized excisive characteristic appearing in $\Gamma_B \Omega \mathrm{Wh}^{\mathrm{PL}}_B(M)$ concerns the existence of topological manifold bundles whose projection maps are homotopic to stabilizations of an arbitrary map of compact topological manifolds.

CHAPTER 3

Characteristics associated with topologically trivial families of h-cobordisms

In this section we formally introduce topologically trivial families of smooth h-cobordisms, with the goal of defining the smooth structure characteristic. We begin in Subsection 3.1 with the definition of a topologically trivial families of smooth h-cobordisms, and introduce the immersed Hatcher construction as the main example. In Subsection 3.2, we identify the moduli space of topologically trivial h-cobordisms as the homotopy fiber of the forgetful map from the space of smooth h-cobordisms to the space of topological h-cobordisms. In Subsection 3.3 we introduce the smooth structure characteristic, a characteristic of topologically trivial families of smooth h-cobordisms defined as a lift of the Becker–Gottlieb transfer. The content of this section is used in Subsection 5.5 and Section 6 where the Theorems of Section 5 are applied to topologically trivial families of smooth h-cobordisms.

3.1. Topologically trivial families of h-cobordisms

In this subsection we define topologically trivial families of h-cobordisms, and then provide examples of such objects.

Definition 1.1. A smooth family of h-cobordisms $p : W \to B$ with boundaries $\partial_0 W := M$ and $\partial_1 W := M'$ given as smooth manifold bundles $p_0 : M \to B$ and $p_1 : M' \to$

B is topologically trivial if there exists a fiberwise homeomorphism $h : W \to M \times I$ over B.

Remark 1.2. When the base is a point, the Whitehead torsion of a topologically trivial h-cobordism must be trivial, and thus by the s-cobordism theorem W must be a cylinder. This is not necessarily the case when the base is not contractible.

Example 1.3. Hatcher's construction takes as input a vector bundle classified by G/O and produces disk bundles that are fiberwise homeomorphic to a trivial disk bundle, but not fiberwise diffeomorphic. The immersed Hatcher construction [**GI14**] utilizes Hatcher's disk bundles to produce topologically trivial h-cobordisms. Briefly, given a smooth manifold bundle $p_0 : M \to B$, they consider the bundle $p_0 \times I : M \times I \to B$ and glue a family of handles parametrized by B on the outgoing boundary $M \times 1 \to B$. This family of handles is essentially one of Hatcher's disk bundles. The result of this construction is a topologically trivial family of h-cobordisms, as the bundle remains fiberwise homeomorphic to $M \times I \to B$. See Sections 6.4.1 and 6.4.2 or [**GI14**] for details of the construction.

3.2. Moduli spaces of h-cobordisms

Given a smooth manifold F, we now define the space of smooth h-cobordisms on F, the space of topological h-cobordisms on F, and the space of topologically trivial h-cobordisms on F. We also introduce notation for the stable versions of these spaces.

Definition 2.1. Let $H^t(F)$ denote the space of topological h-cobordisms on F. This space is defined to be the geometric realization of a simplicial set $H^t_\bullet(F)$. A k-simplex in

$H^t_\bullet(F)$ is a topological manifold bundle $\pi : E \to \Delta^q$ for which each fiber $W_p = \pi^{-1}(p)$ is a topological h-cobordism on F. We denote by $H^t_B(F)$ the mapping space $|H^t_\bullet(F)|^B$.

Definition 2.2. Let $H^d(F)$ denote the space of smooth h-cobordisms on F. This space is defined to be the geometric realization of a simplicial set $H^d_\bullet(F)$. A k-simplex in $H^d_\bullet(F)$ is a smooth bundle $\pi : E \to \Delta^q$ for whcih each fiber $W_p = \pi^{-1}(p)$ is a smooth h-cobordism on F. We denote by $H^d_B(F)$ the mapping space $|H^d_\bullet(F)|^B$.

Definition 2.3. Let $H^{t/d}(F)$ denote the space of topologically trivial h-cobordisms on F. This space is defined to be the geometric realization of a simplicial set $H^{t/d}_\bullet(F)$. A k-simplex in $H^{t/d}_\bullet(F)$ is a pair (π, h) for which the map $\pi : E \to \Delta^q$ is a smooth bundle such that each fiber $W_p = \pi^{-1}(p)$ is a smooth h-cobordism on F. The map h is a homeomorphism from E to $F \times I \times \Delta^k$ over Δ^k. We denote by $H^{t/d}_B(F)$ the mapping space $|H^{t/d}_\bullet(F)|^B$.

Let $\mathcal{H}^X_B(F)$ denote the stabilizations of the spaces $H^X_B(F)$ with respect to stabilization maps $H^X_B(F) \to H^X_B(F \times I)$ for X being any of $t, d,$ or t/d.

Proposition 2.4. The space $\mathcal{H}^{t/d}(F)$ is the homotopy fiber of the forgetful map $\mathcal{H}^d(F) \to \mathcal{H}^t(F)$ over $F \times I$.

Proof. It suffices to see that the unstable space $H^{t/d}(F)$ is the homotopy fiber of the forgetful map $H^d(F) \to H^t(F)$. Let (π, h) be a zero simplex in $H^{t/d}_\bullet(F)$. That is, $\pi : E \to *$ is a smooth manifold bundle with E is a smooth h-cobordism on F, and h is a homeomorphism from $F \times I$ to E. Then π is clearly a point in $H^d(F)$. It remains to show that h is equivalent to a 1-simplex in $H^t_\bullet(F)$. Thus from h we must obtain a one parameter family of topological h-cobordisms on F that starts at $F \times I$ and ends at E. Consider the h-cobordism $F \times I \cup_{\partial_0} E$ in $H^d(F)$, which is diffeomorphic to E, and homeomorphic to $F \times [-1,1]$. Let $\pi' : E \cup F \times I \to [-1,1]$ be the new projection map for this family of topological h-cobordisms, and consider the h-cobordisms given by $\pi^{-1}[-t,1]$ for $t \in [0,1]$. This is the desired one parameter family of topological h-cobordisms. $\square$

3.3. Characteristics of h-cobordisms and the Dwyer–Weiss–Williams pullback square

In this section we introduce the relative Becker–Gottlieb transfer and relative excisive A-theory Euler characteristic on families of smooth and topological h-cobordisms, respectively. We then recall a result from [**DWW03**] which situates these characteristics in a homotopy pullback square.

Associated to a smooth manifold bundle $p : M \to B$ with compact fibers, our characteristics will be points in the section spaces $\Gamma_B Q_B(M_+)$ and $\Gamma_B A^{\%}_B(M)$. Roughly speaking, these spaces are sections of the fiberwise homology bundles obtained by taking a fiberwise smash product with the sphere spectrum $\mathbb{S}$ and the algebraic K-theory of spaces functor evaluated at a point, $A(*)$. Moreover, these section spaces are related by a map $\eta : \Gamma_B Q_B(M_+) \to \Gamma_B A^{\%}_B(M)$ induced by the unit map $\mathbb{S} \to A(*)$.

The Becker–Gottlieb transfer is a section $\mathsf{tr}(p) \in \Gamma_B Q_B(M_+)$, for which the composition of $\mathsf{tr}(p)$ and the inclusion map $Q_B(M_+) \hookrightarrow Q(M_+)$ is the usual transfer $B \to Q(M_+)$. The excisive A-theory Euler characteristic is a section $\chi^{\%}(p) \in \Gamma_B A^{\%}_B(M)$. More precise homotopical formulations of these spaces and exact definitions of these characteristics are given in Section 5.

Given that our purpose is to study h-cobordisms, we also require *relative* versions of these characteristics. When considering a smooth h-cobordism bundle $p : W \to B$ with boundaries $\partial_0 W := M$ and $\partial_1 W := M'$ given as smooth manifold bundles $p_0 : M \to B$ and $p_1 : M' \to B$, we require versions of the Becker–Gottlieb transfer and excisive A-theory Euler characteristics that are relative to $\partial_0 W$.

We denote by $\mathsf{tr}_\partial(p)$ the section $r_* \,\mathsf{tr}(p) - \mathsf{tr}(p_0)$ in $\Gamma_B Q_B(M)$ where r is the retraction of W onto M. We denote by $\chi^\%_\partial(p)$ the section $r_*\chi^\%(p) - \chi^\%(p_0)$ in $\Gamma_B A^\%_B(M)$.

Dwyer, Weiss, and Williams proved the following theorem relating $\mathsf{tr}(p)$ and $\chi^\%(p)$.

Theorem (Index Theorem [**DWW03**])**.** *For $p : M \to B$ a bundle of compact smooth manifolds, $\chi^\%(p) \in \Gamma_B A^\%_B(M)$ is fiberwise homotopic to $\eta \circ \mathsf{tr}(p)$.*

The theorem above implies the commutativity of the diagram in the following stronger theorem about h-cobordisms also proven by Dwyer, Weiss, and Williams.

Theorem (Corollary 12.3 in [**DWW03**])**.** *The following diagram is a homotopy pull-back square:*

(3.1)
$$\begin{array}{ccc} \mathcal{H}^d_B(F) & \xrightarrow{\mathsf{tr}_\partial(p)} & \Gamma_B Q_B(M_+) \\ {\scriptstyle forget}\downarrow & & \downarrow{\scriptstyle \eta} \\ \mathcal{H}^t_B(F) & \xrightarrow{\chi^\%_\partial(p)} & \Gamma_B A^\%_B(M) \end{array}$$

Remark 3.1. This theorem is an essential step in the proof of Dwyer, Weiss, and William's converse Riemann–Roch theorem.

Corollary 3.2. *The homotopy fiber of the left vertical arrow in diagram (3.1) is the space $\mathcal{H}_B^{t/d}(F)$, and the homotopy fiber of the right vertical arrow is the space $\Gamma_B \mathcal{H}_B^{\%}(M)$, the space of sections of the space obtained by taking a fiberwise smash product with $\mathcal{H}(*)$. The induced map on these homotopy fibers, $\theta : \mathcal{H}_B^{t/d}(F) \to \Gamma_B \mathcal{H}_B^{\%}(M)$ is a homotopy equivalence.*

3.4. The smooth structure characteristic

In light of Corollary 3.2, we will now define the smooth structure characteristic for a single family of topologically trivial h-cobordisms. Associated to a topologically trivial family of h-cobordisms is a canonical nullhomotopy of the excisive A-theory Euler characteristic. This nullhomotopy is used to define the smooth structure characteristic, as in the following definition.

Definition 4.1. The smooth structure characteristic of a topologically trivial family of h-cobordisms $p : W \to B$, denoted $\theta(W, \partial_0 W)$, is a section in $\Gamma_B \mathcal{H}_B^{\%}(W)$ canonically determined by the point $\mathsf{tr}_\partial(p) \in \Gamma_B Q_B(W)$ over $\chi_\partial^{\%}(p) \in \Gamma_B A_B^{\%}(W)$, and the canonical path from $\chi_\partial^{\%}(p)$ to $\chi^{\%}(M)$ determined by the fiberwise homeomorphism $h : W \to M \times I$.

Remark 4.2. In keeping with the definition of higher smooth torsion due to [**DWW03**] as a nullhomotopy of the composition of the Becker–Gottlieb transfer with the map $Q(M_+) \to K(\mathbb{Z})$, Definition 4.1 presents the smooth structure characteristic as a nullhomotopy of the excisive A-theory Euler characteristic. Thus, we may think of the smooth structure characteristic as a refinement of the higher smooth torsion.

CHAPTER 4

Review of Generalized Morse Functions

In this section we introduce the theory of generalized Morse functions used in the rest of this paper. In Subsection 4.1 we define generalized Morse functions, give the local properties of these functions as they vary in families, and prove the transversality result that implies that the critical locus of a fiberwise generalized Morse function is a submanifold of the total space. In Subsection 4.2 we introduce *ghost sets*, a perturbation of the critical locus of a fiberwise generalized Morse function in the neighborhood of a birth-death singularity which will be used in subsequent sections to compute characteristics associated to the critical locus. In Subsection 4.3 we introduce stratified subsets as a generalization of the critical locus of a fiberwise generalized Morse function. We define stratified deformations of stratified subsets, and give a general purpose construction that deforms the critical locus into two degrees. This property is used in Subsection 5.5 to prove a fiberwise Poincaré–Hopf theorem for the smooth structure characteristic, and in Subsection 6.2 to prove a duality theorem for the smooth structure characteristic.

4.1. Definitions and transversality properties of generalized Morse functions

We begin with the definition of a *generalized Morse function*.

Definition 1.1. A generalized Morse function on a single manifold M is a function $f : (M, \partial_0 M) \to (I, 0)$ that admits only Morse and birth-death critical points.

In local coordinates, a Morse critical point of the function f can be written in the form

$$f(x) = -x_1^2 - \cdots - x_i^2 + x_{i+1}^2 + \cdots + x_n^2$$

with respect to coordinates $(x_1, \cdots, x_n) \in \mathbb{R}^i \times \mathbb{R}^{n-i}$. At a birth-death critical point, the normal form is as follows:

$$f(x) = -x_1^2 - \cdots - x_{i-1}^2 + x_i^3 + x_{i+1}^2 + \cdots + x_n^2$$

In this paper, we are interested in families of generalized Morse functons. Let $f : (W, \partial_0 W) \to (I, 0)$ be a fiberwise generalized Morse function, meaning that its restriction to each fiber is a generalized Morse function. Igusa proved that such functions always exist on smooth fiber bundles when the dimension of the fiber is at least the dimension of the base [**Igu90**], and this dimensionality condition was later relaxed by independent work of Lurie [**Lur09**] and Eliashberg–Misachev [**EM12**]. Our goal is to recall the local behavior of such functions, and to illustrate the key transversality property enjoyed by their critical loci.

In the parametrized setting, we have the following proposition/summary from [**Igu05**]:

Proposition 1.2. In a generic p-parameter family of generalized Morse functions, birth-death points occur on a codimension one subspace of the parameter space. The family of functions f_t has the form

$$f_t(x) = -x_1^2 - \cdots - x_{i-1}^2 + x_i^3 + t_0 x_i + x_{i+1}^2 + \cdots + x_n^2$$

with respect to parameter coordinates $t_0, \cdots t_{p-1}$ and t-dependent local coordinates $(x_1, \cdots, x_n)$ for M.

The coordinate t_0 in the proposition above is often referred to as the 'unfolding directon' associated to the birth-death critical point. This suggests the following useful depiction of a birth-death critical point, which might be thought of as a 'cancellation' of Morse critical points, or their associated handles.

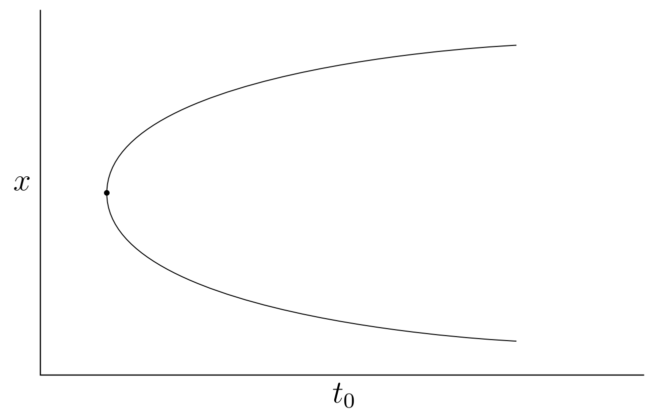

Figure 4.1. A depiction of the critical locus of a fiberwise generalized Morse function in the local neighborhood of a birth-death critical point.

To obtain our desired transversality result, we proceed to compute the Hessian of f at a birth-death critical point.

Let $p : W^m \to B^k$ be a smooth fiber bundle with compact base and fiber F^n. Let $f : (W, \partial_0 W) \to (I, 0)$ be a fiberwise generalized Morse function as before. Then by the proposition above, in local coordinates at a birth-death singularity we have

$$f_t(x) = -x_1^2 - \cdots - x_i^2 + x_{i+1}^3 + t_0 x_{i+1} + x_{i+2}^2 + \cdots + x_n^2.$$

where t_0 is the unfolding direction. The gradient of this function is a section $W \to TW$, and if we take the gradient with respect to fiber coordinates, we have a section of the vertical tangent bundle, $W \to T^\vee W$. We can explicitly compute the map $\nabla f : W \to T^\vee W$ as

$$(x,t) \mapsto \langle -2x_1, \cdots, -2x_i, 3x_{i+1}^2 + t_0, 2x_{i+2}, \cdots, 2x_n \rangle$$

The derivative of ∇f is a map on tangent spaces with the last map in the composition below being the projection off of the nonidentity component.

$$T_{(x,t)}W \mapsto T_{\nabla f(x,t)}(T^\vee W) \cong T^\vee W \oplus T^\vee W \to T^\vee W$$

This map takes the form of a rectangular matrix of size $(2n+k) \times (n+k)$, which is written below. The tangent space in the domain is labeled using coordinates $t_0, \cdots, t_{k-1}$ in the base, and $x_1, \cdots, x_n$ in the fiber. In the target we add labels $\frac{\partial}{\partial x_i}$ for $i, \cdots, n$ for the coordinates in the vertical tangent direction. Keep in mind that in the neighborhood of a birth-death singularity, t_0 is always identified with the 'unfolding' direction.

$$
\begin{array}{r c}
 & \begin{array}{cc} \overbrace{\hphantom{xxxxxxxxxxxxxxxxxxxx}}^{x} & \overbrace{\hphantom{xxxxxxxxxxxxxx}}^{t} \end{array} \\
\begin{array}{r} x\left\{\vphantom{\begin{array}{c}1\\1\\1\\1\\1\\1\\1\end{array}}\right. \\ t\left\{\vphantom{\begin{array}{c}1\\1\\1\\1\\1\\1\\1\end{array}}\right. \\ \frac{\partial}{\partial x}\left\{\vphantom{\begin{array}{c}1\\1\\1\\1\\1\\1\\1\end{array}}\right. \end{array} &
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-2 & & & & & & & & & & & & & \\
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 & & -2 & & & & & & & & & & & \\
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 & & & & & \ddots & & & & & & & & \\
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\end{array}\right) \\
 & \begin{array}{cc} \hphantom{xxxxxxxxxxxxxxxxxxxxxx} & \underbrace{\hphantom{xx}}_{t_0}\hphantom{xxxxxxxxxxxx} \end{array}
\end{array}
$$

We began with a smooth map $\nabla f : W \to T^\vee W$, and now we can check to see whether the image of this map is transverse to the inclusion of the zero section of $T^\vee W$, $i_0 : W \to T^\vee W$. If ℓ is in $\nabla f(W) \cap i_0 W$, then $\nabla f(W)$ is transverse to $i_0 W$ if, for all a, b, ℓ

so that $\nabla f(a) = i_0(b) = \ell$,

$$\mathrm{Im}(D(\nabla f)(a)) \oplus \mathrm{Im}(D(i_0)(b)) \twoheadrightarrow T_p(T^\vee W).$$

It is clear that the intersection $\ell \in \nabla f(W) \cap i_0 W$ is the set of critical points of f, and as these points are either Morse critical points, or birth-death singularities, we handle each of these cases independently. In the event that ℓ is a Morse critical point, it is a standard exercise that the map above is surjective. The case of a birth-death singularity is identical, except in the row labelled by $\frac{\partial}{\partial x_{i+1}}$. At the birth-death singularity, the entry $6x_{i+1}$ vanishes, and if it were not for the 1 in the t_0 entry of the row, there would be no image in the 1-dimensional subspace spanned by $\frac{\partial}{\partial x_{i+1}}$ of the matrix above. So we do have surjectivity and thus transversality, but only because of the derivative in the unfolding direction t_0. So transversality is a direct consequence of the unfolding behavior of a parametrized family of generalized Morse functions. We summarize this discussion in the lemma and corollary below:

Lemma 1.3. For $f : (W, \partial_0 W) \to (I, 0)$ a fiberwise generalized Morse function on a smooth fiber bundle $W \to B$, the section $\nabla f : W \to T^\vee W$ is transverse to the zero section of the vertical tangent bundle of W.

Corollary 1.4. *If Σ_f denotes the critical locus of a fiberwise generalized Morse function f, then the normal bundle $\nu(\Sigma_f) \to \Sigma_f$ to the embedding $\Sigma_f \hookrightarrow W$ is isomorphic to the restriction of the vertical tangent bundle of W to Σ_f, $T^\vee W|_{\Sigma_f}$.*

4.2. Cancelling critical points and ghost sets

In this section we will consider the submanifold Σ_f of W, introduce notation for the submanifolds made up of Morse critical points and birth-death critical points, and discuss how to perturb the critical locus to facilitate the proofs of the fiberwise Poincaré–Hopf theorems appearing later in this paper.

Let Z_i denote the submanifold of Morse critical points of degree i, where the degree is the number of negative eigenvalues of the Hessian of f at any point in Z_i. In general the collection of such critical points may have more than one component, but we will not introduce extra notation for this level of generality. Instead, we assume that Z_i is connected, and note that all proofs in this paper can easily be generalized to accomodate multiple components. The collection of all such Z_i is denoted $\mathscr{S}(\Sigma_f)$.

The submanifolds Z_i and Z_{i+1} share a common boundary which we denote by Z_i^1. The submanifold Z_i^1 contains the birth-death critical of degree i. Again, there may be more than one component of birth-death critical points of degree i, but we elect not to work at that level of generality.

The diagram below depicts how Z_i, Z_{i+1}, and Z_i^1 are arranged in the neighborhood of a birth-death critical point.

The image of the birth-death set in B, $p(Z_1^i)$, is known as the *bifurcation set*. The bifurcation set is a codimension zero submanifold of B. In the neighborhood of a birth-death singularity, the points locally given by $x_i = 0$ and $-\epsilon < t_0 < 0$ are inflection points on which the second derivative of f in the vertical direction vanishes [**Igu05**]. We call

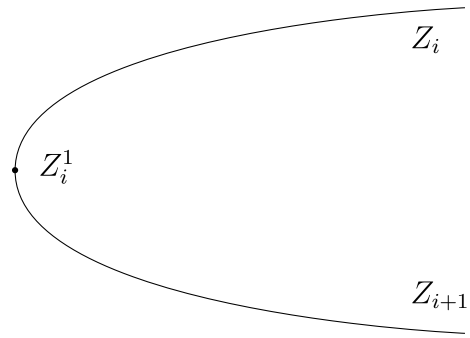


Figure 4.2. The critical submanifolds Z_i and Z_{i+1} share a common boundary Z_i^1.

these points *ghost points*, and they allow us to define a *ghost set*, Z_i^g, which is formally a lift of a one-sided collar neighborhood of the bifurcation set. The ghost set is transverse to Z, as in the following figure.

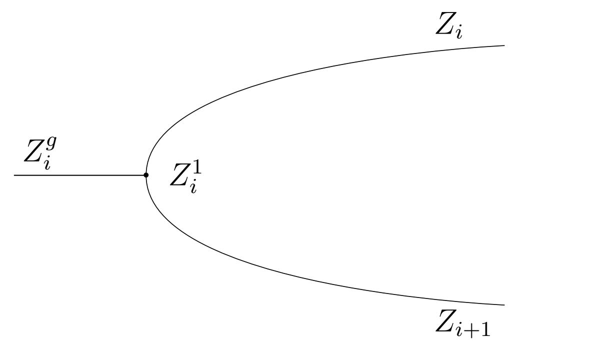


Figure 4.3. The ghost set is transversally attached to the critical locus at the birth-death set.

The ghost set is used to locally perturb Z so that the critical points of the generalized Morse function do not cancel over the ghost set. In particular, we consider manifolds with corners $\widehat{Z}_i := Z_i \cup Z_i^g$ and $\widehat{Z_{i+1}} := Z_{i+1} \cup Z_i^g$ and we smooth both of these manifolds to obtain manifolds $\widetilde{Z}_i$ and $\widetilde{Z_{i+1}}$ that are locally diffeomorphic to B. These are depicted below.

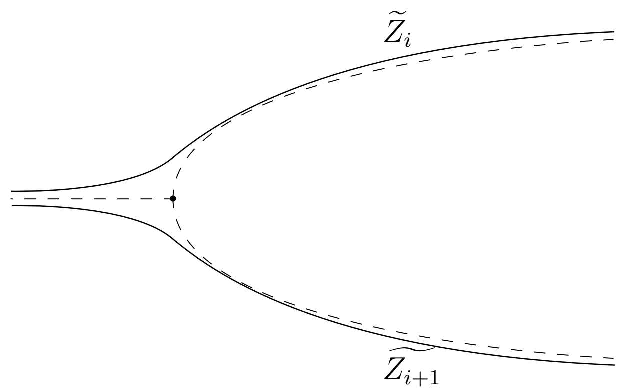


Figure 4.4. The manifolds with corners $\widehat{Z}_i$ and $\widehat{Z_{i+1}}$ are smoothed out to $\widetilde{Z}_i$ and $\widetilde{Z_{i+1}}$.

The outcome of this perturbation is that any sufficiently small simplex in the base which intersects the bifurcation set has the same number of critical points over each point in the simplex. This essential property of ghosts is used in [**Igu05**] in the proof of the 'transfer theorem', and in [**Ohr19**] to give a combinatorial description of the Becker–Gottlieb transfer.

4.3. Stratified deformations of critical loci

In this section we define *stratified subsets* and *stratified deformations*, and we construct a particular stratified deformation that will be used in Section 5.4 to prove Theorems D and E.

Definition 3.1. A stratified subset of a smooth bundle $p : W \to B^k$ is a pair (Σ, ψ) where Σ is a compact smooth k-dimensional manifold together with a map $\rho : \Sigma \to B$

and a tangential structure $\psi : \Sigma \to X$. The map ρ is everywhere smooth, but may admit birth-death singularities locally given by $\rho(x_1, \cdots, x_k) = (x_1^2, x_2, \cdots, x_k)$. These singularities form a $k-1$ dimensional submanifold of Σ.

We give two examples of stratified subsets, the first explains how to obtain the canonical example of a stratified subset from a fiberwise generalized Morse function. The second example introduces a special type of stratified subset called an *immersed lens*. The particular stratified deformation discussed below begins with a critical locus of a fiberwise generalized Morse function and deforms it into a disjoint union of immersed lenses.

Example 3.2. For our purposes the pair (Σ, ψ) will be the stratified subset corresponding to the critical locus of a fiberwise generalized Morse function $f : (W, \partial_0 W) \to (I, 0)$ on the bundle $p : W \to B$. In this case, Σ is the critical locus Σ_f, and the map ρ is the projection p restricted to Σ_f. The map ψ will be a map $\Sigma \to BO \times BO$ classifying the stable negative eigenspace bundle of f in the first component, and the stable positive eigenspace bundle of f in the second component.

Remark 3.3. There are two different stratifications on a stratified subset. The titular stratification refers to the stratification by dimension: each stratum is either of dimension k or dimension $k-1$. Often this will not be the stratification that we are interested in. Instead, we will make use of the degree stratification which distinguishes by the degree of their critical points. For instance, the submanifold of the critical locus containing those critical points of degree i, previously denoted Z_i, is a stratum of the degree-wise stratification of Σ_f. We will denote the set of such strata by $\mathscr{S}(\Sigma_f)$.

The following is an example of a stratified subset concentrated in two degrees.

Example 3.4 (Immersed Lenses - p.70 in [**Igu05**])**.** Let V be a compact connected k manifold with boundary so that V is immersed in B^k. Let $\psi_1, \psi_2 : V \to X$ be continuous maps which are trivial on ∂V. Then the *immersed lens* $L_i(V, \psi_1, \psi_2)$ is defined to be the stratified subset (L, ψ_L) where L is the double of V in indices i and $i+1$, and ψ_L is ψ_1 on the lower stratum and ψ_2 on the upper stratum.

Definition 3.5 (p. 67 in [**Igu05**])**.** A stratified deformation between stratified subsets (Σ, ψ) and (Σ', ψ') of $p : W \to B$ with coefficients in X is a stratified subset (S, Ψ) of $p \times I : W \times I \to B \times I$ with coefficients in X so that the restrictions of (S, Ψ) to $p \times 0 : W \times 0 \to B \times 0$ and $p \times 1 : W \times 1 \to B \times 1$ are (Σ, ψ) and (Σ', ψ'). In the event that (Σ, ψ) and (Σ', ψ') are related by a stratified deformation we say that they belong to the same *stratified deformation class* and use the notation $(\Sigma, \psi) \sim (\Sigma', \psi')$.

Remark 3.6. Note that when considering a stratified deformation of a critical locus of a fiberwise generalized Morse function, the end result of the deformation may not necessarily be realized as the critical locus of a fiberwise generalized Morse function. When referencing the collection of strata of the stratified subset (Σ, ψ) distinguished by their degree (in this case the dimension of the negative eigenspace bundle), we use the notation $\mathscr{S}(\Sigma)$. When Σ is the critical locus Σ_f, $\mathscr{S}(\Sigma)$ is identical to $\mathscr{S}(\Sigma_f)$.

For the remainder of this section we fix (Σ, ψ) as in Example 3.2.

Lemma 3.7. There exists a stratified deformation between stratified subsets (Σ, ψ) and (Σ', ψ'), so that the degree-wise strata of (Σ', ψ') are concentrated in two consecutive degrees. Furthermore, each component of the lower stratum of Σ' lies in a contractible subset of Σ'.

Remark 3.8. Lemma 3.7 is an excerpt from the proof of the transfer theorem in [**Igu05**]. In particular, the statement is identical to Step (c) on p.70, the proof of which appears on pages 71-73.

Briefly, the strategy of the proof is to first add and delete twisted lenses, immersed lenses for which ψ_1 is the same as ψ_2 after composition with the fold map, to concentrate the stratified subset into two consecutives degrees. The stratified deformations obtained by adding and deleting the twisted lenses reduce the number of components in the top degree by one. An inductive argument starting in the minimal stratum will then concentrate all strata in two degrees.

The next task is to prove that the lower stratum lies in a contractible subset of Σ'. To do this, we choose a triangulation of Σ', and do a deformation on each simplex. On the zero simplices the idea is to add a lens above a zero simplex, give a stratified deformation that cancels the lower stratum of this lens to obtain a 'mushroom', and then observe that the mushroom has the desired property: the '$-$' stratum on top of the mushroom (as well as it's boundary) lies in a contractible subset. We give a pictorial version of this stratified deformation in the figure below. This construction, as well as the inductive constructions for higher simplices, also appears with pictures in the proof of Lemma 3.2.1 in [**GI14**].

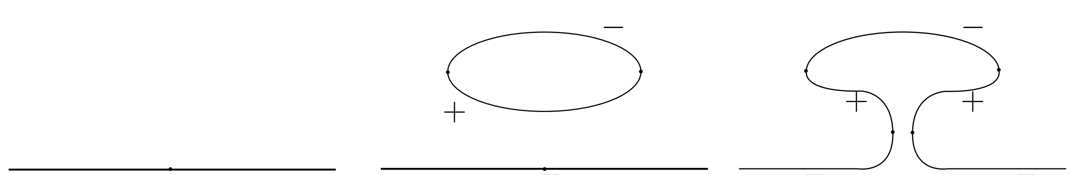


Figure 4.5. The stratified deformation introduces a lens above a designated point in Σ_- and then cancels the + and - strata to obtain a 'mushroom'.

Lemma 3.9 (Lemma 5.7 in [**Igu05**])**.** The stratified subset (Σ', ψ') can be deformed into a stratified subset $(\Sigma_{\mathrm{SD}}, \psi_{\mathrm{SD}})$ presented as a disjoint union of immersed lenses and components on which ψ_{SD} is trivial. Furthermore, ψ_{SD} is trivial on the lower stratum of each of the immersed lenses.

Let Λ denote the component of Σ_{SD} on which ψ_{SD} is trivial.

Lemma 3.10. An integer multiple of the stratified subset $(\Lambda, *)$ is stratified null-deformable.

Proof. Since Λ is concentrated in two degrees, Λ gives a map from B into the configuration space of positive and negative particles which is homotopy equivalent to QS^0. Since QS^0 is rationally trivial in degrees greater than 0, and this map lands in the zero component of $\pi_0(QS^0) = \mathbb{Z}$, some positive integer multiple of $(\Lambda, *)$ must be stratified null-deformable. $\square$

Remark 3.11. In Section 5.4 the stratified deformation between (Σ, ψ) and $(\Sigma_{\mathrm{SD}}, \psi_{\mathrm{SD}})$ is used to prove a fiberwise Poincaré–Hopf theorem that factors the Becker–Gottlieb transfer in terms of $(\Sigma_{\mathrm{SD}}, \psi_{\mathrm{SD}})$. In the proof of Theorem D we make use of ghost sets on Σ_{SD}, which are defined on arbitrary stratified subsets identically to how they are defined on Σ_f.

CHAPTER 5

Fiberwise Poincaré–Hopf Theorems

This section contains several fiberwise Poincaré–Hopf theorems, the main technical results of this paper. Roughly speaking, a fiberwise Poincaré–Hopf theorem computes a characteristic in terms of Morse theoretic data, in analogy with the classical Poincaré–Hopf theorem. The results in this section are preceded in the literature by computations of the Becker–Gottlieb transfer due to [**BM76**] and [**Dou06**].

In Subsection 5.1 we fix notation and introduce the indexing categories that are used to give refinements of our characteristics as Euler sections which should be thought of as the Poincaré duals of the usual characteristics. The constructions of the characteristics appearing in the subsection are used several times in the subsequent subsections.

In Subsection 5.2 we prove a fiberwise Poincaré–Hopf theorem for the Becker–Gottlieb transfer. To be exact, we factor the Poincaré dual of the Becker–Gottlieb transfer, and Euler section $e^d(p)$, in terms of the critical locus of a fiberwise generalized Morse function.

In Subsection 5.3 we prove a fiberwise Poincaré–Hopf theorem for the excisive A-theory Euler characteristic. To be exact once again, we factor the Poincaré dual of the excisive A-theory Euler characteristic, and Euler section $e^t(p)$, in terms of the critical locus of a fiberwise generalized Morse function. We also prove that the results of Subsections 5.2 and 5.3 are compatible.

In Subsection 5.4 we generalize the results of the previous two sections to an arbitrary stratified deformation of the critical locus of a fiberwise generalized Morse function. We also translate the theorems of this section into rational formulas in π_0 for use in Subsection 5.5 and Section 6.

In Subsection 5.5 we use the rational formulas of the previous section, as well as the stratified deformation constructed in Subsection 4.3 to prove a fiberwise Poincaré–Hopf theorem for the smooth structure class. In contrast to the previous fiberwise Poincaré–Hopf theorems, Theorem F is a rational statement.

The contents of this section build towards Theorem F, which is the only theorem from this section used in Section 6.

5.1. Definitions and a recollection of Dwyer–Weiss–Williams index theory

For the subsections that follow we will fix a smooth manifold bundle $p : W \to B$, where W is a compact smooth manifold of dimension m, B is a compact smooth manifold of dimension k, and the fiber of p is a compact smooth manifold F of dimension n. We fix an embedding of W into $B \times \mathbb{R}^d$ over B for d large. We also fix a fiberwise generalized Morse function $f : W \to \mathbb{R}$, with critical locus Σ_f^k and vertical gradient vector field $X := \nabla^\vee f$. We denote by π the restriction of the projection p to Σ_f. The tubular neighborhood of Σ_f in W is a disk bundle $q : D\Sigma_f \to \Sigma_f$, and the restriction of p to $D\Sigma_f$ is a map $\psi : D\Sigma_f \to B$. These choices are summarized in the following diagram. We also fix the notation τ and ν for, respectively, the vertical tangent bundle and the vertical normal bundle of W.

$$\begin{tikzcd}
 & & F^n \arrow[r] \arrow[d] & \mathbb{R}^d \arrow[d] \\
\Sigma_f^k \arrow[r, shift left] \arrow[drr, bend right=10, "\pi"'] & D\Sigma_f^m \arrow[l, shift left, "q"] \arrow[r] \arrow[dr, "\psi"] & W^m \arrow[r] \arrow[d, "p"'] & B \times \mathbb{R}^d \arrow[dl] \\
 & & B^k &
\end{tikzcd}$$

Next we introduce indexing categories associated to the manifolds above, which will be used to construct the characteristics considered in subsequent subsections.

Definition 1.1. The category $\mathsf{Disk}_k^{B/}$ is the category of one point compactifications of q-disks embedded in B. More precisely, an object $U \in \mathsf{Disk}_k^{B/}$ is the one point compactification of an open disk $\mathbb{R}^q$ embedded in B. An open embedding $\mathbb{R}^q \hookrightarrow \mathbb{R}^q \hookrightarrow B$ gives rise to a morphism $U' \to U$ given by the one point compactification of the open embedding $\mathbb{R}^q \hookrightarrow \mathbb{R}^q$.

Remark 1.2. The category $\mathsf{Disk}_k^{B/}$ of Definition 1.1 is similar to the category $\left(\mathcal{D}\mathsf{isk}_n^+\right)^{B_*/}$ used in [**AF19**], with the only difference being that the n-disks used here have only one component.

Definition 1.3. For fixed $U \in \mathsf{Disk}_k^{B/}$, we define the category $\mathsf{Disk}_m^{p^{-1}U/}$ to be the category of one point compactifications of m-disks in $p^{-1}U$. An object V in $\mathsf{Disk}_m^{p^{-1}U/}$ is the one point compactification of an m disk $\mathbb{R}^k \times \mathbb{R}^n \cong \mathbb{R}^m$ embedded in $p^{-1}U$ so that the composition $\mathbb{R}^k \times \mathbb{R}^n \to p^{-1}U \xrightarrow{p} U$ factors through the projection $\mathbb{R}^k \times \mathbb{R}^n \to \mathbb{R}^k$. A morphism from V' to V is given by the one point compactification of an open embedding $\mathbb{R}^k \times \mathbb{R}^n \to \mathbb{R}^k \times \mathbb{R}^n$ that commutes with the projection maps to $\mathbb{R}^k$.

Definition 1.4. The category $\mathsf{Disk}_m^{\psi^{-1}U/}$ is a subcategory of $\mathsf{Disk}_m^{p^{-1}U/}$ containing those disks V for which the embedding $\mathbb{R}^k \times \mathbb{R}^n \hookrightarrow p^{-1}U \hookrightarrow W$ factors through the embedding $D\Sigma \hookrightarrow W$.

The proposition below indicates how the categories $\mathsf{Disk}_k^{B/}$ and $\mathsf{Disk}_m^{p^{-1}U/}$ are used to model the homotopy types of $\Gamma_B Q_B(W)$, $\Gamma_B A_B^{\%}(W)$, and $\Gamma_B \mathcal{H}_B^{\%}(W)$. These are the spaces of sections of the fiberwise homology bundles whose fibers are $Q(F_+) := \Omega^\infty(F_+ \wedge \mathbb{S})$ for $\mathbb{S}$ the sphere spectrum, $A^{\%}(F) := \Omega^\infty(F_+ \wedge A(*))$ for $A(*)$ the algebraic K-theory of spaces functor evaluated at a point, and $\mathcal{H}^{\%}(F) := \Omega^\infty(F_+ \wedge \mathcal{H}(*))$ for $\mathcal{H}(*)$ the stable h-cobordism space of a point.

Proposition 1.5. The following are homotopy equivalences:

$$\Gamma_B Q_B(W) \xrightarrow{\simeq} \operatorname*{\mathsf{holim}}_{U \in \mathsf{Disk}_q^{B/}} \operatorname*{\mathsf{holim}}_{V \in \mathsf{Disk}_m^{p^{-1}U/}} \Omega^\infty(V^\bullet \wedge \mathbb{S}) \tag{5.1}$$

$$\Gamma_B A_B^{\%}(W) \xrightarrow{\simeq} \operatorname*{\mathsf{holim}}_{U \in \mathsf{Disk}_q^{B/}} \operatorname*{\mathsf{holim}}_{V \in \mathsf{Disk}_m^{p^{-1}U/}} \Omega^\infty(V^\bullet \wedge A(*)) \tag{5.2}$$

$$\Gamma_B \mathcal{H}_B^{\%}(W) \xrightarrow{\simeq} \operatorname*{\mathsf{holim}}_{U \in \mathsf{Disk}_q^{B/}} \operatorname*{\mathsf{holim}}_{V \in \mathsf{Disk}_m^{p^{-1}U/}} \Omega^\infty(V^\bullet \wedge \mathcal{H}(*)) \tag{5.3}$$

Proof. These homotopy equivalences are each instances of Poincaré duality, e.g. [**DWW03**] or nonabelian Poincaré duality from [**AF19**]. □

5.1.1. Constructions of characteristics

We will now construct refinements of the Becker–Gottlieb transfer $\mathsf{tr}(p) \in \Gamma_B Q_B(W)$ and the excisive A-theory Euler characteristic $\chi^{\%}(p) \in \Gamma_B A_B^{\%}(W)$. First we recall the Euler sections of [**Bec70**] and [**DWW03**].

Let $\gamma(n)$ be the tautological bundle on $\mathrm{BTop}(n)$, and let $[e^t_n] \in H^{\gamma(n)}(\mathrm{BTop}(n); A(*))$, the cohomology of $\mathrm{BTop}(n)$ with twisted coefficients in the spectrum $A(*)$, denote the generalized Becker–Euler class defined in [**Bec70**]. The class $[e^t_n]$ is refined in [**DWW03**] to a section e^t_n of the bundle with base $\mathrm{BTop}(n)$ and fiber $\Omega^\infty(\gamma(n)^\bullet_x \wedge A(*))$ over $x \in \mathrm{BTop}(n)$. Given a manifold M with tangent Euclidean bundle τ classified by a map $M \to \mathrm{BTop}(n)$, the associated Euler section $e^t_n(\tau)$ is defined as the pullback of e^t_n to a section of the bundle over M with fibers $\Omega^\infty(\tau^\bullet_x \wedge A(*))$ over $x \in M$.

Similarly, let $\epsilon(n)$ be the tautological bundle on $\mathrm{BO}(n)$, and consider the Becker–Euler class $[e^d_n] \in H^{\epsilon(n)}(\mathrm{BO}(n); \mathbb{S})$, the cohomology of $\mathrm{BO}(n)$ with twisted coefficients in the sphere spectrum. The class $[e^d_n]$ is refined in [**DWW03**] to a section e^d_n of the bundle with base $\mathrm{BO}(n)$ and fiber $\Omega^\infty(\epsilon(n)^\bullet_x \wedge \mathbb{S})$ over $x \in \mathrm{BO}(n)$. Given a manifold M with tangent bundle τ classified by a map $M \to \mathrm{BO}(n)$, the associated Euler section $e^d_n(\tau)$ is defined as the pullback of e^d_n to a section of the bundle over M with fibers $\Omega^\infty(\tau^\bullet_x \wedge \mathbb{S})$ over $x \in M$.

For $U \in \mathsf{Disk}^{B/}_k$ and $V \in \mathsf{Disk}^{p^{-1}U/}_m$, let c denote the Thom collapse map $U^\bullet \wedge S^d \to V^\bullet \wedge \mathrm{Th}(\nu|_V)$ associated to the embedding $V \hookrightarrow U \times \mathbb{R}^d$. Let $e^d_n(\tau) : \mathrm{Th}(\nu|_V) \wedge S^0 \xrightarrow{\mathrm{id}\wedge e^d_n(\tau)} \mathrm{Th}(\nu|_V) \wedge \mathrm{Th}(\tau|_V) \wedge \mathbb{S}$ denote the restriction of the Euler section $e^d_n(\tau)$ on M to V. At a point in $\mathrm{Th}(\nu|_V)$ over $x \in V$, the map $e^d_n(\tau)$ is the Euler section at x, i.e. a map $S^0 \to \tau^\bullet_x \wedge \mathbb{S}$. Then we consider the composition below:

$$U^\bullet \wedge S^d \wedge S^0 \xrightarrow{c\wedge \mathrm{id}} V^\bullet \wedge \mathrm{Th}(\nu|_V) \wedge S^0 \xrightarrow{\mathrm{id}\wedge e^d_n(\tau)} V^\bullet \wedge \mathrm{Th}(\nu|_V) \wedge \mathrm{Th}(\tau|_V) \wedge \mathbb{S} \tag{5.4}$$

Identifying $\mathrm{Th}(\nu|_V) \wedge \mathrm{Th}(\tau|_V)$ as $\mathbb{S}^d$ and taking adjoints, we equivalently have a map

$$U^\bullet \wedge S^0 \to \Omega^d \Omega^\infty (V^\bullet \wedge S^d \wedge \mathbb{S}).$$

For simplicity, we further compose with a homotopy equivalence given by the inclusion $\Omega^d\Omega^\infty(V^\bullet \wedge S^d \wedge \mathbb{S}) \to \Omega^\infty(V^\bullet \wedge \mathbb{S})$ to obtain a map

$$U^\bullet \wedge S^0 \to \Omega^\infty (V^\bullet \wedge \mathbb{S})$$

The construction above is natural in V, and thus yields a map

$$U^\bullet \wedge S^0 \to \underset{V \in \mathsf{Disk}_m^{p^{-1}U/}}{\mathsf{holim}} \Omega^\infty (V^\bullet \wedge \mathbb{S})$$

Naturality in U then produces a map

$$S^0 \xrightarrow{e^d(p)} \underset{U \in \mathsf{Disk}_k^{B/}}{\mathsf{holim}} \; \underset{V \in \mathsf{Disk}_m^{p^{-1}U/}}{\mathsf{holim}} \Omega^\infty (V^\bullet \wedge \mathbb{S})$$

which we denote by $e^d(p)$ and refer to as the *Euler section of* p *in* $\Gamma_B Q_B(W)$.

Similarly, considering the composition

$$U^\bullet \wedge S^d \wedge S^0 \xrightarrow{c \wedge \mathrm{id}} V^\bullet \wedge \mathrm{Th}(\nu|_V) \wedge S^0 \xrightarrow{\mathrm{id} \wedge e_n^t(\tau)} V^\bullet \wedge \mathrm{Th}(\nu|_V) \wedge \mathrm{Th}(\tau|_V) \wedge A(*) \tag{5.5}$$

results in a map

$$S^0 \xrightarrow{e^t(p)} \underset{U \in \mathsf{Disk}_k^{B/}}{\mathsf{holim}} \; \underset{V \in \mathsf{Disk}_m^{p^{-1}U/}}{\mathsf{holim}} \Omega^\infty (V^\bullet \wedge A(*))$$

which we denote by $e^t(p)$ and refer to as the *Euler section of* p *in* $\Gamma_B A_B^{\%}(W)$.

5.1.2. Recollections from Dwyer–Weiss–Williams

Proposition 1.6. With the vertical map induced by the unit map $\eta : \mathbb{S} \to A(*)$, the following diagram is homotopy commutative:

$$\begin{tikzcd}
S^0 \arrow[r, "e^d(p)"] \arrow[dr, "e^t(p)"'] & \operatorname*{holim}_{U \in \mathsf{Disk}_k^{B/}} \operatorname*{holim}_{V \in \mathsf{Disk}_m^{p^{-1}U/}} \Omega^\infty(V^\bullet \wedge \mathbb{S}) \arrow[d, "\eta_*"] \\
& \operatorname*{holim}_{U \in \mathsf{Disk}_k^{B/}} \operatorname*{holim}_{V \in \mathsf{Disk}_m^{p^{-1}U/}} \Omega^\infty(V^\bullet \wedge A(*))
\end{tikzcd}$$

Proof. It suffices to construct a path relating the universal Euler sections $\eta_* e_n^d$ and e_n^t. This is imprecisely Theorem 4.10 and precisely Theorem 4.13 in [**DWW03**]. $\square$

The fiberwise Poincaré duality map

$$\mathfrak{p} : \operatorname*{holim}_{V \in \mathsf{Disk}_m^{p^{-1}U/}} \Omega^\infty(V^\bullet \wedge \mathbb{J}) \xrightarrow{\simeq} \Omega^\infty((p^{-1}U)^\bullet \wedge \mathbb{J})$$

is a homotopy equivalence for any spectrum $\mathbb{J}$ (see, e.g. [**AF19**] or [**DWW03**]).

Proposition 1.7. There is a canonical path between $\mathfrak{p}e^d(p)$ and $\mathsf{tr}(p)$ in

$$\operatorname*{holim}_{U \in \mathsf{Disk}_k^{B/}} \Omega^\infty((p^{-1}(U))^\bullet \wedge \mathbb{S}) \simeq \Gamma_B Q_B(W)$$

There is also a canonical path between $\mathfrak{p}e^t(p)$ and $\chi^{\%}(p)$ in

$$\operatorname*{holim}_{U \in \mathsf{Disk}_k^{B/}} \Omega^\infty((p^{-1}(U))^\bullet \wedge A(*)) \simeq \Gamma_B A_B^{\%}(W)$$

Proof. The first sentence is Theorem 5.4 in [**DWW03**]. The second sentence is Theorem 3.18 in [**DWW03**]. □

Proposition 1.8. There is a canonical path betwween $\eta\,\mathsf{tr}(p)$ and $\chi^{\%}(p)$ in $\Gamma_B A_B^{\%}(W)$. In particular, the following diagram is homotopy commutative.

$$\begin{tikzcd}
S^0 \arrow[r, "\mathsf{tr}(p)"] \arrow[dr, "\chi^{\%}(p)"'] & \Gamma_B Q_B(W) \arrow[d, "\eta"] \\
& \Gamma_B A_B^{\%}(W)
\end{tikzcd}$$

Proof. This follows from combining Propositions 1.6 and 1.7. □

5.2. Fiberwise Poincaré–Hopf Theorem for the Becker–Gottlieb transfer

In this section we factor the Becker–Gottlieb transfer in terms of the critical locus Σ_f of the fiberwise generalized Morse function $f : W \to [0,1]$ and the vertical gradient vector field $X := \gamma^\vee f$.

Recall the notation Z_i for the connected submanifold of Σ_f containing those critical points of degree i. The submanifolds Z_i and Z_{i+1} share a boundary Z_i^1 consisting of birth-death critical points. The set $\mathscr{S}(\Sigma_f)$ is defined to be the set of all such Z_i. In this section, we will make use of manifolds $\widetilde{Z}_i$ obtained by perturbing Z_i over the ghost set as in Section 4.2. Note that we will often omit the degree i subscript from $\widetilde{Z}_i$ when it is not essential. We denote by $\pi_{\widetilde{Z}}$ the local diffeomorphism given by the restriction of p to $\widetilde{Z}$.

We begin by introducing two maps associated to $\widetilde{Z}$. The first is an Euler section associated to $\widetilde{Z}$, and the second is an index map associated to $\widetilde{Z}$. After these are defined, we prove the main theorem of this section.

5.2.1. The Euler section associated to $\widetilde{Z}$

We begin by giving definitions of the categories used to approximate $\widetilde{Z}$.

Definition 2.1. The category $\mathsf{Disk}_k^{\widetilde{Z}/}$ is the category of one point compactifications of k disks embedded in $\widetilde{Z}$. More precisely, an object $V \in \mathsf{Disk}_k^{\widetilde{Z}/}$ is the one point compactification of an open disk $\mathbb{R}^k$ embedded in $\widetilde{Z}$. An open embedding $\mathbb{R}^k \hookrightarrow \mathbb{R}^k \hookrightarrow \widetilde{Z}$ gives rise to a morphism $V' \to V$ given by the one point compactification of the open embedding $\mathbb{R}^k \hookrightarrow \mathbb{R}^k$.

Definition 2.2. For $U \in \mathsf{Disk}_k^{B/}$, the category $\mathsf{Disk}_k^{\widetilde{Z}/U}$ is the subcategory of $\mathsf{Disk}_k^{\widetilde{Z}/}$ consisting of those objects V for which the assosicated $\mathbb{R}^k$ embedded in $\widetilde{Z}$ maps into U under the projection map p, which restricts to a local homeomorphism on $\mathbb{R}^k$.

As in the previous section, the composition (5.4) is used to construct a map

$$S^0 \xrightarrow{e^d(\pi_{\widetilde{Z}})} \underset{U \in \mathsf{Disk}_k^{B/}}{\mathsf{holim}} \ \underset{V \in \mathsf{Disk}_m^{\widetilde{Z}/U}}{\mathsf{holim}} \ \Omega^\infty(V^\bullet \wedge \mathbb{S})$$

Aggregate over all $Z \in \mathscr{S}(\Sigma_f)$, we have a map

$$S^0 \xrightarrow{\prod_{Z \in \mathscr{S}(\Sigma_f)} e^d(\pi_{\widetilde{Z}})} \prod_{Z \in \mathscr{S}(\Sigma_f)} \underset{U \in \mathsf{Disk}_k^{B/}}{\mathsf{holim}} \ \underset{V \in \mathsf{Disk}_m^{\widetilde{Z}/U}}{\mathsf{holim}} \ \Omega^\infty(V^\bullet \wedge \mathbb{S})$$

5.2.2. The index map associated to $\widetilde{Z}$

At a point $z \in Z$, we can consider the derivative of the gradient vector field X at z, a map $dX_z : \tau_z \to \tau_z$. This induces a self-map on the one point compactification of τ_z, and thus a map $dX : \mathrm{Th}(\tau_Z) \to \mathrm{Th}(\tau_Z)$. For any $V \in \mathsf{Disk}_k^{\widetilde{Z}/}$, we can restrict the map dX to V to obtain a map $dX|_V : \mathrm{Th}(\tau|_V) \to \mathrm{Th}(\tau|_V)$. Then the local map

$$V^\bullet \wedge \mathrm{Th}(\nu|_V) \wedge \mathrm{Th}(\tau|_V) \wedge \mathbb{S} \xrightarrow{\mathrm{id}\wedge\mathrm{id}\wedge dX|_V\wedge\mathrm{id}} V^\bullet \wedge \mathrm{Th}(\nu|_V) \wedge \mathrm{Th}(\tau|_V) \wedge \mathbb{S} \tag{5.6}$$

induces a map

$$\underset{U\in\mathsf{Disk}_k^{B/}}{\mathsf{holim}}\ \underset{V\in\mathsf{Disk}_m^{\widetilde{Z}/U}}{\mathsf{holim}}\ \Omega^\infty(V^\bullet \wedge \mathbb{S}) \xrightarrow{\mathsf{Ind}^d_{\widetilde{Z}}} \underset{U\in\mathsf{Disk}_k^{B/}}{\mathsf{holim}}\ \underset{V\in\mathsf{Disk}_m^{\widetilde{Z}/U}}{\mathsf{holim}}\ \Omega^\infty(V^\bullet \wedge \mathbb{S}) \tag{5.7}$$

which we denote by $\mathsf{Ind}^d_{\widetilde{Z}}$ and refer to as the index map on $\widetilde{Z}$ with coefficients in $\mathbb{S}$.

Aggregate over all $Z \in \mathscr{S}(\Sigma_f)$ we have a map

(5.8)
$$\prod_{Z\in\mathscr{S}(\Sigma_f)} \underset{U\in\mathsf{Disk}_k^{B/}}{\mathsf{holim}}\ \underset{V\in\mathsf{Disk}_m^{\widetilde{Z}/U}}{\mathsf{holim}}\ \Omega^\infty(V^\bullet \wedge \mathbb{S}) \xrightarrow{\prod_{Z\in\mathscr{S}(\Sigma_f)} \mathsf{Ind}^d_{\widetilde{Z}}} \prod_{Z\in\mathscr{S}(\Sigma_f)} \underset{U\in\mathsf{Disk}_k^{B/}}{\mathsf{holim}}\ \underset{V\in\mathsf{Disk}_m^{\widetilde{Z}/U}}{\mathsf{holim}}\ \Omega^\infty(V^\bullet \wedge \mathbb{S})$$

The following lemma will be used in Section 5.4.

Lemma 2.3. For $Z \in \mathscr{S}(\Sigma_f)$ of degree j, $\mathsf{Ind}^d_{\widetilde{Z}}$ is multiplication by $(-1)^j$ on homotopy groups.

Proof. This is the correction in [**MP89**] to Theorems 2.11 and 3.11 in [**BM76**]. $\square$

5.2.3. Factoring the Becker–Gottlieb transfer

Theorem A. *The diagram below is homotopy commutative.*

$$\begin{tikzcd} S^0 \arrow[r, "e^d(p)"] \arrow[d, "\prod_{Z\in\mathscr{S}(\Sigma_f)} e^d(\pi_{\tilde{Z}})"'] & \operatorname*{holim}_{U\in\mathsf{Disk}_k^{B/}} \operatorname*{holim}_{V\in\mathsf{Disk}_m^{p^{-1}U/}} \Omega^\infty(V^\bullet\wedge\mathbb{S}) \\ \prod_{Z\in\mathscr{S}(\Sigma_f)} \operatorname*{holim}_{U\in\mathsf{Disk}_k^{B/}} \operatorname*{holim}_{V\in\mathsf{Disk}_m^{\tilde{Z}/U}} \Omega^\infty(V^\bullet\wedge\mathbb{S}) \arrow[r, "\prod_{Z\in\mathscr{S}(\Sigma_f)} \mathsf{Ind}^d_{\tilde{Z}}"'] & \prod_{Z\in\mathscr{S}(\Sigma_f)} \operatorname*{holim}_{U\in\mathsf{Disk}_k^{B/}} \operatorname*{holim}_{V\in\mathsf{Disk}_m^{\tilde{Z}/U}} \Omega^\infty(V^\bullet\wedge\mathbb{S}) \arrow[u, "incl"'] \end{tikzcd} \tag{5.9}$$

Proof. We begin by introducing an auxilliary map. Let $e^d_X(p)$ be the *vector field Euler section*, a map

$$S^0 \xrightarrow{e^d_X(p)} \operatorname*{holim}_{U\in\mathsf{Disk}_k^{B/}} \operatorname*{holim}_{V\in\mathsf{Disk}_m^{p^{-1}U/}} \Omega^\infty(V^\bullet\wedge\mathbb{S})$$

given locally as

$$U^\bullet\wedge S^d\wedge S^0 \xrightarrow{c\wedge\mathrm{id}} V^\bullet\wedge\mathrm{Th}(\nu|_V)\wedge S^0 \xrightarrow{\mathrm{id}\wedge e^d_X(\tau)} V^\bullet\wedge\mathrm{Th}(\nu|_V)\wedge\mathrm{Th}(\tau|_V)\wedge\mathbb{S} \tag{5.10}$$

The only difference between the composition above and (5.4) is the map $e^d_X(\tau)$, which we define to be the composition

$$S^0 \xrightarrow{e^d_n(\tau)} \mathrm{Th}(\tau|_V) \wedge \mathbb{S} \xrightarrow{\ell \wedge \mathrm{id}} \mathrm{Th}(\tau|_V) \wedge \mathbb{S}$$

in which the map $\ell : \mathrm{Th}(\tau|_V) \to \mathrm{Th}(\tau|_V)$ sends a point over $y \in V$ to $y + X(y)$. By placing the scaling factor $t \in [0,1]$ as a coefficient in front of X, we obtain a family of maps ℓ_t which is a homotopy between the map ℓ and the identity. Thus, the vector field Euler section $e^d_X(p)$ is homotopic to the Euler section $e^d(p)$ from before. This results in homotopy commutativity of the following diagram:

$$\text{(5.11)} \qquad \begin{tikzcd} S^0 \arrow[r, "e^d(p)"] \arrow[dr, "e^d_X(p)"'] & \underset{U \in \mathsf{Disk}_k^{B/}}{\mathsf{holim}} \ \underset{V \in \mathsf{Disk}_m^{p^{-1}U/}}{\mathsf{holim}} \Omega^\infty(V^\bullet \wedge \mathbb{S}) \\ & \underset{U \in \mathsf{Disk}_k^{B/}}{\mathsf{holim}} \ \underset{V \in \mathsf{Disk}_m^{p^{-1}U/}}{\mathsf{holim}} \Omega^\infty(V^\bullet \wedge \mathbb{S}) \arrow[u, "\mathrm{id}"'] \end{tikzcd}$$

The remainder of this proof is organized in smaller pieces, each of which proves the homotopy commutativity of a triangle in the diagram below.

$$
\begin{tikzcd}
S^0 \arrow[r, "e^d(p)"] \arrow[dr, "e^d_X(p)"'] \arrow[ddr, "e^d_X(\psi)"] \arrow[dddr, "\prod_{Z \in \mathscr{S}(\Sigma_f)} e^d_X(\pi_{\widetilde{Z}})"] \arrow[ddd, "\prod_{Z \in \mathscr{S}(\Sigma_f)} e^d(\pi_{\widetilde{Z}})"'] \arrow[dr, phantom, bend left, "(1)"] \arrow[ddr, phantom, bend left=10, "(2)"] \arrow[dddr, phantom, bend left=10, "(3)"] \arrow[dddr, phantom, bend right=10, "(4)"'] & \underset{U \in \mathsf{Disk}_k^{B/}}{\mathsf{holim}} \ \underset{V \in \mathsf{Disk}_m^{p^{-1}U/}}{\mathsf{holim}} \Omega^\infty(V^\bullet \wedge \mathbb{S}) \\
& \underset{U \in \mathsf{Disk}_k^{B/}}{\mathsf{holim}} \ \underset{V \in \mathsf{Disk}_m^{p^{-1}U/}}{\mathsf{holim}} \Omega^\infty(V^\bullet \wedge \mathbb{S}) \arrow[u, "\mathrm{id}"'] \\
& \underset{U \in \mathsf{Disk}_k^{B/}}{\mathsf{holim}} \ \underset{V \in \mathsf{Disk}_m^{\psi^{-1}U/}}{\mathsf{holim}} \Omega^\infty(V^\bullet \wedge \mathbb{S}) \arrow[u, "\mathrm{incl}"'] \\
\prod_{Z \in \mathscr{S}(\Sigma_f)} \underset{U \in \mathsf{Disk}_k^{B/}}{\mathsf{holim}} \ \underset{V \in \mathsf{Disk}_m^{\widetilde{Z}/U}}{\mathsf{holim}} \Omega^\infty(V^\bullet \wedge \mathbb{S}) \arrow[r, "\prod_{Z \in \mathscr{S}(\Sigma_f)} \mathsf{Ind}^d_{\widetilde{Z}}"'] & \prod_{Z \in \mathscr{S}(\Sigma_f)} \underset{U \in \mathsf{Disk}_k^{B/}}{\mathsf{holim}} \ \underset{V \in \mathsf{Disk}_m^{\widetilde{Z}/U}}{\mathsf{holim}} \Omega^\infty(V^\bullet \wedge \mathbb{S}) \arrow[u, "g"']
\end{tikzcd}
\tag{5.12}
$$

We will now prove the homotopy commutativity of each subtriangle in the diagram above, beginning from the top triangle and proceeding clockwise. We will define the morphisms in each triangle as needed.

(1) The top triangle relating maps $e^d(p)$ and $e^d_X(p)$ is identically diagram (5.11).

(2) The map $e^d_X(\psi)$ is defined locally on a pair $U \in \mathsf{Disk}_k^{B/}$ and $V \in \mathsf{Disk}_m^{\psi^{-1}U/}$ as the composition given in (5.18). Since the characteristics $e^d_X(p)$ and $e^d_X(\psi)$ have identical local definitions, it suffices to see that the composition (5.18) is nullhomotopic on objects of $\mathsf{Disk}_m^{p^{-1}U/}$ that are not also objects in $\mathsf{Disk}_m^{\psi^{-1}U/}$. Since DZ is the unit disk bundle on Z, the vector field X has length greater than 1 at any point $x \notin DZ$. This means that the map $j : \mathrm{Th}(\tau|_V) \to \mathrm{Th}(\tau|_V)$ appearing in the definition of the vector field Euler section $e^d(\tau)$ is nullhomotopic on any $V \in \mathsf{Disk}_m^{p^{-1}U/}$ that is not also an object in $\mathsf{Disk}_m^{\psi^{-1}U/}$. Thus the map $e^d_X(p)$ factors through $e^d_X(\psi)$ as indicated in the diagram above.

(3) The map $e^d_X(\pi_{\widetilde{Z}})$ is defined as the composition $\mathsf{Ind}^d_X \circ e^d(\pi_{\widetilde{Z}})$ so that the bottom triangle commutes.

Consider the functor $R_{\widetilde{Z}} : \mathsf{Disk}_m^{\psi^{-1}U/} \to \mathsf{Disk}_m^{\widetilde{Z}/U}$ that sends a disk V corresponding to an embedding $\mathbb{R}^k \times \mathbb{R}^n \hookrightarrow D\Sigma_f$ to the intersection of the image of this embedding and $\widetilde{Z}$. The map g is induced by the product of functors $R_{\widetilde{Z}}$ over $Z \in \mathscr{S}(\Sigma_f)$.

To see that this triangle commutes, it suffices to see that the local definitions of the characteristics $e^d_X(\pi_{\widetilde{Z}})$ and $e^d_X(\psi)$ agree up to homotopy. There are two cases, either U intersects a bifurcation set in B, or it does not. Assuming that it does not, we have $U \in \mathsf{Disk}_k^{B/}$, $V \in \mathsf{Disk}_m^{\psi^{-1}U/}$, and $R_{\widetilde{Z}}(V) \in \mathsf{Disk}_m^{\widetilde{Z}/U}$ for some $Z \in \mathscr{S}(\Sigma_f)$, and we must check that the local definitions of the characteristics $e_X^d(\pi_{\widetilde{Z}})$ and $e_X^d(\psi)$ agree up to homotopy. Thus we must construct a homotopy that makes the diagram below commute. This diagram is obtained by comparing (5.18) applied to V (the left vertical composition) with (5.4) composed with (5.14) applied to $R_{\widetilde{Z}}(V)$ (the composition along the top).

$$\begin{tikzcd}
U^\bullet \wedge S^d \wedge S^0 \arrow[r, "c\wedge\mathrm{id}"] \arrow[d, "c\wedge\mathrm{id}"] & R_{\widetilde{Z}}(V)^\bullet \wedge \mathrm{Th}(\nu|_{R_{\widetilde{Z}}(V)}) \wedge S^0 \arrow[dr, "\mathrm{id}\wedge e_n^d(\tau)"] & \\
V^\bullet \wedge \mathrm{Th}(\nu|_V) \wedge S^0 \arrow[d, "\mathrm{id}\wedge e_X^d(\tau)"] & & R_{\widetilde{Z}}(V)^\bullet \wedge \mathrm{Th}(\nu|_{R_{\widetilde{Z}}(V)}) \wedge \mathrm{Th}(\tau|_{R_{\widetilde{Z}}(V)}) \wedge \mathbb{S} \arrow[d, "\mathrm{id}\wedge\mathrm{id}\wedge dX|_{R_{\widetilde{Z}}(V)}\wedge\mathrm{id}"] \\
V^\bullet \wedge \mathrm{Th}(\nu|_V) \wedge \mathrm{Th}(\tau|_V) \wedge \mathbb{S} & & R_{\widetilde{Z}}(V)^\bullet \wedge \mathrm{Th}(\nu|_{R_{\widetilde{Z}}(V)}) \wedge \mathrm{Th}(\tau|_{R_{\widetilde{Z}}(V)}) \wedge \mathbb{S} \arrow[ll, "\mathrm{incl}"]
\end{tikzcd} \tag{5.13}$$

Recall that the map $e_X^d(\tau)$ is defined to be the composition

$$S^0 \xrightarrow{e_n^d(\tau)} \mathrm{Th}(\tau|_V) \wedge \mathbb{S} \xrightarrow{j\wedge\mathrm{id}} \mathrm{Th}(\tau|_V) \wedge \mathbb{S}$$

Thus we must show that the map $dX|_{R_{\widetilde{Z}}(V)}$ is homotopic to j. However, this is an immediate consequence of the fact that in the neighborhood of a nondegenerate zero a vector field is homotopic to its derivative.

Next we assume that U does intersect a bifurcation set. Then for some $V \in \mathsf{Disk}_m^{\psi^{-1}U/}$ we have nontrivial $R_{\widetilde{Z}_i}(V) \in \mathsf{Disk}_k^{\widetilde{Z}_i/U}$ and $R_{\widetilde{Z}_{i+1}}(V) \in \mathsf{Disk}_k^{\widetilde{Z}_{i+1}/U}$, and we must see that the local definition of the characteristic $e_X^d(\psi)$ agrees with the wedge sum of the local definitions of the characteristics $e_X^d(\pi_{\widetilde{Z}_i})$ and $e_X^d(\pi_{\widetilde{Z}_{i+1}})$

It suffices to verify that the map $dX|_{R_{\widetilde{Z}_i}(V)} \vee dX|_{R_{\widetilde{Z}_{i+1}}(V)}$ composed with the inclusion

$$\mathrm{Th}(\tau|_{R_i(V)}) \vee \mathrm{Th}(\tau|_{R_{i+1}(V)}) \to \mathrm{Th}(\tau|_V)$$

is homotopic to $j : \mathrm{Th}(\tau|_V) \to \mathrm{Th}(\tau|_V)$. This homotopy is given as the usual homotopy of the vector field X with only one degenerate zero to a vector field with two nondegenerate zeros of degree i and $i+1$.

(4) Recall from the previous step that the map $e^d_X(\pi_{\widetilde{Z}})$ is defined as the composition $\mathsf{Ind}^d_X \circ e^d(\pi_{\widetilde{Z}})$ so that the bottom triangle commutes.

$\square$

5.3. Fiberwise Poincaré–Hopf Theorem for the excisive A-theory Euler characteristic

In this section we prove a fiberwise Poincaré–Hopf theorem for the excisive A-theory Euler characteristic. The proof of this result is largely the same as the proof of Theorem A in the previous section, with the main differences being that the Euler section e^d_n is replaced with e^t_n, and the coefficient spectrum $\mathbb{S}$ is replaced with $A(*)$. All indexing categories used in this section are defined in Section 5.2. However, we give new definitions of the maps and an abbreviated proof to keep the discussion in this section mostly self contained.

5.3.1. Preliminaries

We begin by defining an Euler section and index map associated to Z, as we did in the previous section. The composition (5.5) from Section 5.1 is used to construct an Euler section associated to Z.

$$S^0 \xrightarrow{e^t(\pi)} \underset{U \in \mathsf{Disk}_k^{B/}}{\mathsf{holim}} \ \underset{V \in \mathsf{Disk}_m^{\pi^{-1}U/}}{\mathsf{holim}} \Omega^\infty(V^\bullet \wedge A(*))$$

Aggregate over all $Z \in \mathscr{S}(\Sigma_f)$, we have a map

$$S^0 \xrightarrow{\prod_{Z \in \mathscr{S}(\Sigma_f)} e^d(\pi_{\widetilde{Z}})} \prod_{Z \in \mathscr{S}(\Sigma_f)} \underset{U \in \mathsf{Disk}_k^{B/}}{\mathsf{holim}} \ \underset{V \in \mathsf{Disk}_m^{\widetilde{Z}/U}}{\mathsf{holim}} \Omega^\infty(V^\bullet \wedge A(*))$$

To define the index map, we can consider the derivative of the gradient vector field X at $z \in Z$. This is a map $dX_z : \tau_z \to \tau_z$ which induces a self-map on the one point compactification of τ_z, and thus a map $dX : \mathrm{Th}(\tau_Z) \to \mathrm{Th}(\tau_Z)$. For any $V \in \mathsf{Disk}_k^{Z/}$, we can restrict the map dX to V to obtain a map $dX|_V : \mathrm{Th}(\tau|_V) \to \mathrm{Th}(\tau|_V)$. Then the local map

$$V^\bullet \wedge \mathrm{Th}(\nu|_V) \wedge \mathrm{Th}(\tau|_V) \wedge A(* \xrightarrow{\mathrm{id}\wedge\mathrm{id}\wedge dX|_V\wedge\mathrm{id}} V^\bullet \wedge \mathrm{Th}(\nu|_V) \wedge \mathrm{Th}(\tau|_V) \wedge A(* \tag{5.14}$$

induces a map

$$\underset{U \in \mathsf{Disk}_k^{B/}}{\mathsf{holim}} \ \underset{V \in \mathsf{Disk}_m^{\pi^{-1}U/}}{\mathsf{holim}} \Omega^\infty(V^\bullet \wedge A(*)) \xrightarrow{\mathsf{Ind}_Z^t} \underset{U \in \mathsf{Disk}_k^{B/}}{\mathsf{holim}} \ \underset{V \in \mathsf{Disk}_m^{\pi^{-1}U/}}{\mathsf{holim}} \Omega^\infty(V^\bullet \wedge A(*)) \tag{5.15}$$

which we denote as Ind^t_Z and refer to as the index map on Z with coefficients in $A(*)$. Aggregate over all $Z \in \mathscr{S}(\Sigma_f)$ we have a map

(5.16)
$$\prod_{Z\in\mathscr{S}(\Sigma_f)} \operatorname*{holim}_{U\in\mathsf{Disk}_k^{B/}} \operatorname*{holim}_{V\in\mathsf{Disk}_m^{\widetilde{Z}/U}} \Omega^\infty(V^\bullet \wedge A(*)) \xrightarrow{\prod_{Z\in\mathscr{S}(\Sigma_f)} \mathsf{Ind}^t_{\widetilde{Z}}} \prod_{Z\in\mathscr{S}(\Sigma_f)} \operatorname*{holim}_{U\in\mathsf{Disk}_k^{B/}} \operatorname*{holim}_{V\in\mathsf{Disk}_m^{\widetilde{Z}/U}} \Omega^\infty(V^\bullet \wedge A(*))$$

The following lemma will be used in Section 5.4.

Lemma 3.1. For $Z \in \mathscr{S}(\Sigma_f)$ of degree j, $\mathsf{Ind}^t_{\widetilde{Z}}$ is multiplication by $(-1)^j$ on homotopy groups.

Proof. This follows immediately from Lemma 2.3, since the definition of $\mathsf{Ind}^t_{\widetilde{Z}}$ is identical to that of $\mathsf{Ind}^d_{\widetilde{Z}}$, up to a change of coefficients on which the index map is the identity. $\square$

5.3.2. Factoring the excisive A-theory Euler characteristic

Theorem B. *The diagram below is homotopy commutative.*

$$\begin{tikzcd}
S^0 \arrow[r, "e^t(p)"] \arrow[d, "\prod_{Z\in\mathscr{S}(\Sigma_f)} e^t(\pi_{\widetilde{Z}})"'] & \operatorname*{holim}_{U\in\mathsf{Disk}_k^{B/}} \operatorname*{holim}_{V\in\mathsf{Disk}_m^{p^{-1}U/}} \Omega^\infty(V^\bullet \wedge A(*)) \\
\prod_{Z\in\mathscr{S}(\Sigma_f)} \operatorname*{holim}_{U\in\mathsf{Disk}_k^{B/}} \operatorname*{holim}_{V\in\mathsf{Disk}_m^{\widetilde{Z}/U}} \Omega^\infty(V^\bullet \wedge A(*)) \arrow[r, "\prod_{Z\in\mathscr{S}(\Sigma_f)} \mathsf{Ind}^t_{\widetilde{Z}}"'] & \prod_{Z\in\mathscr{S}(\Sigma_f)} \operatorname*{holim}_{U\in\mathsf{Disk}_k^{B/}} \operatorname*{holim}_{V\in\mathsf{Disk}_m^{\widetilde{Z}/U}} \Omega^\infty(V^\bullet \wedge A(*)) \arrow[u, "incl"']
\end{tikzcd} \tag{5.17}$$

Proof. In this proof we rehash the main steps of the proof of Theorem A, making the necessary changes as needed.

We begin by defining the vector field Euler section $e^t_X(p)$, a map

$$S^0 \xrightarrow{e^t_X(p)} \operatorname*{holim}_{U\in\mathsf{Disk}_k^{B/}} \operatorname*{holim}_{V\in\mathsf{Disk}_m^{p^{-1}U/}} \Omega^\infty(V^\bullet \wedge A(*))$$

given locally as

$$U^\bullet \wedge S^d \wedge S^0 \xrightarrow{c\wedge \mathrm{id}} V^\bullet \wedge \mathrm{Th}(\nu|_V) \wedge S^0 \xrightarrow{\mathrm{id}\wedge e^t_X(\tau)} V^\bullet \wedge \mathrm{Th}(\nu|_V) \wedge \mathrm{Th}(\tau|_V) \wedge A(*) \tag{5.18}$$

The map $e^t_X(\tau)$ is defined to be the composition

$$S^0 \xrightarrow{e^d_n(\tau)} \mathrm{Th}(\tau|_V) \wedge A(*) \xrightarrow{\ell\wedge \mathrm{id}} \mathrm{Th}(\tau|_V) \wedge A(*)$$

in which the map $\ell : \mathrm{Th}(\tau|_V) \to \mathrm{Th}(\tau|_V)$ sends a point over $y \in V$ to $y + X(y)$. By placing the scaling factor $t \in [0,1]$ as a coefficient in front of X, we obtain a family of maps ℓ_t which is a homotopy between the map ℓ and the identity. Thus, the vector field Euler section $e^t_X(p)$ is homotopic to the Euler section $e^t(p)$. This results in homotopy commutativity of the following diagram:

$$\begin{tikzcd} S^0 \arrow[r, "e^t(p)"] \arrow[dr, "e^t_X(p)"'] & \operatorname*{holim}_{U\in\mathsf{Disk}_k^{B/}} \operatorname*{holim}_{V\in\mathsf{Disk}_m^{p^{-1}U/}} \Omega^\infty(V^\bullet \wedge A(*)) \\ & \operatorname*{holim}_{U\in\mathsf{Disk}_k^{B/}} \operatorname*{holim}_{V\in\mathsf{Disk}_m^{p^{-1}U/}} \Omega^\infty(V^\bullet \wedge A(*)) \arrow[u, "\mathrm{id}"'] \end{tikzcd} \tag{5.19}$$

The remainder of this proof is organized in smaller pieces, each of which proves the homotopy commutativity of a triangle in the diagram below.

$$\begin{tikzcd}[column sep=huge, row sep=huge]
S^0 \arrow[r, "e^t(p)"] \arrow[dr, "e^t_X(p)"'] \arrow[ddr, bend right=15, "e^t_X(\psi)"] \arrow[dddr, bend right=10, "\prod_{Z\in\mathscr{S}(\Sigma_f)} e^t_X(\pi)"] \arrow[ddd, "\prod_{Z\in\mathscr{S}(\Sigma_f)} e^t(\pi)"'] & \operatorname*{holim}_{U\in\mathsf{Disk}_k^{B/}} \operatorname*{holim}_{V\in\mathsf{Disk}_m^{p^{-1}U/}} \Omega^\infty(V^\bullet\wedge A(*)) \\
& \operatorname*{holim}_{U\in\mathsf{Disk}_k^{B/}} \operatorname*{holim}_{V\in\mathsf{Disk}_m^{p^{-1}U/}} \Omega^\infty(V^\bullet\wedge A(*)) \arrow[u, "\mathrm{id}"'] \\
& \operatorname*{holim}_{U\in\mathsf{Disk}_k^{B/}} \operatorname*{holim}_{V\in\mathsf{Disk}_m^{\psi^{-1}U/}} \Omega^\infty(V^\bullet\wedge A(*)) \arrow[u, "\mathrm{incl}"'] \\
\prod_{Z\in\mathscr{S}(\Sigma_f)} \operatorname*{holim}_{U\in\mathsf{Disk}_k^{B/}} \operatorname*{holim}_{V\in\mathsf{Disk}_m^{\widetilde{Z}/U}} \Omega^\infty(V^\bullet\wedge A(*)) \arrow[r, "\prod_{Z\in\mathscr{S}(\Sigma_f)} \mathsf{Ind}^t_Z"'] & \prod_{Z\in\mathscr{S}(\Sigma_f)} \operatorname*{holim}_{U\in\mathsf{Disk}_k^{B/}} \operatorname*{holim}_{V\in\mathsf{Disk}_m^{\widetilde{Z}/U}} \Omega^\infty(V^\bullet\wedge A(*)) \arrow[u, "g"', "\simeq"]
\end{tikzcd} \tag{5.20}$$

The map $e^t_X(\pi)$ in the diagram above is defined as the composition $\mathsf{Ind}^t_Z \circ e^t(\pi)$, hence the bottom triangle commutes. The triangle at the top of the diagram above is Diagram (5.19). The proofs of the commutativity of the middle two triangles are identical to the analogous steps in the proof of Theorem A, so long as the Euler sections $e^d_n(\tau)$ are replaced with $e^t_n(\tau)$, and the coefficient spectrum $\mathbb{S}$ is replaced with $A(*)$. $\square$

We conclude this section by giving a generalization of Theorems A and B.

Theorem C. *The following diagram of spaces is homotopy commutative.*

$$\begin{tikzcd}
& S^0 \arrow[rr, "e^d(p)"] \arrow[dl, "\prod_{Z\in\mathscr{S}(\Sigma_f)} e^d(\pi)"'] \arrow[dd] & & \operatorname*{holim}\limits_{U\in\mathsf{Disk}_k^{B/}} \operatorname*{holim}\limits_{V\in\mathsf{Disk}_m^{p^{-1}U/}} \Omega^\infty(V^\bullet\wedge\mathbb{S}) \arrow[dd, "\eta"] \\
\prod\limits_{Z\in\mathscr{S}(\Sigma_f)} \operatorname*{holim}\limits_{U\in\mathsf{Disk}_k^{B/}} \operatorname*{holim}\limits_{V\in\mathsf{Disk}_m^{\pi^{-1}U/}} \Omega^\infty(V^\bullet\wedge\mathbb{S}) \arrow[rr, "\prod_{Z\in\mathscr{S}(\Sigma_f)} \mathsf{Ind}^d_Z"'] \arrow[dd, "\eta"] & & \prod\limits_{Z\in\mathscr{S}(\Sigma_f)} \operatorname*{holim}\limits_{U\in\mathsf{Disk}_k^{B/}} \operatorname*{holim}\limits_{V\in\mathsf{Disk}_m^{\pi^{-1}U/}} \Omega^\infty(V^\bullet\wedge\mathbb{S}) \arrow[ur, "+"] \arrow[dd, "\eta"] & \\
& S^0 \arrow[rr, "e^t(p)"] \arrow[dl, "\prod_{Z\in\mathscr{S}(\Sigma_f)} e^t(\pi)"'] & & \operatorname*{holim}\limits_{U\in\mathsf{Disk}_k^{B/}} \operatorname*{holim}\limits_{V\in\mathsf{Disk}_m^{p^{-1}U/}} \Omega^\infty(V^\bullet\wedge A(*)) \\
\prod\limits_{Z\in\mathscr{S}(\Sigma_f)} \operatorname*{holim}\limits_{U\in\mathsf{Disk}_k^{B/}} \operatorname*{holim}\limits_{V\in\mathsf{Disk}_m^{\pi^{-1}U/}} \Omega^\infty(V^\bullet\wedge A(*)) \arrow[rr, "\prod_{Z\in\mathscr{S}(\Sigma_f)} \mathsf{Ind}^t_Z"'] & & \prod\limits_{Z\in\mathscr{S}(\Sigma_f)} \operatorname*{holim}\limits_{U\in\mathsf{Disk}_k^{B/}} \operatorname*{holim}\limits_{V\in\mathsf{Disk}_m^{\pi^{-1}U/}} \Omega^\infty(V^\bullet\wedge A(*)) \arrow[ur, "+"] &
\end{tikzcd}$$

Proof. Theorem A indicates that the top square commutes, and Theorem B indicates that the bottom square commutes. The back face of the cube commutes by Proposition 1.6.

Homotopy commutativity of each of the remaining vertical faces follows from applying Theorem 4.13 in [**DWW03**] to the local definitions of each map to construct the homotopy locally on each disk. $\square$

5.4. Fiberwise Poincaré–Hopf Theorem for stratified deformations

In this section, our primary goal is to generalize the definitions of Sections 5.2 and 5.3 to arbitrary stratified subsets, so that we may generalize Theorems A, B, and C to the particular stratified deformation constructed in Section 4.3.

We begin by fixing some notation for this section. Let (Σ, ψ) denote an arbitrary stratified subset of $p : W \to B$ with coefficients in $X = BO \times BO$. The map $\psi : \Sigma \to BO \times BO$ composed with projection onto the first factor is denoted γ_-^Σ, and when composed with projection onto the second factor is denoted γ_+^Σ. These choices will ultimately be used to classify the negative and positive eigenspace bundles on Σ, as in Example 3.2. Let $\mathscr{S}(\Sigma)$ denote the collection of strata in the degree-wise stratification on Σ. We denote an element of $\mathscr{S}(\Sigma)$ in degree i by Z_i, but may drop the subscript when it is unnecessary. Stratified subsets admit ghosts just the same as the critical locus of a fiberwise generalized Morse function. Just as before, we use the notation $\widetilde{Z}$ to denote the smooth manifold obtained by perturbing a stratum of $\mathscr{S}(\Sigma)$ over the ghost set as in Section 4.2.

For $Z \in \mathscr{S}(\Sigma)$, the category $\mathsf{Disk}_m^{\widetilde{Z}/U}$ is defined identically as in Definition 2.2. As in Sections 5.2 and 5.3, the composition (5.4) is used to construct a map

$$S^0 \xrightarrow{e^d(\pi_{\widetilde{Z}})} \underset{U \in \mathsf{Disk}_k^{B/}}{\mathsf{holim}} \; \underset{V \in \mathsf{Disk}_m^{\widetilde{Z}/U}}{\mathsf{holim}} \; \Omega^\infty(V^\bullet \wedge \mathbb{S})$$

and the composition (5.5) is used to construct a map

$$S^0 \xrightarrow{e^t(\pi_{\widetilde{Z}})} \underset{U \in \mathsf{Disk}_k^{B/}}{\mathsf{holim}} \; \underset{V \in \mathsf{Disk}_m^{\widetilde{Z}/U}}{\mathsf{holim}} \; \Omega^\infty(V^\bullet \wedge A(*))$$

Aggregate over all $Z \in \mathscr{S}(\Sigma)$, we also have the maps

$$S^0 \xrightarrow{\prod_{Z\in\mathscr{S}(\Sigma)} e^d(\pi_{\widetilde{Z}})} \prod_{Z\in\mathscr{S}(\Sigma_f)} \underset{U\in\mathsf{Disk}_k^{B/}}{\mathsf{holim}} \ \underset{V\in\mathsf{Disk}_m^{\widetilde{Z}/U}}{\mathsf{holim}} \Omega^\infty(V^\bullet \wedge \mathbb{S})$$

$$S^0 \xrightarrow{\prod_{Z\in\mathscr{S}(\Sigma)} e^t(\pi_{\widetilde{Z}})} \prod_{Z\in\mathscr{S}(\Sigma_f)} \underset{U\in\mathsf{Disk}_k^{B/}}{\mathsf{holim}} \ \underset{V\in\mathsf{Disk}_m^{\widetilde{Z}/U}}{\mathsf{holim}} \Omega^\infty(V^\bullet \wedge A(*))$$

In summary, the definitions of the local characteristics on Σ do not depend on Σ being obtained as the critical locus of a fiberwise generalized Morse function.

Next, we must generalize the definitions of the index maps from before. Recall that when working with the critical locus of a fiberwise generalized Morse function, the index map was defined using a family of matrices which provided an automorphism of the one point compactification of the vertical tangent space at each point in the critical locus. That particular matrix was the derivative of the vertical gradient vector field of the function, or the Hessian of the function.

For an arbitrary stratified deformation, our matrix is given by the block sum of negative the identity matrix on the negative eigenspace bundle, and the identity matrix on the positive eigenspace bundle. In the event that Σ is the critical locus of a fiberwise generalized Morse function, this choice clearly agrees with the maps (5.8) and (5.16). Thus, we use the same notation for index maps on arbitrary stratified subsets: $\mathsf{Ind}^d_{\widetilde{Z}}$ and $\mathsf{Ind}^h_{\widetilde{Z}}$.

Let (Σ_f, ψ_f) be the stratified subset obtained from the critical locus of a fiberwise generalized Morse function. Let (S, Ψ) be a stratified deformation between (Σ_f, ψ_f) and (Σ_{SD}, ψ_{SD}). Let (Σ_α, ψ_l) denote the stratified subset given by the slice of (S, Ψ) at time $l \in [0,1]$. For $Z_l \in \mathscr{S}(\Sigma_l)$, consider the composition

$$\begin{tikzcd}
S^0 \arrow[d, "\prod_{Z_l \in \mathscr{S}(\Sigma_l)} e^d(\pi_{\widetilde{Z}_l})"'] & \\
\prod_{Z_l \in \mathscr{S}(\Sigma_l)} \operatorname*{holim}_{U \in \mathsf{Disk}_k^{B/}} \operatorname*{holim}_{V \in \mathsf{Disk}_m^{\widetilde{Z}_l/U}} \Omega^\infty(V^\bullet \wedge \mathbb{S}) \arrow[r, "\prod_{Z_l \in \mathscr{S}(\Sigma_l)} \mathsf{Ind}^d_{\widetilde{Z}}"] & \prod_{Z_l \in \mathscr{S}(\Sigma_l)} \operatorname*{holim}_{U \in \mathsf{Disk}_k^{B/}} \operatorname*{holim}_{V \in \mathsf{Disk}_m^{\widetilde{Z}_l/U}} \Omega^\infty(V^\bullet \wedge \mathbb{S})
\end{tikzcd} \tag{5.21}$$

Varying $l \in [0,1]$ provides a homotopy

$$\left(\left(\prod_{Z \in \mathscr{S}(\Sigma_f)} e^d(\pi_{\widetilde{Z}})\right) \circ \left(\prod_{Z \in \mathscr{S}(\Sigma_f)} \mathsf{Ind}^d_{\widetilde{Z}}\right) \circ +\right) \sim \left(\left(\prod_{Z_1 \in \mathscr{S}(\Sigma_1)} e^d(\pi_{\widetilde{Z}_1})\right) \circ \left(\prod_{Z_1 \in \mathscr{S}(\Sigma_1)} \mathsf{Ind}^d_{\widetilde{Z}}\right) \circ +\right) \tag{5.22}$$

Thus, we have the following theorem.

Theorem D. *With (S, Ψ) a stratified deformation between (Σ_f, ψ_f) and (Σ_{SD}, ψ_{SD}), the following diagram is homotopy commutative.*

$$\begin{tikzcd}
S^0 \arrow[r, "e^d(p)"] \arrow[d, "\prod_{Z \in \mathscr{S}(\Sigma_{SD})} e^d(\pi_{\widetilde{Z}})"'] & \operatorname*{holim}_{U \in \mathsf{Disk}_k^{B/}} \operatorname*{holim}_{V \in \mathsf{Disk}_m^{p^{-1}U/}} \Omega^\infty(V^\bullet \wedge \mathbb{S}) \\
\prod_{Z \in \mathscr{S}(\Sigma_{SD})} \operatorname*{holim}_{U \in \mathsf{Disk}_k^{B/}} \operatorname*{holim}_{V \in \mathsf{Disk}_m^{\widetilde{Z}/U}} \Omega^\infty(V^\bullet \wedge \mathbb{S}) \arrow[r, "\prod_{Z \in \mathscr{S}(\Sigma_{SD})} \mathsf{Ind}^d_{\widetilde{Z}}"'] & \prod_{Z \in \Sigma_{SD}} \operatorname*{holim}_{U \in \mathsf{Disk}_k^{B/}} \operatorname*{holim}_{V \in \mathsf{Disk}_m^{\widetilde{Z}/U}} \Omega^\infty(V^\bullet \wedge \mathbb{S}) \arrow[u, "incl"']
\end{tikzcd} \tag{5.23}$$

Proof. First we apply Theorem A to (Σ_f, ψ_f). Then the homotopy (5.22) is used to make the diagram above commute. $\square$

Remark 4.1. The homotopy (5.22) used in the proof of Theorem D also proves the a general statement: that the composition in question is a *stratified deformation invariant*. Compare to [**Igu05**] Lemma 5.4.

We now have the following generalization of Theorem C:

Theorem E. *The following diagram of spaces is homotopy commutative.*

$$\begin{tikzcd}
& S^0 \arrow[rr, "e^d(p)"] \arrow[dl, "\prod_{Z\in\mathscr{S}(\Sigma_{SD})} e^d(\pi)"'] \arrow[dd] & & \operatorname*{holim}_{U\in\mathsf{Disk}_k^{B/}} \operatorname*{holim}_{V\in\mathsf{Disk}_m^{p^{-1}U/}} \Omega^\infty(V^\bullet\wedge\mathbb{S}) \arrow[dd, "\eta"] \\
\prod_{Z\in\mathscr{S}(\Sigma_{SD})} \operatorname*{holim}_{U\in\mathsf{Disk}_k^{B/}} \operatorname*{holim}_{V\in\mathsf{Disk}_m^{\pi^{-1}U/}} \Omega^\infty(V^\bullet\wedge\mathbb{S}) \arrow[rr, "\prod_{Z\in\mathscr{S}(\Sigma_{SD})} \mathsf{Ind}^d_Z"'] \arrow[dd, "\eta"] & & \prod_{Z\in\mathscr{S}(\Sigma_{SD})} \operatorname*{holim}_{U\in\mathsf{Disk}_k^{B/}} \operatorname*{holim}_{V\in\mathsf{Disk}_m^{\pi^{-1}U/}} \Omega^\infty(V^\bullet\wedge\mathbb{S}) \arrow[ur, "+"] \arrow[dd, "\eta"] & \\
& S^0 \arrow[rr, "e^t(p)"] \arrow[dl, "\prod_{Z\in\mathscr{S}(\Sigma_{SD})} e^t(\pi)"] & & \operatorname*{holim}_{U\in\mathsf{Disk}_k^{B/}} \operatorname*{holim}_{V\in\mathsf{Disk}_m^{p^{-1}U/}} \Omega^\infty(V^\bullet\wedge A(*)) \\
\prod_{Z\in\mathscr{S}(\Sigma_{SD})} \operatorname*{holim}_{U\in\mathsf{Disk}_k^{B/}} \operatorname*{holim}_{V\in\mathsf{Disk}_m^{\pi^{-1}U/}} \Omega^\infty(V^\bullet\wedge A(*)) \arrow[rr, "\prod_{Z\in\mathscr{S}(\Sigma_{SD})} \mathsf{Ind}^t_Z"] & & \prod_{Z\in\mathscr{S}(\Sigma_{SD})} \operatorname*{holim}_{U\in\mathsf{Disk}_k^{B/}} \operatorname*{holim}_{V\in\mathsf{Disk}_m^{\pi^{-1}U/}} \Omega^\infty(V^\bullet\wedge A(*)) \arrow[ur, "+"] &
\end{tikzcd}$$

5.4.1. Rational fiberwise Poincaré–Hopf formulas

In Subsection 5.5 and Section 6, we will make use of the following simplifications of Theorem E. Let $\Sigma_{SD}^{\#}$ be the complement of the component Λ of Σ_{SD} on which ψ_{SD} is trivial (see Lemmas 3.9 and 3.10). Let $\mathscr{A}_j \subset \mathscr{S}(\Sigma_{SD}^{\#})$ contain those elements of degree j corresponding to the lower stratum of the remaining immersed lenses and let $\mathscr{A}_{j+1} \subset \mathscr{S}(\Sigma_{SD}^{\#})$ contain those elements of degree $j+1$ corresponding to the upper stratum of the immersed lenses.

Corollary 4.2. *The following equality holds in* $\pi_0\Gamma_B Q_B(W)\otimes\mathbb{Q}$

$$e^d_\partial(p) = \sum_{Z_i\in\mathscr{S}(\Sigma^{\#}_{SD})} (-1)^i e^d(\pi_{\widetilde{Z}_i})$$

Proof. In $\pi_0\Gamma_B Q_B(W)$, Theorem D reduces to the formula

$$e^d_\partial(p) = \sum_{Z_i\in\mathscr{S}(\Sigma_{SD})} \mathsf{Ind}^d_{\widetilde{Z}}\, e^d(\pi_{\widetilde{Z}_i})$$

By Lemma 2.3, we can replace $\mathsf{Ind}^d_{\widetilde{Z}}$ with $(-1)^i$ as in the formula below

$$e^d_\partial(p) = \sum_{Z_i\in\mathscr{S}(\Sigma_{SD})} (-1)^i e^d(\pi_{\widetilde{Z}_i})$$

By Lemma 3.10, we can eliminate the summands for those Z in $\mathscr{S}(\Lambda)$, and we are left with the following formula in $\pi_0\Gamma_B Q_B(W)\otimes\mathbb{Q}$.

$$e^d_\partial(p) = \sum_{Z_i\in\mathscr{S}(\Sigma^{\#}_{SD})} (-1)^i e^d(\pi_{\widetilde{Z}_i})$$

□

Corollary 4.3. *The following equality holds in* $\pi_0\Gamma_B A^{\%}_B(W)\otimes\mathbb{Q}$

$$e^t_\partial(p) = \sum_{Z_i\in\mathscr{S}(\Sigma^{\#}_{SD})} (-1)^i e^t(\pi_{\widetilde{Z}_i})$$

Proof. In $\pi_0\Gamma_B A^{\%}_B(W)$, Theorem D reduces to the formula

$$e^t_\partial(p) = \sum_{Z_i\in\mathscr{S}(\Sigma_{SD})} \mathsf{Ind}^d_{\widetilde{Z}}\, e^t(\pi_{\widetilde{Z}_i})$$

By Lemma 3.1, we can replace $\mathsf{Ind}^d_{\widetilde{Z}}$ with $(-1)^i$ as in the formula below

$$e^t_\partial(p) = \sum_{Z_i \in \mathscr{S}(\Sigma_{SD})} (-1)^i e^t(\pi_{\widetilde{Z}_i})$$

By Lemma 3.10, we can eliminate the summands for those Z in $\mathscr{S}(\Lambda)$, and we are left with the following formula in $\pi_0\Gamma_B A^{\%}_B(W) \otimes \mathbb{Q}$.

$$e^t_\partial(p) = \sum_{Z_i \in \mathscr{S}(\Sigma^{\#}_{SD})} (-1)^i e^t(\pi_{\widetilde{Z}_i})$$

□

5.5. Fiberwise Poincaré–Hopf Theorem for the smooth structure characteristic

In this section, we return to the setting of Section 3, in which we have a topologically trivial family of smooth h-cobordisms $p : W \to B$ with boundaries $\partial_0 W := M$ and $\partial_1 W := M'$ given as smooth manifold bundles $p_0 : M \to B$ and $p_1 : M' \to B$. The bundle p admits a fiberwise generalized Morse function $f : W \to [0, 1]$. By Theorem 3.3, we have a canonical nullhomotopy of the relative excisive A-theory Euler characteristic $\chi^{\%}_\partial(p)$. It then follows by Proposition 1.7 that we have a nullhomotopy of the Euler section $e^d_\partial(p)$. This nullhomotopy is used in the following definition.

Definition 5.1. For p a topologically trivial family of smooth h-cobordisms $p : W \to B$, the Euler section $e^{t/d}_\partial(p)$ is a map

$$S^0 \xrightarrow{e^{t/d}_\partial(p)} \operatorname*{\mathsf{holim}}_{U \in \mathsf{Disk}^{B/}_k} \operatorname*{\mathsf{holim}}_{V \in \mathsf{Disk}^{p^{-1}U/}_m} \Omega^\infty(V^\bullet \wedge \mathcal{H}(*))$$

defined to be the lift of $e^d_\partial(p)$ obtained from the nullhomotopy of $e^t_\partial(p)$.

In the proof of the theorem below, we express this nullhomotopy in terms of Σ_{SD}, the stratified subset from Subsection 4.3. Recall that the stratified subset is concentrated in two degrees, and is obtained by applying a stratified deformation to the critical locus of f.

For the remainder of this paper, we denote $e^{t/d}(\pi_{\widetilde{Z}}) \in \pi_0\Gamma_B\mathcal{H}^{\%}_B(W)\otimes\mathbb{Q}$ to be the lift of $e^d(\pi_{\widetilde{Z}})$ resulting from the identity $\eta_* e^d(\pi_{\widetilde{Z}}) = e^t(\pi_{\widetilde{Z}}) = 0$.

Theorem F. *For $p : W \to B$ a topologically trivial family of smooth h-cobordisms, the following equality holds in $\pi_0\Gamma_B\mathcal{H}^{\%}_B(W)\otimes\mathbb{Q}$*

$$e^{t/d}_\partial(p) = \sum_{Z_i\in\mathscr{S}(\Sigma^{\#}_{SD})} (-1)^i e^{t/d}(\pi_{\widetilde{Z}_i})$$

Proof. Corollary 4.2 indicates that the following equality holds in $\pi_0\Gamma_B Q_B(W)\otimes\mathbb{Q}$

$$e^d_\partial(p) = \sum_{Z_i\in\mathscr{S}(\Sigma^{\#}_{SD})} (-1)^i e^d(\pi_{\widetilde{Z}_i})$$

From Corollary 4.3, if we apply η_* to both sides we have the following equality in $\pi_0\Gamma_B A^{\%}_B(W)\otimes\mathbb{Q}$

$$e^t_\partial(p) = \sum_{Z_i\in\mathscr{S}(\Sigma^{\#}_{SD})} (-1)^i e^t(\pi_{\widetilde{Z}_i})$$

Since p is a topologically trivial family of smooth h-cobordisms, $e^t_\partial(p) = 0$ in $\pi_0\Gamma_B A^{\%}_B(W)\otimes\mathbb{Q}$. Recall from Lemma 3.7 and Lemma 3.9 that the elements of $\mathscr{S}(\Sigma^{\#}_{SD})$ are concentrated in two degrees, j and $j+1$. Let $\mathscr{A}_j \subset \mathscr{S}(\Sigma^{\#}_{SD})$ contain those elements of degree j and let $\mathscr{A}_{j+1} \subset \mathscr{S}(\Sigma^{\#}_{SD})$ contain those elements of degree $j+1$. Since ψ_{SD} is trivial on each element Z_j in $\mathscr{A}_j$, for all such elements we have that $e^t(\pi_{\widetilde{Z}_j}) = 0$. We are left with the equation

$$\sum_{Z \in \mathscr{A}_{j+1}} e^t(\pi_{\widetilde{Z}}) = 0$$

Since each $Z \in \mathscr{A}_{j+1}$ is of the same degree, there cannot be a relation among these characteristics. Thus, each $e^t(\pi_{\widetilde{Z}})$ for $\widetilde{Z}$ an upper stratum element in $\mathscr{A}_{j+1}$ must be 0. It follows that $\eta_* e^d_\partial(p) = e^t_\partial(p) = 0$, and $\eta_* e^d(\pi_{\widetilde{Z}}) = e^t(\pi_{\widetilde{Z}}) = 0$ for each $Z \in \mathscr{S}(\Sigma^{\#}_{SD})$.

Since the fibration $\Gamma_B \mathcal{H}^{\%}_B(W) \to \Gamma_B Q^{\%}_B(W) \to \Gamma_B A^{\%}_B(W)$ is split by the trace map, we have a short exact sequence

$$0 \to \pi_0 \Gamma_B \mathcal{H}^{\%}_B(W) \otimes \mathbb{Q} \to \pi_0 \Gamma_B Q_B(W) \otimes \mathbb{Q} \to \pi_0 \Gamma_B A^{\%}_B(W) \otimes \mathbb{Q} \to 0$$

It then follows from Definition 5.1 that

$$e^{t/d}_\partial(p) = \sum_{Z_j \in \mathscr{A}_j} (-1)^j e^{t/d}(\pi_{\widetilde{Z}_i}) + \sum_{Z_{j+1} \in \mathscr{A}_{j+1}} (-1)^{j+1} e^{t/d}(\pi_{\widetilde{Z}_{j+1}})$$

The result follows since $\mathscr{A}_j \cup \mathscr{A}_{j+1} = \mathscr{S}(\Sigma^{\#}_{SD})$ $\square$

CHAPTER 6

Calculations of the Smooth Structure Class

In this section we prove the main theorem. We begin in subsection 6.1 by reviewing the setup for the proof of the Rigidity Conjecture. In Subsection 6.2 we use Theorem F to prove a duality theorem for the smooth structure class, Theorem G. In Subsection 6.3 we prove the Rigidity Conjecture, Theorem H. We also give slightly more general statements in Theorem I and Corollary 3.1. In Section 6.4 we explain Hatcher's construction and the immersed Hatcher construction, and we conclude by applying Theorems F and G to recover Goette and Igusa's computation of the smooth structure class for the immersed Hatcher construction.

6.1. Setup for the Proof of the Rigidity Conjecture

Let W be a smooth h-cobordism bundle over B that is *topologically trivial* as in Section 3. This means that $p : W \to B$ is a smooth fiber bundle with two boundary components $\partial_0 W := M_0$ and $\partial_1 W := M_1$ so that $p_0 : M_0 \to B$ and $p_1 : M_1 \to B$ are smooth manifold bundles. In addition, there is a homeomorphism $h : W \to \partial_0 W \times I$ so that $(p_0 \times \mathrm{id}) \circ h$ is a topological manifold bundle and $(p_0 \times \mathrm{id}) \circ h$ restricts to the smooth bundles p_0 and p_1 over $B \times 0$ and $B \times 1$. If the fibers of M_0 and M_1 themselves have boundary, then we additionally require that $\partial^\vee W$ is diffeomorphic to $M_0 \cup \partial M_0 \times I \cup M_0$.

Let $\chi^{\%}(W, \partial_0 W)$ denote the *relative excisive A-theory Euler characteristic* of $(W, \partial_0 W)$, and let $\mathsf{tr}(W, \partial_0 W)$ denote the *relative Becker–Gottlieb transfer* of $(W, \partial_0 W)$. Then the map h provides a path from $\chi^{\%}(W, \partial_0 W)$ to $\chi^{\%}(M_0 \times I, M_0)$. Concatenating this path with the homotopy from $\eta \circ \mathsf{tr}(W, \partial_0 W)$ to $\chi^{\%}(W, \partial_0 W)$ supplied in [**DWW03**], we have a nullhomotopy of the composition $\eta \circ \mathsf{tr}(W, \partial_0 W)$. In Definition 4.1, the *relative smooth structure characteristic* $\theta(W, \partial_0 W) \in \Gamma_B \mathcal{H}_B^{\%}(W)$ is defined to be the lift of $\mathsf{tr}(W, \partial_0 W)$ determined by this nullhomotopy. This is summarized in the following diagram.

$$\begin{tikzcd}
 & \Gamma_B \mathcal{H}_B^{\%}(W) \arrow[d] \\
S^0 \arrow[ur, "\theta(W,\partial_0 W)"] \arrow[r, "\mathsf{tr}(W,\partial_0 W)"] \arrow[dr, "\chi^{\%}(W,\partial_0 W)"'] & \Gamma_B Q_B(W) \arrow[d, "\eta"] \\
 & \Gamma_B A_B^{\%}(W)
\end{tikzcd}$$

We can similarly define $\theta(W, \partial_1 W)$ in $\Gamma_B \mathcal{H}_B^{\%}(W)$.

Proposition 1.1. Given a topologically trivial family of smooth h-cobordisms $p : W \to B$, the smooth structure characteristic $\theta(W, \partial_0 W)$ is fiberwise Poincaré dual to the Euler section $e_\partial^{t/d}(p)$ of Definition 5.1.

Proof. This follows by combining the definition of $\theta(W, \partial_0 W)$ (4.1) as the homotopy fiber of $\mathsf{tr}(W, \partial_0 W)$ over $\chi^{\%}(W, \partial_0 W)$, the definition of $e_\partial^{t/d}(p)$ as the homotopy fiber of $e_\partial^d(p)$ over $e_\partial^t(p)$, and Proposition 1.7. $\square$

To prove the Rigidity Conjecture, we must compute $\Theta(W, \partial_0 W) - \Theta(W, \partial_1 W)$ for a topologically trivial h-cobordism bundle W.

Remark 1.2. See Appendix A for an explanation of why $\Theta(W, \partial_0 W) - \Theta(W, \partial_1 W)$ agrees with the smooth structure class in [**GI14**].

6.2. A Duality Theorem for the Smooth Structure Class

In this section we prove a duality theorem for the smooth structure class using Theorem F. Recall how, in Section 5, we made use of a fiberwise generalized Morse function $f : W \to [0, 1]$ for which $f(\partial_0 N) = 0$ and $f(\partial_1 N) = 1$. In this section we will also make use of the fiberwise generalized Morse function $\overline{f} := 1 - f$. In particular we will prove a duality theorem, Theorem G, by applying Theorem F to f and $\overline{f}$, and then comparing the results.

Theorem G. *For $p : W \to B$ a topologically trivial bundle of smooth h-cobordisms with fiber dimension n,*

$$\Theta(W, \partial_0 W) = (-1)^{n-1}\Theta(W, \partial_1 W)$$

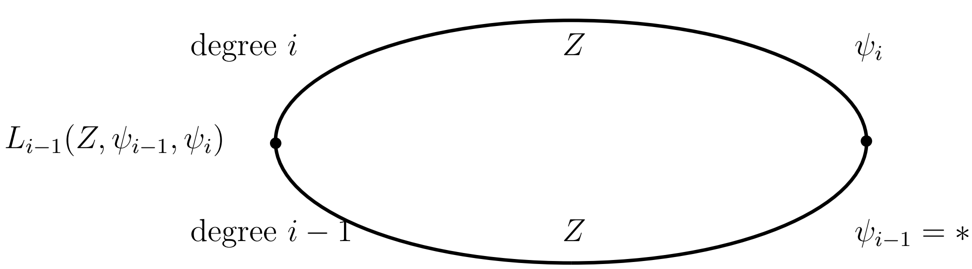
Proof of Theorem G. From Subsection 4.3, we have a stratified deformation of the stratified subset (Σ_f, ψ_f) to the stratified subset concentrated in two degrees (Σ_{SD}, ψ_{SD}). Applying this stratified deformation instead to $(\Sigma_{\overline{f}}, \psi_{\overline{f}})$, we obtain the stratified deformation $(\overline{\Sigma_{SD}}, \overline{\psi_{SD}})$. We take care to point out that while Σ_{SD} and $\overline{\Sigma_{SD}}$ are diffeomorphic, the particular stratum of $\mathscr{S}(\Sigma_{SD})$ may have a different degree in $\overline{\Sigma_{SD}}$, and the bundles ψ_{SD} and $\overline{\psi_{SD}}$ are only equivalent after applying the swap map $BO \times BO \to BO \times BO$, since the positive and negative eigenspaces have been exchanged.

We begin by applying Theorem F to f and $\overline{f}$ to obtain the following formulas.

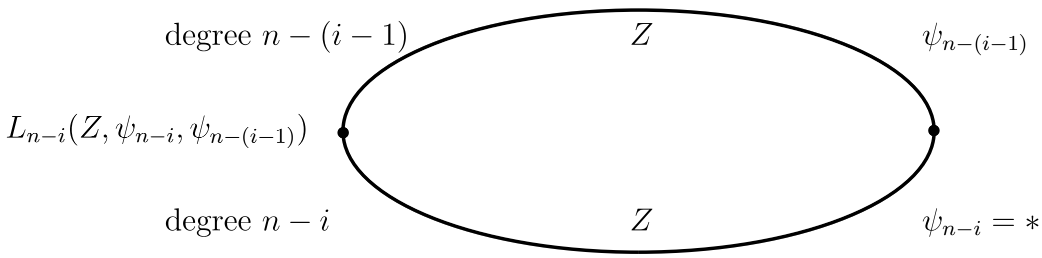
$$e^{t/d}_{\partial_0}(p) = \sum_{Z_i \in \mathscr{S}(\Sigma^{\#}_{SD})} (-1)^i e^{t/d}(\pi_{\widetilde{Z}_i}) \tag{6.1}$$

$$e^{t/d}_{\partial_1}(p) = \sum_{Z_j \in \mathscr{S}(\overline{\Sigma^{\#}_{SD}})} (-1)^j e^{t/d}(\pi_{\widetilde{Z}_i}) \tag{6.2}$$

The stratified subsets (Σ_{SD}, ψ_{SD}) and $(\overline{\Sigma_{SD}}, \overline{\psi_{SD}})$ are each disjoint unions of immersed lenses concentrated in two degrees, and a component on which the tangential data is trivial. We do not consider this extra component because it has a trivial rational contribution, according to Lemma 3.10. Because the critical loci of f and $\overline{f}$ are identical, the submanifolds Σ_{SD} and $\overline{\Sigma_{SD}}$ constructed from the same stratified deformation are identical. In particular, there is a one-to-one correspondence between immersed lenses comprising (Σ_{SD}, ψ_{SD}) and $(\overline{\Sigma_{SD}}, \overline{\psi_{SD}})$. It suffices to consider one such pair, and observe how the bundle data ψ_{SD} and $\overline{\psi_{SD}}$ has changed. Below we depict $L_{i-1}(Z, \psi_{i-1}, \psi_i)$, an immersed lens belonging to (Σ_{SD}, ψ_{SD}).



The immersed lens $L_{i-1}(Z, \psi_{i-1}, \psi_i)$ above corresponds to the immersed lens $L_{n-i}(Z, \psi_{n-i}, \psi_{n-(i-1)})$ belonging to $(\overline{\Sigma_{SD}}, \overline{\psi_{SD}})$, depicted below.



Recall that after the stratified deformation of Lemma 3.7, the bundle data on the lower stratum is trivial, so ψ_{i-1} and ψ_{n-i} are trivial as in the diagrams above. On the upper strata, ψ_{SD} is the map $Z \to BO \times BO$ classifying the stable negative and positive eigenspace bundles, γ_f and γ_{-f}, respectively. These stable bundles γ_f and γ_{-f} have the property that $\gamma_f \oplus \gamma_{-f} \cong T^\vee M|_Z \oplus \epsilon^n$. The section $e^{t/d}(\pi_{\widetilde{Z}})$ is defined in terms of the restriction of the vertical tangent bundle $T^\vee M|_Z$. Thus the summand $(-1)^i e^{t/d}(\pi_{\widetilde{Z}_i})$ for $Z \in \mathscr{S}(\Sigma^{\#}_{SD})$ is the same up to a sign as the summand $(-1)^{n-(i-1)} e^{t/d}(\pi_{\widetilde{Z}_{n-(i-1)}})$ with $Z_{n-(i-1)}$ in $\overline{\Sigma^{\#}_{SD}}$. It then follows that

$$e^{t/d}_{\partial_0}(p) = (-1)^{n-1} e^{t/d}_{\partial_1}(p)$$

Taking Poincaré duals on both sides gives the result in the theorem statement. $\square$

6.3. Proof of the Rigidity Conjecture

We can now prove the Rigidity Conjecture.

Theorem H. *If the fibers of $p_0 : M \to B$ are even dimensional and closed, then for a topologically trivial family of smooth h-cobordisms W from M to M', $\Theta(M \times I, M' \times I)$ is trivial.*

Proof. When the fibers of M are even dimensional, $n-1$ is even. Thus, by Theorem G, $\Theta(W, \partial_0 W) = \Theta(W, \partial_1 W)$. It follows that the smooth structure class of [**GI14**], $\Theta(M \times I, M' \times I)$, is trivial. $\square$

We might also consider the relative case, where the boundaries $\partial_0 W$ and $\partial_1 W$ have corners. Recall that in this case we consider h-cobordisms W from M to M' so that $\partial^\vee W$ is diffeomorphic to $M \cup \partial M \times I \cup M'$.

Theorem I. *If M has boundary and the fiber dimension of M is even, then for any topologically trivial family of smooth h-cobordisms W from M to M' with $\partial^\vee W$ diffeomorphic to $M \cup \partial M \times I \cup M'$, the smooth structure class $\Theta(M \times I, M' \times I)$ is trivial.*

Proof. The proof is word-for-word the same as the proof of Theorem H. $\square$

Applying the two theorems above to a bundle M with closed fibers, we obtain the following corollary.

Corollary 3.1. *If $n + k$ is even and $k \geq 0$, then for any topologically trivial family of smooth h-cobordisms W from $M \times I^k$ to $M' \times I^k$ satisfying the condition $\partial^\vee W = M \times I^k \cup \partial(M \times I^k) \times I \cup M' \times I^k$, the smooth structure class $\Theta(M' \times I^{k+1}, M \times I^{k+1})$ is trivial.*

Remark 3.2. Note that the dependence on k in the corollary above implies that the statement applies just as well when n is odd and k is odd. We explain why this does not contradict the constructions in [**GI14**], which produce topologically trivial h-cobordisms whose boundaries are bundles with closed odd dimensional fibers. If we take such a bundle and stabilize once by multiplying with an interval, we produce an h-cobordism, but the boundary is not a product h-cobordism, and thus Theorem I does not apply. It may be helpful to note that the smooth structure class defined in [**GIW14**] is not a stable invariant with respect to the upper and lower stabilization maps on the h-cobordism space, which maintain a product structure on the boundary. In particular, $\Theta(W, \partial_0 W)$ is an invariant of the lower stabilization map, and $\Theta(W, \partial_1 W)$ is an invariant of the upper stabilization map. Still, the difference $\Theta(W, \partial_0 W) - \Theta(W, \partial_1 W)$ is not an invariant of either stabilization map, but is only an invariant of the stabilization which multiplies the entire h-cobordism by an interval. Theorem I does not apply to h-cobordisms obtained from that type of stabilization.

6.4. Computations for the immersed Hatcher construction

If we apply Theorem G to a topologically trivial smooth h-cobordism bundle $p : W \to B$ with boundaries $p_0 : M \to B$ and $p_1 : M' \to B$ with odd dimensional fibers, the fiber dimension of W, $n-1$, is odd. Thus,

$$\Theta(M \times I, M' \times I) = \Theta(W, \partial_0 W) - \Theta(W, \partial_1 W) = 2\Theta(W, \partial_0 W) \tag{6.3}$$

In [**GI14**], Igusa and Goette used the immersed Hatcher construction to obtain bundles that realize a nontrivial value for the left-hand side. After briefly summarizing Hatcher's construction and the immersed Hatcher construction, we will compute $\Theta(W, \partial_0 W)$ for that particular construction using Theorem F.

6.4.1. Hatcher Construction

Hatcher's example produces a smooth disk bundle over a compact base that is homeomorphic but not diffeomorphic to a trivial disk bundle. The construction proceeds in five steps below. Briefly, the idea is to use a vector bundle in the kernel of the J-homomorphism to obtain a disk bundle for which the corresponding sphere bundle is fiberwise homotopy equivalent to a trivial sphere bundle. After stabilizing so that this homotopy equivalence produces an embedding, we glue this disk bundle into the center of a trivial sphere bundle on which the fiber is a solid torus. The result is a disk bundle with the desired property. There are several descriptions of this construction in the literature, the construction used here is identical to the one used in [**GI14**], and when the base is a $4k$-dimensional sphere a similar construction can be found in [**Goe01**]. We proceed step by step, starting only with a compact manifold B of dimension q, and a continuous map $B^q \to G/O$.

(1) We extract from the map $B^q \to G/O$ an n-dimensional vector bundle whose associated sphere bundle, $S^{n-1}(\xi) \to B$, is fiber homotopy trivial. Here $n \geq q+1$.
(2) Let η be an m-dimensional vector bundle on B complementary to ξ. Construct the bundle $D^n(\xi) \oplus D^m(\eta)$ over B. This bundle contains a subbundle $S^{n-1}(\xi) \oplus D^m(\eta)$. This will make up the boundary of the disk bundle that we later glue inside of the bundle of solid tori.
(3) Construct the trivial sphere bundle $S^{n-1} \times (I \times D^m) \times B \to B$. This is the bundle of solid tori. We group the $(I \times D^m)$ term because it is helpful to think of this as a D^{m+1} disk.
(4) We construct a fiberwise smooth embedding $S(j) : S^{n-1}(\xi) \oplus D^m(\eta) \hookrightarrow S^{n-1} \times D^m \times B$

(5) Now we attach the disk bundle to the solid torus to obtain a smooth disk bundle $E^{n,m}(\xi)$.

$$E^{n,m}(\xi) := D^n(\xi) \oplus D^m(\eta) \cup_{S(j)} S^{n-1} \times D^m \times B$$

The bundle $E^{n,m}(\xi)$ is Hatcher's example.

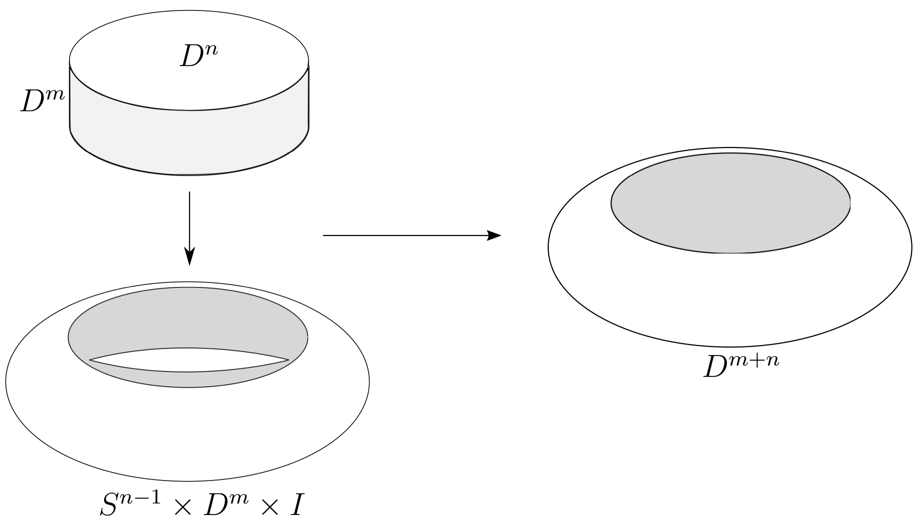


Figure 6.1. Fiberwise surgery in Hatcher's example

Steps (1) and (4) above require justification. The map $B \to G/O$ classifies a stable vector bundle whose spherical fibration is trivial. Since the map $BO(n) \to BO$ is $q+1$ connected when $n \geq q+1$, this stable vector bundle is given by a unique n-plane bundle ξ. See [**GI14**] for a proof that $S^{n-1}(\xi) \to B$ is fiber homotopy trivial (one must check that the composition $B \xrightarrow{\xi} BO_n \to BG_n$ is nullhomotopic). The construction of the embedding $S(j)$ follows from the following crucial lemma.

Lemma 4.1 (Lemma 1.2.1 in [**GI14**])**.** If $m > n > q$ then there is a smooth fiberwise embedding of pairs:

$$j : (D^n(\xi), S^{n-1}(\xi)) \to (D^n, S^{n-1}) \times D^m \times B$$

over B which is the standard embedding over $\partial_0 B$ and which is transverse to $S^{n-1} \times D^m$. Furthermore, if $m \geq q + 3$ then this fiberwise embedding will be unique up to fiberwise isotopy.

Let η be the normal bundle to the embedding j. Then we have a codimension 0 embedding

$$D(j) : D^n(\xi) \oplus D^m(\eta) \hookrightarrow D^n \times D^m \times B$$

from which we obtain $S(j)$ by restricting to the boundary $\partial D^n(\xi) \oplus D^m(\eta)$.

Remark 4.2. The constraint $m > n > q$ implies that $m+n \geq 2q+3$. Thus Hatcher's construction can only be used to produce disk bundles where the dimension of the fiber is $2 \dim B + 3$. Our intent is to perform surgery on manifold bundles by gluing in Hatcher's construction, so these bundles must also have the property that the dimension of the fiber is at least $2 \dim B + 3$.

6.4.2. Immersed Hatcher Construction

The goal of the immersed Hatcher construction is to use Hatcher's disk bundle to produce exotic smoothings of smooth manifold bundles that are supported on embedded disk bundles. Roughly, we attach thickenings of Hatcher's disk bundle along the top $M \times 1$

boundary of $M \times I$ at surgery sites which are tubular neighborhoods of the critical locus of a fiberwise generalized Morse function. Since Hatcher's construction only produces disk bundles whose fibers have dimension at least $2\dim B + 3$, the same is true of the immersed Hatcher construction. The construction proceeds in four steps.

(1) First, given a fiberwise generalized Morse function $f : M \to \mathbb{R}$, let L denote the submanifold of the critical locus consisting of those critical points of even index. L will likely have multiple components and has nonempty boundary consisting of the birth-death critical points. There is an embedding $L \xrightarrow{\lambda} M$ and an immersion $\lambda : L \to B$ given by restricting the projection $p : M \to B$. The reason λ is an immersion and not an embedding is because fibers may have multiple even index critical points. Let T be a tubular neighborhood of L, so that the following diagram commutes

$$\begin{tikzcd} T \arrow[r, "D(\tilde{\lambda})"] \arrow[d, "\pi"] & M \arrow[d, "p"] \\ L \arrow[r, "\lambda"] & B \end{tikzcd}$$

(2) Construct a Hatcher handle $B^{n,m}(\xi, \eta)$ by attaching the top of $T \times I$ to Hatcher's construction $E^{n,m}(\xi)$.

(3) Attach the Hatcher handle to the top of $T \times I$.

$$E_L^{n,m+1}(\xi, \eta) := T \times I \cup B^{n,m}(\xi, \eta)$$

(4) Now we excise $T \times I$ from $M \times I$ and glue in $E_L^{n,m+1}(\xi, \eta)$. The result is

$$E_+^{n,m}(M, \tilde{\lambda}, \xi) = (M - T) \times I \cup E_L^{n,m+1}(\xi, \eta)$$

Figure 6.2. This figure depicts steps 1, 2, and 3 of the immersed Hatcher construction on a single fiber, in which case L is a point.

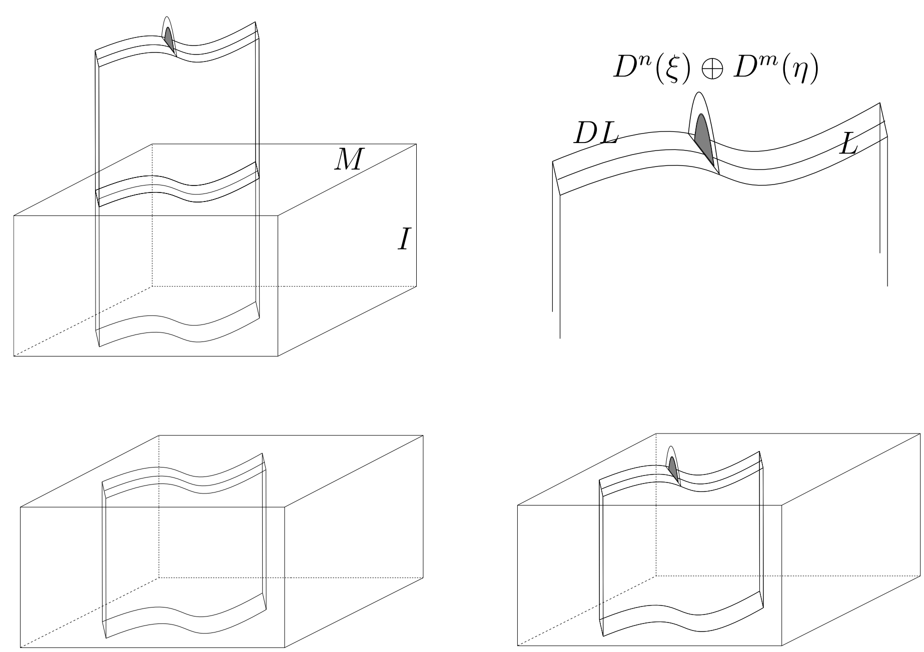


Figure 6.3. This figure depicts steps 3 and 4 of the immersed Hatcher construction when L is one dimensional.

To simplify the notation, let $E := E_+^{n,m}(M, \tilde{\lambda}, \xi)$. The newly constructed h-cobordism bundle $p : E \to B$ is a topologically trivial bundle of h-cobordisms over B. The boundary $\partial_0 E$ is the bundle $p_0 : M \to B$, and the boundary $\partial_1 E$ is the bundle $p_1 : M' \to B$.

6.4.3. Computing the smooth structure class for the immersed Hatcher construction

In this section, our goal is to compute $\Theta(W, \partial_0 W)$. To start we construct a fiberwise generalized Morse function on $p : E \to B$. This construction is extracted from the proof of Theorem 2.4.1 in [**GI14**].

The bundle $E_L = E_L^{n,m}(\xi, \eta)$ admits a fiberwise Morse function $f : E_L \to [0, 1]$ which is projection to I in a neghborhood of the bottom and sides of $T \times I$, and has two critical points over every point $t \in L$, one of y_t of index n and one point x_t of index $n-1$. The vertical tangent bundle of E_L splits as $\epsilon^{n-1}(\eta \oplus \epsilon^1)$ along the section x_t of E_L, where the trivial $n-1$ plane bundle ϵ^{n-1} is the negative eigenspace of $D^2 f_t$ along x_t. The vertical tangent bundle of E_L along y_t splits as $\xi \oplus (\eta_0 \oplus \epsilon^1)$ where the vector bundle ξ, which is homotopically trivial in the sense that $J(\xi) = 0$, is the negative eigenspace bundle. The critical points can be cancelled along $\partial_0 L$, to produce a fiberwise generalized Morse function, which we also denote as f. Let g be the fiberwise generalized Mose function on E obtained by taking projection to I on $(M - T) \times I$, and f on E_L.

Since g has the same Morse theoretic data as f, we can apply Theorem F, with $\mathscr{S}(\Sigma_g) = \{Z_n, Z_{n-1}\}$. Since the vertical tangent bundle is trivial along Z_{n-1}, we are left with

$$e_\partial^{t/d}(p) = (-1)^n e^{t/d}(\pi_{\widetilde{Z}_n})$$

Applying Poincaré duality on both sides, we have

$$\Theta(W, \partial_0 W) = (-1)^n D e^{t/d}(\pi_{\widetilde{Z}_n})$$

From [**GI14**], $e^{t/d}(\pi_{\widetilde{Z}_n})$ is equal to $\tilde{\lambda}_* \widetilde{\mathrm{ch}}(\xi)$, where $\widetilde{\mathrm{ch}}$ is the *normalized Chern character* (see Definition 1.3.8 in [**GI14**]). Thus, by (6.3), we have

$$\Theta(M \times I, M' \times I) = 2 \cdot (-1)^n \cdot \tilde{\lambda}_* D \widetilde{\mathrm{ch}}(\xi)$$

This computation confirms the statement of Theorem 3.0.5 in [**GI14**].

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APPENDIX A

The purpose of this appendix is to explain how the smoothing theory used in [**GIW14**] to define the space of stable exotic smoothings is related to the space of topologically trivial families of smooth h-cobordisms, and to use this relationship to explain and correct work of Goette and Igusa [**GI14**]. Of crucial importance is an old theorem of Burghelea and Lashof stating that the space stable exotic smoothings on a smooth manifold M is a homology theory in M given by smashing with the stable h-cobordism space of a point. Also of fundamental importance is a result of Dwyer, Weiss, and Williams, which identifies the space of topologically trivial h-cobordisms on M as the same homology theory.

In [**GI14**], Goette and Igusa construct topologically trivial families of smooth h-cobordisms and compute an invariant of these manifold bundles known as the smooth structure class. However, in their language, they construct 'exotic smoothings of smooth manifold bundles with closed fibers.' This presents the following contradiction: it follows by classical smoothing theory, or the work of [**GIW14**], that such bundles cannot exist. In fact, exotic smoothings can only exist when the fibers have boundary.

Nevertheless, their constructions and computations are valid. We augment and correct their paper by observing that since their *immersed Hatcher construction* is a topologically trivial family of smooth h-cobordisms, it does give rise to an exotic smoothing of the cylinder $M \times I$, as opposed to an exotic smoothing of a bundle M with closed fibers.

We also redefine their smooth structure class in terms of topologically trivial families of smooth h-cobordisms, so that we may correct a handful of statements in their paper.

A.1. Moduli Spaces of h-cobordisms and exotic smoothings of fiber bundles

In this section we prove that the stable space of exotic smoothings used in [**GIW14**] is homotopy equivalent to the space of topologically trivial families of smooth h-cobordisms. In Subsection A.1.1 we recall the structure spaces of smooth manifolds, linearized manifolds, and topological manifolds used in [**GIW14**]. We also identify these spaces in terms of spaces of lifts of maps classifying tangent bundle data. In Subsection A.1.2 we show that the space of topologically trivial families of smooth h-cobordisms on a smooth bundle $p : M \to B$ is homotopy equivalent to the stable space of exotic smoothings on the same bundle. We also show that both of these spaces are homotopy equivalent to $\Gamma_B \mathcal{H}^{\%}_B(M)$, the space of sections of the bundle obtained by taking a fiberwise smash product with $\mathcal{H}(*)$. These results follow from work of [**DWW03**] on the converse Riemann–Roch theorem, and are descendents of the surprising fact, proven in [**BL77**], that the space of topologically trivial h-cobordisms on a smooth manifold is a homology theory.

A.1.1. Spaces of smooth structures

In this section, we recall the structure spaces defined in [**GIW14**], and identify these spaces as spaces of lifts.

Recall from [**GI14**] the simplicial set $\mathcal{S}^t_\bullet(n)$ whose k-simplices are continuous Δ^k families of compact topological n-manifolds. The geometric realization $|\mathcal{S}^t_\bullet(n)|$ is the classifying space for bundles of compact n manifolds, and satisfies the homotopy equivalence

$$|\mathcal{S}^t_\bullet(n)| \simeq \coprod \mathrm{BHomeo}(M)$$

wherein the disjoint union is taken over all homeomorphism classes of compact n-manifolds.

In the smooth setting, the simplicial set $\mathcal{S}^d_\bullet(n)$ has k-simplices which are smoothings of compact topological n-manifolds immersed into $\mathbb{R}^\infty \times \Delta^k$. The geometric realization $|\mathcal{S}^d_\bullet(n)|$ is the classifying space for bundles of smooth compact n-manifolds, and satisfies the homotopy equivalence

$$|\mathcal{S}^d_\bullet(n)| \simeq \coprod \mathrm{BDiff}(M)$$

wherein the disjoint union is taken over all diffeomorphism classes of smooth compact n-manifolds.

Let $\widetilde{\mathcal{S}}^t_\bullet(n)$ be the simplicial set whose k-simplices are continuous Δ^k families of linearized n-manifolds. A linearized n-manifold is a compact topological manifold M^n with a vector bundle structure on its topological tangent microbundle. In otherwords, we have a lift of the map $M \to \mathrm{BTop}(n)$ classifying the topological tangent bundle on M, to a map $M \to \mathrm{BO}(n)$. The data of a linearization does not include a compatible map $\partial M \to \mathrm{BO}(n-1)$.

The simplicial forgetful map $\mathcal{S}^d_\bullet(n) \to \mathcal{S}^t_\bullet(n)$ factors through $\widetilde{\mathcal{S}}^t_\bullet(n)$ as in the diagram below:

$$\begin{tikzcd}
& \widetilde{\mathcal{S}}^t_\bullet(n) \arrow[dr] & \\
\mathcal{S}^d_\bullet(n) \arrow[ur] \arrow[rr] & & \mathcal{S}^t_\bullet(n)
\end{tikzcd}$$

Below we will work with families of manifolds. Given a simplicial set $X_\bullet$, the notation $|X_\bullet|^B$ refers to the space of maps $\mathrm{Map}(B, |X_\bullet|)$.

Definition 1.1. Suppose $p : M \to B$ is a bundle of compact topological n-manifolds with nonempty boundary classified by a point α in $|\mathcal{S}^t_\bullet(n)|^B$. Then the homotopy fiber of the map $|\mathcal{S}^d_\bullet(n)|^B \to |\mathcal{S}^t_\bullet(n)|^B$ over α is the space of smooth structures on $p : M \to B$, and is denoted by $\mathcal{S}^{t/d}_B(M)$.

Definition 1.2. Suppose $p : M \to B$ is a bundle of compact topological n-manifolds with nonempty boundary classified by a point α in $|\mathcal{S}^t_\bullet(n)|^B$. Then the homotopy fiber of the map $|\widetilde{\mathcal{S}}^t_\bullet(n)|^B \to |\mathcal{S}^t_\bullet(n)|^B$ over α is the space of linear structures on $p : M \to B$, and is denoted by $\widetilde{\mathcal{S}}^{t/t}_B(M)$.

Definition 1.3. Suppose $p : M \to B$ is a bundle of linearized compact topological n-manifolds with nonempty boundary classified by a point β in $|\widetilde{\mathcal{S}}^t_\bullet(n)|^B$. Then the homotopy fiber of the map $|\mathcal{S}^d_\bullet(n)|^B \to |\widetilde{\mathcal{S}}^t_\bullet(n)|^B$ over β is the space of smooth structures on $p : M \to B$, and is denoted by $\widetilde{\mathcal{S}}^{t/d}_B(M)$.

With $p : M \to B$ a bundle of linearized compact topological n-manifolds with nonempty boundary classified by a point β in $|\widetilde{\mathcal{S}}^t_\bullet(n)|^B$, with underlying topological manifold bundle classified by a point $\alpha \in |\mathcal{S}^t_\bullet(n)|^B$, we have the homotopy fiber sequence

$$\widetilde{\mathcal{S}}_B^{t/d}(M) \to \mathcal{S}_B^{t/d}(M) \to \widetilde{\mathcal{S}}_B^{t/t}(M)$$

In the three propositions below, we identify the homotopy types of each of these three spaces in terms of the classifying spaces $\mathrm{BO}(n)$, $\mathrm{BO}(n-1)$, $\mathrm{BTop}(n)$, and $\mathrm{BTop}(n-1)$. At the end of this section, we will use these characterizations to indicate when the homotopy fiber sequence above is useful. For instance, when the fibers are closed the space $\widetilde{\mathcal{S}}_B^{t/d}(M)$ is contractible.

For what follows, we denote by $(\alpha, \partial^\vee \alpha) : (M, \partial^\vee M) \to (\mathrm{BTop}(n), \mathrm{BTop}(n-1))$ the classifying map of the topological vertical tangent bundle on an (underlying) topological manifold bundle $p : M \to B$ classified by a point $\alpha \in |\mathcal{S}_\bullet^t(n)|^B$.

Proposition 1.4. Let $p : M \to B$ be a bundle of compact topological n-manifolds with nonempty boundary classified by a point $\alpha \in |\mathcal{S}_\bullet^t(M)|^B$. Then the space $\mathcal{S}_B^{t/d}(M)$ is homotopy equivalent to the space of pairwise maps $\mathrm{Map}_B((M, \partial M), (\mathrm{BO}(n), \mathrm{BO}(n-1))$ lifting $(\alpha, \partial^\vee \alpha)$.

Proof. This follows from classical smoothing theory, a reference for which is [**KS77a**] Essay V. $\square$

Proposition 1.5. Let $p : M \to B$ be a bundle of compact topological n-manifolds with nonempty boundary classified by a point $\alpha \in |\mathcal{S}_\bullet^t(M)|^B$. Then the space $\widetilde{\mathcal{S}}_B^{t/t}(M)$ is homotopy equivalent to the space of maps $\mathrm{Map}_B(M, \mathrm{BO}(n))$ lifting α.

Proof. This follows from the definition of a linearization as a lift of the classifying map of the topological tangent bundle to a map to $\mathrm{BO}(n)$. $\square$

If the topological manifold bundle $p : M \to B$ classified by α is refined to a linearized manifold bundle $\beta \in |\widetilde{\mathcal{S}}^t_\bullet(n)|^B$, then we already have a lift of the map $\alpha : M \to \mathrm{BTop}(n)$ to a map $\beta : M \to \mathrm{BO}(n)$. Thus, to obtain a smooth structure on $p : M \to B$, it suffices to choose a map $\partial^\vee\beta : \partial^\vee M \to \mathrm{BO}(n-1)$ that fills in the following diagram:

(A.1)

$$\begin{tikzcd}
 & & \mathrm{BTop}(n-1) \arrow[dd] \\
\partial^\vee M \arrow[urr, "\partial^\vee\alpha"] \arrow[r, dashed, "\partial^\vee\beta"] \arrow[ddd] & \mathrm{BO}(n-1) \arrow[ur] \arrow[dd] & \\
 & & \mathrm{BTop}(n) \\
M \arrow[urr, "\alpha"] \arrow[r, "\beta"] & \mathrm{BO}(n) \arrow[ur] &
\end{tikzcd}$$

Proposition 1.6. Let $p : M \to B$ be a bundle of linearized compact topological n-manifolds with nonempty boundary classified by a point $\beta \in |\widetilde{\mathcal{S}}^t_\bullet(M)|^B$. Then the space $\widetilde{\mathcal{S}}^{t/d}_B(M)$ is homotopy equivalent to the space of pairwise maps $\mathrm{Map}_B(\partial M, \mathrm{BO}(n-1))$ compatible with $\beta : M \to \mathrm{BO}(n)$.

Proof. This follows from Propositions 1.4 and 1.5. $\square$

If $p : M \to B$ is a bundle of compact smooth n-manifolds with nonempty boundary, and we consider the space $\widetilde{\mathcal{S}}^{t/d}_B(M)$ constructed as the homotopy fiber of the map $|\mathcal{S}^d_\bullet(n)|^B \to |\widetilde{\mathcal{S}}^t_\bullet(n)|^B$ over β given by the linearized manifold bundle underlying $p : M \to B$, then we refer to $\widetilde{\mathcal{S}}^{t/d}_B(M)$ as the *space of exotic smooth structures* on $p : M \to B$. In this case we think of exotic smooth structures as smoothings of a bundle of manifolds with boundary that extend a smooth structure from the interior to the boundary.

We conclude by illustrating some subtleties in the smoothing theory above.

Example 1.7. If the fibers of M are closed, then the space $\widetilde{\mathcal{S}}_B^{t/d}(M)$ is contractible. This is because a smooth structure on M is equivalent to a lift of the topological tangent microbundle classified by $\alpha : M \to \mathrm{BTop}(n)$ to a vector bundle classified by $\beta : M \to \mathrm{BO}(n)$. Since there is no boundary, no extra compatibility is required on the boundary, and thus, $\mathcal{S}_B^{t/d}(M)$ and $\widetilde{\mathcal{S}}_B^{t/t}(M)$ are equivalent.

Remark 1.8. A fiberwise tangential homeomorphism between two smooth manifold bundles $p_0 : M_0 \to B$ and $p_1 : M_1 \to B$ is a homeomorphism $h : M_0 \to M_1$ over B covered by an isomorphism of the vertical tangent bundles $T^\vee M_0 \to T^\vee M_1$ that is compatible with the topological derivative of the homeomorphism h. In [**GIW14**] and [**GI14**], the authors define the space $\mathcal{S}_B^{t/d}(M_0)$ as the space of smooth manifold bundles $p_1 : M_1 \to B$ that are fiberwise tangentially homeomorphic to $p_0 : M_0 \to B$.

Using Proposition 1.6, we see how a fiberwise tangential homeomorphism determines a point in the space $\widetilde{\mathcal{S}}_B^{t/d}(M)$. Let $(\alpha_0, \partial^\vee\alpha_0)$ classify the topological vertical tangent bundle on p_0, and let $(\beta_0, \partial^\vee\beta_0)$ classify the vertical tangent vector bundle on p_0. Likewise, let $(\alpha_1, \partial^\vee\alpha_1)$ classify the topological vertical tangent bundle on p_1, and let $(\beta_1, \partial^\vee\beta_1)$ classify the vertical tangent vector bundle on p_1. Then, after precomposing with the homeomorphism h, the fiberwise tangential homeomorphism gives a homotopy between β_0 and β_1, α_0 and α_1, and $\partial^\vee\alpha_0$ and $\partial^\vee\alpha_1$. Thus, the linearized manifold bundles underlying p_0 and p_1 are connected by a path in $\widetilde{\mathcal{S}}_B^{t/t}(M)$. However, the maps $\partial^\vee\beta_0$ and $\partial^\vee\beta_1$ need not be compatible, since the isomorphism of n-dimensional vector bundles $T^\vee M_0$ and $T^\vee M_1$ does not necessarily restrict to an isomorphism of $n-1$-dimensional vector bundles on $\partial^\vee M_0$ and $\partial^\vee M_1$. Thus, $p_1 : M_1 \to B$ determines a point in the homotopy fiber $\widetilde{\mathcal{S}}_B^{t/t}(M)$.

We also note that in [**GI14**], the authors incorrectly state that $p_0 : M_0 \to B$ and $p_1 : M_1 \to B$ can be fiberwise tangentially homeomorphic but not fiberwise diffeomorphic even when the fibers of these bundles are closed. We indicate which statements in their paper are affected by this mistake and give corrections in Section A.2.

A.1.2. Characteristics of smooth structures on h-cobordisms

Suppose that $p : M \to B$ is a bundle of smooth compact n-manifolds with nonempty boundary. We begin by defining a map $H_B^{t/d}(M) \to \mathcal{S}_B^{t/d}(M \times I)$.

Let W be an element of $H_B^{t/d}(M)$, i.e. it is a topologically trivial h-cobordism on $M \to B$. Then we must associate to W a one-parameter family of lifts $\gamma_t : (M \times I) \to \mathrm{BO}(n+1)$ of the map $(M \times I, \partial_1^\vee M) \to (\mathrm{BTop}(n+1), \mathrm{BTop}(n))$ classifying the vertical topological tangent bundle of the pair $(M \times I, \partial_1^\vee M)$, in other words a one-parameter family of linearizations of $M \times I$. We must also give a lift $\partial_1^\vee M \to \mathrm{BO}(n)$ that is compatible with γ_1.

Consider the smooth bundle $W \cup_{\partial_1} M \times I$ of h-cobordisms over B. This bundle is fiberwise diffeomorphic to W, and is fiberwise homeomorphic to $M \times [-1,1]$, since W is fiberwise homeomorphic to $M \times I$. After composing the fiberwise homeomorphism with the projection map onto $[-1,1]$, we view the bundle $M \times I \cup_{\partial_1} W$ as a topological manifold bundle over $[-1,1]$. Consider the one parameter family of h-cobordism bundles given by $W_t := \pi^{-1}[-1,t]$ for $t \in [0,1]$. Since each W_t is a submanifold of the smooth manifold bundle $M \times I \cup_{\partial_1} W$, we can restrict the vertical tangent bundle of $M \times I \cup_{\partial_1} W$ to W_t to get a linearization map γ_t on W_t, so that W_0 is fiberwise diffeomorphic to $M \times I$ and W_1 is fiberwise diffeomorphic to W. Thus, we have the desired one parameter family. Since W is a smooth manifold, the vertical tangent bundle of $\partial_1^\vee W$ is classified by a map $\partial_1^\vee W \to \mathrm{BO}(n)$ that is compatible with γ_1.

Thus, to any topologically trivial h-cobordism W on M, we can associate an exotic smoothing of $M \times I$. After stabilizing with respect to maps $M \to M \times I$, we obtain a stable map

$$\mathcal{H}_B^{t/d}(M) \to \widetilde{\mathcal{S}}_B^s(M)$$

Next we define a map $\widetilde{\mathcal{S}}_B^{t/d}(M) \to \Gamma_B \mathcal{H}_B^{\%}(M)$. We can associate to any element of $\widetilde{\mathcal{S}}^{t/d}(M)$ a fiberwise map from $M_+ = M/\partial M$ to the homotopy fiber of the map $\mathrm{BO}(n)/\mathrm{BO}(n-1) \to \mathrm{BTop}(n)/\mathrm{BTop}(n-1)$. Stably, this homotopy fiber is homotopy equivalent to $\mathcal{H}(*)$, and the fiberwise mapping space $\mathrm{Map}_B(M/\partial M, \mathcal{H}(*))$ is homotopy equivalent to $\Gamma_B \Omega^\infty(M_+ \wedge \mathcal{H}(*))$ by Poincaré duality. Thus we have a map $\widetilde{\mathcal{S}}_B^{t/d}(M) \to \Gamma_B \mathcal{H}_B^{\%}(M)$. This construction is compatible with the stabilization maps $\widetilde{\mathcal{S}}_B^{t/d}(M) \to \widetilde{\mathcal{S}}_B^{t/d}(M \times I)$, so we obtain a stable map

$$\theta : \widetilde{\mathcal{S}}_B^s(M) \to \Gamma_B \mathcal{H}_B^{\%}(M)$$

We have arrived at the following theorem:

Theorem (Theorem 1.5.14 in [**GIW14**])**.** *The map* $\theta : \widetilde{\mathcal{S}}_B^s(M) \to \Gamma_B \mathcal{H}_B^{\%}(M)$ *is a homotopy equivalence of infinite loop spaces.*

Remark 1.9. The map $\widetilde{\mathcal{S}}^s_B(M) \to \Gamma_B \mathcal{H}^{\%}_B(M)$ is obtained in [**GIW14**] by inducting over simplices in B, however it is not explicitly defined. For justifictation that our construction of this map agrees with theirs, observe that over a point in B this statement is a theorem of Burghelea and Lashof [**BL77**] which identifies the space $\widetilde{\mathcal{S}}^s_*(X)$ as the homology theory in X corresponding to the homotopy fiber of the map $\mathrm{BO}(n)/\mathrm{BO}(n-1) \to \mathrm{BTop}(n)/\mathrm{BTop}(n-1)$. This homology theory is $\mathcal{H}(*)$.

Next, we have the following theorem, which is an intermediate result in Dwyer, Weiss, and Williams' proof of the *converse Riemann–Roch theorem*.

Theorem (Corollary 12.3 in [**DWW03**])**.** *The diagram below is a homotopy pullback square*

(A.2)
$$\begin{tikzcd} \mathcal{H}^d_B(M) \arrow[r] \arrow[d] & \Gamma_B Q_B(M) \arrow[d] \\ \mathcal{H}^t_B(M) \arrow[r] & \Gamma_B A^{\%}_B(M) \end{tikzcd}$$

Corollary 1.10. *Both maps in the composition*

$$\mathcal{H}^{t/d}_B(M) \to \widetilde{\mathcal{S}}^s_B(M) \to \Gamma_B \mathcal{H}^{\%}_B(M)$$

are homotopy equivalences.

Proof. The second map is a homotopy equivalence by Theorem 1.5.14 in [**GIW14**] (copied above). The composition is the equivalence obtained by taking homotopy fibers over the diagram in Corollary 12.3 in [**DWW03**] (copied above). It follows that the first map is a homotopy equivalence. $\square$

A.1.2.1. Characteristic classes associated to exotic smoothings of fiber bundles. Suppose that W is an element of $H_B^{t/d}(M)$, i.e. it is a topologically trivial h-cobordism on $M \to B$, which forms the ingoing boundary $\partial_0 W$, and suppose that $p' : M' \to B$ is the outgoing boundary $\partial_1 W$. Then W is a point in $\widetilde{\mathcal{S}}_B^{t/d}(M \times I)$ and the homotopy equivalent space $\widetilde{\mathcal{S}}_B^{t/d}(M' \times I)$. In other words, W is an exotic smoothing of both $M \times I$ and $M' \times I$. In the language of [**GIW14**], we have a fiberwise tangential homeomorphism from $M \times I$ to W, from $M' \times I$ to W, and from $M' \times I$ to $M \times I$ so that the following diagram commutes.

$$\begin{tikzcd} & W \arrow[dr] & \\ M \times I \arrow[ur] \arrow[rr] & & M' \times I \end{tikzcd}$$

We can view each of W, $M \times I$, and $M' \times I$ as elements in $\widetilde{\mathcal{S}}_B^{s}(M)$. Define $\theta(W, \partial_0 W)$ to be $\theta(W) - \theta(M \times I)$ and $\theta(W, \partial_1 W)$ to be $\theta(W) - \theta(M' \times I)$. Denote by $\Theta(-,-)$ the π_0 component in which $\theta(-,-)$ lives. In [**GIW14**], given $M' \times I \in \widetilde{\mathcal{S}}_B^{s}(M)$, they define $\Theta(M', M)$, the smooth structure class of M' relative to M, to be the image of M' in $\pi_0 \Gamma_B \mathcal{H}_B^{\%}(M) \otimes \mathbb{R}$. By the discussion above, the class $\Theta(M', M)$ is equal to $\Theta(W, \partial_0 W) - \Theta(W, \partial_1 W)$.

A.2. Corrections to Exotic Smooth Structures on Topological Fiber Bundles II

In [**GI14**], the authors claim to construct exotic smoothings of manifold bundles with closed fibers. However, exotic smoothings of bundles with closed fibers do not exist. The result of this mistake is that the statements of a handful of results in [**GI14**] are false or vacuously true. However, their mistake is largely one of nomenclature: the main construction of [**GI14**] is sound, and the end results of the computations of the smooth structure class and higher Franz–Reidemeister torsion are unaffected by this mistake. In this section we indicate which results are incorrect or vacuously true, and provide corrections.

(1) Theorem 0.1.3 is false as stated: there cannot exist a smooth bundle $p' : M' \to B$ with closed fibers that is fiberwise tangentially homeomorphic to but not fiberwise diffeomorphic to $p : M \to B$. To correct this statement, we observe that the immersed Hatcher construction produces a topologically trivial bundle of smooth h-cobordisms with boundaries given by the smooth bundles p and p', and thus the statement should instead be that $M \times I$ is fiberwise tangentially homeomorphic to $M' \times I$. The relative torsion $\tau^{IK}(M', M)$ should be understood as $\tau^{IK}(M' \times I, M \times I)$.

(2) Conjecture 0.3.3 is vacuously true as stated. In particular, the phrase 'rationally stably, there are no exotic smooth structures on manifold bundles with closed oriented even dimensional fibers' is always true because such exotic smoothings do not exist. However, if M' and M are the boundaries of a topologically trivial

bundle of smooth h-cobordisms then $M' \times I$ is an exotic smoothing of $M \times I$, and it is entirely reasonable to ask if $\Theta(M', M)$ is nonzero, in the sense of Subsubsection A.1.2.1.

Corollary 1.10 indicates that all M' and M for which $M' \times I$ and $M \times I$ are fiberwise tangentially homeomorphic can be obtained as the boundaries of a topologically trivial bundle of smooth h-cobordisms. We provide the following more substantive statment of Goette and Igusa's Rigidity Conjecture.

Conjecture 2.1. *If $p : M \to B$ is a smooth manifold bundle with closed even dimensional fibers, then for any topologically trivial family of smooth h-cobordisms W so that $\partial_0 W$ is the bundle $p : M \to B$ and $\partial_1 W$ is a smooth bundle $p' : M' \to B$ with closed even dimensional fibers, the smooth structure class $\Theta(M', M) = \Theta(W, \partial_0 W) - \Theta(W, \partial_1 W)$ is trivial.*

(3) Corollary 2.2.5 is false as stated. The vertical boundary $\partial^\vee E^{n,m}(\xi, \eta)$ cannot be fiberwise tangentially homeomorphic to the linear sphere bundle $S^{n+m-1}(\eta)$ without also being fiberwise diffeomorphic.

To repair this statement, we first observe that the h-cobordism between $\partial^\vee E^{n,m}(\xi, \eta)$ and $S^{n+m-1}(\eta)$ obtained by deleting a neighborhood of a section of the disk bundle $E^{n,m}(\xi, \eta)$ is a topologically trivial family of smooth h-cobordisms. Thus, $\partial^\vee E^{n,m}(\xi, \eta) \times I$ is fiberwise tangentially homeomorphic to $S^{n+m-1}(\eta) \times I$, and the difference torsion of this pair is defined and can be nonzero. The proof of the corollary goes through since $\tau^{IK}(\partial^\vee E^{n,m}(\xi, \eta)) = \tau^{IK}(\partial^\vee E^{n,m}(\xi, \eta) \times I)$, and $\tau^{IK}(S^{n+m-1}(\eta)) = \tau^{IK}(S^{n+m-1}(\eta) \times I)$.

(4) Corollary 2.3.3 is false as stated. Again, M' and M cannot be fiberwise tangentially homeomorphic and have nonzero difference torsion since both have closed fibers. To correct this statement, we first understand that M' and M bound the topologically trivial bundle of smooth h-cobordisms $E_{+}^{n,m}(M, s, \xi)$ obtained by the immersed Hatcher construction. Thus $M \times I$ is fiberwise tangentially homeomorphic to $M' \times I$. The difference torsion $\tau^{IK}(M', M)$ in the statement should be replaced by $\tau^{IK}(M' \times I, M \times I)$.

In addition to the corrections above, we point out that the main theorem of [**GI14**], Theorem 3.0.5, is correct as stated, but we clarify their statement of the result as follows. The immersed Hatcher construction $W := E_{+}^{n,m}(M, \tilde{\lambda}, \xi)$ is a topologically trivial bundle of smooth h-cobordisms, and M and M' are the boundaries of this bundle. The smooth structure class $\Theta(M', M)$ should be understood as $\Theta(W, \partial_0 W) - \Theta(W, \partial_1 W)$ as in Subsubection A.1.2.1.