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Geometrically Modelling the Lubin-Tate Action

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ABSTRACT

Geometrically Modelling the Lubin-Tate Action

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The action of the automorphisms of a formal group on its deformation space is crucial to understanding periodic families in the homotopy groups of spheres and the unsolved Hecke orbit conjecture for unitary Shimura varieties. This action is called the Lubin-Tate action. We first show sufficient conditions for geometrically modelling this action as coming from an action on a moduli stack, generalizing previous constructions using the moduli stack of elliptic curves. We then construct such a stack satisfying these conditions for height $p-1$ for all odd primes p, and conjecture the correct stack for height $h = p^{k-1}(p-1)$ for all odd primes. These heights capture all topologically interesting information for odd primes. We construct these stacks both locally and globally using inverse Galois theory and Hurwitz stacks. Finally, we relate these stacks to the reduced regular representation of cyclic groups, and use this to compute the action explicitly.

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Introduction

The action of the automorphisms of a formal group on its deformation space is crucial to understanding periodic families in the homotopy groups of spheres and the unsolved Hecke orbit conjecture for unitary Shimura varieties. We can explicitly determine this action by geometrically modelling it as an action on a moduli space, which we construct using inverse Galois theory and some representation theory.

0.1. Homotopic Motivation for the Lubin-Tate Action

Many geometric problems have been reduced to solvable problems in stable homotopy theory. This began with Thom's approach to computing cobordism groups using homotopy theory, continued with Stolz's use of Spin cobordism to study manifolds of positive scalar curvature, and most recently, crescendoed to the work of Hill-Hopkins-Ravenel on the Kervaire invariant (12).

The original problem which has motivated many developments in stable homotopy theory, and which is very far from being understood completely, is the computation of stable homotopy groups of spheres. Bousfield localization informs us that it is sufficient to understand what happens rationally and what happens at each prime p. The p-complete sphere spectrum, although simpler than the sphere spectrum we started with, is still a very complicated object. We need further ideas to break this up into more manageable pieces. The chromatic viewpoint arose in the late 1960s. It is the most successful approach to the structure of stable homotopy, both from a conceptual and a calculational standpoint, connecting algebraic topology of finite complexes with deep patterns in number theory.

After the Adams spectral sequence allowed us to begin computing them, stable homotopy groups of spheres previously seemed chaotic and without pattern. Through the groundbreaking work of Quillen in the late 1960s and its expansions by Morava in the 1970s, we discovered that the stable homotopy groups of spheres have deep structure.

The stable homotopy groups of spheres are like a complicated multilayered signal, and we may construct band pass-filters to read the individual messages. These band-pass filters are extraordinary cohomology theories constructed from one dimensional formal group laws, and they detect periodic families in the homotopy groups of spheres.

The starting point of chromatic homotopy theory is the observation that any cohomology theory for which we have a natural theory of Chern classes gives rise to a smooth one-parameter formal group defined over its ring of coefficients. A formal group is an arithmetic object which can be thought of as a refinement of the concept of a Lie algebra, one which behaves well also in positive characteristic. Surprisingly, the formal group associated to the complex bordism cohomology theory MU turns out to be isomorphic to the universal formal group, setting up a dictionary between homotopy theory and the theory of formal groups. The geometry of the moduli stack of formal groups thus controls the stable homotopy category. Much of what can be proved and conjectured about stable homotopy theory arises from the study of this stack, its stratifications, and the theory of its quasi-coherent sheaves.

At a fixed prime p, there is a single invariant called **height** which stratifies the moduli stack of formal groups. For a prime p and a height h one has the Morava K-theory spectrum $K(h)$ which corresponds to the closed point (h, p) of the Balmer spectrum Spec($\mathbb{S}$), and one has the Morava E-theory spectrum E_h which corresponds to the formal neighborhood of $K(h)$ in the Balmer spectrum.

The latter yields an efficient filtration of the homotopy groups of p-localized spheres $\pi_h(\mathbb{S}_{(p)})$ through localizations $L_h\mathbb{S}$ of the p-local sphere spectrum $\mathbb{S}_{(p)}$ of the relative opens of Spec $\mathbb{S}$. These localizations fit into the chromatic tower

$$\cdots \to L_h\mathbb{S}_{(p)} \to \cdots \to L_1\mathbb{S}_{(p)} \to L_0 \simeq \mathbb{S}_{(p),\mathbb{Q}}$$

and the chromatic convergence theorem of Hopkins and Ravenel (25) implies that the resulting filtration on $\pi_*(\mathbb{S})$ is exhaustive, that is, $\mathbb{S}_{(p)} = \operatorname{holim} L_h\mathbb{S}_{(p)}$.

For a finite CW spectrum X, the difference between $L_h(X)$ and $L_{h-1}(X)$ is measured by $L_{K(h)}(X)$, in the sense that the previous tower arises from an ascending filtration of the (p)-local stable homotopy category Sp,

$$\mathrm{Sp}_{\mathbb{Q}} \simeq \mathrm{Sp}_0 \subset \mathrm{Sp}_1 \subset \cdots \mathrm{Sp}_h \subset \cdots \subset \mathrm{Sp}_{(p)},$$

with filtration quotients equivalent to the category of $K(h)$-local spectra.

Let H be a one dimensional formal group of height h over a perfect characteristic p field k, then $J_h := \mathrm{Aut}(H/k)$ acts on the functions on the deformation space of H, which we call $\mathcal{O}_{LT}$. The action of J_h on $\mathcal{O}_{LT}$ is called the Lubin-Tate action.

If we expand the deformation problem to include units, as discussed in Section , the ring E_* that represents it is the coefficient ring of Morava E-theory.

By the work of Devinantz-Hopkins-Miller (6), we can access the $K(h)$-local category using the J_h-equivariant homotopy theory of E_h,

$$[L_{K(h)}(E_h \wedge X)]^{hJ_h} \simeq L_{K(h)}X.$$

We are thus able to calculate periodic families in the homotopy groups of spheres by a homotopy fixed point spectral sequence which begins with the information of the Lubin-Tate action:

$$H^*(J_h, E_*) \Rightarrow \pi_*(L_{K(h)}\mathbb{S}).$$

The issue now is that directly computing the action of J_h on LT_* is not tractable. By work of Henn (9), we may hope to reduce to computing the action of finite subgroups of J_h. Even computing finite subgroups is recalcitrant and requires tools from algebraic geometry.

Elliptic curves appearing in homotopy theory because they provide a geometric model of the Lubin-Tate action of finite subgroups at height 1 and 2 (10) (14) (1). It is much more difficult to create an abelian variety which geometrically models the Lubin-Tate action at height three or higher: by Cartier duality one must consider abelian varieties of dimension ≥ 3, and then only those for which a 1-dimensional formal group of height h breaks off as a summand of the formal group associated to the abelian variety.

The idea of geometric modelling has been implicit in the literature for some time, but has never been made precise. This thesis describes exactly the conditions a stack must satisfy to model the Lubin-Tate action. Then we investigate this in detail in the case of stacks parameterizing Artin-Schreier curves. We further establish what would be necessary to carry out a similar investigation in the case of stacks parameterizing more general Artin-Schreier-Witt curves. Artin-Schreier curves were previously considered in this context (7) and (26) (27), see also (13).

Some of the past work on geometric models, as in (2), uses unitary Shimura varieties of signature $(1, h-1)$ in order to produce moduli stacks of abelian varieties with CM structure which forces the associated formal group to split into 1-dimensional formal groups. The downside of this approach is that these abelian varieties almost never arise as Jacobians of families of curves, so one is forced to work with a variety of large dimension. In fact by Oort's conjecture, for $g \gg 0$ there should not exist a positive dimensional special subvariety of the moduli of principally polarized abelian varieties of dimension g which is contained in the Torelli locus and which intersects the open Torelli locus nontrivially. Such special subvarieties parameterize abelian varieties with CM structure which actually come from Jacobians of families of curves, and the point of Oort's conjecture is that this can only happen for very small g (conjecturally only for $g \leq 7$) (20).

Geometric models of the Lubin-Tate action give access to certain types of calculation which are otherwise inaccessible. The Kevaire invariant one theorem for $p \geq 5$, was proved using the action of C_p for height $h = p - 1$ (24). It is suspected that the action of C_9 for height $h = p(p-1)$ will detect the remaining $p = 3$ case. For example, for the prime two, height 4, the action of C_8 detects all of the Kevaire elements, but the action of C_2 detects only some of them. The original plan of (12) went through the study $E(4)^{hC_8}$, but it was far too difficult at that time. At the prime two, the homotopy groups of the spectrum $E_4^{hC_4}$ have been calculated (19).

0.2. Arithmetic and Geometric Motivation for the Lubin-Tate Action

The Lubin-Tate action also plays a distinguished role in arithmetic geometry through its relation to the local Langlands correspondence. Although this is not the primary lens through which we study the Lubin-Tate action, it is nevertheless important to explain the broader context into which this action fits.

Let F be a finite extension of $\mathbb{Q}_p$ with valuation ring $\mathcal{O}_F$ and residue field $\mathcal{O}_F/\varpi = \mathbb{F}_q$ where ϖ is a uniformizer. The goal of local class field theory is to describe the

Abelian Galois extensions of F, which is achieved through the local Artin reciprocity morphism $\mathrm{rec}_F : F^\times \to \mathrm{Gal}(F^{\mathrm{ab}}/F)$ which is continuous with dense image; to make this an isomorphism we can either enlarge $\mathrm{Gal}(F^{\mathrm{ab}}/F) = \mathrm{Gal}(F^{\mathrm{ab}}/F^{\mathrm{ur}}) \rtimes \widehat{\mathbb{Z}}$ to the Weil group $W(F^{\mathrm{ab}}/F) = W(F^{\mathrm{ab}}/F^{\mathrm{ur}}) \rtimes \mathbb{Z}$ to obtain the local Artin reciprocity isomorphism $\mathrm{rec}_F : F^\times \xrightarrow{\sim} W(F^{\mathrm{ab}}/F)$, or we can restrict $F^\times$ to the profinite completion $\widehat{F^\times}$ to obtain the local Artin reciprocity isomorphism $\widehat{\mathrm{rec}}_F : \widehat{F^\times} \xrightarrow{\sim} W(F^{\mathrm{ab}}/F)$.

Miraculously, the Abelian extensions of F and the Artin reciprocity morphism can be described explicitly in terms of Lubin-Tate formal groups.

Let $\mathcal{F}_\varpi$ be the set of formal power series $f \in \mathcal{O}_F[[t]]$ such that $f(t) = \varpi t + \mathrm{O}(t^2)$ and such that $f(t) \equiv t^q \mod \varpi$. For $f \in \mathcal{F}_\varpi$ there exists a unique formal group law $F_f(x, y) \in \mathcal{O}_F[[x, y]]$, the Lubin-Tate formal group law, which admits f as an endomorphism. The Lubin-Tate formal group laws over $\mathcal{O}_F$ are precisely those formal group laws over $\mathcal{O}_F$ (of height $h[F : \mathbb{Q}]$ for admitting an endomorphism with derivative at the origin equal to ϖ and which reduces modulo ϖ to the Frobenius endomorphism ϕ_q. Each $a \in \mathcal{O}_F$ defines an endomorphism $[a]_f \in \mathrm{End}(F_f)$ such that $[a]_f(t) = at + \mathrm{O}(t^2)$, which defines an isomorphism $[\cdot]_f : \mathcal{O}_F \xrightarrow{\sim} \mathrm{End}(F_f)$.

For the unramified Abelian extensions we have the following. For μ_n the set of n-th roots of unity in $\overline{F}$ we have $F^{\mathrm{ur}} = \bigcup_{p \nmid n} F(\mu_n)$ with $\mathrm{Gal}(F^{\mathrm{ur}}/F) \simeq \widehat{\mathbb{Z}}$.

For the totally ramified Abelian extensions we have the following. For $f \in \mathcal{F}_\varpi$ consider the $\mathcal{O}_F$-module $\Lambda_f = \{x \in \overline{F} \mid |x| < 1\}$ with group law $x + y = F_f(x, y)$ and action $a \cdot x = [a]_f(x)$, both of which converge in $\overline{F}$. For $h \geq 1$ consider the $\mathcal{O}_F$-submodule $\Lambda_{f,h} = \{x \in \Lambda_f \mid [\varpi]_f^n(x) = 0\}$. Let $F_{\varpi,h} = F[\Lambda_{f,h}]$. Then $F_{\varpi,h}/F$ is totally ramified of degree $(q-1)q^{h-1}$, the action of $\mathcal{O}_F$ on $\Lambda_{f,h}$ defines an isomorphism $(\mathcal{O}_F/\varpi^h)^\times \simeq \mathrm{Gal}(F_{\varpi,h}/F)$, in particular $F_{\varpi,h}/F$ is an Abelian Galois extension and we have $F^{\mathrm{ab}} = \bigcup_{h \geq 1} F_{\varpi,h}$ with $\mathrm{Gal}(F^{\mathrm{ab}}/F) \simeq \widehat{F^\times} = \mathcal{O}_F^\times \times \varpi^{\widehat{\mathbb{Z}}}$.

To summarize this last point, for each $h \geq 1$ the Lubin-Tate formal group law defines a unique up to isomorphism formal $\mathcal{O}_F$-module of height h over $\overline{\mathbb{F}}_q$ (that is a formal group of height $h[F : \mathbb{Q}]$ over $\overline{\mathbb{F}}_p$) whose endomorphism ring in the category of formal $\mathcal{O}_F$-modules over $\overline{\mathbb{F}}_p$ is isomorphic to $\mathcal{O}_D$ where D is the unique division algebra of invariant $\frac{1}{h}$ over F. The torsion points of these Lubin-Tate formal groups, adjoined to F, exhaust the totally ramified Abelian extensions of F, in striking analogy with the theory of complex multiplication for number fields.

What about the non-Abelian extensions of F? For this to make sense, we need to generalize the local Artin reciprocity morphism to the local Langlands correspondence:

$$\mathrm{rec}_F : \begin{Bmatrix} \text{irreducible supercuspidal} \\ \text{representations of } \mathrm{GL}_n(F) \end{Bmatrix} \to \begin{Bmatrix} \text{irreducible n-dimensional} \\ \text{representations of } W_F \end{Bmatrix}$$

Miraculously, the local Langlands correspondence can be realized explicitly in terms of cohomology of the Lubin-Tate tower. We will explain this quickly in the case $F = \mathbb{Q}_p$.

As explained previously, we have the Lubin-Tate space $LT = \mathrm{Spf}(W(\overline{\mathbb{F}}_p)[[u_1, \ldots, u_{h-1}]])$ of formal deformations of a fixed formal group G_0 of height h over $\overline{\mathbb{F}}_p$, where $LT(R)$ parameterizes pairs (G, ι) where G is a 1-dimensional commutative formal group over R and $\iota : G_0 \xrightarrow{\sim} G \otimes_R \overline{\mathbb{F}}_p$. Let $\mathcal{LT}$ be the rigid generic fiber of LT, which is a p-adic rigid open ball of radius 1. We have the Lubin-Tate space LT_n of level p^n, where $LT_n(R)$ parameterizes triples (G, ι, α) where $(G, \iota) \in LT(R)$ and $\alpha : (\mathbb{Z}/p^n\mathbb{Z})^h \xrightarrow{\sim} G[p^n]$ is a Drinfeld level p^n structure. Let $\mathcal{LT}_n$ be the rigid generic fiber of LT_n. Drinfeld showed $\mathcal{LT}_n \to \mathcal{LT}$ is an étale cover of the p-adic rigid open ball of radius 1 with Galois group $\mathrm{GL}_h(\mathbb{Z}/p^n\mathbb{Z})$. At infinite level, the Lubin-Tate tower $\mathcal{LT}_\infty$ (morally this is the inverse limit $\varprojlim_n \mathcal{LT}_n$, although this does not exist as a rigid space) carries an action of $\mathrm{GL}_h(\mathbb{Q}_p) \times D^\times \times W_{\mathbb{Q}_p}$.

For $\ell \neq p$ one can define the ℓ-adic cohomology $H^*_c(\mathcal{LT}_\infty, \overline{\mathbb{Q}}_\ell) = \varinjlim_n H^*_c(\mathcal{LT}_n, \overline{\mathbb{Q}}_\ell)$ using Berkovich's nearby cycle sheaves, which carries an action of $\mathrm{GL}_h(\mathbb{Q}_p) \times D^\times \times W_{\mathbb{Q}_p}$. By the Jacquet-Langlands correspondence we have a canonical bijection:

$$\mathrm{JL} : \left\{ \begin{matrix} \text{irreducible supercuspidal} \\ \text{representations of } \mathrm{GL}_n(F) \end{matrix} \right\} \to \left\{ \begin{matrix} \text{irreducible supercuspidal} \\ \text{representations of } D^\times \end{matrix} \right\}$$

By Harris-Taylor we have a canonical bijection:

$$\mathrm{rec} : \left\{ \begin{matrix} \text{irreducible supercuspidal} \\ \text{representations of } \mathrm{GL}_n(\mathbb{Q}_p) \end{matrix} \right\} \to \left\{ \begin{matrix} \text{irreducible n-dimensional} \\ \text{representations of } W_{\mathbb{Q}_p} \end{matrix} \right\}$$

such that for π an irreducible supercuspidal representations of $\mathrm{GL}_h(\mathbb{Q}_p)$ we have

$$\mathrm{Hom}_{\mathrm{GL}_h(\mathbb{Q}_p)}(\pi, H^*_c(\mathcal{LT}_\infty, \overline{\mathbb{Q}}_\ell)) = \mathrm{JL}(\pi) \otimes \mathrm{rec}(\pi)$$

Much more is known in general: the theorem of Harris-Taylor does not assume $F = \mathbb{Q}_p$, and there are more general theorems of Boyer (3) and Scholze (28) which realize the local Langlands correspondence for irreducible admissible representations of $\mathrm{GL}_n(F)$ (rather than just the irreducible supercuspidal representations) in terms of the cohomology of a similar tower of formal deformations of a fixed p-divisible group of height h over $\overline{\mathbb{F}}_p$. There are now very general theorems of Fargues-Scholze (22) which realize the local Langlands correspondence for irreducible admissible representations of $G(F)$ for any reductive group G in terms of the cohomology of certain local Shimura varieties studied by Rapoport-Zink (cite) and Scholze-Weinstein (21).

It is worth mentioning that many of the above theorems about the local Langlands correspondence are proved using global methods. Roughly speaking, one embeds the Lubin-Tate tower at infinite level into some tower of Shimura varieties at infinite level, then the automorphic representations of $G(\mathbb{A})$ appearing in the cohomology of the tower of Shimura varieties at infinite level restrict to the desired irreducible admissible representations of $G(F)$. For example Deligne (5) realizes the local Langlands correspondence for $\mathrm{GL}_2(\mathbb{Q}_p)$ for $p > 2$ by realizing the global Langlands correspondence for $\mathrm{GL}_2(\mathbb{A})$ in the cohomology of the tower of modular curves at infinite level. More generally Harris-Taylor and Scholze both realize the local Langlands correspondence for $\mathrm{GL}_h(\mathbb{Q}_p)$ by studying the cohomology of a certain tower of unitary Shimura varieties of signature $(1, h-1)$ at infinite level. The role of such unitary Shimura varieties is explained by the fact that locally symmetric spaces for GL_h are not Shimura varieties for $h \geq 3$, and instead one must locate the cohomology of such locally symmetric spaces for GL_h in the boundary cohomology of some other Shimura variety, and the unitary Shimura varieties of signature $(1, h-1)$ do just this.

Finally we should remark that Hurwitz spaces of branched $\mathbb{Z}/n\mathbb{Z}$-covers of $\mathbb{P}^1$ determine unitary Shimura varieties associated to the group algebra of $\mathbb{Z}/n\mathbb{Z}$. The Lubin-Tate spaces which appear in the description of formal deformations of such branched $\mathbb{Z}/n\mathbb{Z}$-covers should therefore be related to some part of the local Langlands correspondence, and the corresponding unitary Shimura varieties allow for global methods to be applied in this situation.

0.3. Division Algebra Finite Subroups: Algebraic vs Topological

As we mentioned, there are two types of things meant when people say Morava stabilizer group. Here we are talking about the maximal order J_h in the division algebra, and not the Galois group extension.

Lemma 0.3.1. *((4) pp. 4, (11)) Let $h = p^{k-1}(p-1)$ where p is odd. Let J_h be the Morava stabilizer group of height h (not extended). This is the only height h in which* *there are p-Sylow subgroups of* J_h. *Further, the p-cyclic subgroup in* J_h *of largest order sits inside of the maximal finite subgroup* $\Gamma := C_{p^k} \rtimes C_{(p-1)^2}$.

Roadmap

Here is an outline of the thesis. We define the notion of geometric modelling in Section 2. In Theorem 2.0.3, we show that for a moduli stack M to geometrically model a finite subgroup G of the Morava stabilizer group acting on the Lubin-Tate space, it is sufficient for it to satisfy two criterion, which we define in Section 1.2.

We construct a few equivalent examples of a moduli stack $\mathcal{M}$ which satisfies these criterion for the maximal finite subgroup of the Morava stabilizer group J_h for height $h = p - 1$, in Sections 5, 6, and in 7. This is done in Theorem 5.0.2.

We further set up a moduli stack $\mathcal{M}_k$ fibered over our $h = p-1$ deformation moduli stack, in Sections 5, 3.3, and 3.5. We conjecture in 4.9.1 that this satisfies the geometric modelling criterion for $h = p^{k-1}(p-1)$, and show that it has h-splitting up to isogeny in the case $h = 6$ and $p = 3$. We explain why we cannot use our previous proof method to immediately show it has h-splitting integrally in Section 3.3. However, there are a few ways we may be able to tweak the proof approach to conclude what we desire, which is also expressed in Section 3.3.

The set up of the constructive part of the thesis is as follows.

- We consider a curve X over a field k of characteristic p such that $G \subset \mathrm{Aut}_k(X)$. [Section 3]
- We consider a moduli stack M on which there is a point $\operatorname{Spec} k \to M$ corresponding to a map $p_y : X \to \mathbb{P}^1_k$. If we consider M to be a deformation moduli stack, then one of the criterion (modelling) is automatically satisfied. Section

5. We can also take integrally defined stacks, as in Section 6 and 7 but this is not needed for geometric modelling. We relate the global and local stacks in 6.

- That leaves us to consider the other criterion (deformation h-splitting). This is done in two steps. We consider how G acts on the Dieduonne module associated to the Jacobian of this curve, $D((X/k)[p^\infty])$. This action splits off a one-dimensional, whose connected component is a one dimensional formal group law of height h, which we call $e_1\hat{J}(X/k)$, discussed in Section 4.6.1, Theorem 4.8.1. It also splits off a universal one dimensional formal group law of height h, which we prove in Section 8.
- We then look at what geometric modelling buys us. We get the following by studying the action of $\mathrm{Aut}(X)$ on $\mathcal{M}$. We may represent the stack $\mathcal{M}$ by a ring Λ, and find that it is the completion of the reduced regular representation in Theorem 5.0.6. This gives us an equivalence to the ungraded Lubin-Tate ring. We then relate this to the graded Lubin-Tate in Lemma 5.0.8. We further speculate the analogous result for $h = p(p-1)$ using tangent spaces, in Section 5.2 and more specifically in Conjecture 5.2.9.
- Finally, we compute the corresponding cohomology of the E_2-page of the homotopy fixed point spectral sequence in Section 9 for $h = p - 1$. The cohomology calculation for $h = 6$ and $p = 3$ is in a forthcoming paper joint with Eva Belmont and is available upon request.

CHAPTER 1

Background for Structural Theorem

1.1. Background on Deformation Functors

Let k be a characteristic p field. Let Art_k be the category of triples $(R, \mathfrak{m}, \iota : k \simeq R/\mathfrak{m})$, where R is an Artinian local ring with maximal ideal $\mathfrak{m}$. We will suppress $\mathfrak{m}$ and ι from the notation.

Definition 1.1.1. *We define the deformation functor of a given map. Given a map $f : X \to Y$ of schemes over a field k, we define a functor $\mathrm{Def}_{f:X\to Y}$ from Art_k to groupoids as follows. Let $\mathcal{X}, \mathcal{Y}$ be schemes over $A \in \mathsf{Art}_k$. Objects are pullback diagrams of the form:*

$$\begin{tikzcd}
X \arrow[r] \arrow[d, "f"'] & \mathcal{X} \arrow[d, "F"] \\
Y \arrow[r] \arrow[d] & \mathcal{Y} \arrow[d] \\
\operatorname{Spf} k \arrow[r] & \operatorname{Spf} A
\end{tikzcd}$$

Morphisms are pullback diagrams of the form:

$$\begin{tikzcd}
 & & \mathcal{X}' \arrow[d, "F'"] \\
X \arrow[r] \arrow[d, "f"'] \arrow[rru] & \mathcal{X} \arrow[d, "F"'] \arrow[ru] & \mathcal{Y}' \arrow[d] \\
Y \arrow[r] \arrow[d] \arrow[rru] & \mathcal{Y} \arrow[d] \arrow[ru] & \operatorname{Spf} A' \\
\operatorname{Spf} k \arrow[r] \arrow[rru] & \operatorname{Spf} A \arrow[ru] &
\end{tikzcd}$$

Definition 1.1.2. *We define the deformation functor of a given map fixing the target. Given a map $f : X \to Y$ of schemes over a field k, we define a functor $\mathrm{Def}_{f:X\to Y}$*

from Art_k to groupoids as the functor above, where we further require the target $\mathcal{Y}$ is isomorphic to the trivial deformation.

Definition 1.1.3. *We define the deformation functor of a map $f : X \to Y$ which fixes the target Y as a specialization of definition definition 1.1.1 to the case where $\mathcal{Y}$ is the product family, i.e., $\mathcal{Y} := Y \times_k S$, where $S = \mathrm{Spf}(A)$.*

Definition 1.1.4. *The stacky deformation functor $\mathcal{D}ef_{f:X\to Y}$ of a map $f : X \to Y$ of schemes is the same as the deformation functor of that map, but we allow the maps to be pullbacks up to an automorphism of the base Y.*

We define this case, which includes deformations of subobjects in a fixed base (called a Hilbert scheme). If we take the example of the base Y being $\mathbb{P}^1_k$, then fixing the base means we do *not* allow bundles of $\mathbb{P}^1$, only the constant family $\mathbb{P}^1_A$ itself as the base.

Definition 1.1.5. *Let B be a stack, and A be an object over B. We define* $\mathrm{Aut}(A, B)$ *to be the* ***pullback automorphism groupoid****. An object in the groupoid* $\mathrm{Aut}(A, B)$ *is a commutative diagram of the form,*

$$\begin{tikzcd} \beta^* A \arrow[r, "\alpha", "\sim"'] \arrow[d] & A \arrow[d] \\ B \arrow[r, "\beta"] & B \end{tikzcd}$$

we denote this as (α, β). We consider only α which are isomorphisms, and only continuous automorphsims of B (automorphisms which preserve the maximal/chosen ideal). Composition of objects (α, β) and (δ, γ) are:

$$\begin{tikzcd} (\gamma \circ \beta)^* A \arrow[r, "\beta^*\delta"] \arrow[d] & \beta^* A \arrow[r, "\alpha"] \arrow[d] & A \arrow[d] \\ B \arrow[r, "\gamma"] & B \arrow[r, "\beta"] & B \end{tikzcd}$$

and morphisms are isomorphisms of diagrams.

Definition 1.1.6. *The deformation functor of a scheme X over a field k is* $\mathrm{Def}_{id:X\to X}$*. That is, objects are of the form*

$$\begin{tikzcd} X \arrow[r] \arrow[d] & \mathcal{X} \arrow[d] \\ \operatorname{Spf} k \arrow[r] & \operatorname{Spf} A \end{tikzcd}$$

This functor carries a natural action by two different groups $\mathrm{Aut}(X/k)$ and $\mathrm{Aut}(X,k)$. The former acts by precomposition, leaving the base field fixed. That is, $j \in \mathrm{Aut}(X/k)$ acts by the following:

$$\begin{tikzcd} X \arrow[r, "j"] \arrow[d] & X \arrow[r] \arrow[d] & \mathcal{X} \arrow[d] \\ \operatorname{Spf} k \arrow[r, equal] & \operatorname{Spf} k \arrow[r] & \operatorname{Spf} A \end{tikzcd}$$

The latter acts by precomposition, allowing for automorphisms of the base field. That is, $(j,h) \in \mathrm{Aut}(X,k)$ acts by the following:

$$\begin{tikzcd} X \arrow[r, "j"] \arrow[d] & X \arrow[r] \arrow[d] & \mathcal{X} \arrow[d] \\ \operatorname{Spf} k \arrow[r, "h"] & \operatorname{Spf} k \arrow[r] & \operatorname{Spf} A \end{tikzcd}$$

Since k is a characteristic p field, these sit in a short exact sequence

$$\mathrm{Aut}(X/k) \to \mathrm{Aut}(X,k) \to \mathrm{Gal}(k/\mathbb{F}_p).$$

1.2. Defining the Lubin-Tate action

Definition 1.2.1. *Let* FGL_R *be the category of formal group laws over a ring* $R \in \mathrm{Art_k}$. *Fix* $H \in \mathrm{FGL_k}$. *The groupoid* $\mathrm{Def}_H(R)$ *has as objects:*

$$\{F \in \mathrm{FGL}_R, \iota : H \xrightarrow{\simeq} F \otimes_R k\},$$

and as morphisms: isomorphisms f *of formal group laws over* R *which reduces to the identity modulo* $\mathfrak{m}$.

Let $\mathrm{Pro}(\mathsf{Art}_k)$ be Artinian local rings with quotient field k. It is a theorem of Lubin and Tate (Theorem 3.1 (16)) that $\mathrm{Def}_H \to \pi_0(\mathrm{Def}_H)$ is an equivalence, and further that Def_H is pro-representable

$$\mathrm{Def}_H(R) \simeq \mathrm{Hom}_{\mathrm{Pro}(\mathsf{Art}_k)}(LT, R).$$

We call $LT \simeq W(k)[[u_1, ..., u_{h-1}]]$ the Lubin-Tate ring, and $\mathrm{Spf}\, LT$ the Lubin-Tate space.

There is another deformation problem, topologists also refer to the ring associated to it as the Lubin-Tate ring. his is simply the periodic version of the above setup, but may be given an explicit moduli theoretic meaning.

Definition 1.2.2. *Let* FGL°_R *be the category of pairs* (F, a) *of* $F \in FGL_R$ *together with a chosen element* $a \in R^\times$, *where morphisms are isomorphisms* $f : (F, a) \to (G, b)$ *such that* $a = f'(0)b$.

There is a natural transformation from $FGL^\circ \to FGL$ sending an object (F, a) to $aF(a^{-1}x, a^{-1}y)$. This is an equivalence with inverse. $\mathrm{Def}^\circ_H(R)$ is also pro-representable, by the ring $LT_* := W(k)[[u_1, ..., u_{h-1}]][u^\pm]$ where u is an element of degree 2. We will dinstinguish between the two Lubin Tate rings by calling one LT and the other LT_*, where $LT = LT_0$. It is the ring LT_* which arises as the coefficient ring of Morava E-theory of height h.

We define $\mathrm{Aut}_{\mathrm{cts}}(A) = \mathrm{Aut}(\mathrm{Spf}\, A)$ for $A \in \mathsf{Art}_k$ to be the automorphisms of A preserving its maximal ideal.

There are two automorphism groups which naturally act on Def_H, discussed at the end of Section 1.1, which we distinguish by calling them $J = \mathrm{Aut}(H/k)$ and $\mathrm{Aut}(H, k)$. Topologists usually distinguish these by calling them the small Morava stabilizer group and the big Morava stabilizer group, respectively. This paper will only use J, so we do not use this language.

Theorem 1.2.3. *(17) Let $FGL_k^{(h)}$ denote the category of height h formal group laws over a field k. Fix $H \in FGL_k^{(h)}$. The group $J_h := \mathrm{Aut}(H/k)$ is the units of a division algebra with Hasse invariant $1/h$.*

We will often drop the height h from the notation when the height is fixed.

Example 1.2.4. *For example, when H is the Honda formal group of height h, it is a theorem of Dieudonné and Tate that $\mathrm{Aut}(H/k)$ may be written in the following way, where $a \in W(k)$ are acted on by Frobenius:*

$$(W(k)\langle S\rangle/(S^h = p, aS = Sa^\sigma)^\times.$$

Definition 1.2.5. *We let $H \in \mathrm{FGL}_k^{(h)}$ and $LT(k, H) := W(k)[[u_1, ..., u_{h-1}]]$. The universal deformation of H is an object $F_{univ} \in \mathrm{FGL}_{LT}$ such that for any $R \in \mathsf{Art}_k$ and any $F \in \mathrm{FGL}_R$ of height h there is a unique local R-algebra homomorphism $\alpha : LT \to R$ such that F is $\star$-isomorphic to $\alpha^*(F_{univ})$. We will also refer to this as "the universal formal group law" for a given height.*

1.3. New Definitions

Definition 1.3.1. *We call a stack "good" if it is either a smooth Artin stack or a smooth formal stack.*

Definition 1.3.2. *Let $\mathcal{M}$ be a good stack and $\mathcal{U} \to \mathcal{M}$ be a curve over $\mathcal{M}$. We further request that $\mathcal{M}_{[X]}$ is smooth.*

*Suppose G acts on $\mathcal{U}$. Then, $[X] : \operatorname{Spec} k \to \mathcal{M}$ **models** G if $[X]$ is fixed by G. In particular,*

- $G \hookrightarrow \operatorname{Aut}(X)$
- $G \hookrightarrow \operatorname{Aut}(\widehat{\mathcal{U}}_{[X]}, \widehat{\mathcal{M}}_{[X]})$*, that is, on a formal neighborhood of* $[X]$.

Lemma 1.3.3. *(Prop 2.5, Page 8, (23)) A stack $\mathcal{M}$ locally of finite type is smooth iff for any $S_0 \hookrightarrow S$ a closed embedding defined by a nilpotent ideal, any morphism $S_0 \to \mathcal{X}$ can be extended to a morphism $S \to \mathcal{M}$. Furthermore, if this condition is true merely for all ideals of square zero, a priori a weaker condition, then the stack is nevertheless smooth.*

Corollary 1.3.4. *The condition of formal smoothness implies that not only do we have the natural restriction map, but we also have an adjoint ι*

$$\begin{tikzcd} \operatorname{Aut}(\widehat{\mathcal{U}}_{[X]}, \widehat{\mathcal{M}}_{[X]}) \arrow[r] & \operatorname{Aut}(X/k) \arrow[l, bend right, "\iota"'] \end{tikzcd}$$

Definition 1.3.5. *Let $(R, I) \in \operatorname{Pro}(\operatorname{Art}_k)$ and $f : \mathcal{X} \to \operatorname{Spec}(R)$ a curve. Suppose G acts continuously on f. Let $\mathcal{J}_e^\wedge$ be the formal group law associated to the Jacobian of $\mathcal{X}$. Then, X has **deformed h-splitting** if there is a G-equivariant isomorphism of* *formal groups, which is a splitting*

$$\mathcal{J}_e^{\wedge} \simeq F_{univ} \times P,$$

where F_{univ} is a universal formal group of height h.

Note that by satisfying Definition definition 1.3.5, it follows that $R \simeq LT_h$.

Definition 1.3.6. *We say the curve has h-splitting if the curve $\mathcal{X}$ is over a field k of characteristic $p > 0$.*

CHAPTER 2

Structural Theorem: Geometric Modelling

Before we can use the notion of geometric modelling we need to understand it. We now prove exactly what is required of a moduli space of curves for it to model the Lubin-Tate action.

Lemma 2.0.1. *For any element X in a $\mathbb{Z}_p$-linear category with an action of G, there is an injective map $\mathbb{Z}_p[G] \hookrightarrow \mathrm{End}(X)$.*

Lemma 2.0.2. *Let G be a finite group. Define $\pi_\chi := \frac{1}{|G|}\sum_{g\in G}\chi(g)^{-1}g$ as an element in $\mathbb{Z}_p[G]$. Then, π_χ is an idempotent element, where χ is a character of G. We denote the idempotent associated to powers of it as χ^i.*

We define some maps that will soon be relevant. Let γ be the map given by the definition of modelling G, let T be the projection of $\mathrm{Aut}(A, B)$ to just $\mathrm{Aut}(B)$, and let the completion with respect to the ideal be the ideal from the definition of deformed h-splitting.

Theorem 2.0.3. *Let G be any finite subgroup of the group J_h for any fixed height $h \geq 1$ and prime p. Let $\mathcal{U}$ be a curve over a moduli stack $\mathcal{M}$ with a point $[X] : \mathrm{Spec}\, k \to \mathcal{M}$ which has deformed h-splitting and models G.*

Then, the following composition map ϕ is the Lubin-Tate action of G on LT.

$$
\begin{tikzcd}
G \arrow[r, hook, "\gamma"] \arrow[rr, bend left, "\phi"] & \mathrm{Aut}(\widehat{\mathcal{U}}_{[X]}, \widehat{\mathcal{M}}_{[X]}) \arrow[r, two heads, "T"] & \mathrm{Aut}(\widehat{\mathcal{M}}_{[X]}) \simeq \mathrm{Aut}_{cts}(LT)
\end{tikzcd}
\tag{2.1}
$$

Remark. Here is an informal statement of the theorem. For any fixed subgroup G of the Morava stabilizer group J_h, if we have a curve X with certain properties (deformed h-splitting and modelling G), then the universal lift of the action of $\mathrm{Aut}(X)$ on the deformations of X is in fact the Lubin-Tate action.

Proof. We may form the following diagram.

- The map c is a map defined in several steps.
  (1) Take the Jacobian of $\widehat{\mathcal{U}}_{[X]}$. We call this $\widehat{\mathcal{J}}_{[X]}$.
  (2) Take the formal completion of $\mathcal{J}$ at the origin e. We call this $\widehat{\mathcal{J}}_{[X],e}$..
  (3) Take the idempotent decomposition of $\widehat{\mathcal{J}}_{[X],e}$ into $\oplus_i e_i \widehat{\mathcal{J}}_{[X],e}$. This splitting is idempotent because all pointed categories are idempotent.
  (4) By the property of deformed h-splitting, there is an e_1 such that $F_{\mathrm{univ}} \simeq e_1 \widehat{\mathcal{J}}_{[X],e}$, where F_{univ} is the universal formal group law of height h defined in definition 1.2.5. We thus project from the full decomposition of $\widehat{\mathcal{J}}_{[X],e}$ onto the e_1 component. We call this map π.

$$\begin{tikzcd}
\mathrm{Aut}(\widehat{\mathcal{U}}_{[X]}, \widehat{\mathcal{M}}_{[X]}) \arrow[r, hook, "\mathrm{Jac}"] \arrow[rrrdd, bend right, "c", red] & \mathrm{Aut}(\widehat{\mathcal{J}}_{[X]}, \widehat{\mathcal{M}}_{[X]}) \arrow[r, hook, "\hat{e}"] & \mathrm{Aut}(\widehat{\mathcal{J}}_{[X],e}, \widehat{\mathcal{M}}_{[X]}) \arrow[d, "\simeq"] & \\
& & \mathrm{Aut}(\oplus_i e_i \widehat{\mathcal{J}}_{[X],e}, LT) \arrow[r, two heads, "\pi"] & \mathrm{Aut}(e_1 \widehat{\mathcal{J}}_{[X],e}, LT) \arrow[d, "\simeq"] \\
& & & \mathrm{Aut}(F_{\mathrm{univ}}, LT)
\end{tikzcd}$$

Steps (2), (3) and (4) in the definition of c are well-defined by definition iff $\mathcal{U}$ has deformed h-splitting.

- The map j is the Lubin-Tate action of $J = \mathrm{Aut}(F/k) = \mathrm{Aut}_k(F)$ on LT.
- The maps ι and ι' are defined due to the moduli problems being formally smooth, so we have not just the restriction map but a map defined the other way (Lemma 1.3.4).
- Let F be a one dimensional height h formal group associated to X, arising from X satisfying h-splitting ($F \simeq e_1 \mathrm{Jac}(X)^{\wedge}_e$). The map $\bar{c}$ is the map c reduced to its special fiber, $\mathrm{Spec}\, k$ (by the canonical quotient map). It does the same thing as c but with $\mathrm{Jac}(X)$, and again only makes sense iff X has h-splitting (which is implied by $\mathcal{U}$ having deformed h-splitting).
- We remind the reader that we use T to denote be the projection of $\mathrm{Aut}(A, B)$ to just $\mathrm{Aut}(B)$.

We now consider the following diagram. We will use this diagram to show the left side is also the Lubin-Tate action.

$$\begin{tikzcd}
G \arrow[r, hook, "\gamma"] & \mathrm{Aut}_k(X) \arrow[r, "\bar{c}"] \arrow[d, hook, "\iota"] & \mathrm{Aut}_k(F) \arrow[d, hook, "\iota'"] \arrow[dd, bend left, "j"] \\
& \mathrm{Aut}(\widehat{\mathcal{U}}_{[X]}, \widehat{\mathcal{M}}_{[X]}) \arrow[r, "c"] \arrow[d, two heads, "T"] & \mathrm{Aut}(F_{\mathrm{univ}}, LT) \arrow[d, two heads, "T"] \\
& \mathrm{Aut}(\widehat{\mathcal{M}}_{[X]}) \arrow[r, equal] & \mathrm{Aut}(LT)
\end{tikzcd}$$

Now, we make a few remarks to conclude that the diagram (2) commutes.

- The upper square commutes since $\bar{c}$ is defined as the specialization of c.
- The lower square commutes by the definition of deformed h-splitting.
- The map $c \circ \iota \circ \gamma$ is injective by Lemma 2.0.1, because any group actions on elements in the category respect the idempotent decomposition of morphisms in the category. In other words, the projection π respects equivariance.

We've shown now that the diagram commutes by construction. This implies that $T \circ \iota \circ \gamma$ coincides with j, the Lubin-Tate action of $J = \mathrm{Aut}(F/k)$ on LT. $\square$

CHAPTER 3

Artin-Schreier-Witt and Harbater-Katz-Gabber curves

In this section we will review some basic properties of Artin-Schreier curves, and more generally of Artin-Schreier-Witt curves, which will be used in subsequent sections.

3.1. Ramification Theory

Definition 3.1.1. *Let $f : X \to Y$ be a morphism of schemes over an algebraically closed field k (or over $\mathbb{Z}$). The ramification divisor R of f is the divisor on X given by*

$$R = \sum_P \mathit{length}(\Omega_{X/Y})_P[P]$$

where the sum is taken over closed points P of X.

Definition 3.1.2. *Let $f : X \to Y$ and $g : Y \to Z$ be two morphisms of affine schemes. Consider the short exact sequence*

$$0 \to f^*\Omega_{Y/Z} \to \Omega_{X/Z} \to \Omega_{X/Y} \to 0$$

. The different ideal $\mathcal{D}_{X/Y}$ is the annihilator ideal of the sheaf $\Omega_{X/Y}$ considered as an $\mathcal{O}(X)$-module.

The exponent of the different at a closed point P of Y is the valuation of $\mathcal{D}_{X/Y}$ at the place P, that is the exponent of the prime P in the primary decomposition of the ideal $\mathcal{D}_{X/Y}$.

We now move to a discussion of local fields. Given an extension of fields L/K, a nonzero prime ideal $\mathfrak{p} \subset \mathcal{O}_K$ is called ramified in L if the ideal $\mathfrak{O}_L$ in $\mathfrak{O}_L$ factors into primes as

$$\mathfrak{p}\mathcal{O}_L = \mathfrak{q}_1^{e_1} \cdots \mathfrak{q}_n^{e_n}.,$$

where $e_i > 1$ for some i. (define wild and tame ram).

If L/K is a finite abelian extension of local fields with Galois group G, there are two different filtrations of G. The "lower numbering" behaves well with respect to subgroups of G, that is, for all normal subgroups $H \subset G$, then $H_\ell = G_\ell \cap H$. The "upper numbering" behaves well with respect to quotients of G, in the sense that for all normal subgroups H of G, one gets: $\forall u \geq -1$, $(G/H)^u = (G^u H)/H$.

Definition 3.1.3. *If L/K is a finite abelian extension of local fields with Galois group G, then the ith ramification group (with lower numbering) is:*

$$G_i := \{g \in G | g \text{ acts trivially on } \mathcal{O}_L/\mathfrak{p}_L^{i+1}\}.$$

Further, this forms a decreasing filtration of G,

$$G = G_{-1} \supset G_0 \supset G_1 \supset \cdots G_m = 0.$$

Note that $G_0 = 1$ iff L/K is unramified, $G_1 = 1$ iff L/K is tamely ramified (ramification index is prime to the residue charactertistic.) $G = G_1$ iff G is a p-group iff L/K is totally wildly ramified.

When we are considering it locally, a nice reformulation:

Definition 3.1.4. *Given $G \subset \mathrm{Aut}(k[[t]])$, then the ith ramification subgroup (with lower numbering) is:*

$$G_i := \{g \in G | g \text{ acts trivially on } k[[t]]/(t^{i+1})\}.$$

Further, this forms a decreasing filtration of G,

$$G = G_{-1} \supset G_0 \supset G_1 \supset \cdots G_m = 0.$$

Definition 3.1.5. *We denote by ℓ_i the lower jumps for L/K or of G, which are numbers satisfying $G_{\ell_i} \neq G_{\ell_i+1}$*

Theorem 3.1.6. *(Stichtenoth) Let K be a local field of char $p > 0$ with perfect residue field. Let L/K be the extension given by the equation $Y^p - Y = a$, for some $a \in K$ and denote $G := \mathrm{Gal}(L/K)$.. Then, if $v_K(a) = -m < 0$ and if m is prime to p, the extension L/K is cyclic of degree p and totally ramified. Further, its ramification groups are given by*

$$G = G^{-1} = \cdots = G^m \text{ and } G^{m+1} = 1.$$

Remark. This can be generalized to Artin-Schreier-Witt curves.

The most important fact about lower numbers is the Hilbert different formula:

Theorem 3.1.7. *(Hilbert's different formula) Consider a Galois extension F'/F of function fields, and a place $P \in \mathbb{P}_F$ and a place $P' \in \mathbb{P}_{F'}$ laying over P. Then, the different exponent $d(P'|P)$ is*

$$\deg(\mathfrak{D}_{L/K}) := d(P'|P) := \sum_{i=0}^{\infty} |G_i(P'/P)| - 1.$$

Theorem 3.1.8 (Wild Riemann Hurwitz with one branch point)**.** . *Let $f : X \to Y$ be a map with ramification divisor R consisting of one point $P' \in X$ which is totally wildly ramified over a point $P \in Y$, then we call $\deg d := \deg_{P'|P}(\mathcal{D}_{X/Y})$, and*

$$g(X) = \frac{1}{2}(\deg(f)(2g(Y) - 2) + \deg d) + 1.$$

3.2. Artin-Schreier Theory

Artin-Schreier theory gives a classification of of $\mathbb{Z}/p\mathbb{Z}$-Galois extensions of fields of characteristic p. For K a field of characteristic p the Artin-Schreier polynomial $x^p - x - f \in K[x]$ for $f \in K$ is irreducible precisely if $f \neq h^p - h$ for any $h \in K$, in which case the splitting field is a $\mathbb{Z}/p\mathbb{Z}$-Galois extension of K. Conversely, every $\mathbb{Z}/p\mathbb{Z}$-Galois extension of K is the splitting field of some Artin-Schreier polynomial.

Specializing to the case where $K = k(x)$ is the function field of $\mathbb{P}^1$, Artin-Schreier extensions have the following geometric incarnation:

Definition 3.2.1. *Let k be a field of characteristic p. An Artin-Schreier curve over k is a smooth projective geometrically connected curve X_1 admitting a morphism $f : X_1 \to X_0 = \mathbb{P}^1_k$ which is a $\mathbb{Z}/p\mathbb{Z}$-Galois cover.*

Such an Artin-Schreier curve admits an affine equation of the form $y^p - y = f$ where $f \neq h^p - h$ for any $h \in k(x)$, and admits two canonical projections: the projection $p_x : X_1 \to \mathbb{P}^1_k$ onto the x-coordinate which is a $\mathbb{Z}/p\mathbb{Z}$-Galois cover, and the projection $p_y : X_1 \to \mathbb{P}^1_k$ onto the y-coordinate.

We will be particularly interested in the following example: consider the Artin-Schreier curve $X_1 = \{y^p - y = x^{p-1}\}$ over $k = \mathbb{F}_{p^{p-1}}$.

Definition 3.2.2. *We now define the curve and main maps that will be used in the rest of this section. The Artin-Schreier curve X over $k = \mathbb{F}_{p^{p-1}}$ with affine form*

$$y^p - y = x^{p-1}$$

has two projection maps to $\mathbb{P}^1_k$). The first map p_x is the $\mathbb{Z}/p$-Galois map coming from its definition. The second map p_y is a degree $p-1$ map.

$$\begin{tikzcd}
 & X \arrow[dl, "p_x"'] \arrow[dr, "p_y"] & \\
\mathbb{P}^1 & {[x:y:z]} \arrow[dl, mapsto] \arrow[dr, mapsto] & \mathbb{P}^1 \\
{[x:z]} & & {[y:z]}
\end{tikzcd}$$

Lemma 3.2.3.

- *This curve is of genus $g = \frac{(p-1)(p-2)}{2}$.*
- *The map p_x is totally wildly ramified at one point, the point infinity on $\mathbb{P}^1$.*
- *The ramfication locus of the map on the right p_y is $\mathbb{F}_p$ plus the point at infinity on $\mathbb{P}^1$, with tame ramification of order $(p-1)$.*

Proof.

- This is proved three times. It is in both Lemma 4.3.3 and Lemma 4.3.1, as well as Lemma 4.7.1.

If one's curve X is embedded in P^2 then for any point $x \in P^2$ there is a map from X to L given by the following. Here $L(x,p)$ is the line intersecting x and p.

$$\pi_x : X \to L$$
$$p \mapsto L(x,p) \cap L$$

If $x \in \mathbb{P}^2$ is not in X, then the degree of the resulting map π_x is equal to the degree d of X. We restrict the domain of our map $\mathbb{P}^2 \backslash \{x\} \to \mathbb{P}^1$ to be $X \to \mathbb{P}^1$. The degree of the resulting map is equal to the degree of X, because a line intersects C in $\deg X$ number of points.

In case of $x \in X$ we have a well-defined map on $X \backslash x$. This map can be continued to a well-defined map on the whole C. We do so as follows. Geometrically, we map x

along the tangent line at x to X, now if you count the degree of the map it becomes $\deg X - 1$. This is because there are $\deg X$ points on every line, including x.

- The point at infinity of $\mathbb{P}^1$ is $[1 : 0 : 0]$. Note that $y^p - y = y(y-1)\cdots(y-(p-1))$. The Galois action acts by $y \mapsto y+1$, leaving x unaffected, thus infinity is fixed. The degree of p_x is determined by the fact that the projection from $[x : z]$ is constructed as a projection from $[0 : 1 : 0]$, which is not a point on X.
- The map p_y is constructed as a projection from $[1 : 0 : 0]$, which is a point on X, thus, it is degree $p-1$. Again, note that $y^p - y = y(y-1)\cdots(y-(p-1))$. The ramification points of this map are then $[0 : i : 1]$ and $[1 : 0 : 0]$, where $0 \leq i \leq p-1$.

$\square$

3.3. Artin-Schreier-Witt Theory

Artin-Schreier theory gives a classification of of $\mathbb{Z}/p^n\mathbb{Z}$-Galois extensions of fields of characteristic p, specializing to the usual Artin-Schreier theory in the case $n = 1$. The idea is to mimic what happens in Artin-Schreier theory using the scheme of truncated Witt vectors.

Let k be a perfect field of characteristic p. For $n \geq 1$ let W_n be the scheme of truncated Witt vectors of length n. We have a morphism

$$\wp : W_n \to W_n$$

$$(x_0, \ldots, x_{n-1}) \mapsto (x_0^p, \ldots, x_{n-1}^p) - (x_0, \ldots, x_{n-1})$$

which satisfies $\wp(x+y) = \wp(x) + \wp(y)$. Addition of Witt vectors is given by

$$x + y = (S_0(x,y), S_1(x,y), \ldots)$$

where $S_i(x,y)$ is a certain polynomial in the components of x and y. For example:

$$S_0(x,y) = x_0 + y_0$$
$$S_1(x,y) = x_1 + y_1 + \frac{1}{p}(x_0^p + y_0^p - (x_0+y_0)^p)$$

Let $C_i(x,y)$ be the part of $S_i(x,y)$ only involving x_i and y_i. For example:

$$C_0(x,y) = x_0 + y_0$$
$$C_1(x,y) = \frac{1}{p}(x_0^p + y_0^p - (x_0+y_0)^p)$$

Now we have the following classification of $\mathbb{Z}/p^n\mathbb{Z}$-Galois covers:

Theorem 3.3.1. *(Bouw-Witt) Let k be a perfect field of characteristic p and let X be a normal variety over k with function field $K = k(X)$. Let $f_0, \ldots, f_{n-1} \in K$ such that $f_0 \neq h^p - h$ for any $h \in K$. Let $f = (f_0, \ldots, f_{n-1}) : X \dashrightarrow W_n$ be the corresponding rational morphism and consider the pullback*

$$\begin{tikzcd} Y \arrow[r, dashed] \arrow[d, "\pi"] & W_n \arrow[d, "\wp"] \\ X \arrow[r, dashed, "f"] & W_n \end{tikzcd}$$

Then $\pi : Y \to X$ is a $\mathbb{Z}/p^n\mathbb{Z}$-Galois cover. Conversely, every $\mathbb{Z}/p^n\mathbb{Z}$-Galois cover of X arises in this way.

Specializing to the case where $X = \mathbb{P}^1$ and $K = k(x)$ is the function field of $\mathbb{P}^1$, we arrive at the following definition:

Definition 3.3.2. *Let k be a field of characteristic p. An Artin-Schreier-Witt curve over k is a smooth projective geometrically connected curve X_n admitting a morphism $f : X_n \to X_0 = \mathbb{P}^1_k$ which is a $\mathbb{Z}/p^n\mathbb{Z}$-Galois cover.*

Such an Artin-Schreier curve admits an affine equation of the form

$$y_0^p - y_0 = f_0$$

$$y_1^p - y_1 = f_1 - C_1(y_0, f_0)$$

$$\vdots$$

$$y_{n-1}^p - y_{n-1} = f_{n-1} - C_{n-1}(y, f_{n-2})$$

and admits two canonical projections: the projection $p_x : X_n \to \mathbb{P}_k^1$ *onto the* x*-coordinate which is a* $\mathbb{Z}/p\mathbb{Z}$*-Galois cover, and the projection* $p_y : X_1 \to \mathbb{P}_k^1$ *onto the* y*-coordinate.*

Example 3.3.3. *We consider the equations for the Artin-Schreier-Witt curve* X_2 *(the general case can be deduced from this but its also discussed in more detail in Wild-by-tame covers (Obus and Pries)). Let* $f_0, f_1 \in k(x)$ *and consider the polynomial*

$$C_1(x,y) = \frac{1}{p}((x+y)^p - x^p - y^p) = \frac{1}{p}\sum_{1\le i\le p-1}\binom{p}{i}x^i y^{p-i} = \sum_{1\le i\le p-1}\frac{(p-1)!}{i!(p-i)!}x^i y^{p-i}$$

Then the Artin-Schreier-Witt curve X_2 *has the following affine equation:*

$$y^p - y = f_0$$

$$w^p - w = f_1 - C_1(y, f_0)$$

We will be particularly interested in the case where $f_0 = x^d$ and $f_1 = 0$.

Definition 3.3.4. *An Artin-Schreier-Witt curve is a smooth projective curve* X *over a field* k *of characteristic* p *which may be considered as a* $\mathbb{Z}/p^k$*-Galois cover of the* $\mathbb{P}^1$*,* $p_x : X \to X/(Z/p^k) \simeq \mathbb{P}^1$*.*

We may consider other maps from our curve to $\mathbb{P}^1$.

$$\begin{tikzcd}
& X_{k-1} \arrow[d] \arrow[dddl, "\mathbb{Z}/p^n-\text{Galois}"'] \arrow[dddr, "\text{degree } p^{n-1}(p-1)"] & \\
& \vdots \arrow[d] & \\
& X_1 \arrow[dl, "\mathbb{Z}/p-\text{Galois}"] \arrow[dr, "\text{degree } p-1"'] & \\
\mathbb{P}^1 & {[x:y:z]} \arrow[dl, mapsto] \arrow[dr, mapsto] & \mathbb{P}^1 \\
{[x:z]} & & {[y:z]}
\end{tikzcd}$$

3.4. Conjectures relating this to the Lubin-Tate action

Let $G = Z/p^k \rtimes Z/(p-1)^2$.

Conjecture 3.4.1. *Let us consider an Artin-Schreier-Witt curve X of minimal genus which is totally wildly ramified at one point with Galois group Z/p^k and whose automorphism group contains G. Let p_x be the map described above. Then, the universal curve over $\mathrm{Def}_{p_x}^{ram}$ has deformed h-splitting, where $p^k - p^{k-1} = p^k(p-1)$.*

Remark. $\mathrm{Def}_{p_x}^{\mathrm{ram}}$ is deformations of the map where we only allow the ramification divisor to deform in one ramification dimension.

Conjecture 3.4.2. *The ring representing $\mathrm{Def}_{p_x}^{ram}$ is $\mathrm{Sym}(\mathrm{Ind}_{\mathbb{Z}/p}^{\mathbb{Z}/p^k}\,\overline{\rho})$ completed at the ideal J defined by X as a G -representation.*

It would then follow from Theorem 2.0.3 that $E_* \simeq \mathrm{Sym}(\mathrm{Ind}_{\mathbb{Z}/p}^{\mathbb{Z}/p^k}\,\overline{\rho})^{\wedge}_J$ as a G-representation, and further, that this commutes with taking Tate cohomology. That is, the following is true, where Δ is the product of the polynomial generators of $\mathrm{Sym}(\mathrm{Ind}_{\mathbb{Z}/p}^{\mathbb{Z}/p^k}\,\overline{\rho})$,

$$H^*(G, E_*) \simeq H^*(G, \mathrm{Sym}(\mathrm{Ind}_{\mathbb{Z}/p}^{\mathbb{Z}/p^k} \overline{\rho}))[\Delta^{-1}]^\wedge_J.$$

3.5. Harbater-Katz-Gabber Curves

These curves, totally wildly ramified at one point, are extremely special. In fact it has been shown that all finite subgroups of $\mathrm{Aut}\, k[[t]]$ are realized by Harbater-Katz-Gabber curves. More specifically, Given an algebraic curve X on which G acts with a fixed point x having residue field k, then G acts on the completion $\hat{\mathcal{O}}_{X,x}$ of the local ring x at X, and $\hat{\mathcal{O}}_{X,x} \simeq k[[t]]$ for any choice of uniformizing parameter t at x. Every finite subgroup G of $\mathrm{Aut}(k[[t]])$ arises in this way from curves of this form.

For the sake of getting our hands on these elusive creatures, let's discuss the properties of our particular example of Artin-Schreier-Witt curves which are also Harbater-Katz-Gabber curves with p-Sylow subgroup $\mathbb{Z}/p^2$.

We consider now curves which are $\mathbb{Z}/p^k$-Galois covers of $\mathbb{P}^1_k$, which are totally wildly ramified at infinity.

Lemma 3.5.1. *Given an Artin-Schreier-Witt curve for $\mathbb{Z}/p^2$ totally wildly ramified at one point over a field of char p. The minimal higher ramification jumps are $u_1 = 1$, $u_2 = p$. The minimal lower ramification jumps are $l_1 = 1$, $l_2 = p(p-1)+1$.*

Proof. Lower jumps are calculated using valuations. □

Lemma 3.5.2. *Given an Artin-Schreier-Witt curve for $\mathbb{Z}/p^2$ totally wildly ramified at one point over a field of char p such that the first equation is $y^p - y = a$, where $v_L(a) = -m < 0$. The higher ramification jumps are $u_1 = m$, $u_2 = m$. The lower ramification jumps are $l_1 = m$, $l_2 = m(p(p-1)+1)$.*

Proof. We take Lemma 3.5.1. Changing the valuation to m can be compared with taking a pullback by the equation $z^m = x$. The total cover then is totally ramified of degree mp^2. The original cover is a quotient of the total one.

Subquotients preserve lower jumps. So the lower jumps of the sub-cover (which is the pullback) are m and $m(p(p-1)+1)$. Using Herbrand's formula again yields m and mp for the upper jumps of the pullback cover. □

Lemma 3.5.3. *$\mathbb{Z}/p^2$-Artin-Schreier-Witt curves X of minimal genus have genus $g(X) = \frac{1}{2}(p^4 - 2p^3 + p^2 - 2p) + 1$.*

Proof. By the Riemann-Hurwitz formula, for a map $f : X \to Y$, we have $g(X) = \frac{1}{2}(\deg(f)(2g(Y) - 2) + \deg d) + 1$. The degree of our ramification divisor, by Hilbert different formula (Stichtenoth) and Lemma 3.5.1 is $d = p^2(p-1) + (u_2 - (p-1))(p-1)$, where $u_2 = (p-1)(p(p-1)+1)$. □

Remark. These curves have quite a large genus. For example, the $\mathbb{Z}/p^2$ curve for $p = 3$ is genus 16, and for $p = 5$ its genus 196. It grows quickly since there is a quadratic term.

Lemma 3.5.4. *The conductor (wrt the $\mathbb{Z}/p^k$-Galois map) of the totally wildly ramified point at infinity on our Artin-Schreier-Witt curve C is $p^k - p^{k-1} + 1$.*

Proof. This follows from the definition of conductor in terms of Herband's lemma.

□

CHAPTER 4

Proof of h-Splitting for Artin-Schreier Curves

4.1. Dieudonné modules

Definition 4.1.1. *Suppose V is a smooth d-dimensional formal variety over A a $\mathbb{Z}_p$-algebra. We define the deRham cohomology of V to be the cohomology of the complex of differentials $\Omega^{\bullet}_{V/A}$, this is the usual deRham complex of the algebra $A(V)$. That is, $\Omega^i_{V/A}$ is the ith exterior power of the A-module of Kähler differentials $\Omega^1_{A(V)/A} \simeq \bigoplus_i A[[T_1]], ..., T_d]]dT_i$.*

Definition 4.1.2. *Suppose F is a formal group over A a complete local $\mathbb{Z}_p$-algebra with residue field k. We define the Dieduonné module $D(F/k)$ to be the cohomology classes $\omega \in H^1_{dR}(F/A)$ which are translation invariant. Let $\Sigma : F \times F \to F$ be the addition law, and let $pr_1, pr_2 : F \times F \to F$ be the projections. Then, $[\omega] \in H^1_{dR}(F/A)$ is translation invariant iff $\Sigma^*(\omega) - (pr_1(\omega) + pr_2(\omega))$ is exact.*

Note that this definition uses the group structure of F, whereas the deRham cohomology does not.

Remark. The more practically useful definition of the Dieudonne module of a formal group F over a perfect field k is $D(F/k) := \mathrm{Ext}^1_{W(k)}(F(W(k)), W(k))$. We then get the exact sequence

$$\mathrm{Ext}^1(F(W(k)), N^{\geq 1}W(k) \to \mathrm{Ext}^1_{W(k)}(F(W(k)), W(k)) \to \mathrm{Ext}^1(F(W(k)), k) \simeq \mathrm{Lie}(F^{\vee}).$$

This is the analogue of the Hodge-to-deRham short exact sequence.

Example 4.1.3. *Let us calculate the Dieudonné module of* $\widehat{\mathbb{G}}_m$*, the formal group of the multiplicative group. This is* $\mathrm{Ext}_{W(k)}(W(k)^\times, W(k)) \simeq W(k)dlog(1+T)$*, in other words* $\frac{dT}{1+T} = dlog(1+T)$*.*

Let us consider a curve X over a perfect field k of characteristic p. Let a lift of X to $R \in \mathsf{Art}_k$ be denoted as $\mathcal{X}$. Further, we consider the Jacobian of this lift, denoted as $\mathcal{J}$, and the formal variety associated to the completions of $\mathcal{X}$ and $\mathcal{J}$ at the point x as $\hat{\mathcal{X}}$ and $\hat{\mathcal{J}}$. We define the Jacobian of a curve $\mathcal{X}$ as $\mathcal{J}(\mathcal{X}/R) := \mathrm{Pic}^0(\mathcal{X}/R)$, which we call $\mathcal{J}$ to simplify notation when $\mathcal{X}$ and R are clear.

Let us consider the Abel-Jacobi map defined by the point $x \in \mathcal{X}$,

$$\psi_x : \mathcal{X} \to \mathcal{J}$$

as $\psi_x(y) =$ the class of the invertible sheaf $\mathcal{I}(y)^{-1} \otimes \mathcal{I}(x)$, where $\mathcal{I}$ denotes the invertible sheaf $y \in \mathcal{X}$ viewed as a Cartier divisor. Let $\hat{X}$ denote the formal completion of X along x, and let $\hat{\mathcal{J}}$ denote the formal completion of $\mathcal{J}$ at $e := \mathcal{O}_X$. This is a pointed Lie variety of dimension one. Since $\phi_x(0) = 0$, ϕ_x induces a map of pointed formal Lie varieties:

$$\hat{\psi}_x : \hat{\mathcal{X}} \to \hat{\mathcal{J}}.$$

Lemma 4.1.4. *(15) (5.9.2) The Abel-Jacobi map induces a map on cohomology. The composite is injective*

$$\begin{tikzcd} D(\mathcal{J}) \arrow[r, hook] \arrow[rr, bend right] & \hat{\mathcal{J}} \arrow[r, "(\hat{\phi}_x)^*"] & H^1_{dR}(\hat{\mathcal{X}}) \end{tikzcd}$$

Lemma 4.1.5. *(15) (Katz, Section 5.6, top of pg 202) The following is an isomorphism compatible with Hodge filtrations:*

$$H^1_{dR}(\mathcal{J}) \simeq D(\hat{\mathcal{J}}) := H^{1,prim}_{dR}(\mathcal{J}).$$

That is, we can just calculate the deRham cohomology of the Jacobian $\mathcal{J}$, and this automatically selects for the primitives in the deRham cohomology of the formal Lie variety associated to $\mathcal{J}$.

This means that we can view the Dieduonné module of $\mathcal{J} \otimes k$ as laying in the Hodge-to-deRham SES. (label this equation)

$$0 \to H^0(\mathcal{J}, \Omega^1_{\mathcal{J}}) \to H^1_{dR}(\mathcal{J}/R) \to H^1(\mathcal{J}, \mathcal{O}_{\mathcal{J}}) \to 0$$

The Abel-Jacobi map induces isomorphisms that are natural in $\mathcal{X}$.

$$H^0(\mathcal{J}, \Omega^1_{\mathcal{J}}) \simeq H^0(\mathcal{X}, \Omega^1_{\mathcal{X}})$$

$$H^1(\mathcal{J}, \mathcal{O}_{\mathcal{J}}) \simeq H^1(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$$

$$H^1_{dR}(\mathcal{J}/R) \simeq H^1_{dR}(\mathcal{X}/R).$$

By Grothendieck-Serre duality,

$$H^0(\mathcal{X}, \Omega^1_{\mathcal{X}}) \simeq H^1(\mathcal{X}, \mathcal{O}_{\mathcal{X}})^\vee.$$

So, by Lemma 4.1.4, we can in fact calculate the Dieduonne module $\hat{\mathcal{J}}$ as $H^1_{dR}(\mathcal{X}/R)$, and write it in the following short exact sequence (which is isomorphic to the above).

$$0 \to H^0(\mathcal{X}, \Omega^1_{\mathcal{X}}) \to H^1_{dR}(\mathcal{X}/R) \to H^1(\mathcal{X}, \mathcal{O}_{\mathcal{X}}) \to 0.$$

4.1.1. Slope decompositions and the Dieudonné-Manin classification

We consider the noncommutative ring arising naturally from Gabriel's theorem,

$$Q := W(k)\{F, V\}/(FV = p).$$

Formal groups are a type of p-divisible group. The Dieudonné module functor gives an embedding of the category of p-divisible of groups of height h into finitely generated Q-modules which are free as $W(k)$-modules of rank h. This is a categorical equivalence.

We define $Q_F := W(k)[1/p]\{F\}/(Fa = \sigma(a)F)$. The category of p-divisible groups up to isogeny has a fully-faithful embedding into the category of finitely generated Q_F-modules.

The category of finitely generated Q_F-modules is semi-simple, and if k is an algebraically closed field, the simple objects are of the form

$$G_{r/s} := Q_F/Q_F(F^r - p^s).$$

The fraction r/s is called the slope and is written in simplest form. The number r is the dimension of the p-divisible group, and the number s is the height.

Remark. For non-algebraically closed fields we would need descent theory to describe the simple modules.

The phrasing "slope a appears in an integral Dieudonné module D" means that in the Dieudonné-Manin decomposition of the rational module $D[1/p]$ there is at least one summand of the form $G_{r/s}$.

4.2. Properties of Artin-Schreier Curves

Lemma 4.2.1. *The Artin-Schreier curve $y^p - y = x^{p-1}$ over $\mathbb{F}_{p^{p-1}}$ has the following properties.*

(1) $\mathbb{Z}/p \rtimes \mathbb{Z}/(p-1)^2 \subseteq \mathrm{Aut}_k(X)$

(2) The Jacobian of this curve has h-splitting for $h = p - 1$.

Proof. We demonstrate the elements that generate this subgroup of the automorphism group. The map $y \mapsto y + 1$ generates $\mathbb{Z}/p$, and the map $f([x : y : z]) = [\alpha^p x : \alpha^{p-1} y : z]$, where α is a $(p-1)^2$st root of unity. The latter property is proved in Proposition 4.8.1. $\square$

4.3. Holomorphic Differentials of Curves

Let X be a curve over a ring in Art_k, and let us decompose it into two affine pieces $X = A \cup B$.

Given a sheaf $\mathcal{F}$ over X, we have the following Mayer-Vietoris sequence.

$$0 \to H^0(X, \mathcal{F}) \hookrightarrow \mathcal{F}(A) \times \mathcal{F}(B) \to \mathcal{F}(A) \cap \mathcal{F}(B) \to H^1(X, \mathcal{F} \to 0.$$

In particular, if take the decomposition of X into the point at infinity and its complement $X = U \cup \{\infty\}$, we get the following sequence, let T be a divisor at ∞.

$$0 \to H^0(X, \mathcal{F}) \hookrightarrow \mathcal{F}(U) \times \mathcal{F}_\infty^\wedge \to \mathcal{F}_\infty^\wedge[\frac{1}{T}] \to H^0(X, \mathcal{F}) \to 0.$$

A global section $H^0(X, \Omega_X^1)$ can be specified by either a basis in terms of $\mathcal{F}(U)$, as in Lemma 4.3.1 or in terms of $\mathcal{F}_\infty^\wedge$, as in Lemma 4.4.2. This is because we are considering $H^0(X, \mathcal{F})$ as a subset of $\mathcal{F}(U) \times \mathcal{F}_\infty^\wedge$.

We include both of these derivations of holomorphic differentials in order to show the reader that we do not need the curve to be planar to work with the holomorphic differentials of that curve explicitly.

Lemma 4.3.1. *A basis of holomorphic differentials of plane curve X defined by an equation P of degree d is*

$$\{x^i y^j dx \colon 0 \leq i + j \leq d - 3\}.$$

Indeed, any choice of distinct (i, j) where both i and j are greater than 0 such that the there are exactly $\frac{(p-1)(p-2)}{2}$ such pairs will also produce a basis.

Proof. A genus of smooth projective plane curve of degree d is $(d-1)(d-2)/2$, so we need to construct that many holomorphic differentials. For a smooth plane curve X given by equation $P(x, y) = 0$ we have $P_x dx + P_y dy = 0$ (by differentiating $P = 0$), where P_x, P_y are the derivatives of P with respect to x and y respectively. Since X is smooth the vector (P_x, P_y) doesn't vanish anywhere on X. Hence we can define a holomorphic nowhere (on X) vanishing 1-form $\omega = \frac{dy}{P_x} = -\frac{dx}{P_y}$. We may also choose any non-zero scalar multiple of ω as our ω.

Let us examine pairs (i, j) such that $0 \leq i+j \leq d-3$. This is choosing ordered pairs of numbers from the set $\{0, 1, ..., d-2\}$, in other words, from a set of $d-1$ numbers. There are then $\binom{d-1}{2} = \frac{(d-1)(d-2)}{2}$ of such (i, j).

We thus get a set of precisely $(d-1)(d-2)/2$ holomorphic forms on X, given by $x^i y^j \omega$ for such (i, j). Further, all such forms are pairwise distinct, making this a basis, though we do not show this here. $\square$

We may also construct a basis for $\Gamma(X, \Omega^1_X)$ for Artin-Schreier curves X without using that they are plane curves. This technique only needs a locally defined basis and the notion of uniformizer, which we have for the more general case. First, we translate our basis to being in terms of the universal.

Lemma 4.3.2. *Given the uniformizer T on the Artin-Schreier curve corresponding to the point at infinity, and x corresponding to the coordinate on $\mathbb{P}^1$*

$$dx = T^{p(p-3)}dT.$$

Proof. Let $T = \frac{y}{x}$, this is a uniformizer of the Artin-Schreier curve at infinity. Further, for $e(T)$ some invertible power series,

$$x = T^{-p}e(T)$$
$$y = T^{-(p-1)}e(T)$$

We may thus expand.

$$\begin{aligned} dT &= d(yx^{-1}) \\ &= \frac{1}{x}dy - y/x^2dx \\ &= (p-1)x^{p-1}dx - y/x^2dx \\ &= (T^{-p(p-3)} - T^{2p-p+1})dx \\ &= T^{-p(p-3)}[1 + T^{p+1+p(p-3)}]dx \end{aligned}$$

Since the latter factor is invertible, we get $dx = T^{p(p-3)}\psi(T)dT$ as claimed. $\square$

Lemma 4.3.3. *A basis of holomorphic differentials the Artin-Schreier curve over $R \in \mathsf{Art}_k$ with affine equation $y^p - y = x^{p-1}$ is*

$$\{z^{-(pi+(p-1)j)+p(p-3)}dz + \textit{ higher order terms}|i, j > 0\},$$

where the exponent of z must be greater than 0.

Proof. Let $k = \mathbb{F}_q$ where q contains enough roots of unity. Let us consider the curve $X = U \cup \infty$. The uniformizer at infinity of this curve is $T := \frac{y}{x}$. We thus have the following pullback diagram:

$$\begin{array}{ccc} \Gamma(X, \Omega_{X/k}) & \longrightarrow & \Omega_{k[[T]]/k} \\ \downarrow & & \downarrow \\ \Omega_{U/k} & \longrightarrow & \Omega_{k((T))/k} \end{array}$$

This is isomorphic to

$$\begin{array}{ccc} \Gamma(X, \Omega_{X/k}) & \longrightarrow & k[[T]]dT \\ \downarrow & & \downarrow \\ Adx & \longrightarrow & k((T))dT \end{array}$$

We now calculate the lower map – which requires us to write dx in terms of dT. We can now calculate the map $Adx \mapsto k((T))dT$. Elements in Adx are of the form ax^iy^jdx + higher order terms. By lemma 4.3.2, their image is $aT^{-(pi+(p-1)j)+p(p-3)}dT$ + higher order terms. This is in the image of $k[[T]]dT \to k((T))dT$ iff $i + j \leq p - 3$. This then gives us the k-basis of $\Gamma(X, \Omega_{X/k})$ that we expect, $\{x^iy^jdx,\ i + j \leq p - 3\}$. $\square$

4.4. Holomorphic Differentials of Artin-Schreier-Witt Curve for $\mathbb{Z}/p^2$ Galois extension

Lemma 4.4.1. *Let u be uniformizer at infinity of the Artin-Schreier-Witt curve for C_{p^2}, then*

$$dx = u^{p^4-2p^3+p^2-2p}du$$

.

Proof. Here is our convienient key:

$$\begin{array}{cccc}
X_2 & w^p - w = g(C_p(x), y) & P" & u \\
\downarrow & & & \\
X_1 & y^p - y = x^{p-1} & P' & T \\
\downarrow & & & \\
X_0 & x & P & \frac{1}{x}
\end{array}$$

We begin by finding a uniformizer for P'. That is, a function $f \in k(Y_1)$ such that $v_{P'}(f) = 1$. We begin by noting that $v_P(x) = -1$, since $P = (\frac{1}{x})$, thus.

$$v_{P'}(x) = -p.$$

further $v_{P'}(x^{p-1}) = pv_P(x^{p-1}) = -p(p-1)$, thus $v_{P'}(y^p - y) = -p(p-1) > 0$, thus,

$$v_{P'}(y) = -(p-1).$$

We now wish to solve for $v_{P'}(x^i y^j) = 1$, we can do this by picking $i = -1, j = 1$, such that $-1(p) + 1(p-1) = p - 1 - p = 1$. So we have

$$T = \frac{y}{x}.$$

When we plug the formulas for x and y in terms of T into the equation for Y_2 : $w^p - w = f(x, y)$, we get, for some d, where d is prime to p

$$w^p - w = T^{-d}.$$

Since x has a larger power of T, the largest power of T will come from the factor $x^{m(p-1)}y$, where $m = (p-1)$ (from $x^m = y^p - y$) So, in general for the C_{p^2} Artin-Schreier-Witt cover, we get:

$$\begin{aligned} w^p - w &= -x^{(p-1)^2}y + other \\ &= -T^{-p(p-1)^2-(p-1)} + other \\ &= -T^{-(p-1)(p(p-1)+1)} + other \\ &= -T^{-(p^3-2p^2+2p-1)} + other \end{aligned}$$

Thus, for us, $d = p^3 - 2p^2 + 2p - 1$. We now repeat the cycle to determine a $u \in k(Y_2)$ such that $v_{P"}(u) = 1$. We note that $v_{P"}(w^p - w) = -pd$, thus $v_{P"}(w) = -d$. Just to have on hand, note that $v_{P"}(x) = -p^2$, and $v_{P"}(y) = -p(p-1)$.

So, to find u, we need to solve for $x^i y^j w^k$ such that $v_{P"}(x^i y^j w^k) = 1$. In other words, we are solving for $ip^2 + jp(p-1) + kd = -1$. For general p, $i = -p, j = 2, k = 1$ works.

$$u = x^{-p}y^2w$$

$$
\begin{array}{cccc}
Y_2 & w^p - w = g(C_p(x), y) & P" & u = x^{-p}y^2w \\
\downarrow & & & \\
Y_1 & y^p - y = x^{p-1} & P' & T = \frac{y}{x} \\
\downarrow & & & \\
Y_0 & x & P & \frac{1}{x}
\end{array}
$$

For some invertible power series $e(u)$

$$x = u^{-p^2}e(u)$$

$$y = u^{-p(p-1)}e(u)$$

$$w = u^{-d}e(u)$$

So, keeping in mind that $u = x^{-p}y^2w$, and $dy = -(p-1)x^{p-2}dx$, and that we only have to consdider dominating term in the polynomial for $w^p - w$.

$$
\begin{aligned}
dw &= d(-x^{(}(p-1)^2)y \\
&= x^{(p-1)^2}dy - (p-1)^2x^{(p-1)^2-1})ydx \\
&= x^{(p-1)^2+(p-2)}dx + u^{-p^2((p-1)^2-1)-p(p-1)}dx \\
&= u^{-p^4+p^3+p^2}dx + u^{-p^4+2p^3-p^2-p}dx \\
&= -u^{-p^4+p^3+p^2}(\epsilon(u))dx
\end{aligned}
$$

We get:

$$\begin{aligned}
du &= d(x^{-p}y^2w) \\
&= x^py^{-2}dw - pwx^{-p-1}y^{-2}dx - 2wx^{-p}y^{-3}dy \\
&= u^{-p(-p^2)-2p(p-1)-(p^4-p^3-p^2)}dx - 2u^{-d+p^3+3p(p-1)}dy \\
&= u^{-p^4+2p^3-p^2+2p}dx \\
dx &= u^{p^4-2p^3+p^2-2p}du
\end{aligned}$$

$\square$

Lemma 4.4.2. *A basis of holomorphic differentials of the minimal genus Artin-Schreier-Witt curve for C_{p^2} totally wildly ramified at one point is*

$$\{au^{-(p^2i+p(p-1)j+dk)+(p^4-2p^3+p^2-2p)}du + \textit{ higher order terms}|i,j,k \geq 0,\ \ j,k \leq p-1\},$$

where the exponent of u must be greater than 0.

4.5. Prequel to Proof of h-splitting

The Dieudonné module of a formal group associated to a variety X can be described as it integral cristalline cohomology $H^1_{cris}(X,k)$, which we will henceforth call H^1_{cris}, this is a $W(k)$-module. This module is flat which means it can be described fiberwise – that is, if we can understand it mod p and over its generic fiber $H^1_{cris}[\frac{1}{p}]$, and $H^1_{cris}/p \simeq H^1_{dR}(X,k)$. In Section 4.6 we pin down the Dieudonné module mod p, and in Section 4.7 we pin it down over its generic fiber. Finally, we put this together in Section 4.8 to understand H^1_{cris} integrally.

4.6. Method of Splitting Dieudonné Module of Jacobian using Idempotent Splitting of Holomorphic Differentials

We fix the following notation for the rest of this subsection. Let $\mathcal{J}$ be the Jacobian of a curve X with affine form $y^p - y =$ over $R \in \mathsf{Art}_k$, and let $\mathcal{J}_e^\wedge$ be its formal group.

Theorem 4.6.1. *The formal group $\mathcal{J}_e^\wedge$ splits into $p-2$ summands of dimensions $1, 2, ..., p-1$ respectively.*

Proof.

- Let ζ be a $(p-1)$st root of unity. Then, we have $f \in \mathrm{Aut}(X)$ such that

$$f : [x : y : z] \mapsto [\zeta x : y : z].$$

The Abel-Jacobi map $\int_\infty : X \to \mathcal{J}$ is constructed with respect to $\infty := [1 : 0 : 0]$, thus the identity section e of $\mathcal{J}$ corresponds to the image of ∞. Since $\mathrm{Aut}(X) \hookrightarrow \mathrm{Aut}(\mathcal{J})$, it is further the case that

$$\mathbb{Z}/p - 1 \subset \mathrm{Stab}_\infty(\mathrm{Aut}(X) \hookrightarrow \mathrm{Stab}_e(\mathrm{Aut}(\mathcal{J})) \hookrightarrow \mathrm{Aut}(\mathcal{J}_e^\wedge).$$

Thus, $\mathbb{Z}/(p-1) \subseteq \mathrm{Aut}(\mathcal{J}_e^\wedge)$.

- Using Lemma 2.0.1, we have an injective map from

$$\bigoplus_i e_i \mathbb{Z}_p[G] = \mathbb{Z}_p[G] \hookrightarrow \mathrm{End}(\mathcal{J}_e^\wedge)$$

where e_i are the idempotents induced by π_χ. This implies that

$$\mathcal{J}_e^\wedge = \bigoplus_i e_i \mathcal{J}_e^\wedge.$$

Let $T_e^*(\mathcal{J})$ be the cotangent space of $\mathcal{J}$, for the same reason, we have

$$T_e^*(\mathcal{J}) = \bigoplus_i e_i T_e^*(\mathcal{J}).$$

- By Lemma 4.6.2, $\bigoplus e_i T_e^*(\mathcal{J})$ is $p-2$ summands of dimension $1, 2, ..., p-2$ respectively.
- It remains to show that

$$\dim e_i T_e^*(\mathcal{J}) = \dim e_i \mathcal{J}_e^\wedge.$$

The image of e_i on the tangent space contains the tangent space of the image of e_i on the formal group, that is

$$T_e(e_i \mathcal{J}_e^\wedge) \subseteq e_i T_e^*(\mathcal{J}).$$

Thus, there is an inequality between dimensions. However, they sum up to an equality for varying i, hence they are all, in fact, equalities.

□

Lemma 4.6.2. *The cotangent space of the Jacobian $\mathcal{J}$ splits into $\bigoplus_i e_i T_e^*(\mathcal{J})$, which is $p-2$ summands of dimension $1, 2, ..., p-2$ respectively.*

Proof. By the Grothendieck-Serre duality of curves, for any curve X,

$$T_e^*(\mathcal{J}) \simeq H^1(X, \mathcal{O}_X) \simeq H^0(X, \Omega_X^1)^\wedge.$$

Let us now examine the action of $\mathbb{Z}/(p-1) \in \mathrm{Aut}(X)$ on $H^0(X, \Omega_X^1$, we call f the generator of $\mathbb{Z}/(p-1)$.

By Lemma 4.3.1 and 4.3.3, for our degree d curve X, we may write a basis of $T_e^*(\mathcal{J})$ as follows

$$\{x^i y^j dx \colon 0 \leq i + j \leq d - 3\}$$

or equivalently

$$\{z^{-(pi+(p-1)j)+p(p-3)}dz + O(z^2) | i, j > 0\}.$$

Recall that

$$f([x : y : z]) = [\zeta x : y : z],$$

where ζ is a $(p-1)$st root of unity. (Truly, it acts by $f([x : y : z]) = [\alpha^p x : \alpha^{p-1} y : z]$, where α is a $(p-1)^2$st root of unity. But we can divide out, to get $[\alpha^{p-(p-1)} x : y : \alpha^{-(p-1)} z]$, thus looking at is as only an action on x, then α^{-1} is a $(p-1)^2$st-root of unity again.) Note that since this action depends only on x, thus on the value of i and not on j, the map f induces following partition.

There are $d-2$ pairs such that $i = 0$, i.e., $(0,1), (0,2), ..., (0, d-2)$. Further, there are $d-3$ pairs such that $i = 1$, and in general, $(d-2)-k$ pairs such that $i = k$. Ending with 1 pair such that $d-3$ is the smallest, i.e., $(d-3, d-2)$.

The differential $\omega_{i,j} := x^i y^j dx$ corresponding to (i, j) is acted on by

$$f \colon \omega_{i,j} \mapsto \zeta^i \omega_{i,j}.$$

Equivalently, since $T = \frac{y}{x}$ and $\mathbb{Z}/(p-1)^2$ acts by $(x, y) \mapsto (\alpha^p x, \alpha^{(p-1)})$ where α is $(p-1)^2$ root of unity. Thus, $T \mapsto \alpha^{-1} dT$, and $TdT \mapsto \alpha^{-2} TdT$.

Thus,

$$e_j T_e^* \mathcal{J} = \langle \zeta^i \rangle T_e^*(\mathcal{J}).$$

$\square$

4.7. Isogeny Decomposition Using Eigenvalues of Frobenius

We now describe its slope decomposition, which is a description of H^1_{cris} up to isogeny. These are the eigenvalues of Frobenius.

Theorem 4.7.1. *(Thm 4.1 (18)) Let us consider ϕ to be a multiplicative character, and χ to be an additive character. Let $t \in \mathbb{F}_p^\times$ and let its lift to $W(k)$ be denoted $\tilde{t}$. Let us define $\phi_i(t) = \zeta^{i\tilde{t}}$ where ζ is a pth root of unity, and $\chi_j(t) = -\tilde{t}^{-j}$. The eigenvalues of Frobenius on the curve $X : y^p - y = x^{p-1}$ are sums of the following form.*

$$\tau(\phi_i, \chi_j) = \sum_{t \in \mathbb{F}_p^\times} \phi_i(t)\chi_j(t).$$

where $1 \leq i \leq p-1$ and $1 \leq j \leq p-2$.

Corollary 4.7.2. *The slopes of the p-divisible group associated to the curve C are $\{1/(p-1), 2/(p-1), ..., (p-2)/(p-1)\}$.*

Proof. We use the eigenvalues above. The key observation of Stickelberger is that for $\lambda = 1 - \zeta$,

$$\tau(\phi_i, \chi_j) = -j^{-1}\lambda^j \mod \lambda^{j+1}.$$

Since $v_p(\lambda) = 1/(p-1)$, this means that $v_p(\tau(\phi_i, \chi_j)) = \frac{j}{p-1}$. So we get $(p-1)$ copies of each $1 \leq j \leq (p-2)$.

Each of these eigenvalues has multiplicity $p-1$. □

We now show how to go from this slope decomposition up to isogeny to an integral decomposition.

4.8. Integral Decomposotion of Dieudonné module

We can rephrase h-splitting of the Artin-Schreier curve in Theorem 4.2.1 as follows:

Theorem 4.8.1. *Let X be the projectivization of $y^p - y = x^{p-1}$. The p-divisible group of* $\mathrm{Jac}(X)$ *splits off a 1-d piece of height $p-1$ and dimension 1.*

Remark. The phrasing 'slope a appears in an integral Dieudonné module D' means that in the Dieudonné-Manin decomposition of the rational module $D[1/p]$ there is at least one summand of the form $K[F]/(F^r - p^s)$ where $a = s/r$ is written in simplest form. Since the slope $1/(p-1)$ has to appear in H^1_{cris}, it appears in one of these summands $(H^1_{cris})^{\chi^j}$.

Proof. Let χ be a multiplicative character of $\mathbb{Z}/(p-1)$. We use theorem 4.6.2, which tells us that $H^0(X, \Omega^1_X) \simeq \oplus_{j \in \mathbb{Z}/(p-1)} H^0(X, \Omega^1_X)^{\chi^j}$ breaks up into all 1-d components, where the components acted on by χ^j are j-dimensional.

Further, by Serre duality,

$$H^0(X, \Omega^1_X) \simeq H^1(X, \mathcal{O}_X)^\wedge.$$

Because of the dual in this isomorphism, the summand acted on by χ^j is sent to the summand χ^{-j}, this is cruicial for what follows. In particular, the summand of $H^1(X, \mathcal{O}_X)^{\chi^j}$ is $(p-1) - j$-dimensional.

$$H^0(X, \Omega^1_X)^{\chi^j} \to H^1(X, \mathcal{O}_X)^{\chi^{-j}}$$

$$H^0(X, \Omega^1_X)^{\chi^{-j}} \to H^1(X, \mathcal{O}_X)^{\chi^j}$$

Since $H^1_{dR}(X)$ is an extension of $H^1(X, \mathcal{O}_X)$ by $H^0(X, \Omega^1_X)$ as a $\mathbb{Z}/p-1$-representation, the summand $H^1_{dR}(X)^{\chi^j}$ has dimension $j + p - 1 - j = p - 1$ for every $1 \le j \le p-2$.

$$H^0(X, \Omega^1_X) \to H^1_{dR}(X) \to H^1(X, \mathcal{O}_X).$$

It follows that in the decomposition $H^1_{cris} = \bigoplus (H^1_{cris})^{\chi^j}$ every summand has rank $p-1$ as a W(k)-module. By Manin's slope formulas (Lemma 4.7.1), slope 1/(p-1) appears at least once in H^1_{cris}. Therefore it appears in some $(H^1_{cris})^{\chi^j}$, which is to say that $(H^1_{cris})^{\chi^j}[1/p]$ has $D_{1/(p-1)}$ as a direct summand.

Since the dimension of $(H^1_{cris})^{\chi^j}[1/p]$ is $(p-1)$ and not larger, we get that $D_{1/(p-1)}$ is actually all of $(H^1_{cris})^{\chi^j}[1/p]$. So $(H^1_{cris})^{\chi^j}$ is the integral Dieudonné module of some p-divisible group, and the sum of all slopes appearing is $(p-1) * 1/(p-1) = 1$ so the dimension of this p-divisible group is 1. □

Corollary 4.8.2. *Let χ be a multiplicative character of $\mathbb{Z}/(p-1)$. In the decomposition $H^0(X, \Omega^1_X) \simeq \oplus_{j \in \mathbb{Z}/(p-1)} H^0(X, \Omega^1_X)^{\chi^j}$, the Dieudonné module of this p-divisible group is the summand corresponding to $j = 1$.*

Proof. We can conclude that $j = 1$, from the proof of Lemma 4.6.2. This is because a 1-dimensional p-divisible group has to have a 1-dimensional Lie algebra, and its Lie algebra is dual to F^1_{Hodge} on $(H^1_{cris})^{\chi^j}/p = (H^1_{dR})^{\chi^j}$ but by Lemma 4.6.2 this F^1_{Hodge} has dimension j. □

Example 4.8.3. *Let us consider the Artin-Schreier curve with affine form $x^4 = y^5 - y$ over $\mathbb{F}_{5^4}$. The collection of holomorphic differentials are then: This breaks up as a $C_5 \rtimes C_4$-rep into 3 indecomposable representations whose basis are as follows.*

$$R_1 := dx \quad ydx \quad y^2dx$$

$$R_2 := xdx \quad xydx$$

$$R_3 := x^2dx$$

In terms of the coordinate z, these blocks of basis are

$$R_1 := z^{10}dz \quad z^6dz \quad z^2dz$$

$$R_2 := z^5dz \quad zdz$$

$$R_3 := dz$$

We are able to explicitly see how the idempotent components of the pairing of holomorphic differentials and the slope decompositions relate. The slopes of the curve, according to Lemma 4.7.1 are $1/4, 2/4, 3/4$, which becomes $G_{1/4} \times 2G_{1/2} \times G_{3/4}$. We further know that $j/4$ is associated to the component $(H^1_{dR})^{\chi^j}$.

- *The 1-dimensional component $H^0(X, \Omega^1_X)^{\chi^1}$ spanned by (dz) maps to $(H^1_{dR})^{\chi^1}$ corresponding to slope $1/4$ which maps to the 3-d component $H^1(X, \mathcal{O}_X)^{\chi^3}$ spanned by the dual of $(z^{10}dz, z^6dz, z^2dz)$.*
- *The 2-dimensional component $H^0(X, \Omega^1_X)^{\chi^2}$ spanned by (z^5dz, zdz) maps to $(H^1_{dR})^{\chi^2}$ corresponding to slope $2/4$ which maps to the 2-d component $H^1(X, \mathcal{O}_X)^{\chi^2}$ spanned by the dual of (z^5dz, zdz).*
- *The 3-dimensional component $H^0(X, \Omega^1_X)^{\chi^3}$ spanned by $(z^{10}dz, z^6dz, z^2dz)$ maps to $(H^1_{dR})^{\chi^3}$ corresponding to slope $3/4$ which maps to the 1-d component $H^1(X, \mathcal{O}_X)^{\chi^1}$ spanned by the dual of (dz).*

4.9. Examples of Newton Slopes from Frobenius Eigenvalues

We provide some examples supporting the following conjecture.

Conjecture 4.9.1. *Given a Harbater-Katz-Gabber curve X with automorphism group $\mathbb{Z}/p^k \rtimes \mathbb{Z}/(p-1)^2$, and considering the map p_y, the Jacobian of X has deformed h-splitting,*

We have the following supportive examples:

Example 4.9.2.

- *$y^3 - y = 1/x^2, w^3 - w = -x^2y^2 - x^4y - y^5 - y^7$ is an ASW curve with genus 29, with slope decomposition $6G_0 \times G_{1/6} \times 2G_{1/3} \times 11G_{1/2} \times 2G_{2/3} \times G_{5/6} \times 6G_1$*
- *$y^3 - y = 1/x^2, w^3 - w = -x^2y^2 - x^4y$ is the minimal genus ASW curve – genus 16, with slope decomposition $G_{1/6} \times 2G_{1/3} \times 4G_{1/2} \times 2G_{2/3} \times G_{5/6}$*

These were obtained using Magma's LPolynomial function. The code is available at `https://github.com/catherineray/newton`. The inverting of x in the example is only in order for the naive homogenization to not require us to remove the line $[y : z]$ to make our homogenized curve irreducible, it isn't playing a major role.

Another conjecture supporting the above conjecture is the following.

Conjecture 4.9.3. *The Jacobians of Harbater-Katz-Gabber curves with automorphism groups $\mathbb{Z}/p^k \rtimes \mathbb{Z}/m$ have complex multiplication with CM field $\mathbb{Q}(\zeta_m)$.*

Combining this with a theorem of Oort relating CM-extensions to heights of sloped components, we get that in general h-splitting should exist.

We also here address an example of why the strategy above does not immediately work for $\mathbb{Z}/p^2$-Galois extensions, as smoothly as it did for $\mathbb{Z}/p$-extensions.

Let us consider the $p = 3$ example in more detail to demonstrate the issue of splitting the basis of holomorphic differentials naively. We consider the curve X whose affine component A is defined by the pair of equations

$$y^3 - y = x^2, \qquad w^3 - w = -x^2y^2 - x^4y$$

Lemma 4.9.4. *We have an action on the curve X by C_4 generated by the map*

$$\begin{pmatrix} x \\ y \\ w \end{pmatrix} \mapsto \begin{pmatrix} i^3 x \\ i^2 y \\ -w \end{pmatrix} = \begin{pmatrix} -ix \\ -y \\ -w \end{pmatrix}.$$

Proof. We know this on the bottom curve $y^3 - y = x^2$, it thus suffices to show it preserves the equation $w^3 - w = -x^2y^2 - x^4y$.

The action sends

$$-x^2y^2 \mapsto -(-ix)^2(-y)^2 = -(i^{10}x^2y^2) = x^2y^2$$

$$-x^4y \mapsto -(-ix)^4(-y) = -(i^{14}x^4y) = x^4y$$

$$-x^2y^2 - x^4y \mapsto x^2y^2 + x^4y.$$

It then follows that the action on w by (-1) preserves the equation. $\square$

We enumerate the basis of this curve using 4.4.2.

i	j	k	30-(9i+6j+14k)	action
0	0	0	30	31
0	0	1	16	17
0	0	2	2	3
0	1	0	24	25
0	1	1	10	11
0	2	0	18	19
0	2	1	4	5
1	0	0	21	22
1	0	1	7	8
1	1	0	15	16
1	1	1	1	2
1	2	0	9	10
2	0	0	12	13
2	1	0	6	7
2	2	0	0	1
3	0	0	3	4

The action of $\mathbb{Z}/4$ on the holomorphic differentials give us the following decomposition, which is insufficient to isolate a one-dimensional piece. We suspect that utilizing higher Hasse-Witt matrices will in fact isolate the one-dimensional piece integrally.

χ^j	$H^0(X, \Omega^1_X)^{\chi^j}$
χ^0	$u^3du, u^7du, u^{15}du$
χ^1	$du, u^4du, u^{12}du, u^{16}du, u^{24}du$
χ^2	$udu, u^9du, u^{21}du$
χ^3	$u^2du, u^6du, u^{10}du, u^{18}du, u^{30}du$

CHAPTER 5

Local Construction: Formal Moduli

This section introduces the moduli stack on which we will apply the structural theorem, shows it is representable, and pins down the action of the group G.

Definition 5.0.1. *We use the definition 1.1.3, to consider the moduli space of deformations of the map $f := p_y$ from the Artin-Schreier We refer to this moduli stack as:* $\mathrm{Def}_{f:X\to\mathbb{P}^1}$ *or* Def_f.

Remark. This is a local neighborhood on an integral Hurwitz stack of genus g curves with a map to $\mathbb{P}^1$ with ramification datum $D = (p-1)(\sum_i^{p-1}[0 : y_i : 1] + [0 : 1 : 0])$. See section .

Theorem 5.0.2. *The moduli functor* $\mathrm{Def}_{p_y:X\to\mathbb{P}^1}$ *models G, and has deformed h-splitting.*

Proof. By Theorem 2.6.1 (29), the automorphism group of the point X lifts automatically to automorphisms of deformations of X (or deformations of maps from X). This implies that modelling is automatically satisfied for moduli stacks which are deformation stacks.

As for showing that the moduli functor has deformed h-splitting, this is shown by combining Theorem 4.8.1 and Theorem 8.0.3. □

We can understand the action of G this deformation stack, described in lemma 5.0.6, which will buy us something enormous. We will now discuss and establish the process of pinning this action down in more generality.

The main observation is that the formal moduli scheme of deformations of a totally tamely ramified morphism $f : X \to \mathbb{P}^1$ can be identified with the formal moduli scheme of deformations of the branch divisor of f in $\mathbb{P}^1$:

Lemma 5.0.3. *Let $f : X \to \mathbb{P}^1$ be a finite morphism which is totally tamely ramified, let $R = \sum_i n_i[y_i]$ be the ramification divisor of f, and let $D = f(R)$ be the branch divisor of f. Then we have an isomorphism of formal schemes*

$$\mathrm{Def}_{f:X\to\mathbb{P}^1} \simeq \mathrm{Def}_{D,\mathbb{P}^1}$$

Proof. Since finite morphisms $f : X \to \mathbb{P}^1$ with tame ramification are uniquely determined by their ramification divisors we have an isomorphism of formal schemes $\mathrm{Def}_{f:X\to\mathbb{P}^1} \simeq \mathrm{Def}_{R,X}$. Under the assumption that $f : X \to \mathbb{P}^1$ is totally tamely ramified of degree d, for $D = \sum_i [x_i]$ we have $R = \sum_i d[y_i]$ where y_i is the unique point above x_i. But then formal deformations of R in X correspond bijectively to formal deformations of D in $\mathbb{P}^1$ and it follows that we have an isomorphism of formal schemes $\mathrm{Def}_{R,X} \simeq \mathrm{Def}_{D,\mathbb{P}^1}$. □

Lemma 5.0.4. *Let $D = \sum_{1\leq i\leq n} n_i[x_i]$ be an effective divisor on $\mathbb{P}^1$. Let $\mathrm{Def}_{D,\mathbb{P}^1}$ be the formal moduli scheme of deformations of the divisor D in $\mathbb{P}^1$. Then we have an isomorphism of formal schemes*

$$\mathrm{Def}_{D,\mathbb{P}^1} \simeq \prod_{1\leq i\leq n} \mathrm{Def}_{n_i\{x_i\},\mathbb{P}^1}$$

Proof. We use the following result of Kodaira: for $Z \hookrightarrow X$ a smooth embedding with normal bundle $N_{Z/X}$ such that $H^1(X, N_{Z/X}) = 0$ (and such that $\mathrm{Ext}^2_X(\Omega^1_X, \mathcal{O}_X) = 0$ and $\mathrm{Ext}^2_Z(\Omega^1_Z, \mathcal{O}_Z) = 0$) then $\mathrm{Def}_{Z,X}$ is unobstructed and the canonical morphism $\mathrm{Def}_{Z,X} \to \mathrm{Def}_X$ is smooth. For $X = \mathbb{P}^1$ and $Z = D$ as above these conditions are satisfied so the canonical morphism $\mathrm{Def}_{D,\mathbb{P}^1} \to \mathrm{Def}_{\mathbb{P}^1} = \mathrm{Spf}(W(k))$ is smooth, hence $\mathrm{Def}_{D,\mathbb{P}^1} \simeq \mathrm{Spf}(W(k)[[t_1, \ldots, t_n]])$ where $n = \dim(H^0(\mathbb{P}^1, N_{D/\mathbb{P}^1}))$. It follows that we have an isomorphism of formal schemes

$$\mathrm{Def}_{D,\mathbb{P}^1} \simeq \mathrm{Spf}(W(k)[[t_1, \ldots, t_n]]) \qquad \square$$

Remark. It is helpful to think about the above lemma in the following way. For $D = \sum_{1\leq i\leq n} n_i[x_i]$ be an effective divisor on $\mathbb{P}^1$ as above, and let $\mathrm{Def}_{n_i[x_i]\},\mathbb{P}^1}$ be the formal moduli scheme of deformations of a point x_i in $\mathbb{P}^1$. Then we have an isomorphism of formal schemes

$$\mathrm{Def}_{D,\mathbb{P}^1} \simeq \prod_{1\leq i\leq n} \mathrm{Def}_{n\{x_i\},\mathbb{P}^1}$$

since for $X = \mathbb{P}^1$ and $Z = \{x_i\}$ the canonical morphism $\mathrm{Def}_{\{x_i\},\mathbb{P}^1} \to \mathrm{Def}_{\mathbb{P}^1} = \mathrm{Spf}(W(k))$ is smooth, hence $\mathrm{Def}_{\{x_i\},\mathbb{P}^1} \simeq \mathrm{Spf}(W(k)[[t_i]])$.

In other words to understand infinitesimal deformations of a divisor in $\mathbb{P}^1$ it is enough to consider infinitesimal deformations of each point separately. Intuitively, one cannot collide separate points in a divisor by applying an infinitesimal deformation.

Lemma 5.0.5. *Let $D = \sum n_i[x_i]$ and $B = \sum[x_i]$ There is a closed inclusion of formal schemes*

$$\mathrm{Def}_{B,\mathbb{P}^1} \hookrightarrow \mathrm{Def}_{D,\mathbb{P}^1}$$

Proof. On the right hand side, we allow for deformations of a given fixed $[x_i]$ with coefficient n to split into $x_i + \epsilon_j$ for $1 \leq j \leq n$. By removing the coefficients in front of x_i we are only allowing for deformation in one infinitesimal direction, that is, $[x_i]$ can only split into $x_i + \epsilon_i$. $\square$

Similarly let $\mathcal{D}ef_{D,\mathbb{P}^1}$ be the formal moduli stack of deformations of the divisor D in $\mathbb{P}^1$ and let $\mathcal{D}ef_{\{x_i\},\mathbb{P}^1}$ be the formal moduli stack of deformations of a point x_i in $\mathbb{P}^1$. Then we have an isomorphism of formal stacks

$$\mathcal{D}ef_{D,\mathbb{P}^1} \simeq \mathrm{Def}_{D,\mathbb{P}^1} // G$$

where $G = \{\varphi \in \mathrm{PGL}_2 \mid \varphi(D) = D\}$ is the subgroup of $\mathrm{Aut}(\mathbb{P}^1) = \mathrm{PGL}_2$ fixing D (i.e. fixing the points of D up to permutation), and we have an isomorphism of formal stacks

$$\mathcal{D}ef_{\{x_i\},\mathbb{P}^1} \simeq \mathrm{Def}_{\{x_i\},\mathbb{P}^1} // \mathbb{G}_m \rtimes \mathbb{G}_a$$

where $\mathbb{G}_m \rtimes \mathbb{G}_a$ is the subgroup of $\mathrm{Aut}(\mathbb{P}^1) = \mathrm{PGL}_2$ fixing x_i. In particular, the formal stacks $\mathcal{D}ef_{D,\mathbb{P}^1}$ and $\prod_{1 \leq i \leq n} \mathcal{D}ef_{\{x_i\},\mathbb{P}^1}$ are not isomorphic unless $n = 1$: for instance $\dim(\mathcal{D}ef_{D,\mathbb{P}^1}) = n - \dim(G)$ whereas $\dim(\prod_{1 \leq i \leq n} \mathcal{D}ef_{\{x_i\},\mathbb{P}^1}) = -n$.

Since $\mathrm{Aut}(\mathbb{P}^1)$ acts 3-transitively on $\mathbb{P}^1$, the group G fixing D is finite as soon as $n \geq 3$. The group G is trivial if the $n \geq 3$ points of D are in general position, but the group G may be nontrivial for certain special configurations of points, as is the case for the Artin-Schreier curve. This immediately gives us the following.

Lemma 5.0.6. *Let $n = p - 1$, for the Artin-Schreier map $f := p_2$ defined in Definition , $\mathcal{D}ef_f$ is represented by*

$$\mathrm{Spf}\, W(k)[[t_1, ..., t_n]] // \mathbb{G}_m \rtimes \mathbb{G}_a \simeq \mathrm{Spf}(\mathrm{Sym}(\overline{\rho})[\Delta^{-1}])^{\wedge}_{I})_0$$

where the ideal I is the special fiber of X $(p, t_1 - 0, ..., t_n - (n-1))$, where $\overline{\rho} := \ker k[Z/p] \to k$ is the reduced regular representation, Delta $= t_1 \cdots t_n$, and the subscript denotes the 0th graded component.

Remark. Considering $H^0(C, N_f)$ as a G rep, we see the following. The generator σ in $\mathbb{Z}/p$ acts on X by sending $y_\infty \mapsto y_\infty$, and $y_i \mapsto y_{i+1}$ for all others, where the subscript is considered mod p. Then, the generator τ in $\mathbb{Z}/(p-1)$ acts by sending $y_i \mapsto y_{\zeta i}$, where ζ is a $(p-1)$st root of unity.

Let $G \simeq \mathbb{Z}/p \rtimes \mathbb{Z}/(p-1)^2$, and let $E(h)_*$ be coefficients of Morava E-theory.

Theorem 5.0.7. *As G-representations,*

$$(E(h)_*)^{\mathbb{Z}/(p-1)} \simeq \operatorname{Sym} \overline{\rho}_I^\wedge,$$

where I is the maximal ideal corresponding to the point X on Def_f. *Further,*

$$H^*(G, \Lambda)_I^\wedge \simeq H^*(G, E_*)^{\mathbb{Z}/(p-1)}.$$

Proof. By Theorem 5.0.2, Def_f satisfies the conditions of Theorem 2.0.3, we may thus conclude that the action of G on the ring representing Def_f is isomorphic as a G-representation to LT, that is,

$$\mathcal{D}ef_f \simeq LT//G.$$

Further, we showed in theorem 5.0.6 that $\mathcal{D}ef_f \simeq \operatorname{Sym}(\overline{\rho})_I^\wedge$. $\square$

Theorem 5.0.8. *Let* $\Lambda := \operatorname{Sym} \overline{\rho}_I^\wedge$. *As G-representations,*

$$E(h)_* \simeq \Lambda[{t_i}^{1/(p-1)}],$$

where I is the maximal ideal corresponding to the point X on Def_f*, and t_i are the generators of* $\mathrm{Sym}\,\overline{\rho}$*. Further,*

$$H^*(G,\Lambda)^\wedge_I \simeq H^*(G,E_*)^{\mathbb{Z}/(p-1)}.$$

Further, $H^*(G,\Lambda)[\Delta^{-1}]^\wedge_I \simeq H^*(G,E_*)$.

Proof. The automorphism group being lifted to $\mathrm{Aut}(\widehat{\mathcal{U}}_{[X]},\widehat{\mathcal{M}}_{[X]})$ requires us to lift to the ring with $(t_i)^{1/p-1}$, as seen in Section 7. This is then required if we are to extend the relationship from LT to LT_*. $\square$

Remarkably enough, Theorem 5.0.8 suggests that the essential complexity of a $K(h)$-local sphere is ultimately discernible in a quotient of the regular representation of a cyclic group.

5.1. Artin-Schreier-Witt p^2 Local deformations

We now make a quick observation about how formal deformations of morphisms behave under composition, and then apply this observation in the case of Artin-Schreier-Witt curves for $\mathbb{Z}/p^2\mathbb{Z}$.

Recall that for $f : X \to Y$ a morphism of schemes over a field k we have the functor $\mathrm{Def}_f : \mathrm{Art}_k \to \mathrm{Grpd}$ sending $A \in \mathrm{Art}_k$ to the groupoid of pullback diagrams of the form

$$\begin{tikzcd} X \arrow[r] \arrow[d, "f"'] & \mathcal{X} \arrow[d, "F"] \\ Y \arrow[r] \arrow[d] & \mathcal{Y} \arrow[d] \\ \mathrm{Spf}\,k \arrow[r] & \mathrm{Spf}\,A \end{tikzcd}$$

For $g : Y \to Z$ another morphism of schemes we obtain a morphism of formal moduli schemes

$$\mathrm{Def}_{f:X\to Y} \times_{\mathrm{Def}_Y} \mathrm{Def}_{g:Y\to Z} \to \mathrm{Def}_{g\circ f:X\to Z}$$

given by pasting pullback diagrams when the morphism $Y \to \mathcal{Y}$ in the first is equal to the morphism $Y \to \mathcal{Y}$ in the second:

$$\left(\begin{array}{ccc} X & \longrightarrow & \mathcal{X} \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle F} \\ Y & \longrightarrow & \mathcal{Y} \\ \downarrow & & \downarrow \\ \mathrm{Spf}\, k & \longrightarrow & \mathrm{Spf}\, A \end{array}\right. , \quad \left.\begin{array}{ccc} Y & \longrightarrow & \mathcal{Y} \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle G} \\ Z & \longrightarrow & \mathcal{Z} \\ \downarrow & & \downarrow \\ \mathrm{Spf}\, k & \longrightarrow & \mathrm{Spf}\, A \end{array}\right) \mapsto \left(\begin{array}{ccc} X & \longrightarrow & \mathcal{X} \\ {\scriptstyle g\circ f}\downarrow & & \downarrow{\scriptstyle G\circ F} \\ Z & \longrightarrow & \mathcal{Z} \\ \downarrow & & \downarrow \\ \mathrm{Spf}\, k & \longrightarrow & \mathrm{Spf}\, A \end{array}\right)$$

The essential image of the morphism $\mathrm{Def}_{f:X\to Y}(A) \times_{\mathrm{Def}_Y} \mathrm{Def}_{g:Y\to Z}(A) \to \mathrm{Def}_{g\circ f:X\to Z}(A)$ consists of those pullback diagrams in $\mathrm{Def}_{g\circ f:X\to Z}(A)$ such that the deformed morphism $H : \mathcal{X} \to \mathcal{Z}$ factors as $G \circ F : \mathcal{X} \to \mathcal{Y} \to \mathcal{Z}$ with $F \in \mathrm{Def}_{f:X\to Y}(A)$ and $G \in \mathrm{Def}_{g:Y\to Z}(A)$ yielding the following pullback diagram:

$$\begin{tikzcd} X \arrow[r] \arrow[d, "f"'] & \mathcal{X} \arrow[d, dashed, "F"] \arrow[dd, bend left, "H"] \\ Y \arrow[r, dashed] \arrow[d, "g"'] & \mathcal{Y} \arrow[d, dashed, "G"] \\ Z \arrow[r] \arrow[d] & \mathcal{Z} \arrow[d] \\ \mathrm{Spf}\, k \arrow[r] & \mathrm{Spf}\, A \end{tikzcd}$$

Similarly we obtain a morphism of formal moduli stacks

$$\mathcal{D}ef_{f:X\to Y} \times_{\mathcal{D}ef_Y} \mathcal{D}ef_{g:Y\to Z} \to \mathcal{D}ef_{g\circ f:X\to Z}$$

given by pasting pullback diagrams, with the same characterization of the essential image. This is all to say that to deform a morphism $g \circ f : X \to Z$ it is enough to deform each morphism $f : X \to Y$ and $g : Y \to Z$ separately, up to a compatiblity condition which allows the pullback diagrams to be pasted.

5.2. Tangent Spaces of Deformation Moduli Problems

5.2

We now recall some facts we will use about tangent spaces of formal moduli schemes, which amounts to understanding first-order deformations rather than formal deformations. This will help give more context to some of the subtleties of the previous section.

We may consider the short exact sequence

$$0 \to T_X \to f^*T_Y \to N_f \to 0,$$

The long exact sequence related to this has the coboundary $\delta : H^0(N_f) \to H^1(T_X)$ which takes a deformation of the map f to the corresponding definition of X, forgetting the map; the kernel consists of deformations of f fixing both X and Y, modulo automorphisms of X.

Lemma 5.2.1. *For $f : X \to \mathbb{P}^1$ where X is genus 2 or higher,*

$$0 \longrightarrow H^0(X, f^*T_{\mathbb{P}^1}) \longrightarrow H^0(X, N_f) \xrightarrow{\delta} H^1(X, T_X) \longrightarrow H^1(X, f^*T_{\mathbb{P}^1}) \longrightarrow 0$$

Remark. We may think of this lemma as telling us the following. Given any cover $f : X \to \mathbb{P}^1$, this cover is determined by (1) The image of the ramification ideal of f in $\mathbb{P}^1$ and (2) An element of the global sections of N_f.

Proof. This can be shown as a consequence of the long exact sequence.

$$\begin{tikzcd}
0 \arrow[r] & H^0(X, T_X) \arrow[r] & H^0(X, f^*T_{\mathbb{P}^1}) \arrow[r] & H^0(X, N_f) \arrow[dll, "\delta"] \\
& H^1(X, T_X) \arrow[r] & H^1(X, f^*T_{\mathbb{P}^1}) \arrow[r] & H^1(X, N_f) \arrow[r] & 0
\end{tikzcd}$$

For genus greater than or equal to 2, $H^0(X, T_X)$ is 0, since $2-2g$ is the degree of T_X. Further, $H^0(X, N_f)$ is related to the ramification points, and $H^1(X, T_X)$ is the infinitesimal deformations of the curve. Higher cohomologies vanish, so we are left with.

$$0 \longrightarrow H^0(X, f^*T_{\mathbb{P}^1}) \longrightarrow H^0(X, N_f) \xrightarrow{\delta} H^1(X, T_X) \longrightarrow H^1(X, f^*T_{\mathbb{P}^1}) \longrightarrow 0$$

$\square$

Definition 5.2.2. *Given a map $f : X \to Y$, denote the normal sheaf of the map f as*

$$N_f = coker(df : T_X \to f^*T_Y)$$

Example 5.2.3. *If f is the inclusion of X as a subvariety of Y, then we get the usual definition of $N_f = T_Y|_X/T_X = (\mathscr{I}_{X/Y}/\mathscr{I}^2_{X/Y})^\wedge$.*

Theorem 5.2.4 (Harris-Morrison Deformation Theory)**.** *If Y is nonsingular, than the tangent space to the deformation functor $f : X \to Y$ keeping only Y fixed is*

$$H^0(X, N_f).$$

We consider the Artin-Schreier-Witt curves and maps p_y defined in 4.2.1.

Lemma 5.2.5. *The tangent space of $\mathrm{Def}_{p_y:X_1\to\mathbb{P}^1_k}$ is isomorphic to the following. Let y_i be the uniformizer of the sheaf at the point i.*

$$H^0(X_1, (N_{p_y})) = \prod_{y_i \in \{0,\ldots,p-1,\infty\}} k[[y_i]]/(y_i)^{p-1}.$$

Proof. The sheaf N_f is a skyscraper sheaf with length the degree of ramification at each ramification point of the map f. We exposit that here.

Let $f : X \to Y$ be a finite morphism of smooth curves. It is useful to have in mind the exact sequence

$$0 \to f^*\Omega_Y \to \Omega_X \to \Omega_{X|Y} \to 0.$$

Note that $\Omega_{X/Y}$ is a torsion sheaf since the two other sheaves are locally free of the same rank (they are line bundles on X). At a point $q \in Y$ and $p \in X$ in the preimage of q, let dx denote a generator for $\Omega_{Y,q}$ as a $O_{Y,q}$-module. Now, $(\Omega_{X|Y})_P = 0$ if and only if f^*dx is a generator of $\Omega_{X,p}$, which happens if and only if f pulls back a local parameter to a local parameter, that is p is unramified.

Moreover, the exact sequence above shows that the ramification index is exactly the length of the sheaf $\Omega_{X/Y}$. □

Lemma 5.2.6. *Given a ramification divisor $B = \sum b_i := \sum n_i p_i$, the map of*

$$\mathrm{Def}_{b_i,\mathbb{P}^1} \to \mathrm{Def}_{p_i,\mathbb{P}^1}$$

induces a map on tangent spaces forgetting the length of the ramification:

$$k[y_i]/(y_i)^{n-2} \mapsto k[y_i]/(y_i)^2.$$

This is worth noting when attempting to compare deformation problems which keep track of the degree of the divisor, and those which do not.

Remark. Considering $H^0(X, N_{p_y})$ as a G rep, we see the following. The generator σ in $\mathbb{Z}/p$ acts on X by sending $y_\infty \mapsto y_\infty$, and $y_i \mapsto y_{i+1}$ for all others, where the subscript is considered mod p. Then, the generator τ in $\mathbb{Z}/(p-1)$ acts by sending $y_i \mapsto y_{\zeta i}$, where ζ is a $(p-1)$st root of unity.

Lemma 5.2.7. *The tangent stack of* $\mathrm{Def}_{p_y : X_2 \to \mathbb{P}^1}$ *is the following.*

$$H^0(X_2, (N_{p_y})) = \prod_{y_i \in \{0,\ldots,p-1} k[[y_i]]/(y_i)^{p-1} \oplus k[[y_\infty]]/(y_\infty)^{M(p-1)}.$$

Where $M := p^4 - 2p^3 - 2p^2 + 3p$.

Proof. We again rewrite dt in terms of du. By Lemma 5.2.8, we found that $dt = u^{p^4-2p^3-p^2+3p}du$, in other words,

$$dt = u^N du.$$

We may trace the uniformizer through the different maps. They are multiplicative and therefore we may divide them.

$$\begin{tikzcd}
X_2 \arrow[d, "M=N-p(p-1)"'] \arrow[dd, bend left, "N=p^4-2p^3-p^2+2p"] \\
X_1 \arrow[d, "p(p-1)"'] \\
\mathbb{P}^1
\end{tikzcd}$$

Thus the uniformizer above infinity $N = M - p(p-1) = (p^4 - 2p^3 - p^2 + 2p) - p(p-1) = p^4 - 2p^3 - 2p^2 + 3p$. $\square$

Lemma 5.2.8. *Let u be a uniformizer at infinity of the Artin-Schreier-Witt curve for $\mathbb{Z}/p^2$ totally wildly ramified at one point, and let t be the uniformizer of $\mathbb{P}^1$ at infinity. Then,*

$$dt = u^{p^4-2p^3-p^2+3p}du$$

Proof. The first step is to express dt in terms of dw. We do so as follows. The dominating term of $w^p - w$ is is $x^{(p-1)^2}y$. Due to a previous calculation, we know that:

$$\begin{aligned} dw &= d(x^{(p-1)^2}y) \\ &= u^{-p^4+p^3+p^2}dx \end{aligned}$$

Thus we may proceed to

$$\begin{aligned} du &= d(x^p y^{-2} w) \\ &= \frac{x^{-p}}{y^2}dw + \frac{-pwx^{-p-1}}{y^2}dx + \frac{-2wx^{-p}}{y^3}dy \end{aligned}$$

The dominating term is $\frac{x^{-p}}{y^2}dw$, so we get:

$$\begin{aligned} du &= \frac{x^{-p}}{y^2}dw \\ &= u^{p^3+2p(p-1)}u^{p^4+p^3+p^2}dx \\ &= u^{-p^4+2p^3+3p^2-3p}dx \\ dx &= u^{p^4-2p^3-3p^2+3p}du \end{aligned}$$

We know that $x = t^{-1}$, and $x = u^{-p^2}$, therefore $t^-2 = u^{-2p^2}$, and $dx = u^{-2p^2}dt$. Thus we conclude:

$$\begin{aligned} du &= u^{-p^4+2p^3+3p^2-3p}dx \\ &= u^{-p^4+2p^3+p^2-2p}dt \\ dt &= u^{p^4-2p^3-p^2+2p}du \end{aligned}$$

□

Conjecture 5.2.9. *Considering $H^0(X_2, N_{p_y})$ as a $G \simeq \mathbb{Z}/p^2 \rtimes \mathbb{Z}/(p-1)^2$ representation, we see the following. The generator for $\mathbb{Z}/p^2$ comes from a combination of two compatible actions. One $\mathbb{Z}/p$ acts on each fixed $H^0(X_2, (N_{p_y})_{y_i})$ by sending y_i to a root of unity times itself, and the other $\mathbb{Z}/p$ is generated by the σ in $\mathbb{Z}/p$ acts on X by sending $y_\infty \mapsto y_\infty$, and $y_i \mapsto y_{i+1}$ for all others, where the subscript is considered mod p. These two actions must be compatible, which restraints from from a $\mathbb{Z}/p \times \mathbb{Z}/p$ action to a $\mathbb{Z}/p^2$ action. The generator of $\mathbb{Z}/(p-1)$ acts the same as in X_1 the generator τ in $\mathbb{Z}/(p-1)$ acts by sending $y_i \mapsto y_{\zeta i}$, where ζ is a $(p-1)$st root of unity.*

CHAPTER 6

Global Construction: Hurwitz Stacks

It is worth spending a moment remarking on the relevance of Hurwitz schemes as a global analogue of the deformation spaces we have studied in the previous section. To that end we quickly recall a rather general construction of Hurwitz schemes which makes sense in characteristic p, following (30) Wewers.

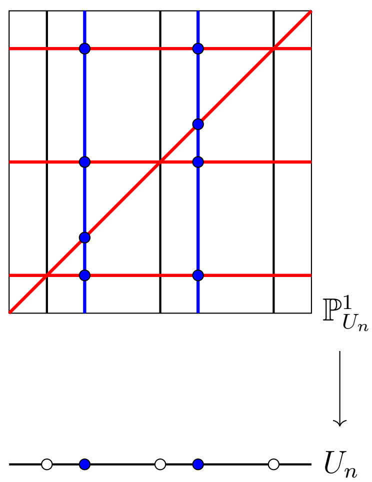
Definition 6.0.1. *Let X/S be a family of smooth projective geometrically connected curves over a stack S. We say $D \subseteq X$ is a marking divisor if it is a closed subscheme of X such that $D \to S$ is a finite étale cover of constant degree. One should think of S as some moduli space of curves and X the universal family of curves over S.*

Even if we are ultimately interested in studying branched covers of a single fiber of X/S, it is often useful to study how these branched covers deform as their branch loci deform, which can be understood by studying branched covers of X/S tamely ramified along D.

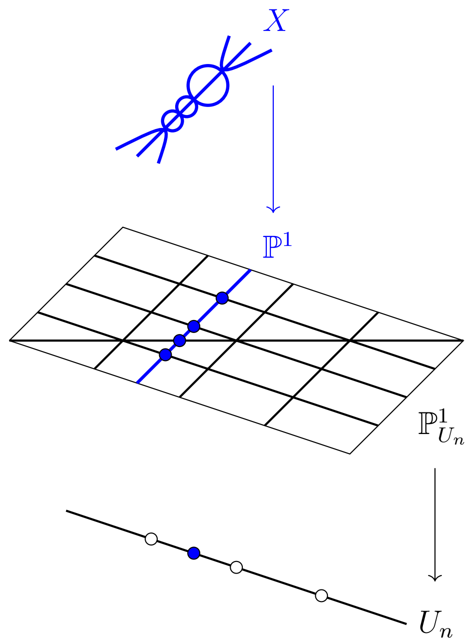
We may take $S = U_n = \mathbb{P}^n - \Delta_n$ where Δ_n is the discriminant locus and $X = \mathbb{P}^1_{U_n}$ the universal family of projective lines, then a point in U_n corresponds to n unordered marked points on $\mathbb{P}^1$ which yields a marking divisor $D \subseteq \mathbb{P}^1_{U_n}$ of degree n. This U_n is the unordered configuration space. In other words, $S = \mathcal{M}_{0,[n]}$ is the moduli of genus 0 curves with n unordered marked points and $X = \mathcal{U}_{0,[n]}$ is the universal family of curves over $\mathcal{M}_{0,[n]}$.

If we wish to study the stack of ordered points, we would consider $O_n = (\mathbb{P}^1)^n - \Delta_n$ where $\Delta_n = \bigcup_{1 \leq i \neq j \leq n} \{x_i = x_j\}$ is the fat diagonal locus.

It may be helpful to visualize the fibers in the universal family of projective lines over U_n: a point in U_n corresponds to n unordered marked points in $\mathbb{P}^1$, the fiber (in blue) over that point in $\mathbb{P}^1_{U_n}$ is the $\mathbb{P}^1$ with that configuration of points. We have also displayed the marking divisors (in red):



The Hurwitz scheme parameterizing branched covers of $\mathbb{P}^1$ naturally lives over U_n. It may be helpful to visualize the fibers: for a point in U_n corresponding to n unorderd marked points in $\mathbb{P}^1$, the fiber over that point in $\mathbb{P}^1_{U_n}$ is the $\mathbb{P}^1$ with that configuration of points, over which we can consider branched covers $X \to \mathbb{P}^1$ with branch locus that configuration of points:



The point of considering such Hurwitz schemes is that we can study branched covers $X \to \mathbb{P}^1$ not only fiber-wise with the branch loci fixed, but also in families as the branch loci are allowed to deform.

Remark. More generally, if we want to study branched covers of genus g curves we may take $S = \mathcal{M}_{g,[n]}$ the moduli of genus g curves with n unordered marked points and $X = \mathcal{U}_{g,[n]}$ the universal family of curves.

Definition 6.0.2. *Now we can define the Hurwitz stack. For an S-scheme $S' \to S$ let $\mathcal{H}_X(S')$ be the groupoid of finite covers $\rho' : Y' \to X' = X \times_S S'$ tamely ramified along $D' = D \times_S S'$.*

If S is smooth, then $\mathcal{H}_X$ is represented by an algebraic stack which is smooth and étale over S. We do not focus on this, for we will be using a slightly more elaborate stack for our purposes.

Remark. The scheme S modulates X, that is, in a fiber we study branched covers of a fixed X.

The Hurwitz stack $\mathcal{H}_X$ encodes information about covers of X/S, with restriction on ramification (only allowing for tame ramification) but without restriction on monodromy. We can control the monodromy of such covers in terms of the tame étale fundamental group as follows. For $\overline{s}$ a geometric point of S and for $\overline{x}$ a geometric point of the fiber $X_{\overline{s}}$ we have an exact sequence of tame étale fundamental groups

$$\pi_1^{D_{\overline{s}}}(X_{\overline{s}}, \overline{x}) \to \pi_1^D(X, \overline{x}) \to \pi_1(S, \overline{s}) \to 0$$

A representation $\phi : \pi_1^D(X/S, \overline{x}) \to \Sigma_n$ gives rise to a cover $\rho : Y \to X$ tamely ramified along D. Note that Y/S has connected fibers precisely if $\phi_{\overline{s}} : \pi_1^{D_{\overline{s}}}(X_{\overline{s}}, x) \to \Sigma_n$ is transitive.

Definition 6.0.3. *Let $n \geq 1$ be an integer, let $G \subseteq \Sigma_n$ be a transitive subgroup, and let $N \subseteq \Sigma_n$ be a subgroup containing G as a normal subgroup. We consider covers $\rho : Y \to X$ coming from representations $\phi : \pi_1^D(X, \overline{x}) \to \Sigma_n$ with $\mathrm{im}(\phi) \subseteq N$ and $\mathrm{im}(\phi_{\overline{s}}) = G$. A G-N-cover is a cover $\rho : Y \to X$ with choice of representation $\phi : \pi_1^D(X, \overline{x}) \to \Sigma_n$ inducing ρ, modulo conjugation by elements of N.*

Example 6.0.4. *For example, if $G \hookrightarrow \Sigma_n$ is the regular representation and $N = G$ then a G-N-cover is a Galois cover $\rho : Y \to X$ along with a choice of isomorphism $\mathrm{Aut}(Y/X) \xrightarrow{\sim} G$.*

Example 6.0.5. *As another example, if $G \hookrightarrow \Sigma_n$ is a faithful transitive representation and N is the normalizer of G in Σ_n then a G-N-cover is simply a cover $\rho : Y \to X$ with* *monodromy group G on each fiber; even if the fibers are G-Galois covers, the Galois action need not be defined over S.*

Definition 6.0.6. *Now we can define the G-N-Hurwitz scheme. For an S-scheme $S' \to S$ let $\mathcal{H}_X^N(G)(S')$ be the groupoid of G-N-covers $\rho' : Y' \to X' = X \times_S S'$ tamely ramified along $D' = D \times_S S'$. Then $\mathcal{H}_X^N(G) \to \mathcal{H}_X$ is (relatively) represented by an algebraic stack over $\mathcal{H}_X$, étale over $\mathcal{H}_X$.*

Theorem 6.0.7 ((30) Theorem 4)**.** *If S is smooth, then $\mathcal{H}_X^N(G)$ is represented by an algebraic stack which is smooth, finite-type, and étale over S, and which is finite over S after base-change to $\mathbb{Z}[\frac{1}{\#G}]$. In fact, $\mathcal{H}_X^N(G)$ is represented by a scheme $H_X^N(G)$ precisely if the centralizer of G in N is trivial.*

Example 6.0.8. *For example taking $S = U_n = \mathbb{P}^n - \Delta_n$ and $X = \mathbb{P}^1_{U_n}$ as before, then $\mathcal{H}_X^N(G) = \mathcal{H}_n^N(G)$ is the Hurwitz scheme parameterizing tamely ramified G-Galois covers of $\mathbb{P}^1$ with n branch points.*

Corollary 6.0.9 ((30) Theorem 3)**.** *A specicialization of the above theorem is then the following. $\mathcal{H}_n^N(G)$ is represented by an algebraic stack which is smooth and finite type over $\mathbb{Z}$, and which is finite étale over U_n after base-change to $\mathbb{Z}[\frac{1}{\#G}]$.*

Returning to the situation of Artin-Schreier covers of $\mathbb{P}^1$, recall that we can regard the Artin-Schreier curve $y^p - y = x^{p-1}$ as either a wildly ramified $\mathbb{Z}/p\mathbb{Z}$-Galois cover of $\mathbb{P}^1$, or as a tamely ramified $\mathbb{Z}/(p-1)\mathbb{Z}$ cover of $\mathbb{P}^1$ which is not Galois. We consider them as the latter in the rest of this section.

Remark. The version of Hurwitz stacks that we have reviewed above only handles tamely ramified covers, defining a wild Hurwitz stack which sees $\mathbb{Z}/p\mathbb{Z}$-Galois covers of $\mathbb{P}^1$ such as the Artin-Schreier cover requires additional work and we do not treat it here.

We may consider the Hurwitz stack $\mathcal{H}_p^N(\mathbb{Z}/(p-1)\mathbb{Z})$ where $\mathbb{Z}/(p-1)\mathbb{Z} \hookrightarrow \Sigma_p$ is a faithful transitive representation and where N is the normalizer of $\mathbb{Z}/(p-1)\mathbb{Z}$ in Σ_p. It is not a priori clear that such a faithful transitive representation always exists, or if there are necessary conditions on p. Granting its existence, $\mathcal{H}_p^N(\mathbb{Z}/(p-1)\mathbb{Z})$ is the Hurwitz stack parameterizing tamely ramified $\mathbb{Z}/(p-1)\mathbb{Z}$-covers of $\mathbb{P}^1$ with p branch points, and the Artin-Schreier curve $y^p - y = x^{p-1}$ defines a point $\mathrm{AS} \in \mathcal{H}_p^N(\mathbb{Z}/(p-1)\mathbb{Z})$.

By corollary 6.0.9 $\mathcal{H}_p^N(\mathbb{Z}/(p-1)\mathbb{Z})$ is represented by an algebraic stack which is smooth and finite type over $\mathbb{Z}$, and which is finite étale over U_n after base-change to $\mathbb{Z}[\frac{1}{p-1}]$.

The isomorphism from the previous section can now be explained as follows:

Theorem 6.0.10. $\mathrm{Def}_{f:X\to\mathbb{P}^1} \simeq \mathrm{Def}_{D,\mathbb{P}^1}$

Proof. The former is a formal neighborhood in $\mathcal{H}_X^N$ and the latter is a formal neighborhood in $\mathcal{U}_N = \mathbb{P}^n - \Delta_n$. The theorem follows map from $\mathcal{H}_X^N(G) \to \mathcal{U}_n$ is finite étale.

The formal neighborhood of the point $\mathrm{AS} \in \mathcal{H}_p^N(\mathbb{Z}/(p-1)\mathbb{Z})$ is represented by the deformation stack $\mathcal{D}ef_{f:X\to\mathbb{P}^1}$ studied in the previous section. The formal neighborhood of the corresponding point $D = \{0, 1, \dots, p-1\} \in U_n$ is represented by the deformation stack $\mathcal{D}ef_{D,\mathbb{P}^1}$, and since $\mathcal{H}_p^N(\mathbb{Z}/(p-1)\mathbb{Z})$ is finite étale over U_n after base-change to $\mathbb{Z}[\frac{1}{p-1}]$, we obtain an identification of these formal neighborhoods. $\square$

CHAPTER 7

Global Construction: Algebraic Moduli

7.1. Definition of Moduli Problem

First, we review a direct generalization/rephrasing of the approach taken by Mahowald and Stojanoswka using explicit curve equations. Note that the Artin-Schreier curve splits as $x^{p-1} = y^p - y = y(y-1)\cdots(y-(p-1))$. We consider a moduli functor of curves of the form $x^{p-1} = (y-e_0)\cdots(y-e_{p-1})$, and then we consider a moduli functor of curves of this form *with* marked points (the roots of the polynomial). The former we refer to as F^{unord} and the latter as F^{ord}. Considering both of these moduli functors allows us to work with schemes more freely.

The curve associated to a generalized Artin-Schreier curve over an algebraically closed field has the following property: X is smooth if the discriminant of $f(x)$ has no repeated roots. Morever, Artin-Schreier curves cannot have isomorphic presentations if they are themselves not isomorphic. Thus, we inject into the open sub-scheme with Δ inverted.

The moduli functors corepresent the moduli stack of curves of the indicated, meaning maps from S into the stack $\mathcal{M}$ are the same as hitting S with the following functor.

Remark. These functors are more generally defined for $\mathbb{Z}[\frac{1}{p-1}]$-schemes, but here for the sake of relation to topology, we use $\mathbb{Z}_{(p)}$-schemes.

Definition 7.1.1. *We consider a moduli functor F defined as follows*

$$F\colon \mathbb{Z}_{(p)} - \mathrm{Sch} \longrightarrow \mathrm{Grpd}$$

$$R \longmapsto objects: \left\{ (X,R) \middle| \begin{array}{c} X \textit{ is a nonsingular family of curves over } R \\ \textit{whose affine expression is} \\ x^{p-1} = y^p + a_1 y^{p-1} + \cdots + a_{p-1}y + a_p.; \\ \textit{where the discriminant is invertible }. \end{array} \right\}.$$

$$morphisms: \left\{ \alpha: \ X(x,y) \mapsto X(\alpha(x,y)) \middle| \ \alpha: \begin{array}{ccc} x & & \mu^{-p}x \\ & \mapsto & \\ y & & \mu^{-p-1}y + c \end{array} \right\}.$$

Lemma 7.1.2. *Let* $Y := \operatorname{Spec} \mathbb{Z}_{(p)}[a_1, ..., a_p][\Delta^{-1}]$*, where* Δ *is the discriminant of* $f_a(x) = x^p + a_1 x^{p-1} + ... + a_{p-1}x + a_p$*, and* G *is the group scheme* $\operatorname{Spec}(\mathbb{Z}_{(p)}[\mu^{\pm}, c])$*. The associated stack* $\mathcal{M}$ *to the moduli functor* F *is equivalent to the quotient stack*

$$\mathcal{M} = EG \times_G Y \simeq Y//G.$$

The algebraic stack $\mathcal{M}$ *is smooth and of dimension* $p-2$ *over* $\mathbb{Z}_{(p)}$

Proof. To show this is the associated quotient stack, we follow (**?**) Chapter 4, Exemple (4.6.1). We first express the isomorphism of the objects as sets $F(S) \simeq \mathrm{Hom}_{\mathbb{Z}_{(p)}-\mathrm{Alg}}(Y, S)$. Let $\psi \in \mathrm{Hom}_{\mathbb{Z}_{(p)}-\mathrm{Alg}}(Y, S)$, then $\psi : (a_1, ..., a_p) \mapsto (\psi(a_1), ..., \psi(a_p))$ where $\psi(a_i) \in S$ are all nonequal. For curves of this form, it is sufficient for the discriminant to be a unit for the curve to be nonsingular. This uniquely specifies a curve of the form $x^{p-1} = y^p + a_1 y^{p-1} + ... + a_{p-1}y + a_p$ with no singularities as desired.

The action groupoid $Y//G$ has isomorphisms when two curves differ by an action of G, which coincides with the definition of the morphisms in the moduli functor F.

Further, we have that

$$EG \times_G Y \simeq Y//G.$$

This is because for G a group scheme acting on a scheme Y, we consider the principle G-bundle EG over BG and take a homotopy pullback of the action groupoid projection $p : Y//G \to BG$ along the identity morphism $id : BG \to BG$.

$$\begin{tikzcd} EG \times_G Y \arrow[r] \arrow[d] & Y//G \arrow[d, "p"] \\ BG \arrow[r, "id"] & BG \end{tikzcd}$$

$\square$

Definition 7.1.3. *We consider a moduli functor F^{unord} defined as follows*

$$F^{unord} \colon \mathbb{Z}_{(p)} - \mathrm{Sch} \longrightarrow \mathrm{Grpd}$$

$$R \longmapsto objects : \left\{ (X, R) \left| \begin{array}{c} X \text{ is a nonsingular family of curves over } R \\ \text{whose affine expression is} \\ x^{p-1} = (y - y_0)(y - y_1) \cdots (y - y_{p-1}); \\ \text{where } y_0, \cdots y_{p-1} \text{ are non-equal elements of} R. \end{array} \right. \right\}.$$

$$morphisms : \left\{ \alpha : \; X(x, y) \mapsto X(\alpha(x, y)) \left| \; \alpha : \begin{array}{ccc} x & & \mu^{-p} x \\ & \mapsto & \\ y & & \mu^{-p-1} y + c \end{array} \right. \right\}.$$

Definition 7.1.4. *We may also defined the corresponding moduli functor F^{ord},*

$$F^{ord} \colon \mathbb{Z}[\frac{1}{p-1}] - \mathrm{Sch} \longrightarrow \mathrm{Grpd}$$

$$R \longmapsto objects: \left\{ (X,R) \left| \begin{array}{c} X \textit{ is a nonsingular family of curves over } R \\ \textit{whose affine expression is} \\ x^{p-1} = (y-y_0)(y-y_1)(y-y_2)\cdots(y-y_{p-1}); \\ \textit{where } y_0, \cdots y_{p-1} \textit{ are non-equal elements of } R \\ P \textit{ is the following ordered set of marked points on } X, \\ P := ([0:y_0:1],[0:y_1:1],\cdots,[0:y_{p-1}:1]). \end{array} \right. \right\}.$$

$$morphisms: \left\{ \alpha: \ X(x,y) \mapsto X(\alpha(x,y)) \ \middle| \ \alpha: \begin{array}{ccc} x & & \mu^{-p}x \\ & \mapsto & \\ y & & \mu^{-p-1}y + c \end{array} \right\}.$$

This moduli functor should be thought of as the analogue of the moduli stack of curves with level structure.

The morphisms in the image of both of these moduli functors $F(S)$ can be thought of quite naturally as the following commutative diagram, that is, as an element of the automorphism groupoid defined in 1.1.5. Any two curves with a map between them are of the form $\beta^* X \to X$.

$$\begin{array}{ccc} \beta^* X & \xrightarrow{\alpha} & X \\ \downarrow & & \downarrow \\ S & \xrightarrow{\beta} & S \end{array}$$

For a more explicit note, given R and α as above,

$$\beta(R) := ([-\mu^{p-1}(y_0 - c) : 0 : 1], [\mu^{p-1}(y_1 - c) : 0 : 1], \cdots, [\mu^{p-1}(y_{p-1} - c) : 0 : 1]).$$

Note that both of these moduli spaces can be recast as local deformation spaces, by changing the source category to be Art_k, and adding the datum that our curves $x^{p-1} = (y-e_0)(y-e_1)\cdots(y-e_{p-1})$ must in fact reduce to $x^{p-1} = y(y-1)\cdots(y-(p-1))$.

Lemma 7.1.5. *Let G be* $\mathrm{Spec}(\mathbb{Z}_{(p)}[\mu^{\pm}, c])$*, and* $Y := \mathrm{Spec}\,\mathbb{Z}_{(p)}[r_2, ..., r_{p-1}][\Delta^{-1}]$*, where* Δ *is the discriminant of* $f(y) = y(y-1)(y-r_2)\dots(y-r_{p-1})$.

The moduli functor F^{unord} *is represented by*

$$\mathcal{M} = EG \times_G Y \simeq Y//G$$

and is smooth and of dimension $p-2$ *over* $\mathbb{Z}_{(p)}$*. In other words,*

$$\mathcal{M} \simeq \mathcal{M}^{unord}.$$

Proof. If the discriminant is invertible there is always an étale extension $R \to S$ such that $y^p + a_1 y^p + \cdots + a_{p-1}y + a_p$ splits over S.

$\square$

Let s_i be the ith symmetric polynomial in the variables r_j.

Corollary 7.1.6. *Let*

$$B := (\mathbb{Z}_{(p)}[r_0, ..., r_{p-1}][\Delta^{-1}])^{\Sigma_p} = \mathbb{Z}_{(p)}[s_0, ..., s_{p-1}],$$

and let $G := \mathrm{Spec}(\mathbb{Z}_{(p)}[\mu^{\pm}, c])$*. The moduli functor* F^{ord} *is represented by*

$$\mathcal{M}^{ord} \simeq E(\mathbb{G}_m \rtimes \mathbb{G}_a) \times_G \mathrm{Spec}\, B$$

Further,

$$\begin{aligned}\mathcal{M}^{ord} &\simeq E\mathbb{G}_m \times_{\mathbb{G}_m} \mathrm{Spec}\,\mathbb{Z}_{(p)}[\lambda_1, \lambda_2, ..., \lambda_{p-1}][\Delta^{\pm}] \\ &\simeq \mathrm{Spec}(\mathbb{Z}_{(p)}[\frac{\lambda_2}{\lambda_1}, \cdots \frac{\lambda_{p-1}}{\lambda_1}][\Delta^{\pm}])\end{aligned}$$

and is smooth and affine of dimension $p-2$ over $\mathbb{Z}_{(p)}$.

7.2. Comparison to Hurwitz Perspective and Discussion of Group Action

We note here that $\mathcal{M}^{ord} \to \mathcal{M}^{unord}$ is a Σ_p Galois cover. In fact, they correspond exactly to the ordered and unordered configuration spaces of points in $\mathbb{P}^1$.

$$\begin{tikzcd}
M^{ord} \arrow[r, "\sim"] \arrow[d, "\Sigma_p"] & (\prod_{i=0}^{p-1}(\mathbb{P}^1 - \Delta)//\mathbb{G}_m \rtimes \mathbb{G}_a \arrow[d, "\Sigma_p"] \\
M^{unord} \arrow[r, "\sim"] & (\mathbb{P}^n - \Delta)//\mathbb{G}_m \rtimes \mathbb{G}_a
\end{tikzcd}$$

Definition 7.2.1. *Let $X : \mathrm{Spec}(\mathbb{F}_q) \to \mathcal{M}$ classify $y^p - y = x^{p-1}$. Let $\mathrm{Aut}(X)$ be the algebraic group $R \mapsto \mathrm{Aut}(X/R)$; here R is an $\mathbb{F}_p$-algebra. In other words, we define the group scheme $\mathrm{Aut}(X)$ as the functor which assigns to $R \in \mathrm{Pro}(\mathsf{Art}_k)$ the morphisms of $\mathrm{Spec}(R)$ over $\mathcal{M}$; that is, diagrams of the following form.*

$$\begin{tikzcd}
\mathrm{Spec}\, R \arrow[dd, "\simeq"] \arrow[dr, "X"] & \\
& \mathcal{M}. \\
\mathrm{Spec}\, R \arrow[ur, "X"'] &
\end{tikzcd}$$

Lemma 7.2.2. *There is a closed immersion of algebraic groups $H := \mathrm{Aut}(X) \to \mathbb{G}_m \rtimes \mathbb{G}_a =: G$.*

Proof. Since $\mathcal{M}$ is a stack, a diagram such as the one in 7.2.1 is specified by a morphism $X \simeq X$ in the groupoid $\mathcal{M}(R)$ from R to itself. $\square$

The map $\mathrm{Spec}\, k \to \mathcal{M}^{unord}$ factors as follows, where $H := \mathrm{Aut}(X)$,

$$\begin{tikzcd}
\mathrm{Spec}(k) \arrow[rr] \arrow[dr] & & M^{unord} \simeq EG \times_G \mathrm{Spec}\, B \\
& BH \simeq EH \times_H \mathrm{Spec}\, \mathbb{F}_q \arrow[ur] &
\end{tikzcd}$$

.

We base change from $\mathbb{Z}_{(p)}$ to $\mathbb{Z}_{(p)}[\zeta]$ where ζ is a $(p-1)$st root of unity, so that $\mathrm{Aut}(X) = H$ remains consistent. We may identify H with the normalizer of $\mathbb{Z}/p$ inside of Σ_p, and get the following pullback diagram.

Lemma 7.2.3.

$$\begin{tikzcd}
\Sigma_p \times \mathrm{Spec}(k) \arrow[r] \arrow[d] & \Sigma_p/H \times \mathrm{Spec}(k) \arrow[r] \arrow[d] & \mathcal{M}^{ord} \arrow[d, "\Sigma_p"] \\
\mathrm{Spec}(k) \arrow[r] & BH \arrow[r] & \mathcal{M}^{unord}
\end{tikzcd}$$

Let $\overline{\rho}$ denote the reduced regular representatation of $\mathbb{Z}/p$, $\overline{\rho} : ker(W(k)[\mathbb{Z}/p] \to W(k))$. This has a natural action of $\mathrm{Aut}(\mathbb{Z}/p) \simeq \mathbb{Z}/(p-1)$, so we may consider it to be a $\mathbb{Z}/p \rtimes \mathbb{Z}/(p-1)$ representation.

Corollary 7.2.4. *The ring Λ representing M^{ord} is $Sym(\overline{\rho})[\Delta^{-1}]$ as a $\mathbb{Z}/p \rtimes \mathbb{Z}/(p-1)$-representation.*

Proof. This follows from the identification of $\mathrm{Aut}_k(X)$ with a subset of Σ_p given by Lemma 7.2.3, the inversion of Δ ensures the curves are smooth. We spell this out explicitly. We will show that two things are equivalent by computing both and seeing that they are the same. First, let us present $\mathrm{Sym}(\overline{\rho})$ explicitly as a $\mathbb{Z}/p$-representation. This looks like:

$$W(k)[d_0, d_1, \cdots, d_{p-1}] \xrightarrow{\epsilon} W(k)$$

$$d_i \mapsto 1$$

The kernel of ϵ is generated by $\delta_i := d_i - d_0$. Let σ be the generator of $\mathbb{Z}/p$. It acts as follows (the subscripts are considered mod p):

$$\sigma(\delta_i) := \sigma(d_i - d_0) = d_{i+1} - d_1 = \delta_{i+1} - \delta_1$$

$$\sigma(\delta_{p-1}) := \sigma(d_{p-1} - d_0) = d_0 - d_1 = -\delta_1.$$

This is the reduced regular representation $\overline{\rho}$.

It remains to observe in our Λ how λ_i are acted on by σ. The original coordinates of the curve are:

$$\begin{aligned} x^{p-1} &= (y - e_0)(y - e_1) \cdots (y - e_{p-1}) \\ &= y(y - (e_1 - e_0)) \cdots (y - (e_{p-1} - e_0)) \\ &:= y(y - \lambda_1) \cdots (y - \lambda_{p-1}) \end{aligned}$$

Thus, the action that sends $e_i \mapsto e_{i+1}$, where the subscripts are considered mod p,

$$\sigma(\lambda_i) := \sigma(e_i - e_0) = e_{i+1} - e_1 = \lambda_{i+1} - \lambda_1$$

$$\sigma(\lambda_{p-1}) := \sigma(e_{p-1} - e_0) = e_0 - e_1 = -\lambda_1.$$

Looks familiar! They are the same. $\square$

Remark. The regular representation of $\mathbb{Z}/p$ would come from the ring representing the moduli stack of curves of the form $x^{p-1} = (y - e_1) \cdots (y - e_{p-2})$ without modding out by the additive action $\mathbb{G}_a$ which normalizes one of the roots to 0. When we mod out by $\mathbb{G}_a$, we get the reduced representation as seen above.

7.3. Action on Special Fiber Lifts to Entire Stack

We wish to show that our global moduli stack $\mathcal{M}$ satisfies the conditions of modelling the Lubin-Tate action. This section is devoted to showing that $[X] : \operatorname{Spec} k \to \mathcal{M}$ models $G \simeq \mathbb{Z}/p \rtimes \mathbb{Z}/(p-1) \simeq N_{\Sigma_p}(\mathbb{Z}/p)$, where the curve X is the Artin-Schreier curve of the form $y^p - y = x^{p-1}$ discussed in detail in Section 3.

We know from Thereom 4.2.1 that $\mathrm{Aut}_k(X)$ contains G, it remains to construct a lift of that action to all of $\mathcal{M}$. This section is devoted to doing so explicitly.

Definition 7.3.1. *We define the order of a group element $(\alpha, \beta) \in \mathrm{Aut}(\mathcal{U}, \mathcal{M})$ to be the number of precompositions until the identity is reached, on top and bottom.*

$$\begin{tikzcd}
(\beta \circ \cdots \circ \beta)^*\mathcal{U} \arrow[r, "(\beta\circ\cdots\circ\beta)^*\alpha"] \arrow[d] & (\beta \circ \beta)^*\mathcal{U} \arrow[r, "\beta^*\alpha"] \arrow[d] & \beta^*\mathcal{U} \arrow[r, "\alpha"] \arrow[d] & \mathcal{U} \arrow[d] \\
\mathcal{M} \arrow[r, "\beta\circ\cdots\circ\beta"] & \mathcal{M} \arrow[r, "\beta"] & \mathcal{M} \arrow[r, "\beta"] & \mathcal{M}
\end{tikzcd}$$

Theorem 7.3.2. *Let $k = \mathbb{F}_{p^{p-1}}$, and let X be the Artin-Schreier curve with affine equation $y^p - y = x^{p-1}$. Then, $[X] : \operatorname{Spec} k \to \mathcal{M}$ models $G \simeq \mathbb{Z}/p \rtimes \mathbb{Z}/(p-1)^2$.*

Proof. By calculation, the general form of $\alpha : \beta^*\mathcal{U} \to \mathcal{U}$ is $\alpha(x, y) = (v^{-p}x, v^{-(p-1)}y + c)$. Let's look this map acting on a fixed fiber. Let's look at the fiber with ordered roots $([0:0:1], [1:0:1], ..., [r_{p-1}:0:1])$ which we abbreviate as $(0, 1, \cdots, r_{p-1})$. The action is then:

$$\alpha(x,y) = \left\{ \begin{matrix} x \mapsto v^{-(p-1)}x + c \\ y \mapsto v^{-p}y \end{matrix} \right\} : \{C \colon x^{p-1} = y(y-1)(y-r_2)\cdots(y-r_p)\}$$

$$\mapsto \{C' \colon x^{p-1} = y(y-1)(y-r_2')\cdots(y-r_p')\}$$

$$\beta\colon (0,1,...,r_{p-1}) \mapsto (\beta(\Sigma(0)),...,\beta(\Sigma(r_{p-1}))) =: (r'_0,...,r'_{p-1}),$$
$$= (-v^{p-1}c, v^{p-1}(1-c),...,v^{p-1}(r_{p-1}-c))$$

where Σ is an element of $\mathbb{Z}/p \rtimes \mathbb{Z}/(p-1)^2$ sitting inside of the symmetric group on p-elements.

To find a lift of a symmetric group element $\sigma \in G \simeq \mathrm{Aut}_k(C) \simeq N_{\Sigma_p}(\mathbb{Z}/p)$, we are reduced to solving the following:

$$\beta \circ \sigma \begin{pmatrix} 0 \\ 1 \\ r_2 \\ \vdots \\ r_{p-1} \end{pmatrix} = \begin{pmatrix} -v^{p-1}c \\ v^{p-1}(1-c) \\ v^{p-1}(r_2-c) \\ \vdots \\ v^{p-1}(r_{p-1}-c) \end{pmatrix}.$$

For example, for a fixed order p symmetric group element, say $\sigma := (23\cdots p1) \in \Sigma_p$, we must solve:

$$\beta_\sigma \begin{pmatrix} 1 \\ r_2 \\ \vdots \\ r_{p-1} \\ 0 \end{pmatrix} = \begin{pmatrix} -v^{p-1}c \\ v^{p-1}(1-c) \\ v^{p-1}(r_2-c) \\ \vdots \\ v^{p-1}(r_{p-1}-c) \end{pmatrix},$$

which may be more simply put as:

$$\beta_\sigma \begin{pmatrix} 1 \\ r_i \\ 0 \end{pmatrix} = \begin{pmatrix} -v^{p-1}c \\ v^{p-1}(r_{i+1} - c) \\ v^{p-1}(r_{p-1} - c) \end{pmatrix}.$$

Solving this, we find $\beta_\sigma(r_i) := \frac{r_{p-1} - r_{i-1}}{r_{p-1}}$, and $\alpha_\sigma(y) = v^{-(p-1)}y + c = -r_{p-1}y + r_{p-1}$. We take the convention that $r_0 = 0$, and $r_1 = 1$, and the indexes of the r_i are considered mod p.

Remark. This morphism β_σ is indeed of order p by calculation, which one can see by looking at the form of the jth composition:

$$(\beta_\sigma)^j(r_i) = \frac{r_{p-j} - r_{i-j}}{r_{p-j} - r_{p-j+1}}.$$

For a nice sanity check, we can also see if $\alpha_\sigma(x, y) \equiv (y, x + (p-1)) \bmod (p, r_i - i)$. Indeed, $r_{p-1} \equiv (p-1)$, and $-r_{p-1} \equiv 1$. For context in our earlier categorical setup, $\gamma(\sigma)$ is the pair $(\alpha_\sigma, \beta_\sigma)$.

To find a lift of the order $(p-1)^2$ element, we must solve the system:

$$\tau \begin{pmatrix} r_{p-1} \\ r_2 \\ \vdots \\ r_{p-2} \end{pmatrix} = \begin{pmatrix} v^{p-1} \\ v^{p-1}r_2 \\ \vdots \\ v^{p-1}r_{p-1} \end{pmatrix},$$

which may be simplified to:

$$\tau \begin{pmatrix} 1 \\ r_i \end{pmatrix} = \begin{pmatrix} v^{p-1}r_2 \\ v^{p-1}r_{i+1} \end{pmatrix}.$$

Solving this, we find $\beta_\tau(r_i) = \frac{r_{i+1}}{r_2}$, and $\alpha_\tau(x, y) = ((r_2)^{p/(p-1)}y, r_2x)$. $\square$

Remark. Let's look at $\alpha_\tau(x,y) \mod (p, r_i - i)$. Firstly, $r_2 \equiv 2$, and $2^{(p-1)} \equiv 1 \mod p$. Thus, we may consider 2 as a $p-1$th root of unity, and we could denote it as ζ^{p-1}, where ζ is a $(p-1)^2$ root of unity. In other words:

$$\alpha_\tau(x,y) \equiv (\zeta^p x, \zeta^{p-1} y) \mod (p, r_i - i).$$

CHAPTER 8

Universality of 1-d Component

The other condition we need to check to see if our stack satisfies the condition of Theorem 2.0.3 is that of deformed h-splitting. We must compare the formal group law that results from the construction outlined in Theorem 2.0.3 applied to our curve $[X]$ in the stack $\mathcal{M}^{ord}$.

To do this, we first must first determine the invariant differential of the formal group law F_1 that comes up from the construction in Lemma 8.0.4, and then compare the invariant differential of F_1 with a universal one, and it suffices to show they agree up to a unit and mod I^2.

The generators of LT as discussed in Theorem (Lubin-Tate) are $u_1, ..., u_{h-1}$. These u_i come from a universal p-typical formal group. The p-series of this formal group is

$$[p]_u(x) = u_1 x^p +_u u_2 x^{p^2} +_u \cdots .$$

We now set up some lemmas to express a recongition result which allows us to show that a given formal group law is universal.

Lemma 8.0.1. *Let F be a formal group law over a $\mathbb{Z}_{(p)}$-algebra with invariant differential*

$$\eta_F = (x + a_1 x^2 + \cdots)dx = (\sum a_{i-1} x^i)dx.$$

Then there is an isomorphism $e : F \to G$ of formal group laws so that G is p-typical and

$$\eta_G = (\sum_j a_{p^j-1}x^{p^j})dx.$$

Proof. The isomorphism e is the Cartier idempotent. See the discussion following Definition A.1.22 of Ravenel's Green Book, especially (A2.1.24). The invariant differential is the derivative of the logarithm. □

Lemma 8.0.2. *Let F be a p-typical formal group law over a $\mathbb{Z}_{(p)}$-algebra and with invariant differential*

$$\eta_F = (\sum_j a_{p^j-1}x^{p^j})dx.$$

Then the p-series of F can be written

$$[p]_F(x) = x +_F v_1x^p +_F v_2x^{p^2} +_F \cdots$$

with

$$a_{p^j-1} = \sum_{i=0}^{j} p^{j-i}a_{p^i-1}v_{j-i}^{p^i}$$

Proof. This follows from (A2.2.22) and (A2.2.4) of Ravenel, once we note $p^i\ell_i = a_{p^i-1}$. This follows Theorem A2.1.27, right before part (d). □

This implies (up to a unit in $\mathbb{Z}_{(p)}$)

$$v_j \equiv p^{j-1}a_{p^j-1} \quad \text{modulo} \quad (v_1, \ldots, v_{j-1}).$$

We now discuss the recognition result. Let $R = W(\mathbb{F}_q)[[u_1, \ldots, u_{n-1}]]$ and let F be a formal group law over R. We may assume F is p-typical. Write v_i for $v_i(F)$ and $v_0 = p$ Then F is a universal deformation if and only if the v_i form a regular sequence, the ideal $(p, v_1, \ldots, v_{n-1})$ is the maximal ideal and v_n is unit. We thus have the following result.

Proposition. *Let F be a p-typical formal group law over a R and with invariant differential*

$$\eta_F = (\sum_j a_{p^j-1} x^{p^j}) dx.$$

Let $a_0 = p$. Suppose

(1) the a_{p^j-1} form a regular sequence;

(2) modulo $(a_{p-1}, \ldots, a_{p^j-1})$, the coefficient a_{p^j-1} is not divisible by p^j; and,

(3) $(p, a_{p-1}, \ldots, a_{p^n-1}) = R$.

Then F is a universal deformation.

Theorem 8.0.3. *The formal group law f_1 has universal h-splitting for $h = p - 1$.*

Proof. Our formal group law is defined by formally integrating a differential $\omega(z)$. The next three lemmas/propositions sets up the coordinates (Lemma 4.3.1), splits the formal group law (Theorem 4.6.2), isolates $\omega(z)$ (Lemma 4.6.1), and finally presents $\omega(z)$ in such a way that its easier to compare to the universal holomorphic differential. □

Corollary 8.0.4. *Let $\mathcal{J}$ be the Jacobian of the universal curve with p marked points over $\mathcal{M}^{unord}$. The invariant differential spanning the one dimensional idempotent piece of $\mathcal{J}$ is $\omega(x) := x^{p-3}dx = zdz$.*

Proof. Follows from Lemma 4.3.1 and Lemma 4.6.2. The same splitting occurs in char 0 since the action by $\mathbb{Z}/(p-1)$ is coprime to p. □

We now proceed with the comparison. We take $\mathcal{U}$, the universal curve with p marked points over $\mathcal{M}^{\text{ord}}$, and we consider the symmetrization:

$$\begin{aligned} y^{p-1} &= x(x-1)(x-r_2)\cdots(x-r_{p-1}) \\ &= x^p + u_1 x^{p-1} + ... + u_{p-2}x^2 - (1 + u_1 + ... + u_{p-2})x. \end{aligned}$$

We call these u_i as they will end up playing the role of u_i in the sense of the coordinates of the universal formal group.

Remark (Rephrasing)**.** We take $\mathcal{U}$, the universal curve with p marked points over $\operatorname{Spf} A$, defined in Section 7. As a matter of convenience in this section, we change coordinates to the symmetrization of the marked points. We forget the marked points. In other words, we consider the etale map

$$\mathbb{A}^{p-2} \xrightarrow{s} \operatorname{Sym} \mathbb{A}^{p-2}$$

which takes an ordered tuple of points $(1, 2, ..., p-2)$ to a divisor $[1] + [2] + ... + [p-2]$. We may also think of this as sending r_i to $u_i := (-1)^i s_i(r_2, ..., r_{p-1})$, where s_i is the ith elementary symmetric polynomial.

Lemma 8.0.5. *We may express $\omega(z)$ from Corollary 8.0.4 as a geometric series*

$$\omega(z) = \frac{1}{1 - (\sum_{i=1^{p-1}} i u_i z^{p-i} w^{i-1} + (p-1)bzw^{p-2})}.$$

Proof. We allow ourselves new coordinates, $z = \frac{x}{y}$, and $w = \frac{1}{y}$. Our curve is

$$y^{p-1} = x^p + u_1 x^{p-1} + ... + u_{p-2}x^2 + bx,$$

or after coordinate change,

$$w = z^p + u_1 w z^{p-1} + ... + u_{p-2} w^{p-2} z^2 + b w^{p-1} z.$$

Next, we differentiate $\omega := \omega(z)$ with respect to z, and get $\omega' = w' f + g$, where,

$$f := u_1 z^{p-1} + 2u_2 w z^{p-2} + ... + (p-2) u_2 w^{p-3} z^2 + (p-1) b w^{p-1}$$

$$g := p z^{p-1} + (p-1) w z^{p-2} + ... + 2u_{p-2} w^{p-2} z + b w^{p-1}$$

By assumption, the logarithm of f_1 is generated by the integral of $\frac{dx}{y} = (1 - \frac{z}{w} w')dz$. Now, $1 - \frac{z}{w} w' = 1 - \frac{z}{w} \frac{1}{1-f} = \frac{w(1-f) - zg}{w(1-f)}$.

Then,

$$\omega(1-f) = \omega - u_1 w z^{p-1} - 2u_2 w^2 z^{p-1} - ... - (p-2) u_2 w^{p-2} z^2 - (p-1) b w^{p-1}$$

$$-gz = -p z^{p-1} - (p-1) w z^{p-1} - (p-2) u_2 w^2 z^{p-1} - ... - 2u_2 w^{p-2} z^2 - b w^{p-1} z$$

Thus, $w(1-f) - gz = w - pw$, or

$$1 - \frac{z}{w} w' = \frac{1-p}{1-f}.$$

□

Corollary 8.0.6. *((8), Section 25) $\omega(z)$ is strictly isomorphic to the invariant differential associated to a universal formal group of height $h = p - 1$.*

CHAPTER 9

Cohomology Calculation for $h = p - 1$

We start with the results.

Theorem 9.0.1. *The Tate cohomology of the G-module $\Lambda = \mathrm{Sym}(\overline{\rho})[\Delta^{\pm}]$ is:*

$$\hat{H}^*(G, \Lambda) \simeq \mathbb{Z}_p[\alpha, \beta, \Delta^{\pm 1}]/\alpha^2,$$

where

$$|\alpha| = (1, 2(p-1)),\ |\beta| = (2, 2p(p-1)),\ \textit{and}\ |\Delta| = (0, 2p(p-1)^2).$$

We restate this in the notation that naturally arises from the computation:

Theorem 9.0.2. *Denote by R the graded ring $\mathbb{Z}/p[c, b^{\pm(p-1)}, d^{p-1}]/(c^2)$. Then, the Tate cohomology of the Γ-module Λ is:*

$$\hat{H}^*(\Gamma, \Lambda) \simeq R \oplus Rbd.$$

The element $b^{(p-1)}$ is in degree $(2(p-1), 0)$, d^{p-1} is in degree $(0, p(p-1)^2)$, and c is in degree $(1, (p-1))$.

Remark. The classes c and bd represent α and β in the homotopy groups of spheres, up to a unit. That is why we notate them as such in Theorem 9.0.1

9.1. Setting up the Short Exact Sequence

Let R be a $\mathbb{Z}_p$-algebra. There is an action of $G = C_p \rtimes C_{p-1} \simeq (\mathbb{F}_p, +) \rtimes (\mathbb{F}_p^*, \times)$ on the group ring $R[\mathbb{F}_p] \simeq R[t]/(t^p - 1)$ by $(c, m) \cdot t^x \mapsto t^{c+mx}$. Let $\sigma := (1, 1)$ be the element of order p in G. If $a \in \mathbb{F}_p^\times$ is an element of order $(p-1)$, then, let $\tau := (0, a)$ be the corresponding element of order $p-1$.

Let $\Lambda_0 := ker(f : R[C_{p^k}] \to R[C_{p^{k-1}}])$. We wish to compute the C_p-cohomology of $\Lambda := \mathrm{Sym}(ker(f : R[C_{p^k}] \to R[C_{p^{k-1}}]))$. One would think that the most natural short exact sequence to use would be the exact sequence

$$\ker f \to R[C_p] \xrightarrow{f} R,$$

this naturally inherits a $C_p \rtimes C_{p-1}$-action from $R[C_p]$. Unfortunately, it breaks when we apply Sym, that is, the map $\mathrm{Sym}(\ker f) \to \mathrm{Sym}(R[C_p])$ is no longer an injection. The kernel of $\mathrm{Sym}(f)$ is also not equal to $\mathrm{Sym}(\ker f)$.

We have to be a bit more creative to construct an exact sequence. Let $A := \mathrm{Sym}(R[C_p])$, we consider the element $s_1 := 1 + \sigma + \cdots + \sigma^{p-1}$ in A. Then, let V be the R-submodule generated by s_1, and let I be the ideal in A generated by s_1. Note that $\Lambda \simeq A/I$.

The short exact sequence we will be working with is thus:

$$0 \to I \mapsto A \mapsto A/I \to 0.$$

9.2. Calculating using SES

We're all set up now to calculate cohomology. Firstly, we'll need a short exact sequence. We take the map f from above of G-representations and extend and complete it to a map of graded G-algebras over $\mathbb{Z}_p$:

$$\mathrm{Sym}(f) : A \to \Lambda,$$

where $\Lambda := \mathrm{Sym}(\Lambda_0)$, and $A := \mathrm{Sym}(A_0)$.

We may consider $A \simeq \mathbb{Z}_p[x_0, x_1, ..., x_{p-1}]$, where $x_i := \sigma^i \otimes y$. The kernel of $\mathrm{Sym}(f)$ is generated by $s_1 := x_0 + \cdots + x_{p-1}$ since $1 + \sigma + \sigma^2 + ... + \sigma^{p-1} = 0$, and $\sigma(x_i) = x_{i+1}$. So we get a short exact sequence:

$$0 \to As_1 \to A \to \Lambda \to 0.$$

We will compute the cohomology of A and As_1 to compute the cohomology of Λ.

The orbit of each monomial of A under σ is free except for $s_p := x_0 \cdots x_{p-1}$ which is fixed (thus the orbit is trivial), and its corresponding cohomology class d lies in degree $(0, p(p-1))$. Therefore, A splits as a sum of a G-module F with a free C_p-action and $\mathbb{Z}_p[s_p]$ which has trivial C_p-action,

$$F \oplus \mathbb{Z}[s_p]$$

.

Next, we take the ring map from $\mathbb{Z} \to A$, this gives a map from $H^*(C_p, \mathbb{Z})$ to $H^*(C_p, A)$. Recall that $H^*(C_p, \mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } * = 0 \\ \mathbb{Z}/p & \text{if } * \text{ even} \\ 0 & \text{if } * \text{ odd} \end{cases}$, i.e., $H^*(C_p, \mathbb{Z}) \simeq \mathbb{Z} \oplus \mathbb{Z}/p\langle b \rangle$, where b has bidegree $(2, 0)$. Let $N : A \to H^0(C_p, A)$ be the norm map. Then, we have an exact sequence

$$A \xrightarrow{N} H^0(C_p, A) \to \mathbb{Z}/p[b, d] \to 0.$$

The Tate cohomology of A is then

$$\hat{H}^*(C_p, A) \simeq H^*(C_p, A)[b^{-1}] \simeq \mathbb{Z}/p[b^{\pm}, d].$$

Let $\widetilde{b} := s_1 b$, and $\widetilde{d} := s_1 d$. Similarly, the Tate cohomology of As_1 is $\hat{H}^*(C_p, As_1) \simeq \mathbb{Z}/p[\widetilde{b}^{\pm}, \widetilde{d}]$.

From our short exact sequence we get a long exact sequence, which is zero at the ends for degree reasons:

$$0 \to \hat{H}^{2k-1}(C_p, \Lambda) \to \hat{H}^{2k}(C_p, As_1) \to \hat{H}^{2k}(C_p, A) \to \hat{H}^{2k}(C_p, \Lambda) \to 0.$$

The middle map in this long exact sequence is zero, because it is induced by multiplication by s_1, which is in the image of the additive norm on A, and norms are modded out by Tate cohomology.

It follows that

$$\hat{H}^*(C_p, \Lambda) \simeq \mathbb{Z}/p[b^{\pm 1}, c, d]$$

where c is the element of bidegree $(1, (p-1))$ which maps to $\widetilde{b} \in \hat{H}^2(C_p, As_1)$.

Let us now examine the invariants of the C_{p-1}-action on $\hat{H}^*(C_p, A)$.

Lemma 9.2.1. $\hat{H}^*(C_p \rtimes C_{(p-1)^2}, \Lambda) \simeq \hat{H}^*(C_p \rtimes C_{(p-1)}, \Lambda) \simeq \hat{H}^*(C_p, \Lambda)^{C_{(p-1)}}$

Proof. Since the action of $C_{(p-1)^2}$ on C_p factors through C_{p-1}, which is prime to p. Since the action of all $x^{(p-1)} \in C_{(p-1)^2}$ is trivial on C_p, $H^*(G, \Lambda) \simeq H^*(C_p, \Lambda)^{C_{p-1}} \simeq H^*(C_p, \Lambda)$.. $\square$

Lemma 9.2.2. *The action of a generator* $\tau \in C_{p-1}$ *on* $\hat{H}^*(C_p, A) \simeq \mathbb{Z}/p[b^{\pm 1} d]$ *is as follows:*

$$\tau : b \mapsto \zeta^{-1} b$$

$$d \mapsto \zeta d$$

Proof. Let a be a chosen primitive root of $\mathbb{Z}/p$. Let ζ be a $p-1$ root of unity. We begin with b the generator of $H^2(C_p, A)$.

Recall that $H_1(C_p, \mathbb{Z}) \simeq C_p$, here τ acts by multiplication by a.

Then, $H^1(C_p, \mathbb{Z}/p) \simeq \mathrm{Hom}(H_1(C_p, \mathbb{Z}), \mathbb{Z}/p))$. The dual of the action by the 1x1 matrix a, is the 1x1 matrix a again, it still acts by a. Via the Bockstein map, $H^1(C_p, \mathbb{Z}/p) \simeq H^2(C_p, \mathbb{Z})$. Thus, $\tau(b) = ab$.

We now examine how τ acts on $d = x_0 x_1 \cdots x_{p-1}$. Note that τ acts on $x_i := \sigma^i \otimes y$.

$$
\begin{aligned}
\tau(x_i) &:= \tau(\sigma^i \otimes y) \\
&= \tau(\sigma^i) \otimes y \\
&= \sigma^{ia} \otimes \tau(y) \\
&= \sigma^{ia} \otimes \zeta y \\
&= \zeta(\sigma^{ia} \otimes y) = \zeta x_{ia}
\end{aligned}
$$

Therefore, $\tau(d) = \zeta^p(\prod_a x_{ia}) = \zeta d$. Lastly, in order for Y to be a basis which spans Λ as a $p-1$ Z_p-module, $|\zeta| := a^{-1}$. $\square$

Lemma 9.2.3. *The action of a generator $\tau \in C_{p-1}$ on $\hat{H}^*(C_p, s_1 A) \simeq \mathbb{Z}/p[\widetilde{b}^{\pm 1}, \widetilde{d}]$ is as follows:*

$$
\begin{aligned}
\tau : \widetilde{b} &\mapsto \widetilde{b} \\
\widetilde{d} &\mapsto \zeta^2 \widetilde{d}
\end{aligned}
$$

Proof.

$$\tau : \widetilde{b} = s_1 b \mapsto (\zeta s_1)(\zeta^{-1} b) = \widetilde{b}$$

$$\widetilde{d} = s_1 d \mapsto (\zeta s_1)(\zeta d) = \zeta^2 \widetilde{d}.$$

$\square$

The last step is to connect the cohomology of A and $s_1 A$ together. Let c be the element in bidegree $(1,(p-1))$ which maps to $\widetilde{b} \in \hat{H}^2(C_p, As_1)$ equivariantly. Thus, the action is the same as that of $\widetilde{b}$, that is:

$$\tau : c \mapsto c.$$

Finally, this concludes the proof.

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