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Spatial Thinking and the Learning of Mathematics in the Game of Go

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Abstract

Although there has been profound evidence showing the positive correlation between spatial abilities and math performances, we still know very little about how and why spatial thinking facilitate the learning of mathematics. This dissertation unpacks several aspects of mathematics that are embedded in learning and playing an ancient and rich game of Go and sheds light on how and why spatial thinking might contribute to the learning of mathematics. Go is a two-player, turn-based strategy game that originated in ancient China. It is infused with numerous spatial and math activities. Players take turns to put down black or white stones to surround more spaces on the board than the opponent. I draw on the sociocultural perspectives which consider learning as mediated by cultural tools to explore the potential of Go to influence the way Grade 2-3 students learn and do mathematics. My study identifies a variety of mathematical reasoning practices emerging from the teaching and learning of Go patterns, such as conjecturing, justifying, and generalizing, and uncovers how the teacher and students use spatial modeling in a dynamic and fluid way to facilitate the reasoning processes. Additionally, I found various score counting strategies at the end of games, which involve using different kinds of spatial numerical representations to solve math problems embedded in the game. Therefore, Go does not facilitate just one type of math strategy but allows players to choose from multiple types of spatial numerical representations, which yield different strategies to solve emerging math problems in the game. My study thus sheds light on an alternative perspective on how spatial thinking might facilitate math learning.

Acknowledgements

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I also would like to thank my collaborator, Mr. Xinming Guo, who is the inventor of the Go and Math Curriculum, and the teacher who taught every Go class I studied. Mr. Guo's passion about the game of Go and math education was what ultimately sparked the idea to do my dissertation on this topic. Mr. Guo is one of the most creative and reflective teachers I have ever met. He shared with many of his insights on the design of the curriculum and his reflections on students' learning, all of which contributed to how I developed this dissertation.

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Chapter 1: Introduction

The Learning Sciences has witnessed tremendous growth in research on how games can support learning. Thousands of new educational games have been invented, and many use technologies such as digital touch screens. My focus here is on the use of a game to facilitate learning, but the game is neither new nor reliant on technology. This dissertation is about the game of Go, which was invented three thousand years ago and is played by millions of people, particularly in Asia, but also in other areas as well.

Go is a board game with simple elements: line and circle, stone and wood, and simple rules—one wins by surrounding more spaces than one's opponent (Figure 1.1). Yet, these simple elements create challenges that have enthralled millions of players across the world for three thousand years. The game involves two players taking turns to strategically add black or white stones onto the board, aiming at constructing and expanding one's own territories with connected stones while trying to hinder or deconstruct those of the opponent. Although Go has existed for a very long time, here is great value in investigating this game as a tool for learning. As I argue below, Go has made a strong impact on Asian cultures and might offer a unique platform from which to investigate the potential of games for learning.

The history game of Go is interwoven with Asian history and cultural development. For many centuries this game was extremely popular among governors, politicians, and strategists in the military. It was built into the curriculum of the educated and privileged because it was thought to support complex reasoning, decision making, persistence, patience, and other leadership characteristics. Now, parents in many countries enroll their children in Go classes for the same intended benefits, that is, to boost intellectual capacities.

Figure 1.1

A Traditional Go Game Board and Stones

Note. Image retrieved from “a traditional Go gameboard” by T. Osato, in *The Mystery of Go, the Ancient Game That Computers Still Can't Win*, by A. Levinovitz, 2014, WIRED, (https://www.wired.com/2014/05/the-world-of-computer-go/).

Although many claims have been made about the benefits of go, there is relatively little in the way of rigorous research that has evaluated Go as a tool for cognitive development and learning (e.g., Chen et al., 2003; Duan et al., 2014; Jung et al., 2018; Lee et al., 2010; Reitman, 1976). Here I hope to expand research on Go by addressing some aspects of its potential to facilitate spatial and mathematical thinking. Next, I will discuss theoretical perspectives on games and learning, which can shed light on the ways of studying the kinds of learning that might occur in the process of learning and playing Go.

Games and Learning in Social Cultural Context

There is growing interest in studying learning in the context of game play. Play is an important way that young children learn (Singer et al., 2006). There have been many approaches to studying learning in game contexts. My work is generally informed by sociocultural perspectives, which view children's learning and cognitive development from the vantage point of historically situated activities that are mediated by cultural artifacts and tools (Cole, 1996; Rogoff, 1991; Saxe, 1981, 1991; Wertsch, 1991).

Sociocultural analyses highlight the importance of social activities like game play as contexts for children's learning and development (Bransford et al., 1999; Lave & Wenger, 1991). Nasir (2005) described games as an ideal setting where a significant amount of learning occurs through interactions with cultural norms and artifacts. As she noted,

> "Games are inherently artifacts of culture through which cultural roles, values, and knowledge bases are transmitted… Games can also be viewed as a microcosm in which culture operates in explicit ways to organize activity. Indeed, games are rich in cultural artifacts and are guided by implicit and explicit rules, norms, and conventions. Additionally, social interaction among players is often integral to game play. These features of game play can scaffold learning and cognitive processes in critical ways—Both through organizing learning in the context of social interactions with other players and affording particular kinds of problems and the strategies to solve them through the organization of the physical environment and the game rule structure." (p.6-7)

In short, Nasir (2005) suggests that games is an setting where learning becomes salient, because in game contexts, we can see how individuals actively contribute to their own learning and how the learning is influenced by the sociocultural norms (e.g., roles, game structures) and cultural tools. Therefore, there is great value in viewing Go as a cultural tool for learning, as this lens might help us understand the role of the individual players in their own learning and the role of the cultural tools shaped by the game and to explore their relations. Next, I share some general theoretical perspectives on math learning as mediated by cultural artifacts and tools.

Cultural Tools as Means to Engage in Informal Mathematical Practices

Learners in various cultures can develop complex mathematical thinking practices in informal everyday situations. Viewing learning as mediated by cultural tools may capture mathematics learning in these informal contexts in broadly two ways. First, the sociocultural perspective sheds light on how cultural tools influence how individuals think mathematically. For examples, symbols and artifacts that represent numbers have affected how individuals do calculations or reason about patterns in daily lives (e.g., Saxe, 1981; Stigler, 1984).

Second, cultural tools can be viewed as a means to engage learners in various forms of mathematical practices (Gutiérrez & Rogoff, 2003; Lave & Wenger, 1991; Rogoff et al., 2003). Learning is interpreted through the lens of goals, roles, and tools which emerge from a group of people participating in activities that are central to this community (e.g., the daily and communal practices of trading); learning occurs when people take on new roles or develop new goals, which may result in shifts in their ways of using the tools and participating in the activities. For example, Nasir (2000) showed how high school basketball players learn or engage with the concepts of average and percent in order to calculate their own statistics and the statistics of other players. However, they might not directly apply such practices in school, where the setting differs sharply (Nasir et al., 2008; Saxe, 1991). One of the reasons is that in out-of-school settings, the problems that arise are often practical and applied (Bell et al., 2006), and they arise as participants seek to reach bigger or broader goals that are meaningful to the learners (Nasir, 2000).

Viewing learning in light of cultural tools might offer valuable insights to important questions about learning in informal contexts. One of those important questions is about the relation between spatial reasoning and math learning.

Studying Space and Math in the Context of Go

Although there has been profound evidence showing the positive correlation between spatial abilities and math performances (Casey et al., 2015; Hawes et al., 2019), we still know very little about how and why spatial thinking facilitate the learning of mathematics. This dissertation will unpack several aspects of mathematics that are embedded in learning and playing the game of Go and shed light on how and why spatial thinking might contribute to the learning of mathematics.

Here, I examine the game of Go as a cultural tool to examine its potential to engage children in spatial reasoning and to influence the way children learn mathematics. The game of Go might be an appropriate context to explore specific ways in which mathematical reasoning can be modeled and supported spatially, because it is a game that builds in these spatial approaches in a natural, organic way. Although Go takes a lifetime to master, it has minimal rules, and thus young children can readily learn to play. This game offers opportunities for young children to participate in increasingly complex spatial and mathematical tasks, which otherwise might be too cognitively demanding for children at the Grade 2-3 level. Therefore, this game could be a proper context to study the underlying mechanism of how spatial thinking supports early math learning.

In the next chapter, I review existing literature on spatial reasoning and math learning and discuss how cultural tools as a lens to studying math learning may contribute to our understanding of spatial and math relations.

Chapter 2: Space and Math

In this chapter, I provide a review of literature on the correlation between spatial skills, spatial training, and mathematical performances. I then incorporate the sociocultural perspectives and demonstrated a few examples of cultural tools that might contribute to our knowledge of how spatial reasoning might support math learning.

Having high levels of spatial skills has been shown to help students succeed in a variety of subjects including Science, Technology, Engineering, and Mathematics (STEM) fields (Sorby et al., 2013; Sorby, 2009; Uttal et al., 2013; Uttal & Cohen, 2012). Uttal and Cohen (2012) indicate that spatial abilities predict performances in early STEM learning but become less predicative as STEM contents advance. Therefore, developing high levels of spatial skills are especially important for STEM learning early on.

The evidence linking mathematics learning and spatial ability is now quite strong (Casey et al., 2015; Hawes et al., 2019). Moreover, studies have demonstrated that spatial training can lead to improvements in math(Cheng & Mix, 2014; Ferrini-Mundy, 1987; Lowrie et al., 2017; S. Sorby et al., 2013). Nevertheless, we still know very little about the mechanisms through which spatial thinking supports the learning of mathematics (Mix & Cheng, 2012). Exploring such mechanisms is extremely valuable, because it will not only shed light on underlying cognitive processes that explain how spatial and mathematics activities are linked, but also help us identify appropriate types of spatial training and understand how and when to implement the spatial training for mathematics.

So far, researchers have articulated several spatial accounts of math learning, which remain to be further evaluated (Hawes & Ansari, 2020). This dissertation research focuses on demonstrating and detailing some of these spatial accounts of math learning in the context of an ancient and rich game of Go. By examining how young children engage in spatial thinking and math learning through Go, this study will advance our understanding of the accounts of how spatial thinking might support math learning. Next, I review some existing spatial accounts of math learning.

Why Space and Math are Related

There are different spatial accounts of math learning; some concern neuro-processing and working memory, while others focus on mental models. I will discuss two accounts of the relation between spatial skills and mathematics, both of which can be illustrated through a case study of young children learning to play Go and could potentially advance our knowledge of the spatial mechanism for math learning. The two accounts are (a) the spatial numerical account, and (b) the spatial modelling account.

The Spatial Numerical Account

The mapping of numbers to space is essential in how we learn and do mathematics. There has been a substantial body of research showing how numbers can be represented spatially. However, fewer studies have examined how and why spatial representations of numbers may yield advantages to learning and doing mathematics.

The number line is a good example to demonstrate how spatial numerical representations might support mathematics understanding. Research on the number line estimation task reveals a consistent and reliable correlation between the task performance and numerical reasoning (Schneider et al., 2018). People who are more accurate at estimating where a given number belongs on a horizonal line are also better at numerical and mathematical reasoning. Moreover, the number line has been used as an effective instructional tool and proven to enhance students' numerical reasoning in the classroom, because they lead to a more refined "mental number line" (Fischer et al., 2011; Ramani & Siegler, 2008; Siegler & Ramani, 2009).

The Spatial Modelling Account

According to the spatial modeling account, spatial visualization is related to numerical reasoning because it provides a "mental blackboard" of which numerical relations and operations can be modeled and visualized. More specifically, spatial visualization has been posited to play a critical role in how one organizes, models, and ultimately conceptualizes novel mathematical problems (Mix et al., 2016; Uttal & Cohen, 2012).

There appear to be few limitations on the types of mathematical relations that can be modeled through visualizations. It is for this reason that it can be difficult to empirically investigate the spatial modeling account (Hawes & Ansari, 2020). How does one reveal the specific type of spatial modeling that occurs in the "mind's eye" of any given individual? Are some types of spatial modeling more conducive to effective mathematical reasoning than others? More studies are needed to show specific spatial models that could be used visualize particular math learning contents and explain how the spatial models might support math learning and problem-solving.

It is important to note that the spatial modeling account overlaps with other theories of numerical and mathematical cognition. In particular, it is closely related to the grounded and embodied accounts of mathematical cognition, which emphasize that mathematical ideas are grounded in embodied interactions in the world (Lakoff & Núñez, 2000; Marghetis et al., 2014; Marghetis & Núñez, 2013; Nathan, 2008). Since Bruner (1966) and Piaget (1964), numerous studies have shown that learners develop abstract mathematical understanding from interacting with objects in the world. Lakoff and Núñez (2000), for example, argue that even the most abstract symbolic understandings are developed through conceptual metaphors which are grounded in bodily interactions with the world. The embodied accounts and spatial modeling are alike in that they both highlight the role of mental processes related to the re-enactment of sensorimotor experiences (e.g., mental imagery) in forming mathematical concepts. The spatial modeling account is a more specific instantiation of mental simulation that deals explicitly with spatial relations.

The above cognitive theoretical perspectives provide some possible ways to explain why spatial thinking and mathematical reasoning are positively related. However, these theories lead to more questions about the mapping between specific types of spatial reasoning and the types of mathematics they can support. Next, I illustrate how the sociocultural perspectives of learning may contribute to our growing knowledge about the spatial mathematical relations.

Cultural Tools as a Lens to Studying Space and Math

The lens of cultural tools can shed light on ways of exploring different aspects of learning; the relation between spatial thinking and math learning is one example. Many kinds of learning mediated by cultural tools evoke spatial reasoning, which play a crucial role in the cultural activities, such as those involved in weaving, crafting, and navigation. Some of these highly spatial cultural tools were invented to produce various forms of numerical representations. These numerical representation systems serve specific practical purposes and influences how people of different cultures think mathematically. For example, Saxe (Saxe, 1981, 1985) showed that indigenous people in Oksapmin communities in a remote area in central New Guinea use a 27-body-part system for numbering, counting, and trading activities. This body-numbering-system can thus be considered as a cultural tool that influences how people in this community engage in mathematical practices. Much of the numbering and counting rely on recognizing and reasoning about spatial relations represented on different bodily locations; arithmetic processes are carried out through bodily movements which also involve spatial reasoning.

Because many cultural activities that are inherently spatial, the lens of cultural tools can potentially provide new ways to understand why and how engaging in spatial activities may facilitate participation in mathematical practices. The question of spatial and mathematical relations may be framed as a question about whether and how interacting with cultural tools may foster spatial ways of participation in diverse forms of mathematical practices. Furthermore, we could explore whether fostering spatial ways of participation might lead to changes in participation—new relations between the learners and the mathematical objects (e.g, learners may develop a new way to use a tool for problem-solving, thereby causing a shift in their roles and a new goal to emerge).

The ways of viewing learning mediated by cultural tools may enrich our understanding of the relationship between spatial thinking and math learnings. Next, I introduce some more detailed examples of cultural tools that have affected how individuals do calculations or reason about patterns in daily lives, and how doing so influenced their numerical understanding. Some of these cultural tools are for numerical representations and might have influenced individuals' mental representations of numbers and their arithmetic skills; the other tools engage people in reasoning with spatial patterns, as a means to solve daily problems that are inherently mathematical.

These examples may support the spatial numerical account of math learning, by illustrating how cultural tools may play a role in shaping how individuals think and reason with numbers in their daily lives. Moreover, these examples may inspire new ways to understand the spatial accounts by demonstrating how math learning can be conceptualized through the lenses of changes in ways of participation mediated by cultural artifacts.

Numerical Representation Systems and Mathematical Thinking

People from different cultures invented various forms of numerical representations. These numerical representation systems serve specific practical purposes and influences how people of different cultures think mathematically.

The abacus is a widely used cultural tool invented in China thousands of years ago and still used today as an instructional tool in math classrooms. The abacus represents numbers in base five and base -ten systems and affords operations of addition and subtractions easily through manipulation, so learning to use the abacus involves representing and manipulating numbers from the traditional base-ten system. Stigler (1984) found that people with the motor skill of working with a physical abacus can transfer that skill into a mental capacity; they can construct a mental image of an abacus and then perform mental calculation by moving the “beads” on their “mental abacus” as they would on a real abacus. As a result, the study found that the level of difficulty (manifested through accuracy, efficiency, and common mistakes) in performing mental abacus is in alignment with the difficulty associated with how the problems are represented on a physical abacus, such as the dealing with numbers exceeding 5 at each column and increasing columns. This finding reveals the striking similarities between physical abacus and mental abacus calculation. In addition, Stigler’s (1984) comparison of mental calculations between Asian and American students shows clear distinctions in response times, accuracy and strategies used, which further suggests the important role that the abacus training played in shaping the way Chinese students think and do arithmetic mathematics. They also highlight the importance and influence of the spatial and motoric representations of number that the abacus affords.

In summary, Stigler's 1984 study of the mental representation used by Chinese children in mental calculation highlights a prominent way in which culture may influence cognitive processes. As he summarized, "not only can culturally specific training alter the strategies a child brings to bear on a cognitive problem, but it also alters the content of the child's thought. Indeed, perhaps the most powerful tools a culture can provide to the developing child will come in the form of specialized mental representations that are passed down through education" (Stigler, 1984, p. 175).

Spatial Patterns and Mathematical Thinking

In many indigenous communities, spatial and mathematical practices were evident in the creation of patterns from weaving practices (Rogoff & Gauvain, 1984; Saxe & Gearhart, 1990). Likewise, Cherinda (2012) suggests that as weavers bring their own contextual knowledge to bear on their creative weaving, they became both inquisitive artisans and mathematical learners at the same time.

Patterns are embedded in many cultural tools and practices; by engaging people with these tools and practices, spatial reasoning and mathematical practices are passed on. Studies have shown that patterning is a powerful learning tool for young children to develop early mathematical ideas (Lowrie et al., 2017; Rittle-Johnson et al., 2013). Pattern knowledge in elementary school is predictive of algebraic proficiency a year later (Lee et al., 2011) and instruction on repeating patterns supports knowledge of growing patterns (Papic et al., 2011), multiplicative thinking (Warren & Cooper, 2007) and general mathematics achievement (Kidd et al., 2013, 2014). In addition, abstraction tasks aimed at identifying underline structures of patterns are found to be more effective for math learning than extending or duplicating tasks, which can be done by matching surface features of objects (Rittle-Johnson et al., 2015).

Even the most basic symmetric pattern can promote math understanding. For example, Tsang et al. (2015) conducted a study that focused on using symmetry to teach integers to fourth graders. They found that recruiting the visual symmetry capacity of the brain, which is familiar to learners, can improve children's understanding of negative numbers – a new and abstract math concept to them. Using symmetry blocks as a tool, the design engages learners to build on their existing knowledge of integers to understand negative integers. The tool also provides a powerful visual illustration of arithmetic problems with positive and negative numbers.

The examples above show how we can understand math learning mediated by spatial reasoning through the lens of cultural tools. In particular, we saw how these cultural tools influence the ways people represent numbers spatially and how they reason with patterns, which in turn affect their math understanding.

Next, I return to the discussion of games as a cultural tool for learning, with an emphasis on their potentials to engage young children in spatial reasoning activities. I review the literature on games and spatial thinking to shed light on potential ways in which the game of Go might facilitate math learning through engaging children in spatial activities.

Spatial Affordances of Games

Games can be considered as cultural tools that might facilitate children to engage in spatial activities. Many studies have shown that playing with spatial toys and engaging in spatial activities may support the development of spatial thinking. Some have related spatial play with spatial skill development (Casey et al., 2008; Jirout & Newcombe, 2015; Levine et al., 2012) and mathematical reasoning (Casey et al., 2015; Cheng & Mix, 2014; Verdine et al., 2014). For example, Jirout and Newcombe (2015) studied a large group of 4- to 7-year-old children and found that those who frequently participated in block play, puzzles, and board games had higher spatial ability than those who participated less often.

Block play has generated a great deal of attention in terms of its potential link to spatial thinking (Casey et al., 2008). There are at least two key types of spatial skills closely related to block building—spatial visualization and mental rotation. Structured block play, in which children build a model of a structure (Verdine et al., 2014), may require the analysis of a spatial representation and may result in more significant improvements in spatial ability than unstructured block play (Casey et al., 2015). The relevant spatial abilities include the ability to segment an object into parts and relate those parts to the overall configuration. Moreover, block play is found to enhance spatial language and thereby promote spatial thinking (Ferrara et al., 2011).

In addition to block play, board games have also been linked to improved spatial processing (Jirout & Newcombe, 2015; Ramani & Siegler, 2008; Siegler & Ramani, 2009). Board games are filled with spatial elements and may require significant involvement with spatial reasoning to play. Some spatially rich board games have the potential to help players learn mathematics, due to their spatial alignment with certain mathematical ideas. Physical materials that are closely aligned with the desired knowledge structures increase analogical transfer and therefore promotes deeper learning (Chen, 1996; DeLoache et al., 1991; Gentner & Markman, 1997). For example, Siegler and Ramani (2009) found that a linear board game that was closely aligned with the linear representation of numerical magnitudes, as opposed to a circular board game, resulted in greater acquisition of the desired linear representation among preschoolers. Laski and Siegler (2014) found that playing a 0–100 number board game with the numbers organized in columns on a 10x10 grid improves kindergartners' knowledge of numerical magnitudes. One of the interpretations they proposed is that 10 x 10 matrix might have helped children learn the base-ten number system, which, in turn, might be important for understanding numerical magnitude of numbers. They concluded that a game board does not need to be strictly linear for acquisition of a linear representation of numbers. This is a great example of a game that influences learners' spatial processing and thereby influences their mental representations and understandings of numbers.

Chapter 3: Studying Space and Math in the Game of Go

Building on prior works that strongly correlate spatial abilities with math performances among young children, this dissertation investigates whether and how young children in Grade 2 and 3 engage and advance in spatial and mathematical practices through playing Go. I plan to specify what those practices are, examine whether they lead to math improvements, and if so, how spatial thinking mediates such improvements.

One crucial feature of the Go game is that it is highly accessible to young children, because the rules are very simple to begin with. Yet the game becomes increasingly complex as players advance. I want to examine whether this game has the affordance of making high level spatial and mathematical thinking accessible to young children.

My general hypothesis is that playing Go can facilitate the development of spatial and mathematical thinking, by grounding numerical and arithmetic concepts in spatial forms, and by situating advanced math practices in the context of learning and playing the game.

This project examines a Go and math curriculum designed specifically to teach young children how to play the game, while fostering spatial and math skills at the same time. It differs from traditional Go curricula in that it highlights the alignments between learning Go and early mathematics, with a particular focus on engaging learners in spatial thinking. The focus of this dissertation is on this curriculum and its influences on children's mathematics learning. Specifically, there are three main purposes of this study: (a) to examine the range of mathematics that could take place in the game of Go; (b) to advance our understanding of the mechanisms through which spatial thinking supports math learning.

The Emergent Goals Framework for Games and Math Learning

My approach to analyses is influenced by prior works that have closely examined artifacts, tools, and practices in various cultures that shaped the ways people think mathematically. To better understand boardgames as a cultural tool and make sense of its affordances for spatial thinking and mathematics learning, I adopted an emergent-goals framework that many have used to analyze mathematics learning in boardgames from a sociocultural perspective. The framework was developed by Saxe and colleagues in 1996 from analyses of everyday mathematics in several indigenous communities. The framework provides three principal constructs for analyzing the development of novel mathematical understanding from cultural practices: (a) the analyses of *goals* that emerge from the situations and shaped by the players, (b) the *forms*, which include ways of representing mathematical information that may yield different approaches to solving problems, like the indigenous number systems, and (c) the *functions* such as counting and arithmetic which utilize the forms to serve the emergent goals. Saxe et. al. (1996) suggested that the relation between forms and functions is reciprocal in nature; learners adjust their use of forms to create novel functions in order to realize goals that emerge in context. The reciprocal relation supports the emergence of more advanced mathematical thinking. Evidence of learning is found when learners repurpose or appropriate existing cultural forms to serve new functions in order to solve emerge problems. Next, I will illustrate how this framework has been used in a few prior studies to examine the affordances of games for learning (Guberman & Saxe, 2000; Nasir, 2005).

The Treasure Hunt Game

Guberman and Saxe (2000) used the emergent goals framework to examine the learning that occurs in the case of treasure hunt games. The study illustrated the ways in which children' roles in the game and mathematical goals are connected. They found that different thematic roles emerged in the children's play. These thematic roles give rise to the formation of particular mathematical goals, and these goals are distributed in the game situations. This study indicates that games may support learning by engaging children in various ways of participation in mathematical practices; as the games promote changes in their roles, the players also generated more and more complex mathematical goals that contributed to their learning.

This study utilized the emergent goals framework to depict how young children take on increasingly complex roles and develop new mathematical goals during the game. Thus, this study demonstrates that by studying learning as participation, we can learn a lot about how young children learn in a game setting. This study shed light on how we can understand math learning in the Go context, by examining whether and how children develop new mathematical goals in the game situations.

The Game of Dominoes

Nasir's (2005) study of children learning the game of dominoes provides both an effective illustration of Saxe's framework as well as another important example of the value of games for learning. She explored how players draw on the cultural resources in the game to actively structure plays in dominoes and how the nature of play strategies in the game shifts developmentally; she also examined the relation between the individual and the sociocultural setting through the lens of form–function shifts.

She identified the primary forms and functions that players at different age levels develop and utilize during their game plays of dominoes. She illustrated that players at elementary, high school, and adult levels utilized different forms and functions to meet growingly complex goals in the game play, which were established as the players integrated more prior knowledge into the play and developed increasing agency to restructure the game environment and define new problems and roles. She also demonstrates how people engage with sophisticated forms of addition, subtraction, and multiplication, in the context of help-seeking and help-offering around strategies and goals.

This study highlights many “form functional shifts” in dominos playing that occur across players of different ages and levels. As players become older and more advanced, they are able use different components of the game, such as game pieces, players’ hands, and knowledge of certain game compositions, to serve particular mathematical functions, and therefore enabled to achieve more advanced game goals. The game study showcased how examining the “form functional shifts” may provide deep insights into the progression of math understanding as players become more advanced. The study also highlights how spatial activities mediate such learning progressions. Thus, this study shed light on how we can understand the spatial mathematical relations in the game of Go by depicting how the spatial patterns of Go might be utilized for mathematical functions and how the use of spatial patterns might evolve as players develop more complex game goals.

Chapter 4: Go as a Cultural Tool: Its Rules, Characteristics, and Affordances for Learning

In this chapter, I introduce the fundamentals of Go, including its rules and some most important patterns, and the process of learning to play Go. The game is played on a board with a 19x19 grid, although novice players often use a smaller board to begin with. The basic playing pieces in Go are called stones (although many are now made of plastic). Each stone has the same value and obeys the same rules. There are two distinctly colored sets of stone; the colors are usually black and white. At the beginning of the game, players determine who will play the white and who will play the black stones. On each turn, the player places a stone on an intersection of the board. Stones of the same color that are connected by the grid of the board form a group. The groups of connected stones define the territories of each player. The ultimate goal of the game is to occupy more territories than one's opponent. That is, the player who takes over more than 50% of the intersections of the board (i.e., 181 out of 361) wins the game.

The Duality of Go: Simple vs. Complex

Go is perhaps one of the most fascinating board games ever created because of its combination of simplicity and complexity. Go is very simple to begin with. Since all the stones are the same, they can be placed freely on any vacant and unbounded intersections on a Go board, and they do not move around once they are placed; there are no varieties of stones with different rules for players to remember. Therefore, Go has a low entry point—even very young children can learn to play this game.

At the same time, the minimal restrictions and the simplicity of the rules lead to an enormous degree of variation, making it the most challenging game for human beings and computers alike. One of the best ways to understand the complexity of Go is to compare it with its western counterpart—chess, which is also a popular traditional board game played with black and white pieces on a grid. In chess, each stone has its designated rules to move, and these rules greatly constrain the number of possible moves. A standard Go board consists of 19 horizontal lines and 19 vertical lines, resulting in 361 intersections. The number of possible positions is in the order of 10^{170}, much more than the number of atoms in the whole universe, 10^{80}. Therefore, until a few years ago, mathematicians, experienced players and programmers agreed that it was extremely difficult to write software that would satisfactorily play Go: Their prediction was that it would take another decade to emulate the “skill” of the best chess programs now able to beat even world champions.

However, they were wrong. A new programming approach led to the defeat of the World Champion in 2015. How can the computer suddenly fill the time gap that was predicted at the time to still be a decade wide? Put simply, the computer was programmed to play more like a human—to look for, recognize, and exploit patterns of stones that alert the player to opportunities and risks. The critical role of patterns in Go is discussed in the next section.

The Critical Role of Patterns

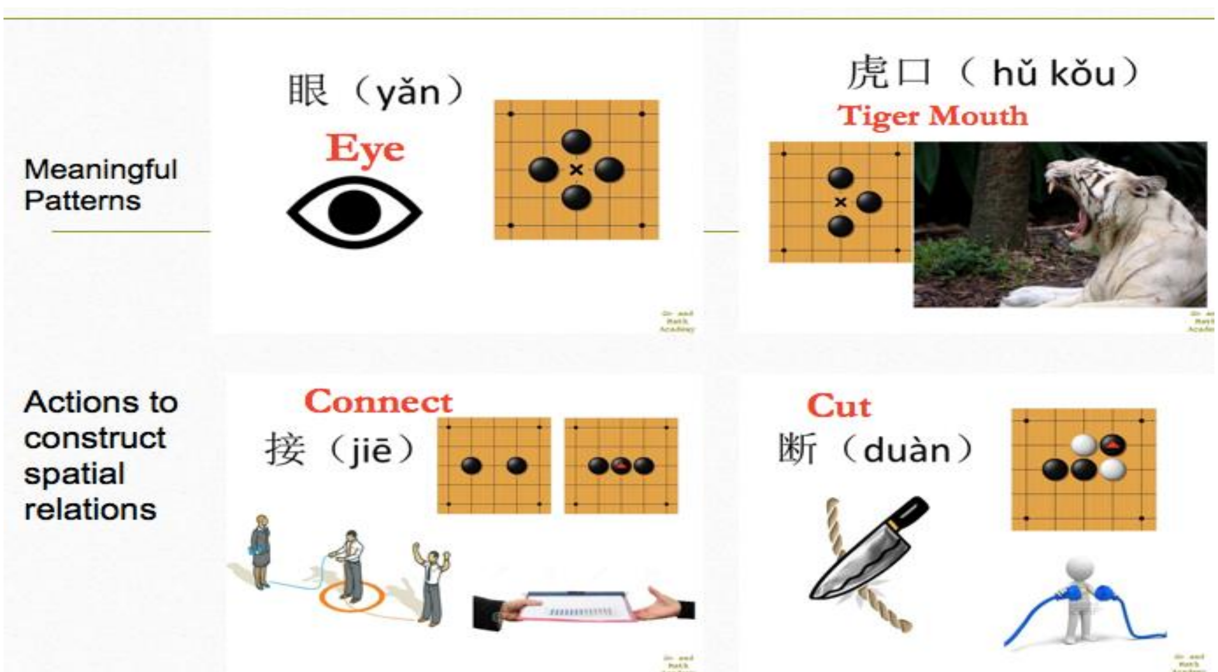
The game of Go presents an abstract landscape in which it is crucial to recognize and build patterns. The secrete to becoming a skilled Go player is to recognize and construct patterns that convey advantages to the player or the opponent; that is, the patterns may serve to expand one’s territory or to destruct the opponent’s territory.

Patterns fall into two broad categories: shapes and actions (See Figure 4.1). The first category includes labels for shapes and familiar objects in the world that serve meaningful functions. These labels are used to define certain configurations of stones to help the player recognize potential advantages for themselves or for the opponent. For example, as shown in the first row of Figure 4.1, the "eye" shape encloses a protected space; the "tiger's mouth" creates a potential to "eat" the opponent's stone.

The second category describes actions that can be performed with a set of stones, which convey strategic actions to attack or defense. For example, the second row in Figure 4.1 enlisted examples of actions ("connect" and "cut") that the player takes either to connect two groups of stones to enlarge and secure their territories or to prevent the opponent from doing so.

Figure 4.1

Basic Go Patterns



Note. The first row are examples of patterns labeled as familiar objects to describe advantageous stone configurations. The second row exemplifies moves labeled as strategic actions, such as those taken to connect their own or to separate their opponent's stones. Adapted from "Cover 60% Math Contents in K-3 with Just One Game—Weiqi/Go", by X.Wu, & X. Guo, 2018, at conference workshop, *Metropolitan Mathematics Club of Chicago*, Lisle, IL.

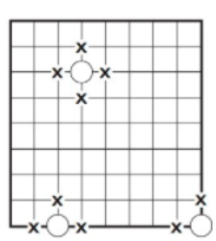
The Basic Patterns

Having learned the critical role of patterns in the game of Go, we now turn to the most fundamental concepts and patterns in the game of Go. As a reminder, the ultimate goal of the game is to occupy more territories than one's opponent. But to secure their territories, players must make sure that their own groups of stones cannot be destroyed by their opponent. Thus, Go always involves a combination of thinking offensively and defensively. A group of stones can be destroyed and taken away by the opponent if the group of stones are completely surrounded by the opponent's stones. Under such circumstances, the surrounded group of stones are "dead". Therefore, the goal of the Go game can be described as to keep more of one's stones "alive" and to take away the "life" of the opponent's stones, by surrounding them with your stones. To understand the concept of "life and death" in the game of Go, players must first understand the idea of Qi, and consecutively the basic pattern of the "eye" which is derived from the idea of Qi.

Qi. The most fundamental idea in Go is "liberty" or "Qi" (pronounced "Chee"). "Qi" in Chinese means air, the breath of life, or energy. Just as the air surrounds all living beings, Qi metaphorically describes all the unoccupied intersections that are connected to a stone or a chain of stones, which serve to keep them alive. For example, one single stone placed at the center of a board has four units of Qi; one single stone placed on a border line has three units of Qi; and a stone at a corner has two units (see Figure 4.2). This concept is illustrated below:

Figure 4.2

The Qi of a Stone on Different Locations of the Board



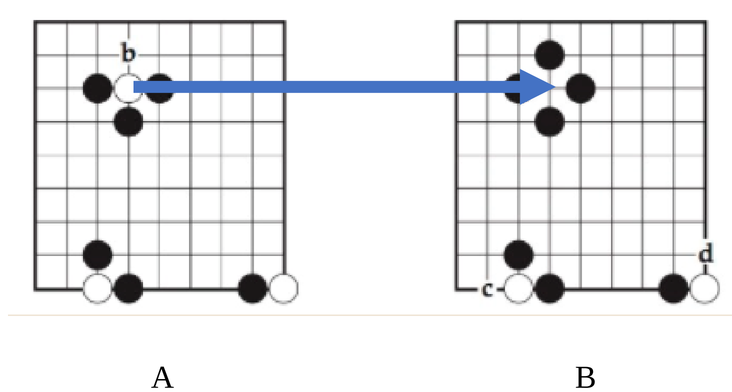
Note. The three white circles represent the placement of stones on three different locations of the board (inside, corner, side). The "x" marks the locations of Qi that are connected with the stones. Based on their locations, the stone on top has more units of Qi than the stones on the boarder and at the corner. Reprinted from *An Introduction to Go*, by British Go Association, (https://www.britgo.org/intro/intro2.html).

The players must attend to the Qi of their own stones and the opponent's stones at any moment, because the presence of Qi determines whether a group of stones is alive or dead. If a stone (or a chain of connected stones) is completely surrounded by the opponent's stones, then it has no Qi left and therefore it is dead. The opponent captures the stones and take them away. Metaphorically, the stones with no air left are suffocated and would consequently die. In other words, the function of Qi is to keep the stones alive.

The "Eye" Shape. The "eye" shape is the most basic shape of Go; it is derived directly from the concept of Qi and is usually formed through the process of occupying the Qi of the opponent's stones. Figure 4.3A and 4.3B illustrate the process of forming an "eye" shape: The black takes over the Qi of the white stones, resulting in the removal of the white; now, the empty point surrounded by the black stones becomes one "eye" for the black.

Figure 4.3

The Creation of the "Eye"



Note. Adapted from *An Introduction to Go*, by British Go Association, (https://www.britgo.org/intro/intro2.html).

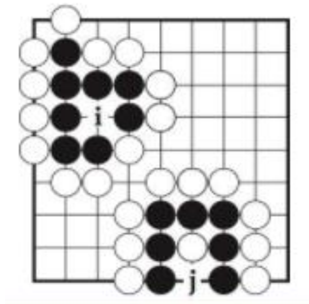
Figure 4.3A shows how each white stone has only one unit of Qi left. If black plays at position *b*, the white stone has no remaining Qi and therefore is captured (shown by the blue arrow). The empty point surrounded by the four black stones becomes an "eye" for the black, as shown in Figure 4.3B. If black plays at *c* and *d*, those white stones will also be dead and taken away. The resulting two empty points will become eyes for the black—an eye at the border line surrounded by three black stones and an eye created by two black stones at a corner.

Let us call this player who formed an "eye" shape player A. Usually, the other player, player B, can no longer place a stone in this eye of player A, because there is no "Qi" for player B's stone, being completely surrounded by the opposite color; if player B places a stone inside that "eye" shape, it will be immediately captured by player A.

In the process of exploring the "eye" shape and it functions, the players may encounter one important exception. If the empty point within this eye is the last Qi left for this chain of stones of player A, then player B can fill in that eye and capture all of player A's stones in that chain (Figure 4.4). That is to say, a group of stones that has only one eye left and no other chance to make more eyes is dead because it has only one last Qi or breath of life left. If the opponent fills in that eye, the group will be completely surrounded and the last Qi or breath of life is taken. In that case, all the stones in that group will be captured and removed from the board.

Figure 4.4

The One "Eye" Exception

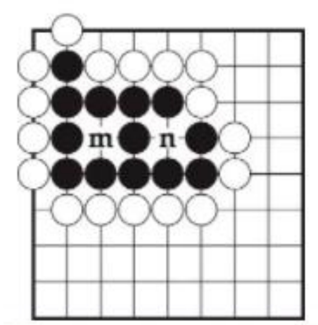


Note. If white plays at i and j, all the blacks will be captured. Both black groups are dead because they each only has only one eye. Reprinted from *An Introduction to Go*, by British Go Association, (https://www.britgo.org/intro/intro2.html).

Double Eyes: The Safety Structure. We have learned that a group with only one eye is not guaranteed to survive. The other player can surround the group and capture all the stones as shown in Figure 4.4. However, if a group of stones are connected to two or more eyes, it cannot be dead because the opponent cannot fill the eyes all at once with only one stone. In that case, this group is "alive" (Figure 4.5). We call it a *live group*.

Figure 4.5

Double Eyes: The Safety Structure



Note. Black has two eyes (centered at m and n) and therefore is alive forever. White cannot play at both m and n to take away all Qis of black. Playing at either m or n is a suicide move for white, because there is no Qi left for white, and therefore it cannot kill the black. Reprinted from *An Introduction to Go*, by British Go Association, (https://www.britgo.org/intro/intro2.html).

The double-eyes structure is one of the most critical in the game of Go. It lies at the center of any Go curriculums for beginners. However, the significance of this structure does not need to be directly imposed on the learners. Instead, I will use the emergent goals framework to illustrate how the double eye safety structure may be derived from learning the basic concepts of Qi and the “eye” shape. The derivation can be driven by the player’s game goals which naturally emerge and advance as they learn more patterns and strategies. Learners may be guided to explore a sequence of problem scenarios (as illustrated in Figure 4.3, 4.4, and 4.5) which will lead them to the discovery of this significant structure.

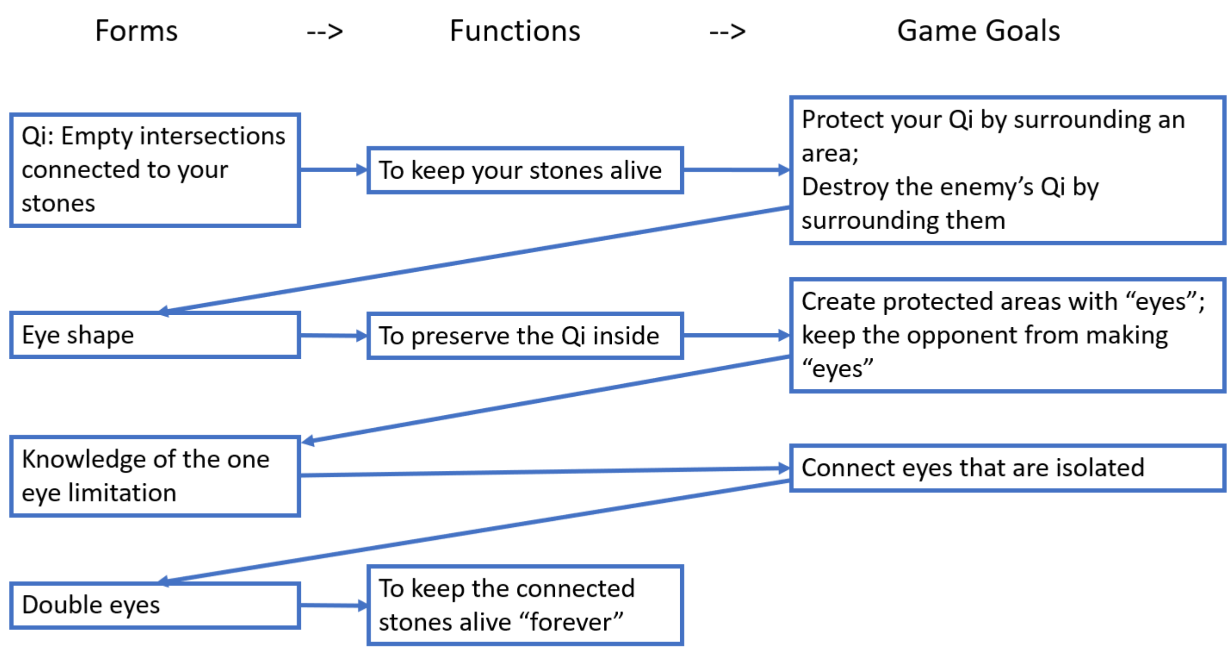
Learning Processes and Affordances

What learning processes are involved in people exploring the patterns of Go? From a sociocultural perspective, the processes that leads to the learning of patterns may involve rich interactions between learners, the emergence of new patterns, and appropriation of a new tool to address new problems and goals, etc. The emergent-goals framework developed by Saxe (1996) provides a systematic way to understand how players discover and learn new patterns as they learn to play the game.

There are three critical elements in this framework: the *forms*, which correspond to the pattern learning in Go; the *functions*, which correspond to the meaning entailed by those patterns or the proposes they serve; and the *goals* that emerge during the game, which are set by the players and can be increasingly complex as they learn more forms and functions. Figure 4.6 shows how players learn some of the most basic patterns of Go in light of this framework.

Figure 4.6

The Learning of Basic Go Patterns Explained by the Emergent Goals Framework



Note. The basic forms of Go, including Qi and the eye shape, serve important game functions and give rise to the double eyes shape. The arrows explain the reciprocal relations between forms, functions, and goals. For example, the discovery of the one eye limitation creates a shift in the game goals and thereby leads to a new form—the double eyes shape, which serves as the ultimate safety structure.

Qi is the first form players encounter in their learning process. Upon learning the function of Qi, which is to keep one's stones alive, a novice player (player A) may generate a goal to destroy the Qi of the opponent's (player B) Qi by placing stones around player B's stone. Accomplishing this goal may result in the formation of an "eye" shape for player A. Player A may also learn that this newly emerged shape serves as a means to occupy spaces on the board. Consequently, new goals emerge as Player A comes to identify these new forms and learn their functions and implications. For example, Player A may decide to create many “eye” shapes while trying to destroy Player B’s Qi.

In the process of exploring the “eye” shape and it functions, the players may encounter one important exception. The situation illustrated in Figure 4.4 shows that a group of stones cannot ultimately survive if it has only one “eye”. Therefore, learning the limitation of the functions of the “eye” shape may cause a new goal to emerge—players may begin to explore whether there is an ultimate safety structure that sustain the live of a group of stones, so that the stones may “survive forever”. This leads to the introduction of the “double eyes” shape, which is the next critical form of Go derived from the basics of Qi and Eye.

Having learned this new form and its function, the novice player may now develop a new game goal: to construct the double eye shapes and connect each group of stones to at least two eyes to keep those stones alive; they may also set a goal of keeping the opponent from creating double eye shapes.

The Affordances of Patterns for Learning to Reason. Using the derivation of the double eye shape as an example, I have shown one of the most important characteristics of this game; a few basic forms give rise to many emergent patterns with increasingly complex functions to be discovered. I suggest that this characteristic may potentially engage learners to participate in a series of reasoning processes, like the process of deriving the double eye shape from Qi and Eye. Such reasoning processes may be similar to the reasoning processes of learning many mathematics concepts. In fact, many mathematicians consider the final position of a game as a mathematical conjecture and the play as the logical steps necessary to prove that conjecture (Bo, 2019). I will further examine the reasoning processes involved in learning and applying Go patterns and discuss its links to mathematical reasoning in Chapter 6.

The Affordances of Patterns for Spatial Thinking. I suggest that that one crucial affordance of Go is making visual patterns the means to represent and solve problems, and thus making the problem-solving or sense-making activities accessible for young children. For example, the process of learning the double eye shape as the safety structure demonstrates the advantage of reasoning in terms of spatial patterns. In contrast, the learning could be more difficult if the problems and the reasoning process were represented in terms of abstract symbols. Moreover, I suggest that the game of Go share many similarities with games that have been previously shown to foster spatial abilities. For example, playing Go requires the same spatial ability as in block play (Casey et al., 2008; 2015) to recognize parts (e.g., the eye shape) from a whole and to put together purposeful spatial configurations (e.g., connect a group of stones to a safety structure) and therefore foster spatial abilities. Moreover, learning the moves of Go and their labels (e.g., "connect" and "cut" as shown in Figure 4.1) creates a learning environment infused with spatial language (Ferrara, et al., 2011) and opportunities to explore various spatial relations, which may improve children's spatial skills. For example, while "connect" refers an action to construct a spatial relationship relative to oneself, "cut" refers an action to construct a spatial relationship relative to the opponent. In some cases, the names of the patterns are also relative to the positions on the board – regarding adjacency to the boarders. Thus, one can see that recognizing such patterns is not a simple pattern recognition task. Instead, it engages learners in rather complex spatial reasoning practices at multiple dimensions, by evaluating its relationship to oneself, to the opponent, and to the board.

Higher Level Game Strategies and Reasoning Processes

Because the ultimate goal of Go is to occupy more territories than one's opponent, Go players do not simply focus on attacking and defending any individual stones. Instead, the key to the game is achieving a balance between enlarging/securing one's own territories and undermining the opponent's territories. What this means for the game strategy is that the players need to evaluate the efficiency of each move; that is, before they decide where to play, they might consider multiple options and determine which option may lead to the best outcome for winning (i.e., occupying more than 50% area of the board). Sometimes, the benefit of a move to take the lead in one area might outweigh the urgency to defend an immediate attack in another area. Under such circumstances, players might choose to "sacrifice" the group of stones under attack, and instead play a more advantageous move for controlling a larger space on the board. In addition to evaluating a move in terms of its efficiency, players also need to turn to the opponent's perspectives to predict their best next moves and play a few turns ahead in their imagination before physically playing a move. The intricate reasoning processes illustrated above are referred to as "reading" in professional Go practices.

Although the in-depth "reading" and analyses are common practices for advanced players, they are quite challenging for novice players, especially young children. Nevertheless, learning these higher-level reasoning processes not impossible with facilitation. In fact, we frequently see these reasoning processes being externalized and rehearsed in Go classrooms with the assistance of the teacher. The teacher of Go may play an important role guiding the students to engage in reasoning practices that are difficult for students alone but can be achieved with some guidance. These guidelines may involve externalizing the reasoning processes and inviting students to take part in those processes, such as exploring various move options by playing a few successive moves on a physical Go board to reveal and evaluate their outcomes.

Score-counting: Another Essential Component of the Game

In addition to the endless emergent patterns that could promote reasoning, another critical feature of Go ties the game directly to both spatial and mathematical practices—the counting of spaces. This critical feature concerns the ultimate game goal: to occupy more spaces on the board than the opponent. Unlike chess and many other board games which come to a clear-cut ending when an action is completed, such as eliminating a certain piece or reaching the finish line, in the game of Go however, the counting of each player's occupied spaces is used to determine who wins, who is leading during a game, and when to resign.

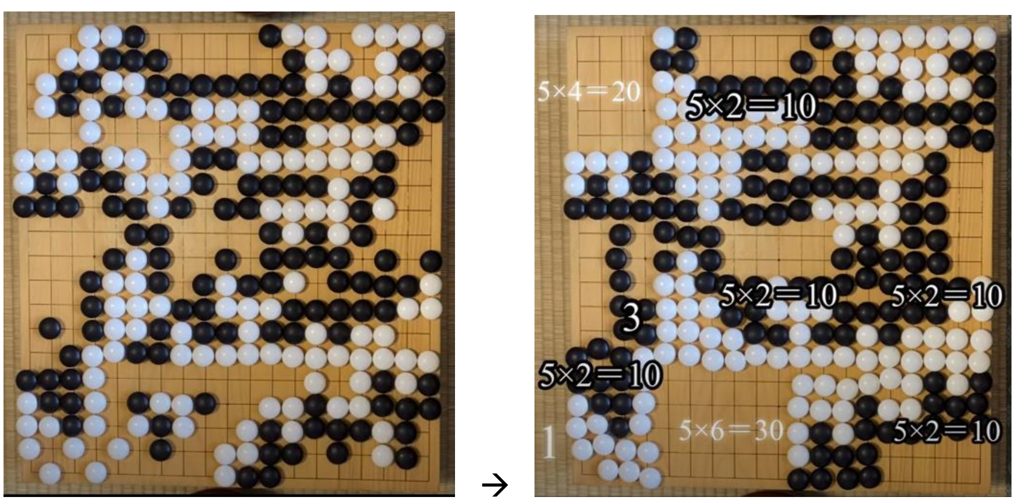
The counting is critical because players' scores are often incredibly close; at competitive levels, the winners sometimes lead by just one or two points on a 19x19 board, which makes battling over a small area exciting. Moreover, experienced players not only count their occupied spaces at the end of games but also mentally keep track of their emerging territories during the game and adjust their game strategies accordingly, such as being more aggressive or defensive at different phases of the game.

Go players' final scores consist of the total number of live stones they constructed on the board plus the empty points in their territories, within which their opponents have zero chance to construct their own spaces; the steps to fill those empty spaces are thus eliminated but treated as if they were filled. Since Go boards typically contains hundreds of spaces to be occupied (the standard 19x19 board contains 361 spaces), the final scores need to be counted strategically and efficiently at the end of each game. Go players themselves are responsible for rearranging their stones in certain ways so that both players as well as the observers can all quickly recognize and agree upon the finals scores of each player.

To count the scores efficiently, players have developed various ways to flexibly utilize features of the board, like the grid, and rearrange stones into certain Go configurations that represent numbers in group of 10s, 20s, etc., to facilitate counting (see Figure 4.7). Panel A shows the board at the end of a game. Panel B shows the board after reorganization, with a pattern that has been optimized to help the players quickly determine who has more stones. The reorganization resulted in square-shaped territories of 10, 20, and 30, which was much easier to count than a less structured pattern would be. In this case, black and white both have total square-shaped territories of 50; therefore, in counting, the two players can cancel out 50 stones. Afterwards, black has remaining score of 3, and white has 1 left. Therefore, black won by 2 on the board.

Figure 4.7

An Example of Reorganization and Final Score-Counting



A B

Note. An end-of-game Go board before and after reorganization. Adapted from *Territory Rearrange Competition*, by X. Guo, 2021, (https://www.youtube.com/watch?v=Knoj1xQuk7I).

Therefore, this frequently needed Go activity engages players in spatial and mathematical practices constantly. For professional players, score counting can be done skillfully in a matter of seconds. Nevertheless, mastering such skills takes time and practice, and has potentials to cultivate young children's spatial and mathematics skills. To study such potentials, I will explore how young children develop spatial and mathematics skills through engaging in score-counting activities in Chapter 7.

Summary

In this chapter, I introduced two of the most unique characteristics of Go—1) endless patterns which requires active reasoning to learn and apply; 2) score-counting which play a role throughout the game and require skillful spatial and mathematical practices. The simple rules of Go not also provide a low entry point for beginners but also give rise to numerous patterns and endless possibilities that promote complex reasoning. My examinations of the rules, characteristics, and the process of learning Go patterns shed light on several affordances for developing mathematical reasoning skills and spatial skills. In addition, score-counting also plays an important role in Go. The game requires efficient score-counting strategies which has potential to cultivate spatial and mathematics skills. These insights motivate my approaches to studying the game of Go as a tool for learning which I will elaborate in the next chapter.

Chapter 5: Overview of the Dissertation Research

In this dissertation, I consider the game of Go as a cultural tool to examine its potential to engage children in spatial reasoning and influence the way children learn mathematics. I view mathematical practices in the game of Go as situated in activities in which learners develop mathematical understanding by appropriating Go pieces to create meaningful patterns, to solve emerging problems, to take on new roles in the learning and playing, and through interacting with other players. In addition, mastering the game requires constantly playing with others, reflecting and discussing with a community, solving dynamic problems, discovering new meanings from Go patterns, and repurposing the Go patterns to address new goals. I will use the emergent goals framework to capture these rich interactions with other participants, artifacts (e.g., Go board and Go pieces that constituent meaningful patterns), and norms (e.g., goals, strategies, and reasoning processes involved in learning and applying Go patterns).

Research Foci

To understand how Go may engage young children in spatial and mathematical practices, I will examine two important aspects in the teaching and learning of Go: 1) the reasoning process in learning Go patterns; 2) the end of the game score-counting process. Understanding these two important aspects of Go may contribute to our knowledge of spatial thinking and math learning—the mechanism of which spatial thinking facilitate math learning.

First of all, studying how the game of Go engage children in reasoning with patterns might contribute to the spatial modeling account of math learning (Hawes & Ansari, 2020). Reasoning with patterns is an important part of mathematics at the elementary level. While there are other games, like chess, that engage children in reasoning with patterns, Go might offer more opportunities for beginners, especially young children, to reason with patterns, because Go has minimal rules and all Go pieces are the same, As I introduced in Chapter 4, a few basic rules give rise to countless patterns that entice players to explore over a lifetime. I used the derivation of the "double eyes" safety structure from the basic concept of "Qi" as an example to illustrate the importance of reasoning involved in the learning and application of patterns. By examining how young children reason with patterns as they learn Go, I will identify specific ways in which the reasoning processes can be modeled spatially, such as through manipulating the Go pieces, and identify any potential features of the spatial models which could potentially provide an explanation for why modeling the reasoning process spatially might facilitate math learning.

Second, also introduced in Chapter 4 was another essential component of the game—score-counting. These activities are not only mathematical, but also very spatial. This is a good example that support the spatial representation of numbers' account (Hawes & Ansari, 2020), as the game introduces a variety of spatial representations of numbers as tools and promotes creative and efficient spatial solutions to score-counting problems. Therefore, examining how students arrange their stones for final score counting and how those practices change over the course are critical for understanding the ways in which learning the game of Go may foster spatial and mathematical practices for young children.

The Go and Math Curriculum

The Go and Math curriculum (Wu & Guo, 2018) that I investigated incorporated a large portion of the Common Core math standards (CCSS, 2010) in its design to facilitate children to think mathematically as they work on counting their scores. It is designed and implemented by Xinming Guo, a professional mathematics and Go instructor. He won the teacher-of-the-year award from the American Go Association in 2015 for his development of this innovative game-based Go and Math curriculum.

The curriculum involves activities with Go that align with many common core math standards at the elementary level, which include counting and understanding numbers, arithmetic operations, and algebraic thinking, etc. I will discuss the curriculum more specifically in Chapter 7. I will only be focusing on the alignment ats the Grade 2 level for the purpose of this study. Because this is a straightforward counting task that would be familiar to most children of this age (regardless of their knowledge of Go), and because the counting strategies largely overlap with the mathematics being learned at Grade 2, I can investigate whether and how children's mathematics skills associated with counting may be affected by their participation in Go.

Site Information and Participants

My research has taken place at three sites across the greater Chicago area. In all cases, the preliminary research has been a partnership with Xinming Guo. He is the teacher, following his Math and Go curriculum, and I am in the class as an observer, researcher, and assistant to Guo.

The total number of participants across the three sites is 57. Participants are students either in Grade 2 or 3, who enrolled in the Go and Math course in one of the sites listed below. The age range of students is between 7 and 9 years old.

Site A

The Go and Math curriculum was offered as a Saturday enrichment program in Fall 2016. The students in Grade 2-3 enrolled in an eight-week series of Go game activities. There were 6 participants at this site.

Site B

Site B is an elementary school. All four Grade 2 math teachers had their classes participate in the study during 2017 Fall. There were 44 participants at this site. Approximately 10 students from each classroom attended the session. We coordinated with the teachers to form two combined classrooms, each consisting of two classes, around 20 participants. Mr. Guo taught once per week at each combined classroom, for 10 weeks. Each class session lasted for one hour.

Site C

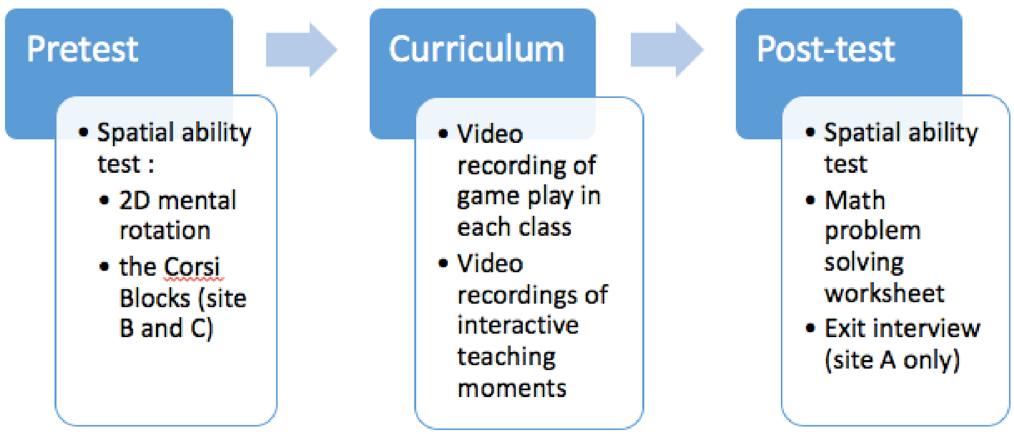
A Go and Math course was offered at a weekend cultural enrichment program which features the Chinese language and arts, Go, and other aspects of Chinese culture. All the 7 students who enrolled in the course participated in the study. The Go and Math course was aimed at Grade 2 and 3 students who are interested in learning the game of Go while practicing math at the same time. The course took place every Sunday for one hour in 2017 Fall, for 12 weeks.

Study Design

The study consists of three phases: pre-assessment, curriculum, and post-assessment. Figure 5.1 is an illustration of the content of the three phases.

Figure 5.1

Study Design and Content of Each Research Phase



The pretest and the posttest on spatial abilities were administered to assess the influences of spatial ability on math outcomes and to determine whether spatial skills changed over the course of the class.

Pre-Assessment

Students individually took a brief test on paper during the second week of instructions. The test included spatial thinking items as well as algebra problems. The spatial test included a classic 2D mental rotation task (PMA; Thurstone & Thurstone, 1962, 2002) and the Corsi Blocks—a traditional spatial working memory test (Corsi,1972; Kessels, et.al., 2000). The 2D

MRT items were retrieved from the Spatial Intelligence and Learning center website and uploaded as supplementary files (refer to "Rotation Task Form B and BA"). The Corsi blocks were projected on a large screen in the classroom when the students recorded the order of highlighted blocks with pen and paper. The test items were downloaded from http://www.psytoolkit.org/experiment-library/corsi.html.

Data Collection in the Classroom

Throughout the course, I took notes in an observation book. During each class, each student had opportunities to play the Go game with a partner. I took pictures or video recordings during each class. In addition, I also recorded Mr. Guo's teaching each week. These recordings included interactive learning moments, such as students answering questions, and students coming up to interact with Mr. Guo's Go demo board.

Post-Assessment

During the last class, I gave the same spatial tests to students as in the pre-test. I also gave a math problem solving worksheet with transfer problems mostly concerned with pattern recognition. In addition, due to the smaller class size at site A, I was able to ask the students there to participate in an exit interview. I assigned students into two groups of three. I asked each group a few questions concerning whether and how they see math in the game of Go. Each interview took 6-8 minutes to complete. The interviews were audio recorded.

Chapter 6: Exploring Spatial and Mathematical Practices in Learning to Play Go

In this chapter, I analyzed a Go and Math curriculum for Grade 3 students which was implemented at a weekend enrichment program to shed light on how Go may engage young children in spatial and mathematical reasoning practices. Because this was a small group (only 5 students) and because each session lasted for two hours, a significant amount of time was dedicated to reasoning about patterns, which took place in the rich interactions between the teacher and the individual students. This site therefore became an ideal setting to investigate the kinds of reasoning involved in the teaching and learning of Go patterns and the spatial features of Go that might facilitate those reasoning processes. My hypothesis is that the reasoning processes can be externalized by manipulating spatial patterns, and thereby making the reasoning processes accessible to young children.

To explore what kinds of reasoning are involved in teaching and learning of Go, I drew on the mathematical reasoning literature to identify practices emerging from the teaching and learning of Go which are similar to reasoning processes that occur in mathematical practices and learning. To examine whether and how the spatial features of Go might facilitate the reasoning processes, I utilized the emergent goals framework, which has been used to study other boardgames like dominoes as cultural tools that foster the learning of mathematics. This framework allowed me to identify the use of *spatial forms* in the game of Go that served as an important means to structure the reasoning activities involved in the teaching and learning of Go. The spatial forms refer to the "shapes" (meaningful structures composed of a few Go pieces, as exemplified in Chapter 4) the teacher set up on the demo-board, in which new problems were embedded.

To understand whether and how spatial forms might facilitate the reasoning processes, I examined the following questions: What and how were spatial forms used in the teaching of Go? What were the mathematical reasoning practices being facilitated by those spatial forms? What did the students learn from interacting with the spatial forms? To address these questions, my analyses consist of three parts: the teacher's acts during teaching and learning, student-teacher interactions during teaching and learning, and students' interactions during cooperative play.

Method

Site and Participants

This course was implemented in the northern suburbs of Chicago. It was offered as part of a gifted children's program, which was operated by a local university and conducted on the weekends. Only children who were admitted to the gifted program according to its selection criteria were eligible to take this course. The course was advertised as a game-based curriculum for third grade students to improve their mathematics skills through playing a board game. Five third-grade students, three males and two females, enrolled in this enrichment course. All the students participated in the study.

Implementation

The Go and Math curriculum consisted of eight weeks of classes. The class took place once every Saturday. Each class lasted for two hours. The two hours of class consisted of a combination of interactive instructions, game play, and other related activities. Please refer to the Appendix I for a weekly description of the curriculum.

Interactive Instruction. In each class, the teacher taught some new game strategies or meaningful Go structures. Most of instructions were interactive. The teacher took a learner-centered approach to his teaching; the content and pace of teaching was partially influenced by the students' responses, such as their questions and the levels of capabilities they demonstrated. The interactive instructions were conducted on a 11x11 demonstration board most of the time, and on a computer occasionally. The demonstration board held magnetic stones. Thus, the teacher was able to hold the board vertically facing the students, and flexibly demonstrate the stone structures/moves. The computer had the Go playing program installed and had its screen projected in the classroom so that all the students could see the Go program. Both platforms were very inviting for the students to interact with during instructions; most students frequently and spontaneously went up to the demo-board or the computer to demonstrate a point being made or to test out their thoughts in response to the teacher's questions during instructions.

Number-Telling Activity. In 4 out of the 8 classes, the teacher prepared a series of number-telling activities for the students. The number-telling activities asked the students to quickly recognize the total number of black or white stones on a Go board shown on the computer screen. Each screenshot of the Go board was presented for only a few seconds, so that the students had to use more efficient strategies than counting by ones or twos. This activity was intended for students to practice number-telling by remembering and recognizing certain shapes associated with numbers, taking advantage of arrays, or mentally adapting shapes to make arrays, etc. Although this highly visuospatial and mathematical activity was not directly related to learning Go strategies, it was meant to cultivate a skill frequently used by Go players to estimate who was winning during a game, and to determine each player's final score at the end of games.

Game Play. Playing the game of Go took multiple forms in this course. Sometimes, the students played in pairs as individuals against each other. Sometimes, the students played their individual games against the teacher, as the teacher took turns to play against each of them. At other times, the students formed a team of two or three to play together against the computer or the teacher; I refer to this kind of play in teams as *cooperative* play.

Other Related Activities. The course also included a series of other related activities such as watching a Go themed anime, playing a trivia game, and a picture-based reasoning game invented by the teacher.

Data Collection

Data come from video recordings of each Go class. I set up a camera facing the teacher and the front of the classroom during instructions and other teacher-led activities, to capture the interactions between the teacher and the students around the demo-board and on the computer. I moved about in the classroom and video recorded a game from each pair/group of students.

Data Analyses

Data Selection. I selected the episodes of interactive instructions and cooperative play for in-depth analyses. These episodes contain rich interactions between the teacher and the students around Go patterns and reveal the reasoning processes involved in learning and applying Go patterns. The number-telling activities and other activities had other focuses and were therefore eliminated from the analyses.

Interactive Instructions. Interactive instructions occurred in weeks 3, 5, 6, and 7. The first 2 weeks were only observed and not video recorded because a 2-week procedure was required to obtain consent for video recording. Minimum interactive instructions occurred during these initial two weeks as the content was mainly introductory (e.g., the "Qi" concept and the basic rules). Week 4 was not included as it was led by me instead of the teacher. Week 8 was the last week of the course and the parents' visiting week. No new content was taught that week and much time was given for the children to teach their parents. During week 3, 5, 6, and 7, I recorded and analyzed the interactive instructions, which focused on learning new moves, strategies, or meaningful stone configurations. The episodes often began by the teacher saying, “today we will learn a new structure”, or “let’s look at this new situation”, etc. The episodes then were followed by a series of questions proposed by the teacher, and students often came up to the demo-board or computer to demonstrate their thoughts. The episodes usually ended when the teacher said “now, it’s time to play the game”, as he put the demo-board away and transitioned to Go playing sessions. These episodes lasted between 20 – 40 minutes each week.

Cooperative Play. Although Go playing in this course took multiple forms (e.g., between individuals, individuals against the teacher, etc.), I only selected the cooperative play episodes in which two or three students formed a team to play against the computer or the teacher for analyses. In these situations, the students spontaneously communicated their thoughts aloud and explored multiple choices of moves before reaching a consensus on the next move. In the teacher’s turns, he also verbalized his strategies and occasionally guided the students by evaluating a move for them or giving a hint. Therefore, I was able to capture rich data from these episodes about the reasoning processes that led to learning.

Transcription. I first transcribed the selected episodes of interactive instructions and cooperative play. I used the turn of talk as the unit of analysis. A code was assigned to each turn of talk. A turn of talk started when a student or the teacher began to talk and ends when they finished the talk or when the talk was taken over by another student. Some turns of talk were accompanied by an action on the Go board; in such circumstances, I would write a line of description about the action and the context; the descriptions were placed in parenthesis after the talk. In some other cases, a student would come to the board to make a move without saying anything. Under such circumstances, a line of description about what the student did would be included in the transcript, put in parenthesis, and treated as a turn of talk when coding.

Coding. My coding was conducted in several phases. First, I used a bottom-up grounded theory approach in my initial coding. I paid close attention to the surroundings, the materials, the situational cues s, etc., which were available to the students in the moment, and the patterns of interactions that emerged from my data.

In my subsequent analyses, I drew on the mathematical reasoning literature (Jeannotte & Kieran, 2017; Knuth et al., 2019; Lannin, 2005; Mata-Pereira & da Ponte, 2017; A. J. Stylianides, 2007; G. J. Stylianides & Stylianides, 2009) to specify the practices emerging from the teaching and learning of Go that are similar to reasoning processes that occur in mathematical practices and learning. I also utilized the emergent goals framework to analyze whether and how certain spatial forms that emerged from my data serve as a way to aid the reasoning processes.

Finally, I drew a parallel between the students' practices that emerged from interactive instructions and those embedded in cooperative play. By comparing the students' behaviors across the two settings, I could explore whether students developed certain practices from learning Go and transferred those practices into their own play.

Initial Coding. My initial coding focused on describing the actions of the teacher or the student at each turn, which were as simple as "Teacher Question", "Student Response", "Student Play", etc. In addition, when applicable, I also noted the content of the questions and responses, which were mostly about a pattern or a shape. For example, if the turn of talk involved a teacher's question about the "eye" pattern, I coded the turn of talk as "teacher question on eyes". Please refer to Table 6.1 for a sample of my initial coding.

Table 6. 1

Sample Initial Coding of An Interactive Instructional Excerpt

Shape 1. "L" shape with three empty points	
Teacher: OK. So How can this red group make two eyes?	Teacher question on "eyes"
(Students raised hands)	Student response
Teacher: OK Bob wants to play.	Teacher invites Student to play
(Bob played)	Student play
Teacher: Do you agree?	Teacher seeking consensus
Students: Yeah.	Student response
Teacher: This shape, remember it?	Teacher guiding students to observe a shape
Students: It's a L.	Student response
Teacher: How many empty points? (finger going over the empty points)	Teacher question
Students: three.	Student response
Teacher. Yes, when there are three empty points –	Teacher confirmation
Jay: -you can make two eyes.	Student bring up "eyes"
Teacher: You have two shapes. One is a shape like this, another shape is this (drawing a straight line and an "L" on the black board). Which one can match this? This first one or the second?	Teacher guides students to match the L shape with a class of shapes that share the same feature.
Student : Second, second, second..	Student response
Teacher: Yes, the second one can match this. Now, take a look at this shape	Teacher transitioning to another shape

As my coding continued, I noticed that some codes appeared repeatedly and seemed to serve an important role. The underlined codes in the right column of Table 6.1 present examples of codes that I used repeatedly. These codes denote purposeful interactions with shapes. For example, the codes "Teacher transitioning to another shape", and the associated teacher acts (e.g., switching between similar shapes), were used frequently to present and connect the Go patterns being taught. These observations led me to develop several categories of codes to capture emerging themes in the data.

Subsequent Coding. I subsequently refined the scope of questions to focus on one key learning objective/tool—the "shapes" (i.e., a series of stone configurations with the same pattern that entail meanings and strategies) and three aspects of the interactions: 1) the role of the teacher in structuring the teaching and learning of Go patterns; 2) the reasoning practices emerging from the student-teacher interactions; 3) the spontaneous reasoning practices emerging from the corporative play among students. All three aspects of the interactions were associated with the reasoning processes of deriving meanings and strategies from interacting with the shapes. The "shapes" emerged not only as a key learning objective but also as a tool because they served as critical means to externalize the reasoning processes involved in the teaching and learning.

To examine the use of "shapes" in the three types of interactions, I drew on the emergent-goals framework to identify specific uses of the "shapes" and analyze their roles in each type of interactions—whether and how the use of those shapes served as means to externalize the reasoning processes and facilitate the interactions among the teacher and the students. (Please refer to Figure 4.6 in Chapter 4 for a theoretical illustration of the process of learning the "double eye" shape using the emergent-goals framework.) In addition, I drew on the mathematical reasoning literature to specify the reasoning practices emerging from the teaching and learning of Go patterns that are similar to those that occur in mathematical practices and learning.

Identifying Spatial and Mathematical Reasoning Practices. Through multiple rounds of coding of teacher acts, teacher-student interactions, and student practices during corporative play, I noticed that there were a series of Go learning and playing practices which resembled mathematical reasoning practices. To identify and examine those practices that emerged from my data, I drew on the mathematical reasoning literature (Jeannotte & Kieran, 2017; A. J. Stylianides, 2007; G. J. Stylianides & Stylianides, 2009) to guide my subsequent coding. Mathematical reasoning is broadly interpreted to include the range of activities used to make sense of and establish mathematical knowledge (Jeannotte & Kieran, 2017; Mata-Pereira & da Ponte, 2017). These activities include observing patterns, classifying, making conjectures (Reid, 2002), testing examples (Knuth et al., 2019), justifying and generalizing (Ellis, 2007; Lannin, 2005; Widjaja et al., 2020), etc. Table 6.2 presents the definitions of mathematical and spatial reasoning practices according to existing literature in parallel with those emerged in the context of my Go analysis.

Table 6.2

Spatial and Mathematical Reasoning Coding Table

Parameter	Code	Definition (Literature)	Definition in Go	Example
Spatial Reasoning	Pattern Observation (PO)	By searching for similarities and differences, infer a narrative about a relation between spatial/mathematical objects.	Observing that several specific Go structures share the same common features. This can include comparing a few structures and discussing their similarities and differences.	Teacher: "Look at this situation. How many empty points are contained in the red?" Students: 3 Teacher: "Look at this new situation, how many empty points?" Students: 3
	Classifying	By searching for similarities and differences between mathematical objects, infer a narrative about a class of objects based on their mathematical properties.	Grouping together a set of Go structures based on their common features and infer a narrative about the group of Go structures based on their properties.	Teacher: "Is this real eye or fake eye?" Students: "Fake!"
Mathematical Reasoning	Conjecturing	Conjecturing involves developing statements	Making a statement about the properties	Teacher: "In this situation, can red

		about a mathematical relation that can be validated to be true or false. Conjecturing launches justifications.	of a Go structure, or a set of structures that share common features, which are thought to be true and subject to testing.	survive?" Student: "Yes! It can make two eyes!"
	Justifying	Justifying refers to the process of searching for data, warrant, or backing to validate whether a conjecture is true.	Using examples, counterexamples, or a set of arguments to validate that a conjecture is true. An articulative social process so that more than one person can accept the conjecture to be true, or refutable.	Teacher: "Can you verify?" Student: "If the red goes here, blue will go here; if the red goes here, blue will go here; if the red goes here, blue will go here… "(showed the outcome of all possible moves).
	Generalizing	The transportation of mathematical relations from given sets to a larger set (Stylianides, 2008).	Producing a statement about a rule which applies to all the Go features that share a set of common features. The rule is obtained from exploring a subset of Go structures which share the common features.	Teacher: "So, whenever you have 3 empty points, you can survive." Teacher: "Which move is the most important?" Students: "The middle point!" Teacher: "Yes, whenever you have a situation like this, you should play at the middle point."

Spatial Foci. The mathematical reasoning literature highlights pattern observation and classification as important steps in the reasoning process. In my study of Go the classroom, I found a series of repeated interactions with "shapes" (meaningful Go configurations) which can be categorized as observation and classification activities. Because these activities concerned observing and classifying spatial patterns, I used "spatial reasoning" as the parameter to define the pattern observation and classifying activities that occurred in the teaching and learning of Go.

The parameter may help us interpret the potential role of spatial reasoning in contributing to the mathematical reasoning practices in this context.

Mathematical Foci. The mathematical reasoning practices in my study focuses on three interrelated actions: conjecturing, generalizing, and justifying. These are the fundamental reasoning practices that play a vital role in the learning of mathematics at the elementary level (Lesseig, 2016). *Conjecturing* involves developing statements about a mathematical relation which can be validated as true or false (Lannin, et al., 2011). *Generalizing* involves identifying commonalities across cases or extending mathematical reasoning to consider a broader range of objects (Ellis, 2011). *Justifying* is the act of developing arguments to demonstrate the truth (or falsehood) of a claim using mathematical reasoning (Staples et al., 2012).

Conceptualizing the spatial and mathematical activities of Go in this way has great value: First, it highlights how exploration and inductive reasoning in the learning of Go can support sense making around core math ideas and can lead to more formal justifications (this is especially true in Grade 2-3 classrooms). Second, this definition focuses on the explanatory and discovery roles of mathematical reasoning which demand more attention in mathematics education.

Analyses of the Teacher's Acts: How were Spatial Forms Used in the Teaching of Go?

The learning of Go mainly involves recognizing meaningful patterns, exploring their implications, and subsequently constructing and responding to these patterns in the play. As I discussed in Chapter 4, an essential part of teaching Go is to support learners' reasoning about Go patterns, which could be difficult for beginner players alone but can be achieved with guidance. Thus, in my analyses of interactive instructional episodes, I first examined the codes assigned to the teacher independently from the codes of the students and explored how the teacher facilitated the students to learn and reason about each new pattern.

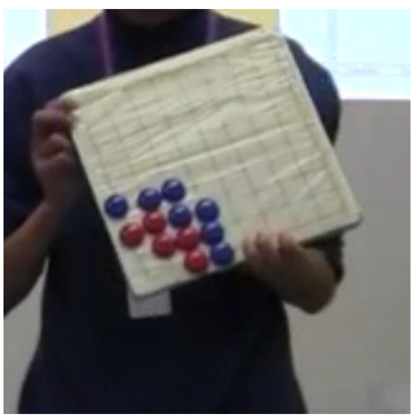
The coding revealed that the majority of teacher acts evolved around shapes that the teacher constructed on the demo-board. The coded teacher acts around shapes included “drawing attention to the feature of a new shape”, “transitioning to a different shape”, and “comparing shapes”, etc. These codes were similar in that they all suggested that the use of shapes played a major role in the teaching. Moreover, I found that these teacher acts around shapes were mostly dynamic and fluid. That is, the teacher frequently made slight changes to existing shapes by moving just a few pieces around or by switching a few Go pieces back and forth. In doing so, the majority of existing shape features were carried into the new shape and the differences were highlighted. Because these dynamic and fluid teacher acts around shapes emerged as a predominant pattern in my analyses, I categorized those teacher acts as *shape transformation.*

To examine whether and how the teacher’s use of shapes and the shape transformation method might support students in engaging in mathematical reasoning practices, I analyzed how a Go pattern was taught during each instructional episode in several steps. First, I extracted the sequences in which different shapes were presented to the students and uncovered the reasoning processes embedded in those sequences. Second, I drew on the spatial and mathematical reasoning framework, using codes such as “pattern observation” and “classifying”, to examine what functions the teacher’s use of shapes and shape transformation might serve in facilitating students in the reasoning activities involved in learning Go patterns.

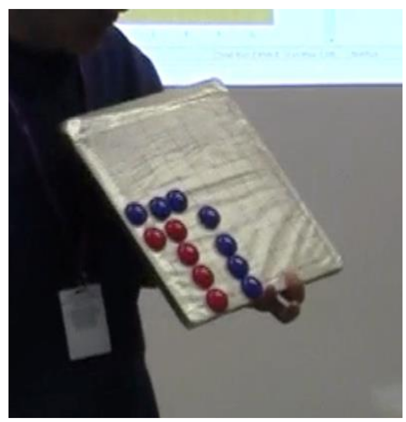
Next, I present an interactive episode where I illustrate the different uses of shape transformation and the various goals they served in teaching about shapes and the underlying reasoning process. This episode occurred during Week 4 of the curriculum. The teacher's use of shape transformation was underlined throughout the excerpts of the episode. The episode focused on a series of teaching and learning around one of the most critical patterns for beginner learners. Although this pattern takes on multiple forms on a Go board, all the forms share the same feature of having three or more connected stones surrounded by the same color of stones (Figure 6.1). Such a pattern requires that the player place a piece at the middle point to form a safety structure (the double "eyes" shape, refer to Chapter 4), so that this group could not be surrounded and hence will survive for the rest of the game; the opponent, on the other hand, must occupy the middle point to overturn the situation.

Figure 6.1

Two Similar Shapes



Shape 1: "L" Shape



Shape 2: "Line" Shape

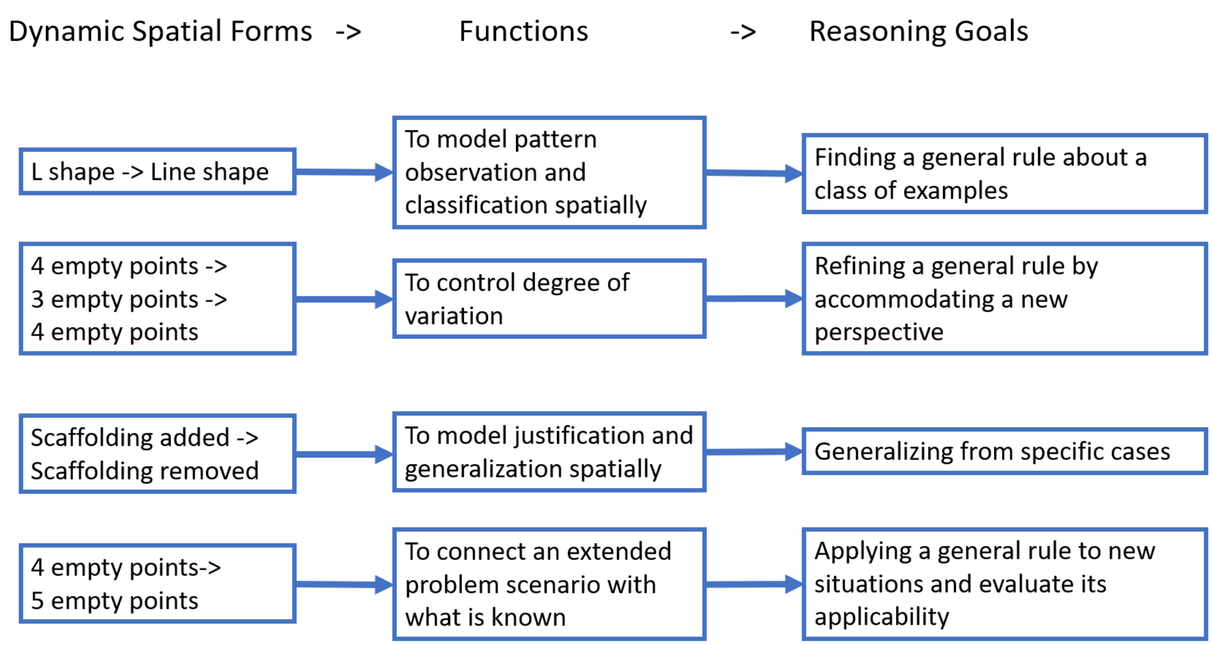
Note. Shape 1 on the left is a "L" shape; shape 2 on the right is a straight-line shape. The "L" and the straight line referred to the negative spaces—the empty points surrounded by the red pieces. Both shapes contain three empty points inside the red's boundary.

I aimed to uncover the reasoning processes involved in learning the general rule entailed by the pattern and the role of shape transformation in facilitating the reasoning processes. I adapted the emergent goals framework to organize my findings about the functions of the shape transformation method and the goals they served in the teaching (see Figure 6.2).

Next, I present my analyses in four parts in accordance with the four functions and reasoning goals shown in Figure 6.2; each part concentrates on illustrating one aspect of the reasoning process involved in developing/applying a general rule and the role of shape transformation in facilitating the teaching and learning of it.

Figure 6.2

The Functions of Shape Transformation in Serving Various Teaching Goals

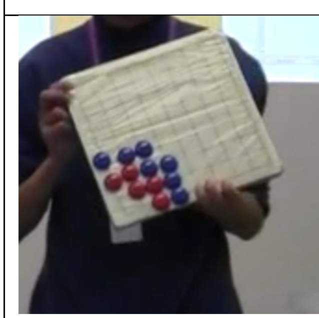


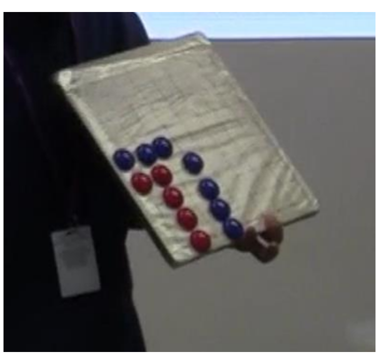
Note. The "dynamic spatial forms" shown in the left column specified the shapes being transformed as the teacher modeled the reasoning processes spatially. The arrows indicate the order in which the shapes were presented to the students, which was achieved by moving just a few pieces from a former shape. The "functions" highlight what the shape transformations afford in facilitating the teaching of reasoning processes shown in the "reasoning goals" column.

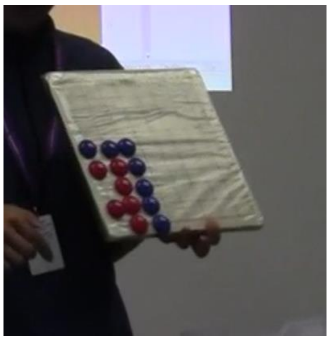
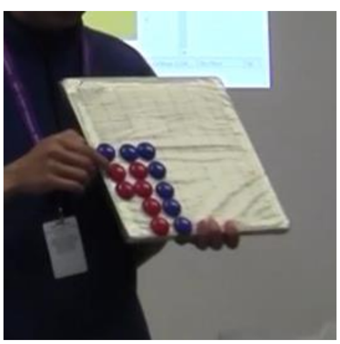
Part 1. Observing Patterns and Classifying to Find a General Rule

In this section, I discuss how the teacher used shape transformation to create multiple similar shape problems, guide students to identify the similarity between shapes, and ultimately discover a general rule which applies to all the shapes that share the same feature. Prior to this episode, the students did a preparatory activity, in which they explored how many shapes could be created by three connected pieces on a Go board. They found two shapes, an "L" and a straight line. The teacher drew those two shapes on the blackboard. Then the teacher constructed those two shapes (Figure 6.1) on the demo board and presented those shapes consecutively to the students and guided them to discover the general rule about where to play, which applies to both of these two shapes. In Excerpt 1, I present the detailed teacher-student interactions around these two shapes. The teacher's turns of talk and actions around a certain shape are underlined.

Excerpt 1.

Transcript	**Code**
 **Shape 1. "L" shape with three empty points** Teacher: OK. So How can this red group make two eyes? (Students raised hands) Teacher: OK Bob wants to play. (Bob played)	Teacher question on "eyes" Student response Teacher invites student to play Student play

Teacher: Do you agree? Student: yeah. Teacher: This shape, remember it? Student: It's a L. Teacher: How many empty points? (finger going over the empty points) Student: three. Teacher. Yes, when there are three empty points – Jay: -you can make two eyes. Teacher: You have two shapes. One is a shape like this, another shape is this (pointing to the straight line and the "L" on the blackboard). Which one can match this? This first one or the second? Student: Second, second, second.. Teacher: Yes, the second one can match this. Now, take a look at this shape.  **Shape 2. Straight line with three empty points**	Teacher seeking consensus Student response Teacher guides student to observe a shape; Student response Teacher guides pattern observation Student response Teacher confirmation Student response Classifying: Teacher guides Student to classify two shapes (L and straight line) in one category (both encompassing 3 empty points). Student response Shape transformation: Teacher transitioning to another shape

Jay: It can make two eyes. Teacher: can this shape make two eyes? Dan, come to try. (Dan came to play) Teacher: Do you agree? (looking at other students) Yes, now you have two eyes. 	Student bring up "eyes" Teacher question on "eyes" Student play Teacher seeking consensus
Teacher: So when this situation happens, what's the score for red? How many pieces of red on board? … Teacher: <u>OK if the shape looks like this,</u> (moved one piece up) – 	Teacher question on score Student response <u>Shape transformation: transitioning to another shape which has only two empty points inside</u>
Jay: You cannot make two eyes; you cannot make two eyes. Teacher:- can this group make two eyes?	Student bring up "eyes" Teacher question on "eyes"

Student: No you cannot.	Student response
Teacher: you cannot make two eyes. That means. so what's the score for red?	Teacher confirm+ question
Student: Zero!	Student response
Teacher: Zero! (nodding) You have to make two eyes.	Teacher confirmation
<u>Teacher: OK, so we have explored this shape (L) and this shape (straight line) (pointing to the drawing on blackboard).</u>	<u>Classifying: Teacher matches the two shapes explored to the same class of shapes that share the same feature and rule</u>

Understanding the Reasoning Process. Now, let us uncover the reasoning process involved in learning the general rule and find out how the teacher used shape transformation to facilitate the process. First, prior to the excerpt, the teacher asked the students to explore all the shapes consisting of three connected points, which include Shape 1, Shape 2, and no other possibility. That is:

Shapes that contain three empty points inside → Shape 1 "L" and Shape 2 "Line"

Then, in this excerpt, the teacher first presented the "L" shape in a defense scenario, where blue had surrounded all the red pieces; therefore, if red does not respond appropriately, blue can capture all the red pieces. The appropriate defensive move is for red to place a piece at the middle point of the three connected empty points (the lower, left-hand quarter), so that red can form a safety structure ("two eyes"). This move blocks blue's attempt to fully encircle the red stones. The teacher also guided the students to observe the pattern (coded as PO) by asking them to observe the number of empty points inside.

PO (Pattern Observation)

Shape 1 → Rule A (a critical defensive move): play at the middle point to survive

After finding the critical move for the “L” shape, the teacher used shape transformation (moving a few pieces around to modify an existing shape) to present a similar problem and drew out the similarities between both shapes. Specifically, the teacher transitioned the problem scenario from the “L” shape to the line shape by moving just a few pieces around; the “L” shape was altered into a straight line, surrounded by the red. The teacher again asked the students to observe the shape and find the critical move in this situation.

PO

Shape 2 → Rule A: play at the middle point to survive

As the result of exploring Shape 1 and 2, the students learned to play at the middle point to make the red group alive in both situations. Moreover, after finding the rule based on Shape 1 and Shape 2, the teacher purposefully guided the students to match the shape they just explored with the class of shapes (containing three empty points inside), which they examined earlier. I coded those teacher acts as classifying.

Classifying

Shape 1 and 2 → Shapes that enclose three connected empty points

Because Shape 1 and Shape 2 encompass all the problem scenarios of shapes that contain three empty points inside, this teacher-structured exploratory process logically led to a general rule which applies to all shapes that exhibit the same feature.

General Rule: Shapes that enclose three connected empty points → Rule A

From the analysis above, we can see the teacher utilized a set of similar shapes, presented in a logical sequence, to externalize the reasoning process involved in deriving the general rule. The reasoning process was inductive because the teacher purposefully guided the students to explore all possible situations (Shape 1 and 2) of shapes that enclose three connected empty points in order to derive the general rule. Pattern observation and classifying activities are important steps in the reasoning process because the reasoning relies on identifying and relating similar shape features and outcomes.

Understanding the Role of Shape Transformation. Pattern observation and classifying activities are naturally embedded in the interactions with shapes. The teacher used shape transformation to create multiple similar situations for the students to explore, drew the students' attention to a new shape (pattern observation), and pointed out the key feature (three empty points) that applied to all the shape variations (classifying).

The use of the shape transformation was very fluid because of how the teacher made transitions between shapes. There was always a sense of connectivity in these transitions—always building on an existing shape by just moving a few pieces around and highlighting the shape features that stay consistent or become different. These fluid shape transformations might facilitate the students' recognition of the similarities and make connections. Therefore, I suggest that the teacher’s shape transformation method might assist the students in pattern observation and classifying activities, which play an important role in the reasoning process.

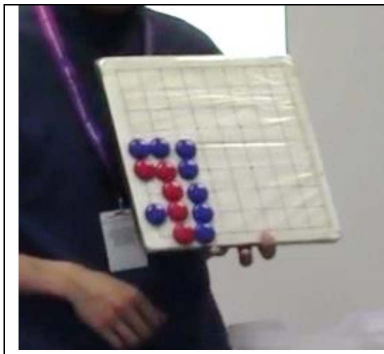
Part 2. “Laddering” to Refine a General Rule

As the episode continued, the teacher intended to transition to a new problem scenario—Shape 3, a structure with four empty points connected in a straight line, which was an extension of Shape 2 (three empty points, same structure). The teacher only asked the students to consider the perspective of the player using the red stone. The rule they developed so far was only formulated as a defensive move by the player on one side—the “red”. However, as they started exploring Shape 3, one student spontaneously brought up the perspective of the “blue” opponent (*italicized* in Excerpt 2). In response to the students’ awareness, the teacher switched back to Shape 2, as the entry point to discuss the opponent’s perspective. Through consideration of both the red and the blue perspectives, the class was able to refine and articulate the general rule they found about Shape 2. These events are presented in Excerpt 2; the events presented in two immediately followed those presented in Except 1 above. Again, the teacher’s turns of talk and actions coded as shape transformation were underlined.

Excerpt 2.

Transcript	Code
Shape 3. Line with four empty points 	

Teacher: Next, we will move from three empty points to four (empty) points. OK. Take a look at this shape. (added new pieces) OK. Can red make two eyes? (finger going through the empty points).	Shape transformation: Teacher transitioning to another shape; PO Teacher question on "eyes"
Students: Yes.	Student response
Students: No, it can make three.	Student response
Jay: it can make two eyes, but it has to make two moves, but *blue can make two moves*.	Student response (alternative perspective)
(Bob came to the board to play, spontaneous. Bob put two red pieces on board to make two eyes) 	Student play
Teacher: Ok, very good. let's go back to this situation (restored the board to 3-empty- points situation) **Back to Shape 2 alternation 1: String with three empty points + a blue in the middle (turned to the opponent perspective)**	Teacher confirms Shape transformation: Teacher transitioning back to shape 2

	
Teacher: If blue plays here, can red make two eyes?	Teacher question on “eyes”
Students: yeah, no, no, NO (reaching consensus).	Students reaching consensus
Teacher: No? (Nodding) Emily?	Teacher question
Amy: (Nodding.)	Student response
Teacher: OK, so where is the most important move you see in this situation?	Teacher question
Students: The middle one.	Student response
Teacher: Good. The middle point is the most important. (demonstration) you can make two eye here. And then if blue plays here, you cannot make two eyes.	Teacher justifying why this move is important for both red and blue
Students: … (something about who plays first).	Student response
Teacher: Yeah, depends on who’s turn. Well, It’s not a real game, but it’s an example. OK, so this move, we call it a “kill” move.	Teacher naming the move (articulate the general rule)
Students: a what?	Student question
Bob: it’s a suicide.	Bob presents a different view
Teacher: Well, yes, that looks like suicide, but in fact, it’s not	Teacher response +justifying

a suicide. When you play this move, you kill that group. It's a really good move, because-	
Students: Killing red.	Student comment
Teacher: -if red play here, red score is 6; if blue play here, then red score is zero.	Teacher justifying (cont.)
(Bob went up to board to play an alternative move)	Student play
Teacher: Ok Bob just said, red can play here, and then, where is a good move for blue?	Teacher response+ question
(Mary went up)	
Teacher: show me, (handing piece to Mary.)	Teacher invites Student to
(Mary played.)	play
Mary: Blue play here, and then (taking red pieces off the	Student play (testing the
board).	alternative move and
Teacher: Bob you know what happened?	demonstrating the outcome)
Bob: yes.	Teacher question
Teacher: Mary is right. So when this situation happens, this	Student response
group is dead and the score is zero.	Teacher confirmation

Understanding the Reasoning Process. By considering both players' perspectives, the teacher guided the students to generalize their knowledge about where to play next in a situation with three surrounded empty points (L shape and line shape)—one should always play at the middle point. If it were the red's turn, the red should play at the middle point to make two eyes so that the red group would be alive. In contrast, if it were the blue's turn, the blue should play there to kill the red group. The teacher named the move as a "kill" move, justified it on the demo board, and allowed the students to test alternatives, which contributed to the process of learning and refining the general rule:

Shapes that enclose three connected empty points → Player A: Play at the middle to survive

→ Player B: Play at the middle to "kill"

Understanding the Role of Shape Transformation. The analyses above highlight the importance of the shape transformation strategy. The teacher first transformed the shape containing three empty points (Shape 2) to four empty points (Shape 3), articulated this transformation in words, and asked the students to solve the same problem—whether and how to make two eyes for the red group to survive. Shape transformation was used to guide students to extend the general rule and apply it in the new situation, which is slightly more complex because Shape 3 introduced a degree of variability by adding another empty point.

However, after a student spontaneously brought up the blue's perspective, the teacher restored the board to the Shape 2 situation for the students to consider the blue's perspective. The teacher's action suggested that he intentionally restored the shape back to the familiar and less complex form (variability reduced), so that the student could explore their newly emerged question in more easily. As the episode continued, we see that the teacher eventually guided the students to explore this question with the more complex Shape 3 setup, having prepared them through Shape 2. Thus, the shape transformation method was used to_as a scaffold to facilitate learning—the teacher first reduced a degree of variability and then increased it, which allowed the students to explore increasingly complex problems step-by-step, building on their existing knowledge. Again, I suggest that the use of shape transformation as a ladder highlighted fluidity in the ways teacher structured his teaching, which might facilitate learning.

Thus far, I have highlighted two ways that the teacher adeptly utilized shape transformation to facilitate students' understanding. First, by moving a few pieces around, he set up multiple similar shapes to engage the students in pattern observation and classification, which led them to find a general rule. Second, the teacher used shape transformation as a scaffold to control the level of difficulty, so that the students could explore increasingly complex questions within their range of capability, building on what they had just learned.

Part 3: The Use of Scaffolding for Justifying and Generalizing

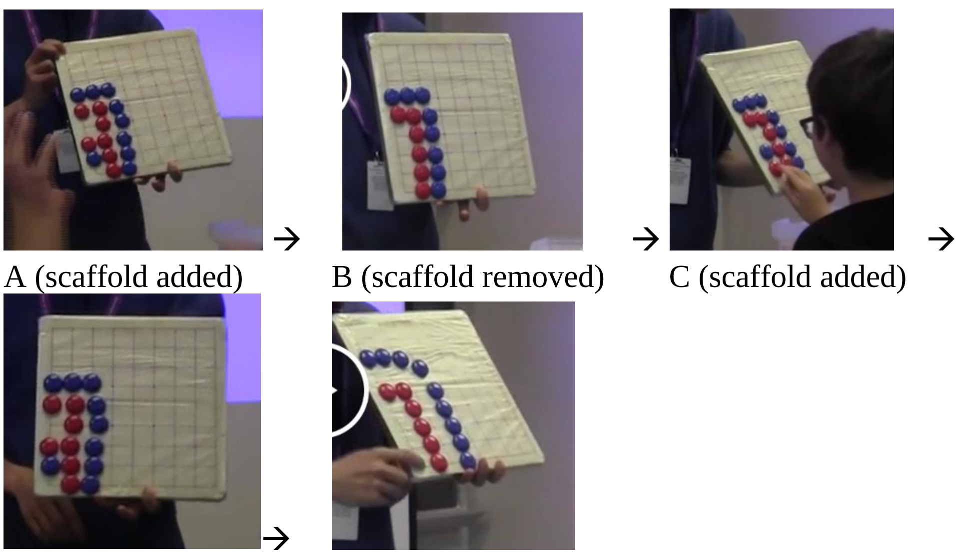
In Part 3 of the episode, the teacher first used shape transformation to create Shape 3 (with four empty points) on the board, the same setup as the beginning of Excerpt 2. Having prepared the students through working with Shape 2, the teacher now asked the students to consider both players' perspectives with Shape 3, instead of considering the red alone. This shape was more complex than Shape 2, because the addition of one empty point opened up more possible moves for the blue and the red.

There were rich teacher-student interactions in this part of the episode that signified the ways in which the students actively participated in the reasoning process. I will present the detailed excerpt and the coding later in the section on teacher-student interactions. Here, I mainly focus on analyzing the teacher's strategic use of shape transformation to support his teaching. Therefore, I presented the teacher's sequential acts through a series of related figures (see Figures 6.3 A-E), which demonstrated how the teacher used shape transformation to externalize the reasoning process leading to general rule about Shape 3.

The teacher’s first goal was to help the students recognize, with scaffolding (Figure 6.3A), that Shape 3 (the red structure shown in Figure 6.3B) would be safe, *even if* the blue had played at the middle. To aid the students’ exploration, the teacher began by adding a blue and a red piece in the middle of the structure (as shown in Figure 6.3A), which was a specific example that illustrated why the red structure would be safe. I considered this set-up as a scaffold because the implications of those middle pieces built on the students’ understanding of the middle point from Shape 2.

Figure 6.3

Sequence of Stone Configurations Constructed in Forming a General Rule About Shape 3



A (scaffold added) → B (scaffold removed) → C (scaffold added) →

D (scaffold added) → E (scaffold removed)

Note. Figure A presents a Shape 3 setup with scaffolding (a blue and a red in the middle), which exemplified one way to secure the red group even though the blue attacked the red by playing a move in the middle; Figure B presents the general setup with scaffolding removed. Figure C and D present two specific ways to make the red group alive that the students tested, following the blue’s attempts to kill the red group. These two specific cases thus justify that the red structure (four enclosed empty points) is safe forever.

Figure E presents the general setup again with scaffolding removed.

Understanding the Reasoning Process. Through interacting with Shape 3 with a scaffold (Figure 6.3 A), the students recognized that the red could make two eyes and therefore was safe. However, would the students be able to learn that Shape 3 was safe forever, without having to see it through a specific example (Figure 6.3B)? The teacher’s next goal was to guide the students to generalize this knowledge about Shape 3—a structure with four connected empty points is safe forever-- by following the reasoning process—even if the opponent attack by placing a stone in the middle, the player can still defend and secure the structure. To assist the students to generalize, the teacher removed the scaffold (blue and red pieces in the middle) and asked the students to evaluate whether the original Shape 3 (Figure 6.3B) was safe forever.

The removal of the scaffold opened up many possibilities for the students to test and evaluate. The remainder of the Part Three episode consisted of a series of repeated interactions where the teacher used the shape transformation method to guide the students to explore many similar situations and tested out different moves (Figure 6.3C and D). Specifically, in these interactions, he altered the situation slightly, by switching the middle blue piece up and down, or by adjusting the blue layer on the outside. Each time, the teacher repeatedly asked the same questions: “if blue plays here, where should red play?” or “can blue kill red?”.

As a result, the students explored this question in multiple similar formats and tested out different moves and outcomes. In the end, they were able to create the safety structure from all the situations they explored. That is, they were always able make two eyes for the red in time (by placing a red piece in the middle), no matter where the blue had played. Therefore, with the help of the teacher, they generalized the knowledge about Shape 3—a shape with four empty connected pointed would be safe forever, because the opponent could not kill it in any way. This process, which was used to discover a general rule, resembled the inductive reasoning process uncovered in Part One—by testing all possible moves (alternation 1-2, where the blue played a move inside the enclosed space ahead of the red) and examining their outcomes.

PO, Classifying	**Conjecturing and justifying**	**Generalizing**
Shape 3 Alternation 1		
Shape 3 Alternation 2	→	Shape 3: Safe forever
Scaffolding Added		**Scaffolding Removed**

Understanding the Role of Shape Transformation. In this part of the instruction, the shape transformation method was used to add and remove scaffolding. With the use of scaffolding, the teacher utilized Shape 3 as the base to create multiple similar situations by altering it slightly, which prompted the students to systematically test all possible moves and outcomes associated with the given situation. The removal of scaffolding guided the students to gather their knowledge of specific cases and generalize their findings. The fluidity in the use of shape transformation was reflected in how the teacher used scaffolding to transition back and forth between reasoning about specific cases and generalizing.

Part 4. Using Shape Transformation to Extend a General Rule

After discovering the general rule about four connected empty points, the teacher asked the students to explore whether the general rule would extend to other shapes. Their interactions with the new shapes are captured in Part Four, the final part of this interactive instruction episode. The teacher's talks and actions that initiated a shape transformation were underlined.

Excerpt 3.

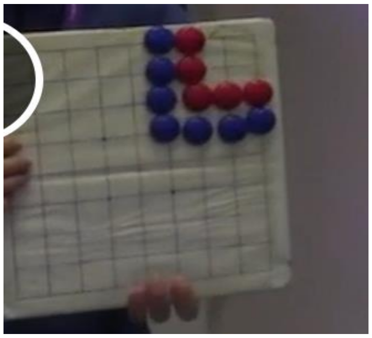
Transcript	Coding
Shape 4. Line with five empty points 	
Teacher: right yes, Ok. Now, take a look at this (new shape, 5 empty points). (Then rotated the board, finger went over the empty points within red multiple times).	Shape transformation: Transition to Shape 4). PO (pattern observation).
Jay: It's the same exact and exact thing.	Jay recognized the similarity.
Students: Oh yeah.	Student affirm
(Mary came up)	
Teacher: Mary, is this group dead or alive?	Teacher question
Students: It's kinda alive.	Student response
Mary: It's alive.	Student response
Teacher: Yes, it's alive. We already know that this shape is alive (switch from Shape 4 to Shape 3), now you have more space (add more empty points inside, switch to Shape 4) you are alive.	Shape transformation: transition between Shape 3 to 4, to extend a general rule. PO: compared shape features.
(Jay went up to demonstrate all possible moves in the	Student justified a general

given situation and showed that the red can make two eye no matter the blue would play.)	statement by verifying all possible moves and outcomes.
Jay: Here, there, here…	
Teacher: Jay have you verified?	Teacher question (highlighted Jay's practice)
Jay: yes.	Student response

Based on the episode so far, we could see that the teacher continued to use the shape transformation method in a similar way as he did in Part 2 and the beginning of Part 3, where he fluidly switched back and forth between Shape 2 and 3 to control the degrees of variation, so that the students can explore a new idea step-by-step. In Part 4, the teacher first increased the complexity of the problems by presenting Shape 4, which was created by extending one more empty point from the original structure of Shape 3. Based on the student's expression that "it is the exact same thing!", we could see that the students were able to recognize that the general rule about Shape 3 could apply to Shape 4 as well, although the teacher modified the shape and even rotated it.

After the students explored Shape 4, which had the same string feature as Shape 3, the teacher again modified the shape to Shape 5, which has a square feature while keeping four empty points.

Excerpt 4.

Transcript	Coding
Shape 5. Square with four empty points 	
<u>Teacher: OK now let's look at this shape.</u>	<u>Teacher shape transformation</u>
Jay: it's kind of half alive and half dead, I don't know how to explain. It's complicated.	Student conjecturing
Mary: Let's see. (Mary came up to the board to explore by putting pieces on board).	Student justifying
Teacher: Ok can you make two eyes?	Teacher question
Mary, if it's red's turn, it can play here and then here (playing…)	Student testing
Teacher: can it make two eyes?	Teacher question
Mary: no.	Student response
Jay: It's horrible, horrible!	Student comment
Teacher: when you play here, where should blue play?	Teacher question
Mary: Right there (pointing).	Student response
Teacher: Yes, yes! That's a kill move. Can this red group	Teacher named a critical

make two eyes?	move; question about "eyes".
Students: NO.	Student response
Jay: Yes but blue can stop it.	Student response
Teacher: **Wherever red play, blue can play here and kill that group**. So can this group be alive? Who have not tried this question. Is this a dead or alive or it depends?	Teacher generalizing Teacher asks students to justify
Students: It's dead.	Student response; consensus.
Teacher: Even red play first, red cannot make two eyes.	Teacher confirmation
Students: What if blue plays first?	Student question
Teacher: Well, do we need to talk about whether blue plays first?	Teacher question
Students: No.	Student response
Teacher: No. Even if red play first, red is dead, then-	Teacher justifying
Jay: If blue plays first, red is MORE dead.	Student justifying
Teacher: More dead, yes.	Teacher confirmation

Shape 5 was the last shape introduced to the students in this interactive instruction episode. It was a complex and novel shape because its square structure differed from the string shape and the "L" shape they had learned. However, based on the excerpt above, we could see that the students discerned the difference and made a reasonable judgement about this new shape rather quickly, after just a few turns of talks and trying. It was also clear that the student considered both players' perspectives in their judgment, building on what they learned in Part 2; their considerations of both players' perspectives in making a judgement was effective and efficient. For example, at the end of this excerpt, the students and the teacher discussed whether who played first would alter the outcome, and they quickly arrived at the conclusion that “if blue plays first, red is MORE dead”, without having to examine any cases where blue played first (because *even if* the red had an advantage to play first, the red could not survive). The student’s conclusion that “red is MORE dead” here is enlightening. By using the term “more dead”, the student expressed his reasoning that the conclusion of “death” drawn from an advantages situation for the red could logically apply to a disadvantageous situation. Although the underlying logic was actually substantial, the conclusion was drawn efficiently and expressed simply and vividly. Given the students’ demonstration of their newly learned reasoning skills, the teacher’s use of the shape transformation strategy was quite effective.

Interim Summary

So far, my analyses of the teacher’s acts showed that the teacher used the shape transformation method in various ways to guide the students to derive and apply general rules associated with a class of similar or related patterns. By using the shape transformation method, the teacher externalized the reasoning processes underlying the general rules. The teacher also assisted the students to participate in those reasoning processes by using the shape transformation method to promote pattern observation and classifying activities in Part 1 and facilitate generalization in Part 3. Moreover, the shape transformation method was used as a scaffold in Part 2 and 4 to adjust the level of difficulty, which allowed the students to explore increasingly complex ideas step-by-step, building on what they had just learned.

The findings about the shape transformation method signified dynamics and fluidity in the teaching of shapes. I suggest that this feature might have contributed to the teaching and learning because the fluid transitions between shapes might highlight not only shape similarities for the students, but also foster a sense of connectivity or continuation in the practices of reasoning. The “more dead” expression, for example, illustrates this point: because the students fluidly transitioned from blue’s advantageous situation to the red’s, they quickly transferred their reasoning about the previous situation to the new situation by only considering the difference. For another examples, one of the students quickly realized that “”it is the exact same thing” , when the teacher transformed the shape from having three empty points to four, and therefore the same indications apply. Thus, because of the fluidity, the shape transformation method might support the students to carry the reasoning practices they developed in learning a basic shape to increasingly complex situations, so that they could learn more complex ideas about Go patterns.

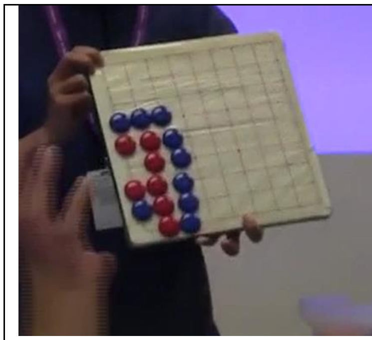
Analysis of Teacher-Student Interactions: What mathematical reasoning practices emerge from those interactions?

Having examined the teacher’s acts, I now turn to the detailed interactions between the teacher and the students within the interactive instruction episodes. The turn-by-turn coding of teacher-student interactions led to the emergence of a unique pattern, which highlights the students’ spontaneous contribution to the reasoning processes underlying the learning of Go patterns. The analyses also show that the students’ spontaneous contributions were mediated by the use of spatial forms.

The following was the excerpt from Part 3 of the episode, in which the teacher guided the students to reason about Shape 3 with scaffolding. There were four students who took turns to interact with the teacher and the board. The first three students repeated the same actions on the board while the fourth student gave a different response. The first student, Jay, spontaneously went to the board to show how the red could make two eyes by capturing the blue piece, assuming the red would play next. Then, the teacher restored the board and asked, “can blue capture red?”. The second student Mary and the third student Dan answered no and demonstrated their reasoning by playing a red piece and taking the blue piece away, the same moves as Jay made. Then, the teacher rephrased the question slightly, seeking an answer in another way. Finally, student Bob came up to the board and demonstrated why the blue could not capture red; not by taking it away (a single example), but by pointing out all the possible moves for the blue and showing that all of those would be suicide moves. The part that included students’ spontaneous actions of justifications are underlined.

Excerpt 5.

Transcript	**Coding**
Back to Shape 3. alternation 1: String with four empty points + a blue piece and a red (scaffolding) in the middle (turned to opponent perspective)	
Teacher: Now, go back to this situation (restoring demo board to 4-empty-points, with a blue piece and a red in the middle) Ok, take a look at this.	Shape transformation: Teacher transitioning to another shape; PO (pattern observation)
Jay: it can make two eyes... (raised hand and went up to board, played a red capture move and took away the blue piece, leaving two eyes for red).	Jay conjecturing about “eyes” Jay justifying by demonstrating the process of making two eyes.

	
Teacher: Ok. The first question is can blue capture this red group?	Teacher call out to whole group
Students: NO.	Student response (conjecture)
Teacher: NO?	Teacher question
(Mary went up to replay a red capture move-> two eyes; same as what Jay did)	Student play (justifying)
Teacher: so, can blue capture this red group?	Teacher question (scaffolding removed)
Students: yes, no.. sort of.	Student response (conjecturing)
Dan: No, blue can't because red will go here (demonstrated on board).	Dan justifying with a specific example
Teacher: OK. My question is. If red doesn't play here, can blue capture red? (if blue play next instead of red)	Teacher question
…Bob: Suicide, Suicide (Bob came up, put blue in two other places, demonstrated those two moves were suicide moves.).	Bob justify by demonstrating the outcome of all possible moves for blue.
Teacher: So, is this red group safe forever?	Teacher question

	(generalizing)
Students: Yes yes yes yes (consensus).	Student consensus
Teacher: Ok, safe forever (nodding), so red has one eye and another space to make another eye.	Teacher confirms Generalization
Bob: So blue can play everywhere over here (Bob went up and pointed to the other areas on the board outside the surrounded area.)	Bob play

An important pattern of teacher-student interactions emerged from Excerpt 5. The pattern starts with a teacher question about a specific case shown on the demo-board ("Can blue capture this red group?"); the question is followed by a student conjecture (either a yes or a no), which is followed by a student justification with a specific example (playing a red move to capture the blue).

However, Bob's move (underlined) differed from the other players' moves. While taking the blue away was a legitimate immediate defense from the red's perspective, it did not provide the ultimate explanation as to why the blue could not capture the red, because that move assumed that the red had an advantage to defend. In contrast, Bob gave a justification by considering all the possibilities and outcomes when the blue plays next, and named those moves "suicide", which implied that the blue could do nothing to capture the red inside, *even if* it had an advantage to attack first. Moreover, he went further to suggest that blue should play outside the boundaries, given the implication that blue could not capture red in any way. This is a new pattern that differed from the prior student responses which concerned the use of one specific example or counter example. Therefore, I treated this student's response as a *complete justification*, which involved demonstrating all possible moves within a pattern that ultimately leads to a generalization.

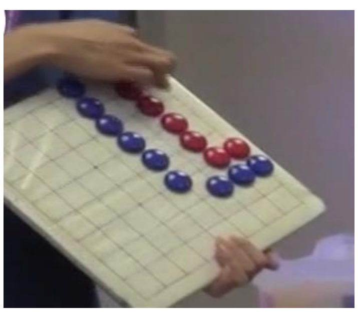
The student's act of complete justification was an integral part of the teacher guided process of generalization illustrated in part 3 of the previous section (same except). The student here externalized the reasoning process through evaluating the outcomes of all possible stone placements (by physically placing stones on the board). In doing so, the student offered a complete justification in response to the teacher's question, which led to a general statement about a given situation.

What is the role of spatial forms in carrying out this reasoning process and engaging students in it? At the surface level, the Go board provided a platform to externalize the reasoning process. At a deeper level, I suggest that the shapes provided *physical constraints* to the generalization problem, which might prompt the students to easily identify all the possible moves and justify their outcomes physically on the board. As shown in Figure 6.3A, the two possible moves of the blue were showed inside the red boundaries (the two empty points). The students could fill in those empty spaces with stones to demonstrate that those were suicide moves, or just visualize those moves and outcomes easily.

Next, I show another excerpt from Part 3, where the student Jay spontaneously made a justification following the teacher's general claim about a pattern. In this interaction, the teacher first guided the students to observe the feature of Shape 3—that is, there are four empty points inside the red territory. Then, the teacher made a generalization about this structure—four empty points make a group alive. Following this generalization, the student Jay spontaneously went up to the board and demonstrated all the possible moves and outcomes for both sides, which justified this generalization. After that, the teacher transformed Shape 3 to Shape 4, by adding one more empty point inside, and asked students to evaluate whether the generalization may be extended to Shape 4. Again, Jay spontaneously went up to the board to justify the generalization about Shape 4. The teacher's general claims were presented in bold. The two incidences of Jay's justification were underlined. Because Jay's justifications involved demonstrating all possible moves within a pattern that leads to a generalization, as opposed to using one specific example or counter example, I coded Jay's acts of justifications (underlined) as *complete justification*.

Excerpt 6.

Transcript	**Coding**
Teacher: Now let's go back. (removed the blue and red pieces in the middle) Can blue kill this group? 	Shape transformation; Scaffolding removed.
Jay: No no no no	Student response
Teacher: OK how many empty points in this situation? How many? (finger going over the empty space) Dan, how many?	Teacher question; PO
Students: five	Student response
Dan: six	Student response
Teacher: no, Jay, how many empty points in this situation?	Teacher question; PO

Jay: four.	Student response
Teacher: Four, one – two- three- four, **four space (empty points) makes your group alive, never be captured.**	Teacher generalization
Jay: And all the situations that happen …	Jay complete justification
(Jay went up to demonstrate his verification)	
Jay: so if, if black goes there then white goes here, if black goes here then white goes here, if black goes there then white goes here, if black goes here and then white goes here.	
(demonstrated the outcome of all situations.)	
Teacher: right yes, Ok. Now, take a look at this. (new shape, 5-empty points. Then rotated the board, finger went over the empty points within red multiple times).	Teacher confirms. Shape transformation; PO
	
Jay: It's the same exact and exact thing.	Jay response; classifying as the same.
Students: Ohh yeah.	Whole group response
(Mary came up)	
Teacher: Mary, is this group dead or alive?	Teacher question
Students: it's kinda alive.	Student response
Mary: it's alive.	Student conjecture

Teacher: Yes, it's alive. **We already know that this shape (**switched to Shape 3**) is alive, you have more space (**switched to Shape 4 by adding another empty point inside**) you are alive.** (Jay went up to verify all the situation.) Jay: here.. there.. here.. Teacher: Jay have you verified? Jay: yes.	Shape transformation; Teacher extends and articulates general rule Jay complete justification Teacher named Jay's spontaneous act as to "verify".

From the excerpts above, we can see that the interactions between the teacher and the students involved many turns of conjecturing and justifying. While Excerpt 5 showed a student's complete justification that emerged in the teacher guided process of forming a generalization, Excerpt 6 showed a student's two spontaneous acts of complete justifications, to complement the teacher's generalizations made directly after pattern observations. In other words, the student has adopted justifications as an integral part of reasoning with patterns,

These acts of complete justification involved demonstrating all possible moves within a pattern that ultimately leads to a generalization. In doing so, the students made a generalization, an abstract form of knowledge, become more concrete, because it showed the process that led to the generalization. Again, the spatial forms (refer to the figures listed in Excerpt 6) provided physical constraints to facilitate their complete justification. It is worth noting that the teacher also interacted with the spatial forms to facilitate their reasoning processes, by highlighting the number of empty points inside the red boundaries.

As the previous section highlighted the teacher's use of shape transformation which facilitated various aspects of reasoning, this section, however, highlighted the students' contributions to the reasoning process, which were also mediated by spatial forms. The analyses showed that the students not only participated in the reasoning practices by simply responding to the teacher's questions, but they also initiated their own practices of providing complete justifications to a general rule, which actually echo the ways the teacher guided the students earlier in the episode to examine all possible moves and outcomes to derive a general rule.

Analyses of Student Interactions During Play

I have examined how the teacher used the shape transform strategies to scaffold the teaching and learning process, which involves exploring various shapes that share a similar pattern and forming a generalization about these shapes. I have also explored how the students actively contributed to their own and other students' learning by initiating the practices of justifications that made the generalization more concrete. Next, I explored the student interactions during their play. I found that the students transferred some practices of conjecturing and justifying that they developed during interactive instructions into their own play.

Next, I show a cooperative play episode where Bob, Jay, and Dan worked as a team to play against the teacher. They exchanged their thoughts and demonstrated their reasoning in order to decide on each move together. This episode occurred in week 5, one week after the interactive instruction episode presented in the previous section. The interactions during cooperative play differed from those during interactive instructions, which were much more structured and more centered on generalizing a rule or a pattern from specific cases. The interactions here involved analyzing a few steps ahead from both sides to choose the best next moves. Therefore, some messy debates between team members are expected. Moreover, I am interested in whether the students chose to bring the reasoning practices they engaged in earlier into these debates, because the emergence of those reasoning practices would not only indicate that the students might have adopted reasoning as their own means to solve problems, but also indicate how much they value these reasoning practices when it comes to decision making as a group.

Excerpt 6.

Transcript	**Coding**
Bob: I think black will play here.	Suggest a move
Jay: Oh, Yeah, yeah.	Affirm
Teacher: if black play here, what will happen to the white?	Teacher prompts to analyze
Dan: Wait, look if we go here, then black will go there; if we go there, they will go there. Then we'll just go there, and then...	Student suggesting a move with reasoning
Jay: Uh... So we go there… I know why we shouldn't go there.	Student elects himself to justify
Dan: Why?	Student welcomes to justify
Jay: Because we are still gonna be captured.	Justifying
Dan: No, we won't because we can escape through here.	Objecting the justification
Jay: Oh my God...	Expressing disagreement
Dan: See, no, look, they'll probably not capture two sides, look, here.	Proposing a conjecture and testing it
Bob: I got it, I got it. So we put it right there. And they capture us, and then we put it right here.	Suggesting a new move based on the analysis so far

Jay: Yeah, yeah. But guys, there's one more thing,	Affirm
Dan: Oh Yeah yeah, Oh we should, cuz then they will have only one eye, so we can kill them.	Affirm + justifying
Jay: Look at this, think about it-	Jay asking others to reason (with him)
Dan: -if the black	Dan cont. his justifying
Jay: - let me tell you something; let me tell you something.	Jay Asking others to hear him
Dan: -If we go here, then black go here, and then they take us, they don't have two eyes, so we can-	Dan cont. his justifying
Jay: -Wait, I wanna talk. We don't wanna go here or here, cuz if you go here,	Jay asking other to hear his justification
(Bob blew the stones away)	
Jay, Dan: Oh No Dude..dude.	Express frustration
(They restored the board)	
Jay: Guys, no no no. Actually, I wanna tell you something. If we go here, we think we can capture them, we Can't. Cuz they'll just gonna go here and capture both of these.	Cont. Justifying; reasoning with both side perspectives
Bob: but they only have one eye.	Raised Objection
Jay: well	
Bob: they need two eyes (to survive).	Explained objection
Jay: It's because, that, when we do this, these will be gone. All will be missing.	Cont. Justifying. (defensive perspective)

Dan: So we have to escape somewhere.	Suggesting alternative actions given analysis so far
Jay: we can’t escape.	Draw a logical conclusion
Dan: Then we have to sacrifice them. That’s the whole point!	Suggesting to “sacrifice”
Jay: Yes. And there’s also reason why we can’t go here.	Offering a justification
Dan: yes we have to sacrifice them, so go here. Because that…	
Jay: yes. But this won’t-	Affirm, justifying
Dan: -but then we go there, and they’ll be surrounded, and then we put one there, and when they make one eye, we’ll go get them.	Suggesting an alternative move
Jay: haha they won’t be surrounded.	Rejecting the move
Dan: So look at this.	
Jay: Dude they won’t be surrounded.	Rejecting the move
Dan: Look, We go here, they go there, They got there, and then we go here, and then they go wherever they’re gonna go.	Justifying
Jay: Oh my god, hahha	Expressing objection
Dan: and then we go here, they take us, and guess what happens, so they take us, and if we go there, and then bang! And look at that thing!.	Justifying
Jay: hahah, And, Go back in time, they won’t go here, they’ll	Challenging Dan’s justification

capture this one.	
Teacher: (came back) remember, you have opportunity to use help. You need help?	Teacher offers help
Jay: Yeah we need help.	Seeking help

In this episode, we saw that after the teacher prompted the student to analyze what would happen if the opponent played certain moves, each student expressed their own opinions and gave their justifications. Most of the justifications were given in forms of “if…then…” statements with demonstrations on the board. (This could suggest that the physical interactions with the Go pieces, along with the cooperative play setting, might prompt students to articulate the justifications in words. Although the current study does not concentrate on this question, the observation here could inspire future studies on the relation between verbal reasoning and physical interactions/spatial reasoning with patterns in a board game setting.) In addition, they actively followed through the other’s justifications, offered suggestions based on those justifications, or challenged others’ justifications by demonstrating alternative moves and outcomes.

Moreover, I found that the student Jay constantly made the effort to articulate and demonstrate his reasoning to the group and asked others to attend to his reasoning. For example, Jay said “Listen, I know why we shouldn’t go there”, “Yes. And there’s also reason why we can’t go here.”, etc. in multiple situations (refer to the underlined sections in Excerpt 6 above), which showed his persistent effort to let his group follow his reasoning. These efforts show that the student highly value the actions of communicating his reasoning to the group of players and involving other players in the reasoning process.

Therefore, from the analysis of student interactions during cooperative play, we can see that the students were able to carry the reasoning practices they developed through interacting with the teacher to the new setting where they work together to find the best moves. From the students' efforts made to communicate their reasoning with their team-mates and their active involvements in evaluating others' reasoning, we can see that they highly value the reasoning practices and used those practices frequently in their cooperative play.

Discussion

This chapter concentrates on the analyses of teaching and learning of Go patterns, which play an essential role in mastering the game of Go. The episodes presented in this chapter illustrate that Go patterns can be derived from a few basic rules and concepts (e.g., the concept of Qi and the associated concept of "life and death", the basic "eye" pattern, etc.) through a series of reasoning processes such as conjecturing (Lannin et al., 2011), justifying (Ellis, 2011), and generalizing (Staples et al., 2012). These reasoning processes are similar to the previously studied mathematical reasoning processes (G. J. Stylianides & Stylianides, 2009). Therefore, my analyses revealed what kinds of reasoning processes are embedded in the process of learning and applying Go patterns.

Moreover, my analyses showed the role of spatial forms in facilitating the teaching and learning of Go patterns, and particularly their roles in externalizing the reasoning processes and supporting the students to participate in the reasoning practices. Using the emergent goals framework, I identified the teacher's dynamic and fluid use of spatial forms as important means to facilitate the teaching and learning of Go patterns and the underlying reasoning processes. The use of spatial forms suggest how spatial thinking might mediate the reasoning processes. For example, the pattern observation and classifying activities, which were heavily spatial, were necessary steps in the derivation of a general rule for shapes that share similar features. Therefore, my analyses suggest that spatial thinking played a role in supporting students' mathematical reasoning practices in the context of learning Go.

Moreover, my analyses of the teacher-student interactions showed that the students not only participated in the reasoning practices by responding to the teacher's questions, but also actively contributed to the reasoning process by spontaneously offering justification following the teacher's claims or generalization. These ways of participations are mediated by spatial forms which might facilitate the reasoning process by providing a platform to externalize the reasoning, and providing visual and physical constraints to identify and test all possible moves and outcomes. Furthermore, I found that the students were able to transfer some of the reasoning practices into their own cooperative play. Those reasoning practices were adopted as the students' own means to solve problems. The analyses also indicate how much the students value the reasoning practices when making decisions as a group.

Chapter 7: Diversifying Mathematical Strategies through Score-counting

In this chapter, I examine whether and how mastering score-counting could be a means to foster spatial and mathematical practices among young children. As I discussed in Chapter 4, score-counting is an essential practice of Go because it concerns the ultimate game goal: to occupy more spaces on the board than the opponent. Thus, counting each player's occupied spaces will reveal the winner of the game. Moreover, experienced players do not limit their counting to the end of the game; they keep a mental tally of the current score and of their projected occupied spaces throughout the game. Players monitor the opponent's and their own emerging territories in real time and adjust their game strategies accordingly. Because score-counting involves working with large numbers (especially when playing on a 19x19 board) and often requires getting a fast result, players need to develop very efficient strategies to count or estimate territories. The process of developing score-counting strategies may provide opportunities for young children to improve their spatial and mathematical and skills.

I collaborated with Xinming Guo, who implemented the Go and Math Curriculum (Wu & Guo, 2018) at two Grade 2 classrooms during the Fall semester of 2017, over the course of 10 weeks. Although the Go contents mostly overlapped with those in the prior study discussed in Chapter 6, the curriculum was augmented to highlight and foster mathematical practices which are embedded in the score-counting activities. Score counting in this curriculum used a basic and straightforward approach, which differed slightly from the professional score counting. Please refer to Appendix II for a discussion about the distinctions. Next, I will use the common core math standards (CCSS, 2010) to illustrate the specific mathematical practices which can be fostered by participating in the score-counting activities.

Alignment of Score-counting Activities and the Common Core Math Standard (CCSC)

The score-counting activities (Wu & Guo, 2018) were designed in accordance with the common core math standards (CCSS, 2010) for Grade 2 and 3, which include understanding numbers and place value, operations and algebraic thinking, etc. Table 7.1. provides a list of common core math standards provided by Guo, which he claimed are supported by the score-counting activities in the curriculum.

Table 7. 1

List of Grade 2-3 Common Core Math Standards Supported by Score-Counting

Grade	Domain	Cluster	Standard Code	Standard
2	OA	Represent and solve problems involving addition and subtraction	2.OA.1	Use addition and subtraction within 100 to solve one- and two-step word problems involving situations of adding to, taking from, putting together, taking apart, and comparing, with unknowns in all positions, e.g., by using drawings and equations with a symbol for the unknown number to represent the problem.
2	OA	Add and subtract within 20	2.OA.2	Fluently add and subtract within 20 using mental strategies. By end of Grade 2, know from memory all sums of two one-digit numbers.
2	OA	Work with equal groups of objects to gain foundations for multiplication	2.OA.3	Determine whether a group of objects (up to 20) has an odd or even number of members, e.g., by pairing objects or counting them by 2s; write an equation to express an even number as a sum of two equal addends.
2	OA	Work with equal groups of objects to gain foundations for multiplication	2.OA.4	Use addition to find the total number of objects arranged in rectangular arrays with up to 5 rows and up to 5 columns; write an equation to express the total as a sum of equal addends.
2	NBT	Understand place value	2.NBT.1	Understand that the three digits of a three-digit number represent amounts of hundreds, tens, and ones; e.g., 706 equals 7 hundreds, 0 tens, and 6 ones. Understand the following as special cases: a. 100 can be thought of as a bundle of ten tens — called a "hundred." b. The numbers 100, 200, 300, 400, 500, 600, 700, 800, 900 refer to one, two, three, four, five, six, seven, eight, or nine hundreds (and 0 tens and 0 ones).
2	NBT	Understand place value	2.NBT.2	Count within 1000; skip-count by 5s, 10s, and 100s.
2	NBT	Use place value understanding and properties of operations to add and subtract	2.NBT.5	Fluently add and subtract within 100 using strategies based on place value, properties of operations, and/or the relationship between addition and subtraction.
2	NBT	Use place value understanding and properties of operations to add and subtract	2.NBT.6	Add up to four two-digit numbers using strategies based on place value and properties of operations.
2	G	Reason with shapes and their attributes	2.G.2	Partition a rectangle into rows and columns of same-size squares and count to find the total number of them.
3	OA	Represent and solve problems involving multiplication and division	3.OA.1	Interpret products of whole numbers, e.g., interpret 5 × 7 as the total number of objects in 5 groups of 7 objects each. For example, describe a context in which a total number of objects can be expressed as 5 × 7.
3	OA	Represent and solve problems involving multiplication and division	3.OA.3	Use multiplication and division within 100 to solve word problems in situations involving equal groups, arrays, and measurement quantities, e.g., by using drawings and equations with a symbol for the unknown number to represent the

Note. Adapted from "Cover 60% Math Contents in K-3 with Just One Game—Weiqi/Go", by X.Wu, & X. Guo, 2018, at conference workshop, Metropolitan Mathematics Club of Chicago, Lisle, IL. Math items in this table are based on *Common Core State Standards for Mathematics* by National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010, http://www.corestandards.org/Math/

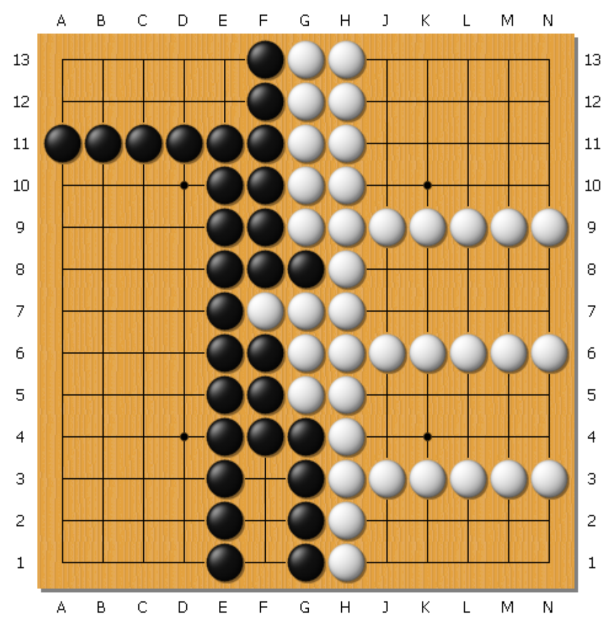
The list covers the majority of the common core standards at the Grade 2 level, which cover 3 domains—*Operations and algebraic thinking (OA)*, *Geometry (G)*, and *Number and operations in base ten (NBT)*, and a few common core standards for Grade 3, which involves the understanding of multiplication. In the next paragraphs, I discuss a few examples of end-of-game board set-up that demonstrate such alignments.

The first example focuses on encouraging Grade 2 students to represent and interact with numbers in arrays during score-counting. This activity aligns with the two aspects of the Grade 2 common core standards: 1) *Operations and algebraic thinking*, which asks students to work with equal groups of objects to gain conceptual foundations for multiplication, and 2) *Number and operations in base ten*, which asks students to understand place values in base ten (10s, 20s, 100s) and use place value understanding and properties of operations to add and subtract.

Figure 7.1 is an example of the end of game set-up where players rearranged the board for score counting by relocating some of the stones to make their territories into rectangular shapes while keeping the sizes of their territories constant.

Figure 7.1

An End-of-Game Go Board Rearranged for Score Counting



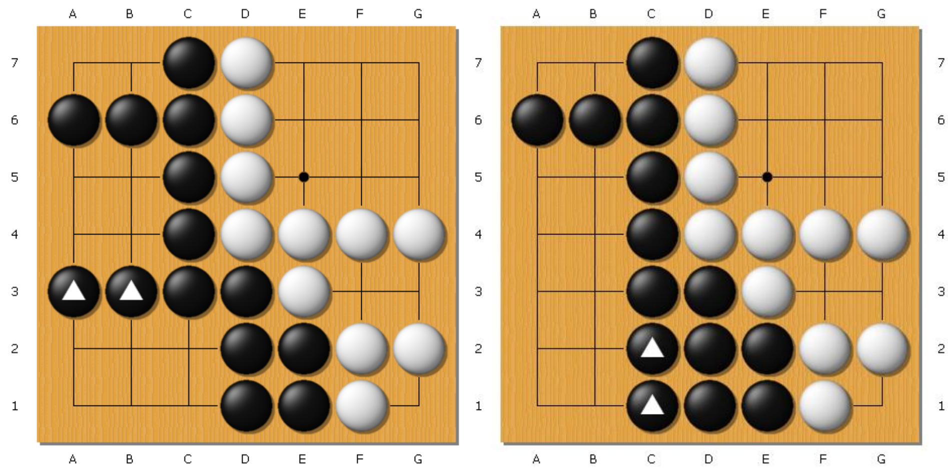
Note. The board is rearranged in arrays and groups of 10s and 5s for easy counting. The white territories on the right side of the board are rearranged into a rectangular group of 20 (4 arrays of 5) at the upper right conner and three rectangular groups of 10s stacked below. The black territories are rearranged into a group of 10 at the upper left corner and 4 columns of 10s below. Adapted from "Cover 60% Math Contents in K-3 with Just One Game—Weiqi/Go", by X.Wu, & X. Guo, 2018, at conference workshop, Metropolitan Mathematics Club of Chicago, Lisle, IL.

The calculation of the territories involves working with equal groups of objects. It is thus consistent with the specific common core math standard for Grade 2 *Operations and Algebraic Thinking* (CCSS.MATH.CONTENT.2.OA.C.4), which states that students should use addition to find the total number of objects arranged in rectangular arrays with up to 5 rows and up to 5 columns; this activity may serve to gain foundations for multiplication. In addition, because the score counting involves counting and adding the groups of 5s, 10s, and 20s, it is also consistent with the standard for *Number and Operations in Base Ten* (CCSS.MATH.CONTENT.2.NBT.B.5), which states that students should fluently add and subtract within 100 using strategies based on place value, properties of operations, and/or the relationship between addition and subtraction. We can also see that the mathematical activities in the example above are achieved with spatial thinking because the activities involve recognizing numbers represented spatially as arrays and being able to mentally combine those spatially represented numbers.

Moreover, since score counting involves not only combining numbers but also carefully and strategically rearranging stone configurations into arrays and groups of 10s that do not change the game outcome, accomplishing such rearrangement activities involves developing and applying fundamental understanding about properties of operations. In the next example, I illustrate how the stone rearrangement activity aligns with the common core math standard of applying properties of operations as strategies to add and subtract (see Figure 7.2).

Figure 7.2

A Rearranged Go Board for Score Counting



Note. The board on the left show the board before rearrangement; the board on the right show the board afterwards. The white arrows indicate the two pieces of stones that were relocated. Adapted from “Cover 60% Math Contents in K-3 with Just One Game—Weiqi/Go”, by X.Wu, & X. Guo, 2018, at conference workshop, Metropolitan Mathematics Club of Chicago, Lisle, IL.

As shown in Figure 7.2, the total score within one’s territory does not change as the outcome of rearrangement. The black territories initially consist of three parts: 2+4+6=12. The rearrangements occurred within the black’s territory and therefore did not affect the outcome. The rearrangement combined two territories (4+6) into a group of 10 and made the calculation easier as 2+10 =12. This rearrangement thus demonstrates the associative property of addition: 2+4+6=2+10=12.

The ability to understand the associative property of addition and apply it to solve problems in the context of Go is consistent with the Grade 1 common core math standard about *Operations and Algebraic Thinking*—apply properties of operations as strategies to add and subtract. When the same rearrangement skill is applied to a larger board (as shown in Figure 7.1) which involves more operations with 2-digit numbers, the alignment extends to the Grade 2 level—fluently add and subtract within 100 using strategies based on place value, properties of operations, and/or the relationship between addition and subtraction. Thus, the score counting activity may provide an authentic problem-solving context for Grade 2 students to connect what they have already learned (i.e., the properties of operations) with what they are learning (i.e., the base-ten place values and operations within 100), so that they may build on their prior knowledge to solve problems and develop new understanding.

The two examples thus show that the Go and Math curriculum may encourage students to apply what they had learned and were learning in Grade 2 mathematics at the time to score-counting. Moreover, some score-counting activities may also be their first exposure to certain mathematics, including multiplication.

The common core standards provide an approach to investigating whether and how the score-counting activities may support the learning of mathematics at the Grade 2 levels. I will examine how the teacher aligned score-counting with the common core mathematics standards in his teaching. I will also investigate what counting strategies the children develop over the course of the curriculum and whether these strategies contributed to their spatial and mathematical skills.

Method

Site and Participants

This course was implemented at an elementary school in the northern suburbs of Chicago. All four Grade 2 teachers at this school had their classes participate in the study during the Fall semester of 2017. There were 44 participants at this site, roughly evenly divided between the 4 classes. We coordinated with the teachers to form two combined classrooms for Go instruction, each consisting of two classes, around 20 participants. The course was implemented once per week during the “extended learning sessions”, at one of the combined classrooms. The extended learning sessions occur daily in the morning for 60 minutes. Students may opt to take a variety of courses, such as Chinese, during that time, or choose to participate in the extended learning sessions. Normally, the extended learning sessions are flexible instruction times, when the teacher may assign some tasks, or help students with homework.

Implementation

The course covered mostly the same contents as those in the prior study described in Chapter 6. Similarly, the course consisted of interactive instruction, number telling activities, and play (please refer to Chapter 6 for a description of these parts of the curriculum). However, several new curricular activities were added to these sites, all of which were aimed at leveraging the score-counting activities at the end of games for mathematical learning outcomes. The pre and post spatial tests were implemented at this site. Please refer to Chapter 5 for an detailed description of the tests.

Data Collection

The data come from video recordings of the Go classes. During interactive instructions, I set up a camera facing the teacher and the front of the classroom during instructions and other teacher-led activities, to capture the interactions between the teacher and the students around the demo-board and on the computer. During play sessions, I moved about in the classroom and video recorded a game from multiple pairs/groups of students. I carefully attended to the moments when students counted their final scores at the end of games. To understand their strategies, I often asked them to illustrate how they counted their scores when they finished their score-counting. These short "interviews" constitute an important part of my data.

Data Analyses

Data Selection. Because this study aimed at understanding how students rearranging their stones for final score-counting and how those practices change over time are critical for this research, I carefully extracted the episodes where students counted their final scores at the end of games for in-depth analyses. These data mainly emerged from two parts of each class—first, during the interactive instructions, when the teacher intentionally asked the students to articulate how they counted the final scores and asked them to come to the demo-board to demonstrate; second, during the student play, when I observed the students' spontaneous counting activities and then asked them to elaborate on their counting strategies.

Coding. To understand what spatial and mathematical strategies are used and the sequence in which they develop, I analyzed the videos directly and coded for counting strategies as they emerged. I paid attention to subjects' gestures in addition to language because gestures may convey what features of the problem subjects attend to and reveal their spatial thinking. I also observed subjects' facial expressions, especially eye gaze, which indicate what features of the problem or representation the subjects focus on moments by moment. The moment-by-moment information about their focus of attention is crucial for understanding the score-counting process.

Through my initial coding, I developed two foci for my subsequent analyses. First, I focused on uncovering the primary ways in which the teacher facilitated the score-counting activities. Second, I focused on uncovering the strategies used by the students for score-counting.

I drew on the emergent goals framework to further analyze these two aspects of the score-counting activities.

I used the emergent goals framework to identify the forms, functions, and goals emerging from the final score counting activities. The framework provides three principal constructs for analyzing the development of novel mathematical understanding from cultural practices: (a) the analyses of *goals* that emerge from the situations, (b) the *forms*, which include new ways of representing mathematical information that yield new approaches to solving problems, (c) the *functions* such as counting and arithmetic which utilize the forms to serve the emergent goals.

Identifying the goals, forms, and functions involved in the score-counting activities has great benefits. First, the coding enabled me to better understand the ways in which the teacher facilitated the score-counting activities and helped the students connect their mathematical understanding with Go. For example, by analyzing how the goals emerged, I was able to find that the teacher not only promoted the students to find multiple strategies, but also encouraged the students to compare multiple strategies in pursuit of the most efficient. Second, by demonstrating the various strategies players used to count their scores, and by keeping track of how their strategies shifted as they played more through the course, I was able to understand what mathematical practices can be facilitated by the spatial forms embedded in the game of Go.

Results

Did Learning to Play Go Improve Young Children's Spatial Abilities?

There were a total of 36 students in the classrooms who participated in both the pre and post mental rotation tests, The pre-test was implemented during the second week while the post test was implemented during the last week. The paired t-test results from the pre-test ($M = 11.9$, SD = 2.3) and post-test (M = 10.9, SD = 2.8) indicate that there was no significant effect of playing Go on the mental rotation tasks, $t(34) = 1.98$, $p = .0559$,

Nevertheless, the use of spatial forms still might play an important role in how young children develop strategies for score-counting and thereby engaging in mathematics learning. The rest of my analyses focuses on identifying what spatial forms were used in the score-counting and what kinds of mathematical thinking might be facilitated through interacting with those spatial forms,

How did the Teacher Facilitate the Score-counting Activities and Foster Mathematical Outcomes?

The teacher played an important role supporting the students' score-counting activities and fostering mathematics learning. My analyses of the score-counting activities during interactive instructions revealed two major ways the teacher facilitated the learning of mathematics embedded in score-counting: 1) introducing and strategizing spatial numerical forms, and 2) diversifying and comparing strategies.

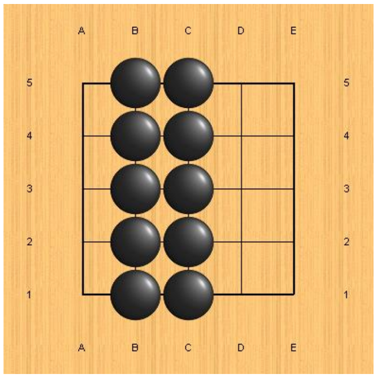
Introducing and Strategizing Spatial Numerical Forms. Building on Saxe's emergent goals framework, I identified multiple spatial forms introduced by the teacher that could serve as an important means to score-counting. I used the term *spatial numerical forms* to refer to those Go configurations which represent numbers as certain shapes and lends themselves to certain spatial arrangements to aid counting. These spatial forms are different from those discussed in Chapter 6, which concerns meaningful patterns and game strategies.

Next, I discuss the specific spatial numerical forms introduced by the teacher that learners could use to count their scores. I also discuss the mathematical understanding involved in learning to use those spatial forms. The spatial numerical forms include number shapes, a color-value system, and the matrix of the board. The uses of spatial numerical forms for score-counting indicate the potentials of playing Go in fostering spatial skills and mathematics learning.

The Number Shapes: Visual Representation of Numbers and the Base-Ten. At the end of each game, players fill all the empty spaces of their own territories until there are only two eyes open. The final score of one player would be the total number of stones placed by this player. At the end of each game, players are encouraged to rearrange all the stones into two basic shapes (Figure 7.3 and 7.4) that consist of ten stones to enable accurate and efficient score counting. Such basic shapes are called *number shapes* (term invented by Guo). These number shapes are adopted from the traditional Go counting practices. Professional players also make these shapes like these to aid fast and accurate counting. The ten-shape shown in Figure 7.3 is called the "rectangular ten" in professional practice, while the ten-shape shown in Figure 7.4 is called the "turtle-ten".

Figure 7.3

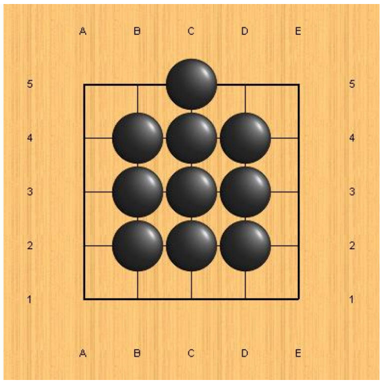
Rectangular Ten



Note. Adapted from "Cover 60% Math Contents in K-3 with Just One Game—Weiqi/Go", by X.Wu, & X. Guo, 2018, at conference workshop, Metropolitan Mathematics Club of Chicago, Lisle, IL.

Figure 7.4

Turtle Ten



Note. Adapted from “Cover 60% Math Contents in K-3 with Just One Game—Weiqi/Go”, by X.Wu, & X. Guo, 2018, at conference workshop, Metropolitan Mathematics Club of Chicago, Lisle, IL.

Figure 7.5

End-of-Game Counting: A Player Arranging Stones in Groups of 10

The use of these spatial forms to count in groups of 10s is consistent with the standard for Number and Operations in Base Ten (CCSS.MATH.CONTENT.2.NBT.B.5), which states that students should fluently add and subtract within 100 using strategies based on place value, properties of operations, and/or the relationship between addition and subtraction.

A Fluid Number Representation System: The Color Value System and Base-Ten. To keep track of student scores, a score keeping system was introduced at Week 5 to allow students to combine their scores from each play. At the end of each game, the students were asked to transfer their stones (after filling in until two eyes open) directly into their score boxes (Figure 7.6 A). However, due to this action, the players would soon realize that their stones for playing quickly run out as they would end up in the score boxes. This learning situation naturally lends itself to problem solving. After the problem emerged, the instructor introduced a color value system on Week 6 to keep track of scores (Figure 7.6 B).

Figure 7.6

Score Boxes and The Color Value System

A B

The blue marbles on top of the picture are used as place values for groups of tens. The other colored marbles represent values of 100s and 1000s. When students accumulated 10 or more stones as their scores, they may go to the teacher, who holds the "trading box" of blue marbles, to trade a group of ten for a blue marble. To trade, the students were asked to first arrange their score pieces into a rectangular ten, in exchange for a blue marble. After receiving the blue marble, they were asked to put the blue marble to the score box and return the ten game stones back to the original player box. Therefore, if a student had 3 blue marbles and 6 stones in their score box, we could easily tell that their score would be 36. As the score accumulates, stones with different colors were introduced through the trading activity: a red marble (100) to replace ten blues, a yellow marble (1000) to replace ten reds, etc.

The two features above, the number shapes and the color-value system, are both in alignment with the base-ten numerical structure. They match particularly well with the following list of common core standards that concerns the understanding of place values in base-ten and reasoning with shapes (partitioning a rectangle into rows and column) to support the understanding of base-ten (Table 7.2).

Table 7. 2

List of Grade 2 Common Core Standards Related to the Color-Value System

Grade	Domain	Cluster	Standard Code	Standard
2	NBT	Understand place value	2.NBT.1	Understand that the three digits of a three-digit number represent amounts of hundreds, tens, and ones; e.g., 706 equals 7 hundreds, 0 tens, and 6 ones. Understand the following as special cases: a. 100 can be thought of as a bundle of ten tens — called a "hundred." b. The numbers 100, 200, 300, 400, 500, 600, 700, 800, 900 refer to one, two, three, four, five, six, seven, eight, or nine hundreds (and 0 tens and 0 ones).
2	NBT	Understand place value	2.NBT.2	Count within 1000; skip-count by 5s, 10s, and 100s.
2	NBT	Use place value understanding and properties of operations to add and subtract	2.NBT.5	Fluently add and subtract within 100 using strategies based on place value, properties of operations, and/or the relationship between addition and subtraction.
2	NBT	Use place value understanding and properties of operations to add and subtract	2.NBT.6	Add up to four two-digit numbers using strategies based on place value and properties of operations.
2	G	Reason with shapes and their attributes	2.G.2	Partition a rectangle into rows and columns of same-size squares and count to find the total number of them.

Note. Adapted from "Cover 60% Math Contents in K-3 with Just One Game—Weiqi/Go", by X.Wu, & X. Guo, 2018, at conference workshop, Metropolitan Mathematics Club of Chicago, Lisle, IL. Math items in this table are based on *Common Core State Standards for Mathematics* by National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010, http://www.corestandards.org/Math/

The color-value system may be considered as a fluid number representation system that aligns the visuals with the representation of numbers according to the decimal (base-ten) system. I call it "fluid" for the following reasons: First, the stones a player successfully kept alive are directly transferred to the score box as the player's score -the basic level score. This transfer activity highlights a *continuity* from concrete objects (stones) to representation (scores). Second, the trading activity highlights the idea of *equivalence* through the exchange of stones and colored marbles. Thus, the concept of face values and the base-ten decimal system are naturally and gradually introduced, as a different color represents a decimal place.

The Matrix System and Multiplication. Most prior studies regarding representations and operations of numbers among young children are associated with the number line representation, which has a linear format. In contrast, Laski and Siegler's (2014) study of a non-linear 0–100 number board game showed that the numbers organized in columns on a 10x10 grid improves kindergartners' knowledge of numerical magnitudes, counting, and numeral identification. They suggested that 10x10 matrix might have helped children learn the base-ten structure of the number system.

The game of Go is an ideal case to study how 2D matrix as a tool can influence children's math learning. Go players engage in counting and operations of stones on a matrix set-up. While the traditional standard Go board is 19x19, in this classroom, players played on a 5x5 board (Week 1 and 2), 6x6 and 7x7 board (Week 3-9), or 9x9 board (Week 10), all of which have a matrix structure.

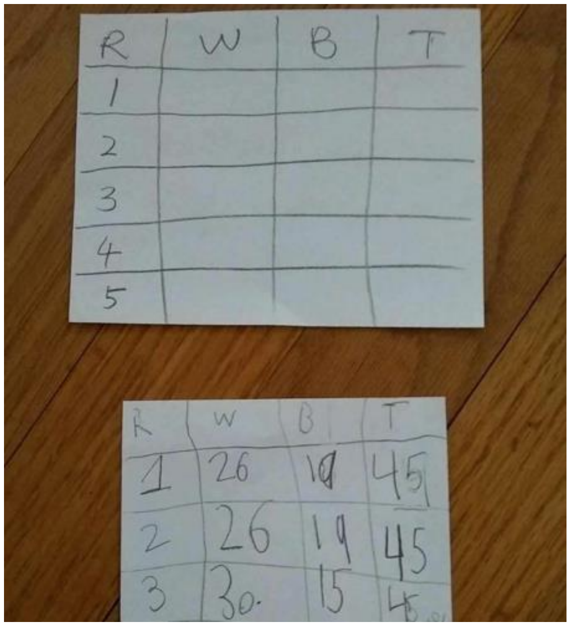
I consider the matrix structure as a type of spatial form, because this structure may afford a range of mathematical practices (Novick, 2006; Novick & Hmelo, 1994), especially in the context of score counting. To be specific, as the goal of the Go game is to maximize one' spaces on the board by constructing live groups of stones, the game would naturally progress toward the formation of several divided sections on a matrix. To determine the scores in each section of the matrix, players may find a set of rectangular shapes, such as the rectangular ten discussed in a previous section or stones arranged in an array, which they can easily apply multiplication; alternatively, the stone arrangement in that section may lend itself to some simple modification to become an array.

In addition, as the players become more familiar with the board, they may gradually become familiar with the total number of stones on a full board or part of the board and integrate these pieces of knowledge to develop new solutions to the score counting problem. Moreover, the teacher introduced the following activity to support the students' understanding of matrix.

As shown in Figure 7.7, a score record card was distributed to learners at each class to record their scores after each round of game. This score-recording activity was designed to support the understanding of matrix: by asking both players to combine their scores, the activity would eventually lead learners to discover that the total score should be consistently the same number and become familiar with the total numbers presented on the 5x5, 6x6, or the 7x7 boards. Thus, I suggest that such experience can be useful for multiplicative thinking.

Figure 7.7

A Score Record Card for Tracking and Exploring the Matrix of The Go Board



Note. R stands for round; w stands for white; B stands for black; T stands for total (black and white scores combined. As showed on the card on the bottom, the total score was consistently 45 after 3 rounds. The students played on a 7x7 board, which has 49 inter-sections. Because each player has to leave two eyes in their territories, the total score was thus 49-2-2=45.

In summary, the number-shapes and the matrix setup of the Go board can all be viewed as *spatial numerical forms,* which are the ways of representing mathematical information that could yield different approaches to solving problems. As young children interact with these forms to count their final scores, they will experience numbers represented in different formats, including those represented according to the base-ten system, those represented by empty spaces as opposed to concrete entities (stones), those represented in rolls, arrays, and matrixes.

In my subsequent analyses, I will demonstrate the various ways players used spatial numerical forms to assist their score-counting, keep track of how their strategies shifted as they became familiar with those forms, and discuss what these shifts suggest about their mathematics understanding.

Diversifying and Comparing Strategies. In addition to introducing spatial numerical forms and demonstrating score-counting strategies with these spatial numerical forms, the teacher also promoted the students to diversify score-counting strategies and to compare those strategies in pursuit of efficiency.

Since the beginning of the curriculum, the teacher often engaged the students in number-telling activities—he often asked students to look at one setup board for one second and tell him the number of stones shown. These number-telling activities might promote the students to align numbers with shapes on a Go board and were intentionally designed to lay foundations for score-counting. Moreover, the teacher also followed up with questions such as "how do you know?" Then, the students articulated multiple strategies they used to find out the number.

These practices were repeated in actual score-counting. After each demo games, the teacher asked the students what the final score for black and white were on the demo board. In these score-counting episodes, I always found the teacher always encouraged students to find multiple strategies by asking "What is the score for black?" "How do you know?" "What is another way to count?" "Can you show me another way", etc. Moreover, as more counting strategies emerged, the teacher would ask the students to compare multiple strategies and consider "which way is better?"

To articulate their counting strategies, students needed to dissemble the stones into multiple smaller groups and then add them together. Such practices open up a new way to think about addition—as assembling and rearranging stones into basic number shapes. Excerpt 7 shows that the teacher created various opportunities in class for counting by visualizing numbers in groups and shapes. After the students articulated their own strategies, the teacher demonstrated a strategy that involved rearranging stones into groups of tens on the board.

Excerpt 7.

Teacher: “What is the score for black?”

Student: “15!”

Teacher: “How do you know?”

Student: “Because there are 5 here, 5 here and 5 here”

Teacher: “Great! any other ways to count?”

Student: “Because this is 8 and this is 7, 8+7 is 15”

Teacher: “Good. Let me show you another way: (moving the stones into a rectangular-ten and a string of 5 stones)”

Teacher: “What is the score here? (using a piece of paper to cover the rest of the board and only showing the rectangular ten)”

Students: “Ten!”

Teacher: “And how many are here (pointing to the group of 5)”

Students: “Five!”

Teacher: “So the final score is ten and five, fifteen.”

Teacher: “There are different ways to count. Which way do you think is better?”

Students: “The last one.”

Teacher: “Right. When we count the final score, we should think about efficiency. When we can reorganize the board into these rectangular shapes, we can count the score very fast; it’s accurate and efficient.”

Excerpt 7 illustrates that at least three strategies were demonstrated and discussed for counting the final score at the end of a game. These score-counting instructions were always present in each week's Go lessons. As the student become more familiar with spatial numerical forms such as the "rectangular-tens", the teacher frequently guided the students to compare different strategies in pursuit of the more efficient.

These teacher-facilitated score-counting activities show that the teacher played an important role guiding students to not only find multiple strategies but also evaluate strategies in terms of efficiency. In light of the emergent-goals framework, I suggest that these discussions around multiple strategies and efficiency might influence the formation of goals and thereby cause shifts in the students' choices counting strategies.

So far, I have examined the spatial forms the teacher presented in the curriculum and discussed their potential relevance to mathematical outcomes which are illustrated in terms of the common core math standards. For example, the number-shapes and the color-value system are associated with the understanding of place values; the rearrangement activities are related to operations and algebra thinking; the matrix may contribute to the understanding of multiplications. Moreover, I have also examined the ways in which the teacher facilitated score-counting activities—by diversifying counting strategies and forming the goal of choosing efficient strategies.

In the next section, I will examine the counting strategies actually being adopted by the students in their score-counting. By studying how their counting strategies evolve, we will understand whether and how the spatial forms and goals may potentially influence the students' counting strategies and contribute to math learning.

The Advancement of Students' Counting Strategies

Identifying Students' Counting Strategies. I first coded for various counting strategies that emerged in the counting of stones at the end of games. These strategies include skip counting, grouping by number-shapes, and multiplying by making arrays, etc. I found that the students incorporated the spatial numerical forms discussed in the previous section, such as number shapes (e.g., rectangular-ten and turtle-ten) and the matrix structure of the board, etc., into their counting strategies. I used the emergent goals framework to categorize counting-strategies. For each strategy, I defined the spatial form being used, its function, and goals it served. I found four types of counting strategies in total (see Table 7.3). The four types of counting strategies occurred in different times over the course of the curriculum. Next, I give a detailed account of each strategy and when they emerged.

Strategy 1. During the first two weeks, the students mostly used their fingers to count one by one or two-by-two on the board. I coded such strategy as Strategy 1. I did not observe any other counting strategies during the first two weeks. The students only played on a 5x5 board during the first two weeks, so this strategy did not seem tedious. However, since some students did not follow specific orders when counting on the board, they sometimes made mistakes or needed to recount to ensure accuracy.

Strategy 2. Beginning on Week 3, the teacher introduced two basic base-ten shapes, including the rectangular ten (2x5) and turtle ten, and demonstrated how these number shapes could be used for score-counting. The students then adopted these number shapes as one of their primary counting strategies.

Specifically, I coded their strategy as Strategy 2 when the students took their stones off the board and made tens from scratch. This is to be distinguished from Strategy 3, in which the students rearranged stones into arrays (including 2x5) without taking the stones off the board. This strategy is associated with the understanding of the base-ten system.

Table 7. 3

Score-counting Strategies Defined by Spatial Forms, Functions, and Emergent Goals

Spatial Forms	**Functions**	**Emergent Goals**
Single stone or pairs of stones	Count by one or two	Find the score by counting the number of actual stones.
Number shapes (e.g. rectangular-ten 2x5)	Count by a visual base-ten system	Make number shapes to find score and reach agreement between players using a shared norm.
Arrays	Skip counting and multiplication	Faster counting; find how a 6x6 board differs from a 5x5 board.
Board Matrix	Multiplication	Find the score without filling in all surrounded spaces with stones (when stones run out); rearrange one's spaces into a matrix on the board.

Strategy 3. This strategy involves relocating a couple of stones to join or dissect from a larger group of stones on the board to make rectangular tens (2x5) or other rectangular arrays on the board (e.g. 3x5, 4x4) and then skip counting each column or just reading it out. Strategy 3 often emerged in situations where the learners noticed that the existing shape of stones on the board could be rearranged into one or a few rectangular shapes plus or minus a few extra stones. Consequently, they rearranged the stones and then counted the rectangular shapes as a whole or

skip counted each column. This is evidence that they learned to utilize the structures that already existed on the board and made use of the matrix.

Strategy 4. This strategy involves deriving the final score based on the size of the board. The total number of intersections on a board is fixed: 25 on a 5x5 board, 36 on a 6x6 board, etc. If a player won the whole 6x6 board, their final score final score would be 36 - 2 = 34 (two eyes subtracted). If a player had the score of 16 on a 6x6 board, the other player's score would be 36-16-2-2=16 (each player has to leave two eyes unoccupied). Therefore, like Strategy 3, this strategy also involves flexibly utilizing existing knowledge about the matrix to solve problems.

In my earlier analyses on the role of the teacher, I identified the spatial forms the teacher introduced in the curriculum, such as the number shapes and the matrix, as means to facilitate score-counting. In this section, I found that the students were able to adapt these spatial forms as their own tools for score counting. Because the students actively interacted with the spatial forms for score counting, they might improve their mathematics understanding that were associated with the spatial forms.

Moreover, in the previous section, I also showed that the teacher guided the students to explore multiple strategies and consider efficiency as they compared different strategies. In light of the emergent goals framework, I suggested that the pursuit of efficiency may be considered as a goal which could potentially motivate students to adopt more efficient strategies, which were associated with more advanced mathematics knowledge including multiplication. In the next section, I examined whether there were advancements in the students' counting strategies, and if so, whether the advancements were related to the pursuit of efficiency or other emergent goals.

Identifying Shifts in Goals and Counting Strategies. I found that the students generally advanced from Strategy 1 (counting by 1s and 2s) to Strategy 2 (base-ten number shapes) and

Strategy 3 (counting by arrays) during Week 3 through Week 6; Strategy 4 (using matrix knowledge) began to emerge later in the course as the students became more familiar with the sizes of boards. To understand what may influence and advance score-counting strategies, I identified the emergent goals in each score-counting episode by attending to what tasks were being promoted at the time (e.g. trading for score marbles, filling the score record card, etc.), the students' prior exposure to certain number-shapes, and situational constrains such as sizes of the board, time limit, etc. I found that the students' counting strategies are related to the goals in the moment and can advance as different goals emerge.

For example, between Week 3 and Week 5, Strategy 2 and Strategy 3 appeared equally frequently, each took up 40% of total score-counting instances I recorded and coded. However, Strategy 2 became the dominant strategy (80% of instances) on Week 6 and afterward, when the color value system was introduced—the students were asked to arrange their stones into rectangular tens in exchange for a blue marble which represented score of 10. Therefore, the introduction of the color-value-system might have encouraged the students to take their stones off the board to make rectangular-tens, causing the shift to Strategy 2.

While Strategy 2 was related to the understanding of place values, Strategy 3 and 4 involved utilizing the matrix of the board and therefore were associated with the understanding of multiplication. Although I found a decrease in the use of Strategy 3 in the latter half of the semester, there were profound evidence that the students developed understanding of multiplication through the use of Strategy 4. Next, I examine how Strategy 4 emerged and identify the goals which might have contributed to the strategy and the learning of multiplication.

Multiplicative Thinking Supported by the Matrix. I found that Strategy 4 initially emerged a situation where a player captured all stones of the opponent (which occurred frequently on small boards for novices) but did not have enough stones to fill the whole board or chose not to. Some players were able to take advantage of their knowledge of the board, such as knowing how many stones there should be in one line of the board, and how many stones there should be when the whole board is filled. They used such knowledge adaptively to find final scores, without putting down and counting physical stones.

Next, I present Excerpt 8 which shows the first instance of Strategy 4 I observed from a student's end-of-a-game score-counting. This event was recorded in Week 6.

Excerpt 8.

> Student: Look! I took over the whole board!
>
> Teacher: Nice job! So what's your final score?
>
> Student: The entire board is 25. I need to make two eyes. So I subtract 2 from the total. So my final score is 23.

In the earlier interactive instructions, this individual came up to the demo board to answer Mr. Guo's question that was related to the total number of stones on the demo board, which was a 5x5 board. In the process, the individual learned that the total number should be 25. However, in Excerpt 8, the board actually used was a 6x6 board. Therefore, the final score should be 6x6-2 which is 34. However, this is a productive mistake that clearly demonstrated that the individual adaptively made use of what he just learned to solve problems in this new situation.

This counting strategy was not only used in a situation where the player won the entire board, but also in a situation in which a player needed to find their own score efficiently without filling in their territories with stones and counting. In Week 8, I recorded a pair of students who just finished a round of game on a 7x7 board and were counting their final score to fill in their score-record card. At that moment, the class was about to end, so the teacher asked them to finish their game and put their stones away. Consequently, only one of the players was able to finish counting his own final score. Nevertheless, the other player was able to derive his own score by subtracting his opponent's score from the total score, which he learned to be 49-2-2=45.

From the two examples above, we can see that that Strategy 4 naturally emerged from the students' score counting activities. Several factors contributed to the strategy. First, this strategy emerged as the students adaptively utilized what was available to them at the moment (e.g., the knowledge of the total score on a board, the opponent's score) as tools for score counting. The strategy was also motivated by the pursuit of efficiency or the demand to work with constraints; it was tedious to fill the entire board in the first example, and the player ran out of time and stones in the second example.

Therefore, this analysis show that the students were able to adaptively utilize the spatial form introduced by the teacher, response to emergent goals, including dealing with physical constraints, and be innovative in their counting strategies. Thus, the score-counting activities may be considered as an dynamic and adaptive learning environment, where the players are active problem-solving agents, who are provided with a set of spatial forms as tools. They may develop new ways to use these tools to solve problems as they emerge. In the process, they may gain new knowledge about mathematics as they learn to use the spatial forms in different ways to meet their problem-solving goals.

To further illustrate this point, I demonstrate another example showing how the students were able to extend what they learned about arrays and matrix to estimate the size of a Go board which they never played with. The following Excerpt 9 illustrates an interaction between the instructor and the students around different sizes of Go boards. This interaction occurred in

Week 9, when the students had become familiar with 5x5, 6x6, and 7x7 boards, through playing, score-counting, and the score tracking activity.

Excerpt 9.

Teacher: How many stones can you place on this board (5x5)?

Student A: 25!

Teacher: What about this board (7x7)?

Student B: 49!

Student C: 30! …

Teacher: If the entire board is yours now, and your opponent have no space to survive, what is your final score? Remember, you have to make two eyes.

Student B: 47!

Teacher: 47? How did you know?

Student B: Because there are 49 points on this board, and you have to have two eyes. 49-2 is 47.

Then, the teacher put out a large board they never played on, and asked: "What is the maximum score possible for this 11x11 board? Can you guess?"

Student D: I guess it is more than 100.

Teacher: It is more than 100. Do you agree?

Student E: Yes!

Teacher: How do you know?

Student E: "If you put 10 stones on each line, and there are 11 lines, so the total score should be more than 110."

Teacher: "Yes! Very good! This is a great way to estimate. Is there another way?"

> Student F: "I did skip-counting by 11, so it's 11, 22, 33… to 99, and then you have to add more…"

From Excerpt 9 above, we can see that through these score counting activities on several boards, students had gained knowledge about the 5x5, 6x6, and 7x7 matrix. They also could utilize the ways of thinking about matrix as a tool to estimate a large number. For example, Student E was able to combine what they learned about calculations with base ten and the strategy to arrange stones into arrays to solving this problem. Similarly, Student F was able to use skip counting and addition of arrays of equal numbers to make the estimate. These strategies indicate that they have adapted the matrix as a mathematics tool and gained foundations for multiplication.

Discussion

Prior studies have shown that physical materials that are closely aligned with the desired knowledge structures increase analogical transfer and therefore promotes deeper learning (Chen, 1996; DeLoache, Kolstad, & Anderson, 1991; Gentner & Markman, 1997). Based on my analysis of the score counting activities, I found that learning Go can potentially facilitate young children to understand place values, solve arithmetic problems that could be difficult for them otherwise, or to solve them in more efficient ways, because this curriculum aligned numerical concepts and operations with basic shapes, colors, and matrix. The spatial forms such us the rectangular tens and the matrix provided the spatial grounding for the base-ten system and multiplicative thinking.

In addition to the spatial numerical alignment supported by the basic shapes, colors values and 2D matrix boards, the counting activities also created opportunities for adaptive learning problem solving. My analyses of the emergent goals in score-counting activities show that the students were able to adaptively utilize the spatial form to solve emergent problems such as dealing with physical constraints and time limit. They may gain new knowledge about mathematics along the way, as they learn to use the spatial forms in different ways to meet their problem-solving goals.

In conclusion, the students gradually adapted the spatial forms in Go as tools to serve the increasingly complex goals. Through this process, the Grade 2 students began to participate in higher-level math practices like multiplication. As children become familiar with the tools, they can adaptively shift the functionality of these tools to meet increasingly complex goals in the game play, thereby practicing higher-level math functions like multiplication.

Chapter 8: Conclusions

In this dissertation, I demonstrated and detailed the relation between spatial thinking and math learning in the context of an ancient and rich game of Go. By examining how young children engage in spatial thinking and math learning through Go, this study advanced our understanding of the accounts of how spatial thinking might support math learning (Hawes & Ansari, 2020) in light of the sociocultural perspectives of learning as mediated by cultural tools (Cole, 1996; Rogoff, 2003; Vygotsky, 1987; Wertsch, 1985).

My approach to studying Go is largely motivated by examining its core features. I identified two primary features of Go which potentially make it a tool for math learning. First, the game of Go has minimum rules to remember and basic stone pieces which have no restrictions on where to play and all perform in the same way. Thus, the game of Go provides a low entry point for beginners. At the same time, the minimum rules and free stone pieces give rise to millions of meaningful patterns which requires reasoning to learn. Therefore, the first feature of Go highlights the role of reasoning in learning and applying Go patterns. Second, score-counting play a critical role in the game of Go. Unlike most board games that have a clear-cut ending when certain action is completed, the game of Go relies on consistently estimating and counting territories to determine when to end a game and adjust strategies during a game. As a result, players need to engage in spatial thinking and utilize certain spatial arrangements on the board to calculate their scores. The skills required by score-counting potentially overlaps with a significant amount of mathematics understanding that young children develop at Grade 2 and 3.

Based on these two core features of Go that I identified, I developed two studies to explore the game of Go as a tool for math learning. In study 1, I examined the reasoning practices Grade 3 students participated in as they learned to explore Go patterns. In study 2, I explored how Grade 2 students developed score counting skills and investigated whether and how these score counting practices might support the learning of mathematics, including place values, operations and algebraic thinking, and multiplicative thinking.

In my first study, I mainly investigated the math learning in the game Go through the lens of participation. I identified the ways in which the teacher engaged students in reasoning practices, and the ways in which the students participated in the reasoning practices during interactions with the teacher and among themselves. I found that the students were able to engage in reasoning practices that frequently occur in learning and doing mathematics, which include conjecturing, justifying, and generalizing (Ellis, 2007; Hanna, 2000; Lannin, 2005; G. J. Stylianides & Stylianides, 2009).

In addition, my first study revealed important ways in which engaging students in spatial reasoning, including pattern observations, and classifying activities, contributed to the reasoning process. I found that the teacher used spatial transformation strategies—presenting multiple patterns in dynamic and fluid ways—to support pattern observation, classifying, and other reasoning processes involved in learning Go patterns. The spatial transformations may foster learning because the fluid transitions between shapes might not only highlight shape similarities, but also foster a sense of connectivity in the reasoning processes underlying similar shapes, so that the students might carry the same line of reasoning they developed in learning a basic shape to increasingly complex patterns. In other words, my analyses indicated that the dynamic and fluid use of similar shapes might support young children to reason about Go patterns.

Thus, my study contributed to the line of research on supporting mathematical reasoning practices (Ellis, 2007; Hanna, 2000; Lannin, 2005; G. J. Stylianides & Stylianides, 2009), by proposing a set of spatial tools which can make reasoning processes easier and more accessible for young children and thereby encourage them to participate in mathematical reasoning practices. Therefore, this study may also provide an alternative approach to understanding why and how developing spatial skills might help young children learn mathematics (Levine et al., 2012; Jirout & Newcombe, 2015; Ramani & Siegler, 2008; Siegler & Ramani, 2009). Specifically, my study contributed to the spatial modeling account of math learning (Hawes & Ansari, 2020), by detailing specific ways of modeling mathematical reasoning spatially and by highlighting “dynamic and fluid” shape transformations as a critical feature of spatial modeling that might facilitate mathematical reasoning.

In my second study, I explored how Grade 2 students developed score counting skills and investigated whether and how these score counting practices might support the learning of mathematics. I approached this question in light of studies on cultural tools that influence numerical representations and arithmetic practices and thereby influence how individual’s think mathematically. I drew on the emergent goals framework to identify spatial forms which were used for numerical representations and operations during score-counting activities. I found that the use of spatial forms, including base-ten number shapes, color values, and the matrix of the board, aligned well with the types of mathematics understanding and skills illustrated by Grade 2 common core standards. In other words, the spatial forms served as a means to connect multiple formal mathematics practices and authentic problem-solving in a game context. Moreover, I found that the score counting activities formed a learning environment where students could turn many different mathematical ideas and practices (e.g., place values, operations and algebraic thinking) they were learning into problem-solving strategies and apply them in the same game context. By adopting these spatial forms as tools for score-counting, the Grade 2 students could even learn new mathematics that they had never experienced with before, such as multiplication.

This second study showcased how a variety of spatial numerical representations, which yield different strategies to solve math problems, could be flexibly created with the same physical materials in the same game context. Because of this, the spatial numerical representations in Go are not conceived as means to facilitate one particular type of math strategy or concept, but rather conceived as particles on a platform, where different ways of representing and organizing numbers are connected and interchangeable. Players can be conceived as active agents in this platform, who are responsible for their own learning as they search for the best strategies to solve emerging problems. Therefore, my study builds on the spatial numerical account of math learning (Hawes & Ansari, 2020), by offering an alternative perspective on why spatial numerical representations might support math learning.

In addition, my study contributes to literature that explores the potentials of board games and other cultural activities in supporting the learning of mathematics. In particular, my study contributed to studies on the spatial alignment between physical materteral and mathematic understanding (Chen, 1996; DeLoache, Kolstad, & Anderson, 1991; Gentner & Markman, 1997). Building on Laski and Siegler's (2014) study of a number board game presented on a 10x10 grid, my study highlighted the 2D matrix (Novick, 2006; Novick & Hmelo, 1994) as a potential tool for young children to gain foundations for multiplication.

Moreover, my analyses of emergent goals (Saxe, 2008) revealed that score-counting activities fostered an adaptive learning environment, where the players were active problem solvers, who creatively used spatial forms to address emergent problems; they might learn mathematics as they adaptive new ways to use spatial forms. For example, the students developed multiplicative thinking as they adopted the knowledge of the size of a familiar board and strategies of counting by arrays to estimate the size of a much larger board which they had never played with before. Therefore, my study showed that the students were not only able to complete score-counting tasks by following the teacher's instructions, but also "internalize" some of the spatial forms and use them creatively. This finding thus resonates with prior studies on mathematic learning with board games (Guberman & Saxe, 2000; Nasir, 2005; Saxe, 1992), which demonstrated how players could engage in sophisticated forms of addition, subtraction, and multiplication, through repurpose or appropriate existing cultural forms in order to solve emerge problems in context.

Furthermore, my study highlighted multiple ways in which the teacher facilitated the reasoning with patterns and the score-counting processes. In study 1, I showed that the teacher used spatial transformations fluidly to support the students' reasoning processes. In study 2, I showed that the teacher not only introduced a series of useful spatial forms that align with core mathematical concepts in Grade 2, but also fostered learning community which encouraged multiple strategies and promoted discussion around the efficiency of different strategies. These insights about the teacher's roles could potentially generate implications for the design of curricula and pedagogical techniques.

In conclusion, I examined Go as a cultural tool and explore its potential to influence the way Grade 2-3 students learn and do mathematics. I studied video data of interactive Go instructions, game play, and end-of-game score counting across three sites where a Go and Math curriculum was implemented. I identified various counting strategies used at the end of games, such as skip counting, grouping, and multiplying, and kept track of how individuals' counting strategies advance over time. I also identified mathematical reasoning practices that emerged in the learning process, which include conjecturing, justifying, and generalizing. Several key features of the Go and math curriculum have potential to facilitate spatial thinking and math learning.

I suggest that Go introduces a representation system in which numbers are presented as visual patterns and shapes constructed by stones and lines of the board. Such a representation system may promote a spatially based way of thinking about and doing mathematics: children can count by recognizing shapes, and do arithmetic by moving, rearranging, and combining different pieces together, both physically and mentally. In addition, the process of learning Go may enable young children to participate in mathematical reasoning practices such as conjecturing, justifying, and generalizing, which might be difficult to partake in otherwise at Grade 2-3 level.

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Appendices

Appendix I

Course Title: The Game of Go and Math (Gr. 2-3)

Course Description:
The game of Go originated in China about 3000 years ago. It is the most ancient strategic board game that is still played. Today, there are more than 40 million Go players worldwide. Go cultivates a wide range of cognitive abilities, including spatial reasoning, problem-solving, decision-making, etc. The game also fosters concentration, discipline, and endurance. Moreover, research suggests promising links between Go and math achievement. Our interdisciplinary curriculum aligns Go with the Common Core standards, including counting and cardinality, number and operations, arithmetic and algebraic thinking. The game-based curriculum makes advanced math concepts accessible for Grade 2 and 3 students. Students can learn math without even noticing, while enjoying the rewarding experience Go offers.

Essential Questions:

- How is Go played?
- How does playing Go develop spatial thinking skills?
- How does playing GO improve mathematics learning and achievement?

Course Learning Outcomes:
Upon successful completion of this course, students will:

a. Improving spatial thinking, such as recognizing patterns and distributions. Developing understanding of number and mathematics, including multiplication and division.
b. Learning how to label patterns. .
c. Solve arithmetic problems including addition, subtraction, multiplication, and division using the Go board and stones.
d. Develop other useful skills and intuitions in preparation for future math learning.

Resources and Materials:

a. Go game sets for the classroom
b. Classroom resource kit
    - 7x7 boards and 9x9 boards printed on paper
    - Black and white Go stone to mark positions on the GO boards
    - Markers, stickers and notebooks for portfolio
    -

Schedule: (Subject to change)

Session	**Today's Content**	**Student Learning Objectives**	**Product Choices**
Week 1	**Know the Go board I:** Coordinates, size, intersection **Basic concepts I:** The concept oft: *"liberty"* -- the number of unoccupied intersection points that are connected with stones. Moves: *capture, connect,* and *cut* **Embodied activity:** students play stone characters to act out the basic concepts.	1. Pre-Assessment 2. Become familiar with the Go board and stones by creating personally meaningful geometric shapes 3. Practice thinking about analogies and geometric design 4. Explore multiple ways of counting the *"liberty"* of stones 5. Apply the basic moves – *connect, cut, and capture* –in a Go game 6. Play the first Go game	A group skit that demonstrates the basic Go moves--*capture, cut and connect*–through body movements. Portfolio: 1. Designs of geometric shapes 2. Record the patterns of basic moves
Week 2	**Know the Go board II**: Corners, edges, the center **Basic concepts II:** Change of liberty, territory, risks and uncertainty. **Strategies:** Counting territories during and after a game Recognizing and responding to risks of being captured	1. Explore ways to count the territory each player owns 2. Learn ways to organize patterns to facilitate counting, addition, multiplication, etc.Recognize simple risks 3. Be able to escape from risks of being captured 4. Master the strategy of "cut" in attacks and defenses 5. Complete multiple games	Rearrangement activity I: Invent a way to rearrange the stones of a finished game, so that it's easy to count territories A group skit that demonstrates risks and escapes from capture
Week 3	**Basic concepts III:** "*Eternal life*" – cannot be captured by the opponent Patterns: "diagonal" "fly" "jump" "Atari "etc. **Strategies:** Guidelines and techniques for *capturing* **Number shapes I:** explore math representations on the Go board	1. Develop a sense for who is winning during a game by mental counting 2. Apply the capture guidelines and techniques 3. Flexibly apply the basic patterns as strategies in a game 4. Recognize the patterns made by the opponent 5. Be able to explain what makes a group of stones "alive"	Portfolio: Record the patterns learned today. Create a group skit demonstrating the process leading to "eternal life" through body movements

Session	Today's Content	Student Learning Objectives	Product Choices
Week 4	**Basic concepts IV:** The "*eyes*" – structure leading to "eternal life" **Strategies:** Finding the "key" stones How to be efficient on moves **Number shapes II:** explore math representations on the Go board	1. Recognize the "key" stones of a game by considering relative position of stones on the board 2. Make simple "alive" groups in a real game by making the "eye" pattern 3. Be able to capture stones in groups. 4. Improve techniques by considering efficiency of moves	Rearrangement activity II: Rearrange the stones of your own finished game to make it easy for counting. Portfolio: construct symmetric, rotated, and mirror images of learned patterns on the go board
Week 5	**Patterns and Strategies:** "*Gates*" and "*Ladder*" The "Ko" rule – the concept, procedure and application **Number shapes III:** Arithmetic problem solving on the Go board	1. Apply new patterns - "*gates*" and "*ladders*" - to capture, escape, and avoid risks 2. Follow the "Ko" rule in real games 3. Discovering the connections between additions and multiplications on the Go board	Portfolio: Record the strategies and patterns learned today. Create a group skit demonstrating the "Ko" rule in action.
Week 6	**More on Strategies:** Variations of the "eye" pattern Explore Symmetry in Go The common practices of Go players: record, replay, and reflect on a game Tournament Round 1	1. Produce "eye" patterns in different sizes and orientations 2. Determine whether an "eye" pattern is "alive" 3. Recognize and make symmetric patterns on the go board 4. Reflect on the process of a game – evaluate the good and bad moves, the efficiency, and consider alternative moves. 5. Become familiar with the game record sheet.	Portfolio: Construct the various learned patterns in different sizes and orientations. Recreate a game on the Go board according to a game record sheet

Session	Today's Content	Student Learning Objectives	Product Choices
Week 7	Introduction to some of the vocabulary of professional GO players Go: moments, stages, short/long term, etc. Review on game strategies Invent an abacus on the Go board Tournament Round 2	1. Develop deeper understanding of Go through review and reflection 2. Improve on Go techniques through tournaments 3. Advance arithmetic thinking by making numeric representations and operations on the Go board 4. Be able to record a game on the record sheet.	Portfolio: Record another pairs' tournament on a game record sheet. Record results of your tournament Fun activity: Using the Go board and stones to invent an abacus system in pairs.
Week 8	Try out on the whole Go board (19x19) Review on game strategies Tournament Round 3 Explore beyond the Go game	1. Understand how the whole board differs from the smaller boards 2. Complete the post-assessment 3. Cultivate creativity by designing a new game	Portfolio: Record results of your tournaments Fun activity: Using the Go board and stones to invent a new game

Appendix ii: The Ways of Counting Final Scores

In traditional Go practices, at the end of each Go game, the players rearrange stones on the board to enable score counting. This activity creates various opportunities for players to exercise math and spatial skills. First I will illustrate how professional players rearrange stones on a professional 19x19 board. Next, I will illustrate the adapted counting system used in this Go and Math curriculum, on smaller boards such as 5x5, 6x6, and 7x7 boards.

In traditional Go practices, there are usually two counting systems – the Chinese counting system and the Korean/Japanese counting system. In this paper, I will only introduce the Korean/Japanese counting system: At the end of a game, players will first remove all dead stones from the board. Next, players will put back the captured stones onto the board by filling in the empty spaces within their own territories. Usually, players would start to fill in those smaller territories and then move on to fill the larger territories. At the same times, in doing also, they would fill around the boarders of the territories so that the empty spaces within territories would appear to be rectangular shapes. Since both players put back their captured stones, there will be equal numbers of black and white stones on the board. Therefore, players can determine who wins a game by comparing the empty spaces within each other's territories. The player with more empty points within one's territory wins the game. It is more accurate and efficient to rearrange the stones such that the empty spaces within territories become rectangular shapes – to apply multiplication rules. Moreover, it would be even better if players arrange the empty spaces into groups of tens, 20s, 30s, and so on.

While this way of counting is efficient, the rearranging might be too demanding for elementary school students. In addition, counting the empty spaces of their territories might not be the most intuitive for the students. Therefore, the counting method is adjusted in the Go and Math curriculum. The students were asked to fill their territories until they only have two eyes reflect. Then they would proceed to the counting. The total number of stones they put on the board would be their final score. This approach is simple and straight forward, because it does not require putting back dead stones onto the board.